Approximating Many-Body Quantum States with Quantum Circuits and Measurements

Lorenzo Piroli,[1] Georgios Styliaris,[2,3] and J. Ignacio Cirac[2,3]

[1]Dipartimento di Fisica e Astronomia, Università di Bologna and INFN Sezione di Bologna, via Irnerio 46, I-40126 Bologna, Italy
[2]Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, 85748 Garching, Germany
[3]Munich Center for Quantum Science and Technology (MCQST), Schellingstraße 4, D-80799 München, Germany

(Received 4 April 2024; revised 26 September 2024; accepted 17 October 2024; published 3 December 2024)

We introduce protocols to prepare many-body quantum states with quantum circuits assisted by local operations and classical communication. We show that by lifting the requirement of exact preparation, one can substantially save resources. In particular, the so-called W and, more generally, Dicke states require a circuit depth and number of ancillas per site that are independent of the system size. As a by-product of our work, we introduce an efficient scheme to implement certain nonlocal, non-Clifford unitary operators. We also discuss how similar ideas may be applied in the preparation of eigenstates of well-known spin models, both free and interacting.

DOI: 10.1103/PhysRevLett.133.230401

Introduction—The preparation of many-body quantum states plays a pivotal role in quantum simulation [1]. On the one hand, some of those states are required to exploit the field of quantum sensing [2], quantum communication [3], or play a crucial role in quantum information theory [4]. On the other, they allow to investigate quantum many-body systems, extracting properties that otherwise are difficult to compute. Furthermore, some of them can be useful to initialize quantum algorithms that prepare ground states [5–7] or thermal states [8–12].

As current noisy intermediate-scale quantum (NISQ) devices [13] are limited in the number of qubits and the coherence time, it is very important to devise efficient preparation schemes making use of the minimum amount of resources. Following early ideas [14,15], an emerging theme is that preparation protocols using unitary circuits can be improved by making use of additional ancillas, measurements, and feed-forward operations, notably in the context of topological order [16–29]. These ingredients are very natural from the point of view of quantum information, where they are called local operations and classical communication (LOCC) [4].

The goal of this work is to introduce protocols that save additional resources as compared to existing schemes. As we show, this is achieved by relaxing the condition of preparing the states exactly and deterministically. This does not cause any disadvantage since for any realistic device exact preparation will never be possible. A cornerstone of our schemes is a nonlocal unitary operation that can be efficiently implemented and that, in contrast to those introduced in Ref. [16], is not Clifford [30,31]. We also show how this operation can help to save resources by creating one-by-one excitations in spin systems.

In this Letter, we identify as resources the depth D of the quantum circuit (QC), the number of experimental repetitions N_r, and the number of ancillas per qubit N_a needed in order to produce an infidelity $I = \varepsilon$. It is important to carefully define the depth of the circuit, which will be done later. We anticipate that, contrary to some of the protocols in Ref. [16], we will only allow for LOCC where all the measurements are executed in parallel. We also note that, in our schemes, one can trade among different resources, but we will be mostly concerned with saving N_a and D, which are arguably more important for the first generation of quantum computers.

Our main result is to show how to prepare the N-qubit *Dicke states* [32]

$$|W(M)\rangle = Z_M^{-1}(S^+)^M|0\ldots0\rangle, \tag{1}$$

where $S^\pm = \sum_{m=1}^N \sigma_m^\pm$, Z_M is a normalization factor, while $\sigma_m^\pm$ are the ladder operators at position m. The states (1) are eigenstates of the Dicke Hamiltonian $H_D = S^+S^- + S^-S^+$, where $M \in [0, N]$ is the number of excitations. They were defined in the Dicke model of superradiance [32,33], and are expected to be useful in different kinds of quantum simulations of that model, see, e.g., Ref. [34]. Our interest in these states is twofold. On the one hand, they play a fundamental role in quantum information science and in particular in metrology. As a consequence, a significant amount of experimental [35–38] and theoretical [39–42] work has studied protocols for their preparation in digital

TABLE I. Summary of our results and comparison with previous work [M: number of excitations; ε: infidelity]. The resources are the depth D (including LOCC, if applicable) the number of ancillas per site N_a and of repetitions N_r. A trade-off is possible in some cases, and we give variants optimizing either D, N_a, or N_r [$\ell_{M,\varepsilon}$ is defined in Eqs. (4) and (6) for Results 3 and 5, respectively]. Ref. [44] allows for $M = O(\sqrt{N})$, while, for arbitrary M, $N_a = O(\mathrm{Poly}(N))$, $D = O(\ln N)$.

