Supplemental Material for "Approximating many-body quantum states with quantum circuits and measurements"

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Here we provide additional details about the results stated in the main text.

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Appendix A: Preparation of the Dicke states

In this section we provide additional details on the preparation of the Dicke states. Denoting by N and M the number of qubits and excitations, respectively, and by $|W_N(M)\rangle$ the Dicke state, we provide a full proof for the following statement:

Proposition 1. *For any $\varepsilon > 0$, there exists a (non-deterministic) protocol which prepares a state $|\Psi_N\rangle$ with*

$$|1 - |\langle W_N(M)|\Psi_N\rangle|^2| \leq \varepsilon\,. \tag{SA.1}$$

The protocol is successful with probability

$$P_{\rm succ} \geq \frac{1}{\sqrt{8\pi M}}\,, \tag{SA.2}$$

it uses $N_a = 1$ ancilla per site, $D = O(\ell)$ and ℓ additional ancillas where

$$\ell = \max\left\{\ln(4M)/\ln 2, 1 + \frac{\ln\ln(\sqrt{8\pi M}/\varepsilon)}{\ln 2}\right\}\,, \tag{SA.3}$$

independent of N.

Note that, because the probability of success is $O(1/\sqrt{M})$, the protocol needs to be repeated, on average, $O(\sqrt{M})$ times, which is the result announced in the main text.

Proof. As in the main text, we set $p = M/N$ and start with the initial state (ommitting the dependence on N)

$$|\Psi(p)\rangle = (\sqrt{1-p}\,|0\rangle + \sqrt{p}\,|1\rangle)^{\otimes N} = \sum_{e=0}^{N}\left[\binom{N}{e}p^e(1-p)^{N-e}\right]^{1/2}|W(e)\rangle\,. \tag{SA.4}$$

Choose ℓ as in (SA.3), and define $\Pi_j^\ell = \sum_{i\in\mathcal{T}_j^\ell}\Pi_i$, where Π_i is a projector onto the eigenspace with i excitations, while

$$\mathcal{T}_j^\ell = \{i : i \equiv j \pmod{2^\ell}\}. \tag{SA.5}$$

We perform a measurement with respect to the projectors $\{\Pi_j^\ell\}$ and repeat the precedure until we obtain the outcome M. In case of success, the state reads

$$|\Psi^{(\ell)}\rangle = \frac{1}{Z_\ell} \sum_{e\in\mathcal{T}_M^\ell} \left[\binom{N}{e} p^e (1-p)^{N-e}\right]^{1/2} |W(e)\rangle \,, \tag{SA.6}$$

where Z_ℓ is a normalization factor. According to Result 2, this measurement can be performed using a circuit with $D = O(\ell)$, $N_a = 1$, and ℓ additional ancillas.

We need to estimate the success probability and the distance between $|\Psi^{(\ell)}\rangle$ and $|W_N(M)\rangle$. The former is

$$\begin{aligned} P_{\rm succ} = Z_\ell^2 &= \sum_{e\in\mathcal{T}_M^\ell} \left[\binom{N}{e} p^e (1-p)^{N-e}\right] \\ &\geq \binom{N}{M} p^M (1-p)^{N-M} \geq \frac{1}{2}\left(\frac{N}{2\pi M(N-M)}\right)^{1/2} \geq \frac{1}{\sqrt{8\pi M}}\,, \end{aligned} \tag{SA.7}$$

where we used

$$\frac{1}{2}\left(\frac{N}{2\pi M(N-M)}\right)^{1/2} < \binom{N}{M} p^M (1-p)^{N-M} < 2\left(\frac{N}{2\pi M(N-M)}\right)^{1/2}\,, \tag{SA.8}$$

which holds for $0 < M < N$ and can be proved using known inequalities for the factorial [1].

For the overlap, we write

$$\sum_{e\in\mathcal{T}_M^\ell} \binom{N}{e} (1-p)^{N-e} p^e \leq \binom{N}{M} p^M (1-p)^{N-M} + \Pr[e \leq M - 2^\ell] + \Pr[e \geq M + 2^\ell]\,. \tag{SA.9}$$

Since $2^\ell > M$, we have $\Pr[e \leq M - 2^\ell] = 0$. Let us analyze $\Pr[e \geq M + 2^\ell]$. The Chernoff inequality gives [2]

$$\Pr[e \geq M + 2^\ell] \leq \exp\left[-ND\left(\frac{M+2^\ell}{N}\middle\|\frac{M}{N}\right)\right]\,, \tag{SA.10}$$

where $D(\cdot||\cdot)$ is the relative entropy

$$D(a\|p) = a\ln\frac{a}{p} + (1-a)\ln\frac{1-a}{1-p}\,. \tag{SA.11}$$

We have

$$\lim_{N\to\infty} ND\left(\frac{M+2^\ell}{N}\middle\|\frac{M}{N}\right) = -2^\ell + (2^\ell + M)\ln[(2^\ell+M)/M]\,, \tag{SA.12}$$

$$\frac{d}{dN} ND\left(\frac{M+2^\ell}{N}\middle\|\frac{M}{N}\right) < 0\,. \tag{SA.13}$$

Eq. (SA.13) implies that $ND\left(\frac{M+2^\ell}{N}\middle\|\frac{M}{N}\right)$ is always larger than its asymptotic value. Therefore

$$ND\left(\frac{M+2^\ell}{N}\middle\|\frac{M}{N}\right) \geq -2^\ell + (2^\ell + M)\ln[(2^\ell+M)/M]\,. \tag{SA.14}$$

Since $2^\ell \geq 4M$, we have

$$-2^\ell + (2^\ell + M)\ln[(2^\ell+M)/M] \geq 2^{\ell-1}\,. \tag{SA.15}$$

Setting $C_{N,M} = \binom{N}{M} p^M (1-p)^{N-M}$, and putting all together, we get

$$\begin{aligned} |\langle W_N(M)|\Psi^{(\ell)}\rangle|^2 = \frac{C_{N,M}}{Z_\ell^2} &\geq 1 - \frac{P[e \geq M + 2^\ell]}{C_{N,M}} \\ &\geq 1 - \sqrt{8\pi M} e^{-2^{\ell-1}}\,, \end{aligned} \tag{SA.16}$$

where we used $1/(1+x) \geq 1-x$.