	Ref.	D	N_a	N_r
W	Result 3	$O(\ln\ln 1/\varepsilon)$	1	$O(1)$
	Result 3	$O(1)$	$O(\ln\ln 1/\varepsilon)$	$O(1)$
	Result 4	$O(1)$	1	$\mathrm{O}(1/\sqrt{\varepsilon})$
	Ref. [44]	$O(1)$	$O(\ln N)$	1
Dicke	Result 3	$O(1)$	$O(\ell_{M,\varepsilon})$	$O(\sqrt{M})$
	Result 3	$O(\ell_{M,\varepsilon})$	1	$O(\sqrt{M})$
	Result 5	$O(M^{1/4}\ell^2_{M,\varepsilon})$	$1+\ell_{M,\varepsilon}/N$	1
	Ref. [44]	$O(1)$	$O(N\ln N)$	1

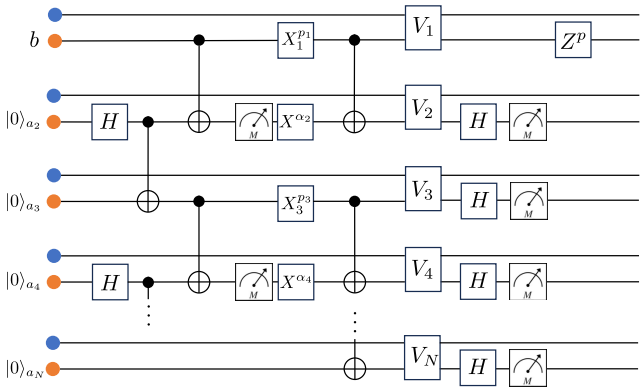

FIG. 1. Quantum circuit implementing the unitary in Eq. (2) (the exchange of bits via classical communication is not shown). Physical and ancillary input qubits are denoted by blue and orange circles, respectively. X and Z are Pauli operators, whose exponents α_j, p_j, and p are defined in the main text, together with V_j. All measurements are in the Z basis.

and analog quantum platforms. On the other hand, Dicke states have resisted previous attempts to devise preparation schemes using finite-depth circuits and a finite number of ancillas per site [16,21,43], raising the question of whether there are some fundamental limitations to achieve this task.

The preparation of Dicke states with LOCC has been previously considered in the literature. In Ref. [16] a protocol was proposed to prepare the W state that uses a QC with $D = O(1)$ but requires sequential use of LOCC, i.e., a $O(N)$ preparation time. In Ref. [44] an ingenious approach was introduced to deterministically prepare the W and Dicke states with constant depth but N_a scaling with N. Instead, the protocols developed in this work allow, for any fixed desired infidelity and a constant number of excitations, N-independent resources (Table I). Our approach is very different from that of Ref. [44] and arguably simpler. The physical intuition behind our protocol is that the Dicke state may be obtained by measuring the total number of excitations, starting from some suitable unentangled (and thus, easily prepared) initial state. This strategy is very natural, as it relies on the interpretation of Dicke states as made of quasiparticle excitations. In fact, we note that a similar idea has been first followed in Ref. [45], in a very different analog setting. In our work we solve the nontrivial problem of implementing this idea using finite-depth circuits and LOCC.

We also discuss how similar ideas may be useful to prepare certain states of interest in many-body physics. We consider the eigenstates of the XX Hamiltonian and present a deterministic preparation protocol with $D = O(MN)$, where M is the number of excitations. While our protocol is less efficient than the state-of-the-art unitary algorithm requiring $O(N)$ depth [46–48], our method is of interest as it is in principle applicable to more general states and could lead to further improvement or generalizations. Finally, we also discuss how extension of our ideas may allow one to prepare eigenstates of interacting spin chains, including the so-called Richardson-Gaudin model [49,50].

Non-Clifford unitaries from QCs and LOCC—We consider N qubits in one spatial dimension. The associated Hilbert space is $\mathcal{H} = \mathcal{H}_2^{\otimes N}$, with $\mathcal{H}_2 \simeq \mathbb{C}^2$, while we denote by $\{|0\rangle, |1\rangle\}$ the computational basis. We attach to each qubit N_a ancillas. Then, we define the local QCs as the unitaries $W = W_\ell \ldots W_2 W_1$, where each "layer" W_n contains quantum gates acting on disjoint pairs of nearest-neighbor qubits and possibly the associated ancillas. In between each layer, we allow for LOCC consisting of a round of measurements executed in parallel, classical processing of the outcomes and local corrections (executed in parallel). We define the circuit depth as the total number of unitary layers and LOCC steps.

We begin by showing how to implement non-Clifford operations of the form

$$V = |0\rangle_b\langle 0| \otimes U^{(0)} + |1\rangle_b\langle 1| \otimes U^{(1)}, \tag{2}$$

where $U^{(k)} = \bigotimes_{j=1}^N U_{k,j}$ and $U_{k,j}$ act on system qubit j, with $k = 0, 1$. Here, $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, while b is the ancilla placed at position 1. The form (2) includes quantum fan-out gates, which are useful in quantum computing [44,51,52]. We prove the following:

Result 1—V can be implemented deterministically (i.e. by a single repetition $N_r = 1$), using $N_a = 1$ and $D = 6$. Given the (unnormalized) joint input state $|0\rangle_b|\psi_0\rangle + |1\rangle_b|\psi_1\rangle$, the QC implementing V is depicted in Fig. 1 and detailed below. In the first layer, the circuit creates maximally entangled pairs between neighboring ancillas $|\Phi^+\rangle_{2j,2j+1}$ $(j = 1, 2, \ldots, N/2-1)$. Second, $\mathrm{CNOT}_{2j-1,2j}$ gates are applied over pairs of ancillas, except for the last one, $j = 1, \ldots, (N/2-1)$. This layer is followed by a

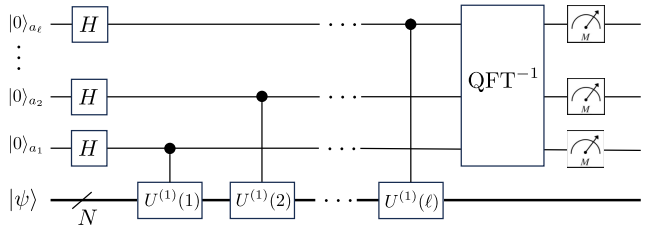

FIG. 2. Quantum circuit with $D = O(\ell)$ implementing the measurements corresponding to $\{\Pi_j^\ell\}_j$. The bottom thick line corresponds to the physical Hilbert space of N qubits, while ℓ ancillas are attached to the first qubit. Each control-U operation is implemented with depth $O(1)$ via the unitary V in Eq. (2). All measurements are performed in the Z basis.

LOCC step: we measure all even ancillas in the Z basis, obtaining measurement outcomes $\alpha_{2j} \in \{0, 1\}$, and apply local Pauli corrections $X_k^{p_k}$ over all odd ancillas $k \geq 3$, where $p_k = \sum_{2j<k} \alpha_{2j}$. At the same time, the decoupled even ancillas are rotated to the $|0\rangle_{2j}$ state. We then apply another layer of $\mathrm{CNOT}_{2j-1,2j}$, $j = 1, \ldots, N/2$ to all ancilla pairs, yielding the joint state $|0\rangle^{\otimes N}|\psi_0\rangle + |1\rangle^{\otimes N}|\psi_1\rangle$, and proceed by applying to each ancilla and system qubit the control unitary $V_j = |0\rangle\langle 0| \otimes U_{0,j} + |1\rangle\langle 1| \otimes U_{1,j}$. Finally, we perform a LOCC step: we measure all ancillas except b in the $|\pm\rangle$ basis, yielding the outcomes $\{\beta_j\}_{j=2}^N$ and apply Z_b^p, where p is the parity of $\sum_j \beta_j$. This yields $|0\rangle_b U^{(0)}|\psi_0\rangle + |1\rangle_b U^{(1)}|\psi_1\rangle$ [53].

Measuring the number of excitations—The unitary (2) is the key ingredient to our preparation protocol for the Dicke state, as it allows for an efficient measurement of the number of excitations. Consider the state $|\psi\rangle$ and let us define the excitation number $N_e = \sum_j n_j$, where $n_j = (1 - \sigma_j^z)/2$. Denoting by Π_j the projector onto the eigenspace of N_e associated with the eigenvalue j, we wish to implement the corresponding measurement. It turns out that it is possible to implement a closely related measurement using shallow QCs and LOCC, corresponding to the projectors $\Pi_j^\ell = \sum_{i \in \mathcal{T}_j^\ell} \Pi_i$, where $\mathcal{T}_j^\ell$ is the set of indices i such that $i \equiv j \pmod{2^\ell}$. In particular, we obtain the following:

Result 2—The measurement corresponding to the set $\{\Pi_j^\ell\}_j$ can be implemented using a circuit with $D = O(\ell)$, $N_a = 1$ and ℓ additional ancillas.