Finally, using

$$\ell \geq 1 + \frac{\ln \ln(\sqrt{8\pi M}/\varepsilon)}{\ln 2} . \tag{SA.17}$$

we get $|| \langle W_N(M)|\Psi^{(\ell)}\rangle |^2 - 1| \leq \varepsilon$. Therefore, setting $|\Psi_N\rangle = |\Psi^{(\ell)}\rangle$ we obtain the statement. ■

Next, we prove that the protocol can be slightly modified to trade the depth with the number of ancillas.

Proposition 2. *For any $\varepsilon > 0$, there exists a (non-deterministic) protocol which prepares a state $|\Psi_N\rangle$ with*

$$|1 - | \langle W_N(M)|\Psi_N\rangle |^2| \leq \varepsilon . \tag{SA.18}$$

The protocol is successful with probability

$$P_{\text{succ}} \geq \frac{1}{\sqrt{8\pi M}} , \tag{SA.19}$$

it uses $D = O(1)$, $N_a = O(\ell)$ ancilla per site, and ℓ additional ancillas, where ℓ is defined in Eq. (SA.3).

Proof. Compared to the protocol explained in Prop. 1, we need to reduce the depth of the circuit to $D = O(1)$. To this end, we need to remove the inverse of the quantum Fourier transform (QFT) in the measurement of the excitations (which requires a depth scaling with the number of ancillas) and parallelize the application of the operators $U^{(1)}(x)$. Our parallelization scheme is closely related to the one of Ref. [3].

We proceed as follows. We define ℓ as in Eq. (SA.3), and append $\ell - 1$ ancillas per site, plus $\ell - 1$ additional ancillas to the first site (so, in the first site we have $\ell - 1 + \ell = 2\ell - 1$ ancillas). All ancillas are initialized in $|0\rangle$. Suppose the initial state of the system is

$$|\psi\rangle = \sum_{\{j_k\}} c_{j_1 \ldots j_N} |j_1 \ldots j_N\rangle . \tag{SA.20}$$

We perform a controlled operations in each local set consisting of one system qubit and $\ell - 1$ ancilla qubits, mapping

$$|j_k\rangle \otimes |0\rangle^{\otimes(\ell-1)} \mapsto |j_k\rangle \otimes |j_k\rangle^{\otimes(\ell-1)} , \tag{SA.21}$$

yielding the state

$$|\Psi\rangle = \sum_{\{j_k\}} c_{j_1 \ldots j_N} |j_1 \ldots j_N\rangle^{\otimes \ell} . \tag{SA.22}$$

The step (SA.21) corresponds to parallel application of fan-out gates and takes constant depth using LOCC [4]. Next, we apply a Hadamard transformation to each of the ℓ ancillas in the first site. Then, for each of the ℓ ancillary systems, we apply a unitary $V^{(x)}$ which acts on the x-th copy of the system and the x-th ancilla in the first site. Each unitary is of the form (2) with $U^{(0)} = \mathbb{1}$ and

$$U^{(1)} = U^{(1)}(x) = e^{2i\pi(N_e - M)/2^x} , \tag{SA.23}$$

where $x = 1, \ldots, \ell$ corresponding to each ancilla. These operations can be performed in parallel as they act on distinct qubits. Next, we act with the inverse of (SA.21), apply a Hadamard transformation to each ancilla and measure them in the Z-basis. The protocol is successful if we obtain the outcome 0 for all ancillas. In this case, it is easy to see that the state after the measurement is proportional to the state (SA.6). Note that the unitary (SA.23) is different from that used in the measurement procedure explained in the main text (Result 2). Indeed, while the final state in the case of success is the same as in the previous Proposition, Eq. (SA.6), the outcome is not equal to a projection onto Π_j^ℓ for different measurement outcomes.

It is immediate to show that the probability of success and the infidelity of the output state are the same as computed in Prop. 1, which proves the statement. ■

Appendix B: Dicke states from amplitude amplification

In this section we provide additional details for the preparation of the Dicke state using the amplitude-amplification protocol. We start by recalling the precise statement of the latter.

Lemma 1 (Amplitude amplification)**.** *Let*

$$|\psi\rangle = \sin\alpha\,|\psi_1\rangle + \cos\alpha\,|\psi_2\rangle\ , \tag{SB.1}$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal states, and $|\tilde{\psi}\rangle$ be the state orthogonal to $|\psi\rangle$ in the subspace generated by $|\psi_1\rangle$, $|\psi_2\rangle$. Let $S_1(\omega)$, $S_2(\omega)$ be two families of unitary operators such that

$$S_1(\omega)\,|\psi\rangle = e^{i\omega}\,|\psi\rangle\ , \qquad S_1(\omega)\,|\tilde{\psi}\rangle = |\tilde{\psi}\rangle\ , \tag{SB.2}$$

$$S_2(\omega)\,|\psi_1\rangle = e^{i\omega}\,|\psi_1\rangle\ , \qquad S_2(\omega)\,|\psi_2\rangle = |\psi_2\rangle\ , \tag{SB.3}$$

and define

$$Q(\phi,\varphi) = -S_1(\phi)S_2(\varphi)\,. \tag{SB.4}$$

Then, if the number $m^* = \pi/(4\alpha) - 1/2$ is an integer, we have

$$Q^{m^*}(\pi,\pi)\,|\psi\rangle \propto |\psi_1\rangle\ . \tag{SB.5}$$

Otherwise, there exist two values $\phi^*, \varphi^* \in \mathbb{R}$ such that

$$Q(\phi^*,\varphi^*)Q^{\lfloor m^*\rfloor}(\pi,\pi)\,|\psi\rangle \propto |\psi_1\rangle\ , \tag{SB.6}$$

where $\lfloor\cdot\rfloor$ is the integer floor function.