The circuit implementing this measurement is represented in Fig. 2. Attaching all ℓ ancillas, initialized in $|0\rangle$, to the first site, the circuit applies to each of them, sequentially, a controlled operator consisting of the unitary operation V in Eq. (2) with $U^{(0)} = \mathbb{1}$ and $U^{(1)} = U^{(1)}(x) = e^{i2\pi N_e/2^x}$, where $x = 1, \ldots, \ell$ corresponding to each ancilla. At the end of the circuit, an inverse quantum Fourier transform (QFT) is applied to the ℓ ancillas. This unitary requires depth $D = O(\ell)$ [54] (even assuming 1D locality constrains). It is easy to see that these operations map a state $|\psi\rangle$ into $\sum_{i_1,\ldots,i_\ell=0}^1 |i_1, \ldots, i_\ell\rangle \otimes \Pi_i^\ell |\psi\rangle$, where $i_1 \cdots i_\ell$ is the binary representation of i. The desired measurement, with the expected probability distribution, is then achieved by performing a projective measurement onto the ℓ ancillas. Note that $U^{(1)}(x) = [U^{(1)}(\ell)]^{2^{\ell-x}}$, and thus the protocol is the same of the phase estimation algorithm [55], with the difference that $|\psi\rangle$ is not an eigenstate for $U^{(1)}(\ell)$. We note that a similar construction to measure the number of excitations was first given in Ref. [45].

Preparation of Dicke states—We are now in a position to describe our protocol for the preparation of the Dicke state $|W(M)\rangle$. Fixing $M \leq N/2$, set $p = M/N$ and define $|\Psi(p)\rangle = (\sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle)^{\otimes N}$ which can be trivially prepared with $D = 1$. Now, if we could perform a measurement of the number of excitations and force the outcome to M, then we would obtain $|W(M)\rangle$. This is because of the identity

$$|\Psi(p)\rangle = \sum_{e=0}^{N} \left[\binom{N}{e} p^e (1-p)^{N-e} \right]^{1/2} |W(e)\rangle, \quad (3)$$

which implies $\Pi_M |\Psi(p)\rangle \propto |W(M)\rangle$. Based on this observation and our previous results, it is easy to devise a preparation scheme. The idea is to perform a measurement corresponding to the projectors $\{\Pi_j^\ell\}_j$ for sufficiently large ℓ, and repeat the procedure N_r times until we get the desired measurement outcome M. At the end of this procedure we obtain a final state $|\psi_\ell\rangle \propto \Pi_M^\ell |\Psi(p)\rangle$. The accuracy of the protocol is controlled by the infidelity $I = |1 - |\langle W(M)|\psi_\ell^2\rangle|^2|$, while the number of repetitions depends on the probability P_M of obtaining the outcome M. By inspection of the state (3), we find $I \sim e^{-2^\ell}$, $P_M \sim M^{-1/2}$ [56], and we arrive at the following:

Result 3—Preparation of Dicke states. Up to an infidelity $I = \varepsilon$, the Dicke state $|W(M)\rangle$ can be prepared with $N_r = O(\sqrt{M})$, $N_a = 1$, $D = O(\ell_{M,\varepsilon})$ and $\ell_{M,\varepsilon}$ additional ancillas, where

$$\ell_{M,\varepsilon} = \max\left\{ \log_2(4M), 1 + \log_2 \ln(\sqrt{8\pi M}/\varepsilon) \right\}. \quad (4)$$

Alternatively, by slight modifications of the protocol, it is not difficult to show that one can trade the depth with the number of ancillas, realizing a circuit with $N_a = O(\ell_{M,\varepsilon})$, $N_r = O(\sqrt{M})$, $D = O(1)$ [56]; cf. Table I. Note that both the number of repetitions and the depth of the circuit do not scale with the system size. In addition, note that we assumed that $\ell_{M,\varepsilon}$ is smaller than $\log_2(L)$. Indeed, for $D = \log_2(L)$ the circuit in Fig. 2 performs a measurement of N_e, so the Dicke state is prepared exactly.

We stress that a small infidelity (independent of N) automatically guarantees an accurate description of correlation functions. Indeed, denoting by $\langle \mathcal{O} \rangle_\psi = \langle \psi | \mathcal{O} | \psi \rangle$,

we have $|\langle\mathcal{O}\rangle_\psi - \langle\mathcal{O}\rangle_\phi| \leq 2(1-|\langle\psi|\phi\rangle|^2)^{1/2}\|\mathcal{O}\|_\infty$, where $\|\mathcal{O}\|_\infty$ is the operator norm. Since the latter equals one for any product of Pauli matrices, we obtain that correlation functions in the prepared state will be arbitrarily close to those of the Dicke state. Finally, in some cases one may need to obtain the Dicke state up to an exponential accuracy. In this case, as mentioned, we can run our protocol implementing the measurement of N_e exactly, leading to an overall depth $O(\log L)$.

The W state—For $M=1$, the above construction gives us an efficient protocol for the W state. In this case, there exists an alternative construction which, while less efficient, is simpler and could be of interest for implementation in NISQ devices. The idea is to prepare the product state $(\sqrt{1-\delta/N}|0\rangle + \sqrt{\delta/N}|1\rangle)^{\otimes N}$ and simply measure the parity of the excitations. The protocol is successful if the outcome is odd, in which case it yields a state which we call $|\Phi(\delta)\rangle$. Denoting by $|W\rangle$ the W state, it is easy to see that $|1-|\langle W|\Phi(\delta)\rangle|^2 \leq \delta^2/4$ and that the probability of success is larger than $\delta/2$. On the other hand, the measurement of the parity corresponds to the set $\{\Pi_j^\ell\}_j$ with $\ell=1$, so it can be done efficiently using Result 2. Therefore, we have the following:

Result 4—Up to an infidelity $I=\varepsilon$, the W state can be prepared with $N_r = O(1/\sqrt{\varepsilon})$, $N_a=1$, $D=O(1)$.

Improved scheme via amplitude amplification—Using our previous protocol, the average preparation time of the Dicke state scales as $N_r = O(\sqrt{M})$, because we have to do this number of repetitions to have a high probability of success. The reason is that, given the initial state $|\Psi(p)\rangle$, the probability of having M excitations scales as $1/\sqrt{M}$. We now show how we can exploit the Grover algorithm [or its practical version, named amplitude amplification protocol (AAP) [60–63]] to improve this result. It is important to notice that a direct application of that algorithm makes the resources dependent on the system size, N, something that we want to avoid. Thus, we have to devise an alternative method, which is consistent with the approximation, that circumvents this obstacle.

We recall that, given $|\psi\rangle = \sin\alpha|\psi_1\rangle + \cos\alpha|\psi_2\rangle$, and denoting by $|\tilde{\psi}\rangle$ the state orthogonal to $|\psi\rangle$ in the subspace generated by $|\psi_1\rangle$ and $|\psi_2\rangle$, the AAP allows one to obtain $|\psi_1\rangle$ by applying a product of $O(1/\alpha)$ unitaries $S_1(\omega_j)$, $S_2(\omega_j)$ (for α small), which act as follows

$$S_1(\omega)|\psi\rangle = e^{i\omega}|\psi\rangle, \qquad S_1(\omega)|\tilde{\psi}\rangle = |\tilde{\psi}\rangle, \tag{5a}$$

$$S_2(\omega)|\psi_1\rangle = e^{i\omega}|\psi_1\rangle, \qquad S_2(\omega)|\psi_2\rangle = |\psi_2\rangle, \tag{5b}$$

where $\omega_j \in \mathbb{R}$ depend on α. Writing $|\Psi(p)\rangle = \sin\alpha|W(M)\rangle + \cos\alpha|R\rangle$, we see that if $S_1(\omega)$, $S_2(\omega)$ can be implemented with circuits of constant depth, then the AAP gives us a deterministic algorithm to obtain $|\psi_1\rangle$ with $D=O(M^{1/4})$, thus reducing the preparation time.

Realizing the operators in Eqs. (5) exactly could be done by known methods using $N_a = \log_2(N)$ and $D=O(1)$ [44]. Instead, we show that, applying ideas similar to those developed so far, an approximate version of them can be realized using a finite amount of resources [56]. This leads to the following improved version of Result 3:

Result 5—Improved scheme via amplitude amplification. Up to an infidelity $I=\varepsilon$, the Dicke state $|W(M)\rangle$ can be prepared deterministically ($N_r=1$), with $N_a=1$, $\ell_{M,\varepsilon}$ additional ancillas, and $D=O(M^{1/4}\ell_{M,\varepsilon}^2)$, where

$$\ell_{M,\varepsilon} = \log_2\left\{\frac{1}{\ln(4/3)}[2M(\ln 2M + 9/2) + \ln(\mathrm{Poly}(M)/\varepsilon^2)]\right\}. \tag{6}$$