Proof. The proof can be found in Refs. [5–7], see in particular Sec. 2.1 in Ref. [7]. Note that the lemma states that we can deterministically obtain the state $|\psi_1\rangle$, provided that we can implement the operators $Q(\phi,\varphi)$. They need to be applied a number of times growing as $\sim 1/\alpha$. ■

Next, we show that the amplitude amplification protocol may be carried out even when the unitaries $S_1(\omega)$ and $S_2(\omega)$ can only be implemented approximately.

Lemma 2 (Approximate amplitude amplification)**.** *Let*

$$|\psi\rangle = \sin\alpha\,|\psi_1\rangle + \cos\alpha\,|\psi_2\rangle\ , \tag{SB.7}$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal states, and $|\tilde{\psi}\rangle$ be the state orthogonal to $|\psi\rangle$ in the subspace generated by $|\psi_1\rangle$, $|\psi_2\rangle$. Fix $1 > \delta > 0$ and let $T_1(\omega)$, $T_2(\omega)$ be two families of unitary operators such that

$$T_1(\omega)\,|\psi\rangle = e^{i\omega}\,|\psi\rangle\ , \qquad T_1(\omega)\,|\tilde{\psi}\rangle = |\tilde{\psi}\rangle + \varepsilon_1\,|v\rangle\ , \tag{SB.8a}$$

$$T_2(\omega)\,|\psi_1\rangle = e^{i\omega}\,|\psi_1\rangle\ , \qquad T_2(\omega)\,|\psi_2\rangle = |\psi_2\rangle + \varepsilon_2\,|w\rangle\ , \tag{SB.8b}$$

where $0 < |\varepsilon_1|, |\varepsilon_2| < \delta/2$, while $|v\rangle$, $|w\rangle$ are normalized states. Finally, set

$$P(\phi,\varphi) = -T_1(\phi)T_2(\varphi)\,. \tag{SB.9}$$

If the number $m^* = \pi/(4\alpha) - 1/2$ is an integer, define

$$|\chi\rangle = P^{m^*}(\pi,\pi)\,|\psi\rangle\ , \tag{SB.10}$$

otherwise, define

$$|\chi\rangle = P(\phi^*,\varphi^*)P^{\lfloor m^*\rfloor}(\pi,\pi)\,|\psi\rangle\ , \tag{SB.11}$$

where $\lfloor\cdot\rfloor$ is the integer floor function and ϕ^*, φ^* are chosen as in Lemma 1. Then

$$\left|\,|\langle\psi_1|\chi\rangle|^2 - 1\right| \leq 4(\lfloor m^*\rfloor + 1)\delta. \tag{SB.12}$$

Proof. First, note that

$$
\begin{aligned}
P(\phi,\varphi)\left|\psi_1\right\rangle &= -e^{i\varphi}T_1(\phi)\left|\psi_1\right\rangle = -e^{i\varphi}T_1(\phi)\left[\zeta_1\left|\psi\right\rangle+\zeta_2|\tilde{\psi}\rangle\right] = -\zeta_1 e^{i\varphi}e^{i\phi}\left|\psi\right\rangle - \zeta_2 e^{i\varphi}|\tilde{\psi}\rangle - \zeta_2 e^{i\varphi}\varepsilon_1\left|v\right\rangle \\
&= Q(\phi,\varphi)\left|\psi_1\right\rangle + \delta_1\left|v\right\rangle \ ,
\end{aligned} \tag{SB.13}
$$

where $|\delta_1| \leq \delta/2 < \delta$, while Q is defined in (SB.4). Here we introduced the coefficients $\zeta_1 = \langle\psi|\psi_1\rangle$, $\zeta_2 = \langle\tilde{\psi}|\psi_1\rangle$. Similarly,

$$
\begin{aligned}
P(\phi,\varphi)\left|\psi_2\right\rangle &= -T_1(\phi)(\left|\psi_2\right\rangle+\varepsilon_2\left|w\right\rangle) = -T_1(\phi)(\xi_1\left|\psi\right\rangle+\xi_2|\tilde{\psi}\rangle) - \varepsilon_2 T_1(\phi)\left|w\right\rangle \\
&= -\xi_1 e^{i\phi}\left|\psi\right\rangle - \xi_2|\tilde{\psi}\rangle - \varepsilon_1\xi_2\left|v\right\rangle - \varepsilon_2 T_1(\phi)\left|w\right\rangle = Q(\phi,\varphi)\left|\psi_2\right\rangle + \delta_2\left|u\right\rangle \ .
\end{aligned} \tag{SB.14}
$$

Here $|u\rangle$ is a normalized vector, while

$$
|\delta_2| \leq (|\varepsilon_1\xi_2|^2 + |\varepsilon_2|^2 + 2|\varepsilon_1\varepsilon_2\xi_2\left\langle v|T_1(\phi)|w\right\rangle|)^{1/2} \leq \delta \,. \tag{SB.15}
$$

Therefore, we have

$$
\prod_{j=1}^{n} P(\phi_j,\varphi_j)\left|\psi\right\rangle = \prod_{j=1}^{n} Q(\phi_j,\varphi_j)\left|\psi\right\rangle + \sum_{k=1}^{n}\left[c_k\prod_{j=1}^{k-1}P(\phi_j,\varphi_j)\left|u\right\rangle + d_k\prod_{j=1}^{k-1}P(\phi_j,\varphi_j)\left|v\right\rangle\right] \ , \tag{SB.16}
$$

where $|c_j| \leq \delta$, $|d_j| \leq \delta$. The statement then follows immediately using Lemma 1 and that $||\left\langle\psi_1|\chi\right\rangle|^2 - 1| \leq 2||\left\langle\psi_1|\chi\right\rangle| - 1|$. ■

We will also use the following result

Lemma 3. *Consider the unitary operation defined by*

$$
F_{\varphi}^{[\ell;m]}\left|i_1\dots i_N\right\rangle = e^{i\varphi f_{\ell,m}(i_1\dots i_N)}\left|i_1\dots i_N\right\rangle \ , \tag{SB.17}
$$

where

$$
f_{\ell,m}(i_1\dots i_N) = \begin{cases} 1 & \text{if } (\sum_{j=1}^{N} i_j) - m \equiv 0 \pmod{2^\ell} \\ 0 & \text{otherwise}\,. \end{cases} \tag{SB.18}
$$

Then, $F_{\varphi}^{[\ell;m]}$ can be implemented using ℓ total ancillas and a circuit of depth $O(\ell^2)$.

Proof. We attach all ℓ ancillas to the first site. We prepare them in the state $|+\rangle$, and apply to each of them, sequentially, the unitary operation V in Eq. (2) with $U_0 = \mathbb{1}$ and $U_1 = e^{i\pi(N-m)/2^x}$, where $x = 1,\dots,\ell$ corresponding to each ancilla. This can be done by a circuit of depth $O(\ell)$. After that, we apply an inverse QFT to the ℓ ancillas, which requires depth $D = O(\ell)$ [8]. This transforms an input state $|\psi\rangle$ as

$$
W : |+\dots+\rangle\otimes|\psi\rangle \to \sum_{i_1,\dots,i_\ell=0}^{1} |i_1,\dots,i_\ell\rangle\otimes\Pi_{i+m}^{\ell}|\psi\rangle \tag{SB.19}
$$

where Π_{i+m}^{ℓ} is the projector onto the subspace with a number of excitations e (namely, a number of values for which $i_k = 1$) satisfying $e \equiv i + m \bmod (2^\ell)$, namely $e - m \equiv i \bmod (2^\ell)$, and where $i_1 \dots i_\ell$ is the binary decomposition of i. We can now apply a unitary to the ancillas in the first site mapping $|0\dots0\rangle \mapsto e^{i\varphi}|0\dots0\rangle$ and acting as the identity on the other basis states. This operation can be implemented by a local circuit of depth $O(\ell^2)$ [9]. We can finally apply the inverse $W^\dagger$ of the unitary (SB.19), yielding the desired result. ■

Finally, we prove our main result of this section.

Proposition 3 (Preparation of Dicke states)**.** *Let $N \geq 4M$ and $M \geq 1$. For any $0 < \delta < 1$, there exists an efficient preparation protocol to realize a state $|\Phi\rangle$ such that*

$$
||\left\langle\Phi|W_N(M)\right\rangle|^2 - 1| \leq 4\delta\,. \tag{SB.20}
$$

The protocol applies a sequence of $2n_M$ unitary operators which are either $F_\omega^{[\ell,0]}V^\dagger$ or $F_\omega^{[\ell,M]}$, where $V = e^{-i\theta S_y}$ is a product of local unitaries (which can be implemented in parallel),

$$n_M \leq \frac{\pi(8\pi M)^{1/4}}{2}\,, \tag{SB.21}$$

while

$$\ell = \log_2\left\{\frac{1}{\ln(4/3)}\left[2M(\ln 2M + 9/2) + \ln\left(\frac{\mathrm{Poly}(M)}{\delta^2}\right)\right]\right\}\,, \tag{SB.22}$$

with

$$\mathrm{Poly}(M) = \frac{8\pi e^2 (8\pi M)^{1/2}}{M\ln(4/3)\left(1 - [8/(3\pi M)]^{1/2}\right)}\,. \tag{SB.23}$$

Proof. Define

$$|\theta\rangle = V\,|0\rangle^{\otimes N}\,, \tag{SB.24}$$

$V = e^{-i\theta S_y}$ with $\cos(\theta) = \sqrt{1-p}$ and

$$p = M/N\,. \tag{SB.25}$$

We start with the identity

$$|\theta\rangle = \sin\alpha\,|W(M)\rangle + \cos\alpha\,|R\rangle\,. \tag{SB.26}$$

Here, $|W(M)\rangle$ is the normalized Dicke state with M excitations, while

$$|R\rangle = \frac{1}{Z_R}\sum_{e\neq M}\left[\binom{N}{e}p^e(1-p)^{N-e}\right]^{1/2}|W(e)\rangle\,, \tag{SB.27}$$

where Z_R is a normalization factor, and

$$\sin\alpha = \left[\binom{N}{M}p^M(1-p)^{N-M}\right]^{1/2}\,. \tag{SB.28}$$

Choose ℓ as in (SB.22) and define

$$T_1(\omega) = F_\omega^{[\ell,0]}V^\dagger \qquad T_2(\omega) = F_\omega^{[\ell,M]}\,. \tag{SB.29}$$

Note that V (and hence $V^\dagger$) is a product of local unitaries, while $F_\omega^{[\ell,m]}$ can be implemented efficiently thanks to Lemma 3. In the following, we will show

$$T_1(\omega)\,|\theta\rangle = e^{i\omega}\,|\theta\rangle\,, \qquad T_1(\omega)\,|\tilde\theta\rangle = |\tilde\theta\rangle + \varepsilon_1\,|v\rangle\,, \tag{SB.30a}$$
$$T_2(\omega)\,|W(M)\rangle = e^{i\omega}\,|W(M)\rangle\,, \qquad T_2(\omega)\,|R\rangle = |R\rangle + \varepsilon_2\,|w\rangle\,, \tag{SB.30b}$$

where we denoted by $|\tilde\theta\rangle$ the state orthogonal to $|\theta\rangle$ generated by $|W(M)\rangle$ and $|R\rangle$, while $|v\rangle$, $|w\rangle$ are normalized states, with

$$|\varepsilon_1|, |\varepsilon_2| \leq \frac{\delta}{\pi(8\pi M)^{1/4}}\,. \tag{SB.31}$$

Combining Lemmas 1 and 2, we see that this is enough to prove the statement. Indeed, if (SB.31) holds, we can implement the approximate amplitude amplification algorithm applying $T_1(\omega)$ and $T_2(\omega)$ a number of times

$$n_M \leq \frac{\pi}{4\alpha} + \frac{1}{2} \leq \frac{\pi(8\pi M)^{1/4}}{4} + \frac{1}{2} \leq \frac{\pi(8\pi M)^{1/4}}{2}\,, \tag{SB.32}$$

where we used $\sin\alpha \le \alpha$ for $0 \le \alpha \le 1$, and $(\sin\alpha)^{-1} \le (8\pi M)^{1/4}$ [which follows from Eqs. (SA.8) and (SB.28)]. By Lemma 2, this gives us the desired state $|W(M)\rangle$ up to an infidelity $I = \varepsilon$ with

$$\varepsilon \le 4\left(\frac{\pi}{4\alpha} + \frac{1}{2}\right)\frac{2\delta}{\pi(8\pi M)^{1/4}} \le 4\delta\,. \tag{SB.33}$$

Let us prove (SB.30), starting with the action of $T_2(\omega)$. First, it is obvious that $T_2(\omega)\,|W(M)\rangle = e^{i\omega}\,|W(M)\rangle$. Next, we have

$$\begin{aligned} T_2(\omega)\,|R\rangle &= \frac{1}{Z_R}T_2(\omega)\left[\sum_{e\in\mathcal{T}_M^\ell\setminus\{M\}}\left[\binom{N}{e}p^e(1-p)^{N-e}\right]^{1/2}|W(e)\rangle + \sum_{j\neq 0}\sum_{e\in\mathcal{T}_{M+j}^\ell}\left[\binom{N}{e}p^e(1-p)^{N-e}\right]^{1/2}|W(e)\rangle\right] \\ &= \frac{1}{Z_R}\left[e^{i\omega}\sum_{e\in\mathcal{T}_M^\ell\setminus\{M\}}\left[\binom{N}{e}p^e(1-p)^{N-e}\right]^{1/2}|W(e)\rangle + \sum_{j\neq 0}\sum_{e\in\mathcal{T}_{M+j}^\ell}\left[\binom{N}{e}p^e(1-p)^{N-e}\right]^{1/2}|W(e)\rangle\right] \\ &= |R\rangle + |w\rangle\ . \end{aligned} \tag{SB.34}$$

where $\mathcal{T}_M^\ell$ is defined in Eq. (SA.5), while

$$|w\rangle = \frac{(e^{i\omega}-1)}{Z_R}\sum_{e\in\mathcal{T}_M^\ell\setminus\{M\}}\left[\binom{N}{e}p^e(1-p)^{N-e}\right]^{1/2}|W(e)\rangle\ . \tag{SB.35}$$

We can bound the norm of $|w\rangle$ using

$$Z_R^2 = 1 - \binom{N}{M}p^M(1-p)^{N-M}\,, \tag{SB.36}$$

and the results of Appendix A, cf. Eqs. (SA.9), (SA.10). Using $N \ge 4M$, we obtain

$$\langle w|w\rangle \le 4\frac{e^{-2^{\ell-1}}}{Z_R^2} \le \frac{4e^{-2^{\ell-1}}}{1-\left(\frac{8}{3\pi M}\right)^{1/2}} \le \frac{\delta^2}{\pi^2(8\pi M)^{1/2}}\,, \tag{SB.37}$$

This inequality holds if

$$\ell \ge 1 + \log_2\left\{\ln\left[\frac{4\pi^2(8\pi M)^{1/2}}{\delta^2\left(1-\left(\frac{8}{3\pi M}\right)^{1/2}\right)}\right]\right\}, \tag{SB.38}$$

which is true if ℓ is chosen as in (SB.22) (this is easily established with the help of numerical inspection).

Next, let us consider $T_1(\omega)$. Again, it is obvious that $T_1(\omega)\,|\theta\rangle = e^{i\omega}\,|\theta\rangle$. To prove the second identity in Eq. (SB.30a), we start from

$$V^\dagger\,|\tilde\theta\rangle = \frac{1}{Z}V^\dagger(1-|\theta\rangle\langle\theta|)\,|W(M)\rangle\ , \tag{SB.39}$$

where

$$Z^2 = 1 - |\langle W(M)|\theta\rangle|^2\,. \tag{SB.40}$$

It follows from the results of Appendix C that

$$V^\dagger\,|W(M)\rangle = \sum_{s=0}^{N} c_s\,|W(s)\rangle\ , \tag{SB.41}$$

cf. Eq. (SC.8). Therefore, we can write

$$V^\dagger\,|\tilde\theta\rangle = \frac{1}{Z}\left(\sum_{s=0}^{2^\ell} c_s\,|W(s)\rangle + \langle\theta|W(M)\rangle\,|0\rangle^{\otimes N}\right) + |\tilde w\rangle\ , \tag{SB.42}$$

where $|\tilde{w}\rangle$ has more than 2^ℓ excitations. Using the results of Sec. C and $N \geq 4M$, we can bound its norm as

$$\begin{aligned}\langle\tilde{w}|\tilde{w}\rangle = \frac{1}{Z^2}\sum_{s\geq 2^\ell+1}|c_s|^2 &\leq \frac{1}{1-[8/(3\pi M)]^{1/2}}\frac{2e^2}{M\pi\ln(4/3)}\exp\left[-2^\ell\ln(4/3)+2M(\ln(2M)+9/2)\right] \\ &\leq \frac{\delta^2}{4\pi^2(8\pi M)^{1/2}}\,,\end{aligned} \tag{SB.43}$$

where we used that

$$2^\ell \geq \frac{1}{\ln(4/3)}\left[2M(\ln 2M+9/2)+\ln\left(\frac{\mathrm{Poly}(M)}{\delta^2}\right)\right]\,, \tag{SB.44}$$

with

$$\mathrm{Poly}(M) = \frac{8\pi e^2(8\pi M)^{1/2}}{M\ln(4/3)\left(1-[8/(3\pi M)]^{1/2}\right)}\,. \tag{SB.45}$$

Now, in the space generated by states with at most 2^ℓ excitations, $F_\omega^{[\ell,0]}$ only multiplies the phase $e^{i\omega}$ to the state $|0\rangle^{\otimes N}$, leaving the rest of the basis states invariant. Therefore, we arrive at the final result

$$T_1(\omega)]\,|\tilde{\theta}\rangle = F_\omega^{[\ell,0]}V^\dagger\,|\tilde{\theta}\rangle = |\tilde{\theta}\rangle + |v\rangle\ , \tag{SB.46}$$

where $|v\rangle = (-\mathbb{1}+F_\omega^{[\ell,0]}\,|\tilde{w}\rangle)$, and therefore

$$\langle v|v\rangle \leq \frac{\delta^2}{\pi^2(8\pi M)^{1/2}}\,. \tag{SB.47}$$

■

Appendix C: Technical computations

The goal of this section is to analyze the state

$$V^\dagger\,|W(M)\rangle \tag{SC.1}$$

where $|W(M)\rangle$ is the normalized Dicke state with M excitations, while $V = e^{-i\theta S_y}$ with $\cos(\theta) = \sqrt{1-p}$ and $p = M/N$. Throughout this section, we will assume $N \geq 4M$.

We start by introducing the unnormalized Dicke states

$$|U(M)\rangle = \sum_{i_1<\ldots<i_M}\sigma_{i_1}^+\cdots\sigma_{i_M}^+\,|0\rangle^{\otimes N}\ , \tag{SC.2}$$

and note that we can also write

$$|W(M)\rangle = \frac{1}{\sqrt{\binom{N}{M}}}\frac{1}{(N-M)!M!}\sum_{\pi\in S_N}|\underbrace{1\ldots1}_{M}\underbrace{0\ldots0}_{N-M}\rangle\ , \tag{SC.3}$$

where the sum is over all permutations of qubits. Therefore, we can compute

$$\begin{aligned}V^\dagger\,|W(M)\rangle &= \frac{1}{\sqrt{\binom{N}{M}}}\frac{1}{(N-M)!M!}\sum_{\pi\in S_N}e^{+iS_y\theta}\,|\underbrace{1\ldots1}_{M}\underbrace{0\ldots0}_{N-M}\rangle \\ &= \frac{1}{\sqrt{\binom{N}{M}}}\frac{1}{(N-M)!M!}\sum_{\pi\in S_N}(\sqrt{p}\,|0\rangle+\sqrt{1-p}\,|1\rangle)^{\otimes M}(\sqrt{1-p}\,|0\rangle-\sqrt{p}\,|1\rangle)^{\otimes(N-M)}\end{aligned} \tag{SC.4}$$

We can rewrite

$$\begin{aligned}
&\sum_{\pi\in S_N}(\sqrt{p}\,|0\rangle+\sqrt{1-p}\,|1\rangle)^{\otimes M}(\sqrt{1-p}\,|0\rangle-\sqrt{p}\,|1\rangle)^{\otimes(N-M)}\\
&=\sum_{\pi\in S_N}\left[\sum_{e=0}^{M}p^{(M-e)/2}(1-p)^{e/2}\,|U(e)\rangle\right]\left[\sum_{f=0}^{N-M}(1-p)^{(N-M-f)/2}(-p^{1/2})^{f}\,|U(f)\rangle\right]\\
&=\sum_{e=0}^{M}\sum_{f=0}^{N-M}p^{(M-e)/2}(1-p)^{e/2}(1-p)^{(N-M-f)/2}(-p^{1/2})^{f}\sum_{\pi\in S_N}|U(e)\rangle\,|U(f)\rangle
\end{aligned} \tag{SC.5}$$

Next, we use

$$\begin{aligned}
\sum_{\pi\in S_N}|U(e)\rangle\,|U(f)\rangle&=\binom{M}{e}\binom{N-M}{f}\sum_{\pi\in S_N}|\underbrace{1\dots1}_{e+f}\underbrace{0\dots0}_{N-e-f}\rangle\\
&=\binom{M}{e}\binom{N-M}{f}\sqrt{\binom{N}{e+f}}(e+f)!(N-e-f)!\,|W(e+f)\rangle
\end{aligned} \tag{SC.6}$$

Introducing the variable $s=e+f$, we finally get

$$\begin{aligned}
V^{\dagger}\,|W(M)\rangle=&\sum_{s=0}^{N}|W(s)\rangle\,\frac{\sqrt{s!(N-s)!}}{\sqrt{M!(N-M)!}}\\
&\times\sum_{e=0}^{M}\sum_{f=0}^{N-M}\delta_{f+e,s}\binom{M}{e}\binom{N-M}{f}p^{(M-e)/2}(1-p)^{e/2}(1-p)^{(N-M-f)/2}(-p^{1/2})^{f}\,.
\end{aligned} \tag{SC.7}$$

Therefore,

$$V^{\dagger}\,|W(M)\rangle=\sum_{s=0}^{N}c_s\,|W(s)\rangle\ , \tag{SC.8}$$

where

$$c_s=\frac{\sqrt{s!(N-s)!}}{\sqrt{M!(N-M)!}}\times\sum_{e=0}^{M}\binom{M}{e}\binom{N-M}{s-e}p^{(M-e)/2}(1-p)^{e/2}(1-p)^{(N-M-(s-e))/2}(-p^{1/2})^{s-e}\,. \tag{SC.9}$$

Next, we bound $|c_s|$ for $s\geq 3M$. We have

$$|c_s|\leq\frac{\sqrt{s!(N-s)!}}{\sqrt{M!(N-M)!}}\sum_{e=0}^{M}\binom{M}{e}\binom{N-M}{s-e}p^{(M+s-2e)/2}(1-p)^{(N-M-(s-2e))/2}\,. \tag{SC.10}$$

Using

$$\binom{N-M}{s-e}\leq\frac{(N-M)^{s-e}}{(s-M)!}\,, \tag{SC.11}$$

we obtain

$$\begin{aligned}
|c_s|&\leq\frac{\sqrt{s!(N-s)!}}{\sqrt{M!(N-M)!}}\frac{1}{(s-M)!}p^{(s-M)/2}\frac{(N-M)^{s}}{(N-M)^{M}}(1-p)^{(N-M-s)/2}\sum_{e=0}^{M}\binom{M}{e}(N-M)^{M-e}p^{M-e}(1-p)^{e}\\
&=\frac{\sqrt{s!(N-s)!}}{\sqrt{M!(N-M)!}}\frac{1}{(s-M)!}p^{(s-M)/2}\frac{(N-M)^{s}}{(N-M)^{M}}(1-p)^{(N-M-s)/2}\left[(M+1)\left(1-\frac{M}{N}\right)\right]^{M}\,.
\end{aligned} \tag{SC.12}$$

Taking the square, using Stirling's inequality $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n<n!<\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$ and rearranging, we arrive at

$$|c_s|^2\leq\left(1+\frac{1}{M}\right)^{2M}\frac{1}{\pi}\left(\frac{s(N-s)}{M(N-M)(s-M)^2)}\right)^{1/2}\left(\frac{sM(N-M)}{N-s}\right)^{s}\left(1-\frac{s}{N}\right)^{N}\exp\left[2(s-M)(1-\ln(s-M)\right]\,. \tag{SC.13}$$

Now we use that for $s \geq 3M$ one has

$$\left(\frac{sM(N-M)}{N-s}\right)^s \left(1-\frac{s}{N}\right)^N \exp\left[2(s-M)(1-\ln(s-M)\right] \leq e^M (2M)^{2M} \left[\frac{3(N-M)}{4(N-3M)}\right]^s . \tag{SC.14}$$

This inequality can be established as follows. Denoting the lhs by $g(s)$, we note that the logarithmic derivative $d\ln g(s)/ds$ is a monotonically decreasing function of s. Therefore, for $s \geq 3M$, we have $d\ln g(s)/ds \leq d\ln g(s)/ds|_{s=3M} =: \Gamma$. This implies $g(s) \leq g(3M)e^{s\Gamma}$, from which the above inequality follows. Finally, we have

$$\left(\frac{s(N-s)}{M(N-M)(s-M)^2)}\right)^{1/2} = \left[\frac{s}{M}\left(\frac{1}{(s-M)^2} - \frac{1}{(s-M)(N-M)}\right)\right]^{1/2} \leq \frac{2}{M} , \tag{SC.15}$$

and also

$$\left[\frac{(N-M)}{(N-3M)}\right]^s = \left(1+\frac{2M}{N-3M}\right)^s \leq \exp\left[\frac{2Ms}{N-3M}\right] \leq e^{8M} , \tag{SC.16}$$

where we used $N \geq s$ and $N \geq 4M$. Putting all together, we obtain

$$|c_s|^2 \leq \frac{2e^2}{\pi M} e^{9M} (2M)^{2M} \left[\frac{3}{4}\right]^s = \mathrm{Poly}(M) \exp\left[-s\ln(4/3) + 2M(\ln(2M)+9/2)\right] . \tag{SC.17}$$

Therefore, we arrive at the final result

$$\sum_{k\geq s} |c_k|^2 \leq \int_{s-1}^{\infty} dk |c_k|^2 = \frac{2e^2}{M\pi\ln(4/3)} \exp\left[-(s-1)\ln(4/3) + 2M(\ln(2M)+9/2)\right] . \tag{SC.18}$$

Appendix D: Eigenstates of the XX chain

We consider the XX Hamiltonian with open boundary conditions

$$H = -\sum_{k=1}^{N-1} (\sigma_k^x \sigma_k^x + \sigma_k^y \sigma_k^y) . \tag{SD.1}$$

Introducing the fermionic modes via the Jordan-Wigner mapping

$$a_k = \left(\prod_{j=1}^{k-1} \sigma_j^z\right) \sigma_k^- , \qquad a_k^\dagger = \left(\prod_{j=1}^{k-1} \sigma_j^z\right) \sigma_k^+ , \tag{SD.2}$$

the Hamiltonian (SD.1) can be rewritten as

$$H = -\sum_{k=1}^{N-1} (a_j^\dagger a_{j+1} + \text{h.c.}) . \tag{SD.3}$$

Note that

$$\{a_j^\dagger, a_k\} = \delta_{j,k} . \tag{SD.4}$$

Based on this mapping, a standard result states that the eigenstates of the model read

$$|\Psi_M\rangle = A_M^\dagger \cdots A_1^\dagger |0\rangle^{\otimes N} \tag{SD.5}$$

with

$$A_\alpha = \sum_{k=1}^{N} c_k^\alpha \left(\prod_{j=1}^{k-1} \sigma_j^z\right) \sigma_k^- , \tag{SD.6}$$

where $\{c_k^\alpha\}$ are pairwise distinct sets of numerical coefficients, $\{c_k^\alpha\} \neq \{c_k^\beta\}$. These operators satisfy the canonical anticommutation relations

$$\{A_\alpha, A_\beta\} = 0\,, \qquad \{A_\alpha^\dagger, A_\beta\} = \delta_{\alpha,\beta}\,, \tag{SD.7}$$

and so one has the constraint

$$\sum_{j=1}^{N} \overline{c_j^\alpha} c_j^\beta = \delta_{\alpha,\beta}\,. \tag{SD.8}$$

We prove the following statement

Proposition 4. *The eigenstates $|\Psi_M\rangle$ can be prepared by a circuit (with LOCC) of depth $O(NM)$ and a single ancilla per site.*

Proof. First, note that

$$|\Psi_M\rangle = e^{i\frac{\pi}{2}(A_M + A_M^\dagger)}|\Psi_{M-1}\rangle\,. \tag{SD.9}$$

This is because $(A_M + A_M^\dagger)^2 = 1$ and also $A_M|\Psi_{M-1}\rangle = 0$. The last equality follows from the anticommutation relations (SD.7) and the fact that $A_M \neq A_j$ for $j < M$.

Next, let A and B two anticommuting operators, such that $\{A, A^\dagger\} = \{B, B^\dagger\} = 1$. Then, for $\alpha, \theta \in \mathbb{R}$, we have the identity

$$e^{i\alpha[\cos(\theta)(A+A^\dagger)+\sin(\theta)(B+B^\dagger)]} = e^{i\beta(B+B^\dagger)} e^{i\gamma(A+A^\dagger)} e^{i\beta(B+B^\dagger)}\,, \tag{SD.10}$$

where

$$\cos(2\beta)\cos(\gamma) = \cos(\alpha)\,, \tag{SD.11}$$

$$\sin(2\beta)\cos(\gamma) = \sin(\alpha)\sin(\theta)\,, \tag{SD.12}$$

$$\sin(\gamma) = \sin(\alpha)\cos(\theta)\,, \tag{SD.13}$$

which can be simply derived expanding the exponential functions. Note that the third equation can be derived from the first two, so that there is always a solution to the above system. By applying iteratively this relation, we obtain

$$e^{i\theta(A_M + A_M^\dagger)} = R_N \dots R_2 R_1 R_2 \dots R_N, \tag{SD.14}$$

where $R_j = e^{i\theta_j(c_j^M a_j^\dagger + \text{h.c.})}$, where θ_j can be easily computed through iteration of (SD.10). Note that the coefficients c_j^M are in general complex, but Eq. (SD.10) can be applied as the phases are reabsorbed in the definition of the operators A and B.

Finally, we notice that

$$R_j = V_j X_j V_j^\dagger\,, \tag{SD.15}$$

where, $X_j = e^{i\theta_j(c_j^M \sigma_j^+ + \text{h.c.})}$ and

$$V_j = |0\rangle_j\langle 0| \otimes \left(\otimes_{k=1}^{j-1} U_{0,k}\right) + |1\rangle_j\langle 1| \otimes \left(\otimes_{k=1}^{j-1} U_{1,k}\right)\,, \tag{SD.16}$$

with $U_{0,k} = 1$ and $U_{1,k} = \sigma_z$. It is easy to see that it performs the Jordan Winger transformation, *i.e.*

$$V_j \sigma_j^+ V_j^\dagger = \sigma_1^z \otimes \dots \otimes \sigma_{j-1}^z \otimes \sigma_j^+\,. \tag{SD.17}$$

Note that Eq. (SD.16) is of the form (2), where the control qubit is at site j, and can thus be implemented with a circuit of depth $D = O(1)$ using LOCC.

Finally, using

$$X_N \dots X_{j+1} V_j = V_j X_N \dots X_{j+1}\,, \tag{SD.18}$$

$$V_j X_{j+1} \dots X_N = X_{j+1} \dots X_N V_j\,, \tag{SD.19}$$

and that $V_j = V_j^\dagger$, $[V_j, V_k] = 0$, we obtain

$$e^{i\theta(A_M + A_M^\dagger)} = T L_N L_{N-1} \dots L_1 \tilde{L}_2 \dots, \tilde{L}_N T, \tag{SD.20}$$

where $T = \left[\prod_{j=1}^N V_j\right]$, and

$$L_N = X_N, \quad L_{N-1} = V_N X_{N-1}, \dots, L_2 = V_3 X_2, \quad L_1 = V_2 X_1 V_2, \tag{SD.21}$$

$$\tilde{L}_N = X_N, \quad \tilde{L}_{N-1} = X_{N-1} V_N, \dots, \tilde{L}_2 = X_2 V_3\,. \tag{SD.22}$$

Therefore, putting together all the excitations, and denoting by $\mathcal{L}_k = L_N L_{N-1} \dots L_1 \tilde{L}_2 \dots, \tilde{L}_N$ the operator corresponding to the k-th excitation (*i.e.*, depending on the parameters c_j^k), we have

$$|\Psi_M\rangle = T \mathcal{L}_M \dots \mathcal{L}_1 |0 \dots 0\rangle. \tag{SD.23}$$

Since each L_j and V_j can be implemented by a circuit of depth $D = O(1)$, we immediately obtain the statement. ■

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