Eigenstates of the XX Hamiltonian—Going beyond the Dicke model, the previous ideas have ramifications for other Hamiltonians whose eigenstates are labeled by the number of excitations. As a first example, we discuss the well-known XX spin chain $H = -\sum_{k=1}^{N-1}(\sigma_k^x\sigma_k^x + \sigma_k^y\sigma_k^y)$. This model can be solved via the Jordan-Wigner (JW) transformation $a_k = \left(\prod_{j=1}^{k-1}\sigma_j^z\right)\sigma_k^-$, mapping it to a noninteracting Hamiltonian $H = -\sum_{k=1}^{N-1}(a_j^\dagger a_{j+1} + \mathrm{H.c.})$, where $\{a_j^\dagger, a_k\} = \delta_{j,k}$. Accordingly, the eigenstates read $|\Phi(M)\rangle = A_M^\dagger \cdots A_1^\dagger|0\rangle^{\otimes N}$ with

$$A_\alpha = \sum_{k=1}^{N} c_k^\alpha \left(\prod_{j=1}^{k-1}\sigma_j^z\right)\sigma_k^-. \tag{7}$$

Here, $\{c_k^\alpha\}$ are distinct sets of coefficients, such that $\{A_\alpha, A_\beta\} = 0$, $\{A_\alpha^\dagger, A_\beta\} = \delta_{\alpha,\beta}$ [64], while $M=0,\ldots,N$.

The form of the eigenstates is superficially similar to that of the Dicke states, but it is more complicated due to nonuniform coefficients c_k^α and the string operators $\prod_j \sigma_j^z$. Yet, the anticommutation relations of A_α allows us to devise an efficient preparation protocol. Indeed, the latter implies that $|\Phi(M)\rangle = W_M \cdots W_1|0\rangle$, where $W_j = e^{i\pi(A_j + A_j^\dagger)/2}$. The W_j are unitary and, using our previous constructions, we find that they can be realized deterministically with depth $D=O(N)$ [56]. Therefore, the eigenstates of the XX Hamiltonian with M excitations can be prepared deterministically ($N_r=1$) by a QC with LOCC of depth $D=O(NM)$ and $N_a=1$.

The preparation of spin states which can be mapped onto free (or Gaussian) fermionic states has been considered before [46–48,65–72]. Reference [46] finds a unitary algorithm preparing arbitrary Gaussian operators with depth $O(N)$, yielding a more efficient protocol. However, our approach also allows us to prepare states which are not Gaussian and in principle out of the reach of previous work.

For instance, we could prepare $A^\dagger_{\alpha_1}\cdots A^\dagger_{\alpha_n}|\phi_0\rangle$, where $|\phi_0\rangle$ is any linear combination of Gaussian states (assuming $|\phi_0\rangle$ can be prepared efficiently). We also expect that our method could be further improved and generalized to more interesting situations.

Eigenstates of interacting Hamiltonians—As a final example, we consider general states of the form

$$|\Psi_M\rangle = B^\dagger_M \ldots B^\dagger_1 |0\rangle^{\otimes N}, \tag{8}$$

where $B^\dagger_\alpha = \sum_{j=1}^N c^\alpha_j \sigma^+_j$ are interpreted as creating spin excitations. These states are quite general, including the Dicke states and the eigenstates of the so-called Richardson-Gaudin spin chain [49,50], an interacting integrable model. Without assumptions on the coefficients c^α_j, efficient preparation of (8) is challenging. Here, we will assume that we are in the "low excitation regime," namely $M \ll N$, and that $\sum_{j=1}^N \bar{c}^\alpha_j c^\beta_j = \delta_{\alpha,\beta} + O(M/N)$. If $|\psi\rangle$ has at most M excitations, this implies

$$[B_\alpha, B^\dagger_\beta]|\psi\rangle = \delta_{\alpha,\beta}|\psi\rangle + O(M/N). \tag{9}$$

Namely, $B^\dagger_\alpha$ act as creation operators, up to a $O(M/N)$ error. This allows us to devise a simple preparation protocol, sketched below, and estimate the number of resources needed. We postpone a more detailed analysis of the states (8) to future work, including a full study of the Richardson-Gaudin eigenstates.

First, suppose that (9) holds exactly, i.e., without the term $O(M/N)$. Then, we create the state (8) by induction. Assuming we have prepared $|\Psi_{M-1}\rangle$, we apply

$$e^{i\theta(B_M + B^\dagger_M)}|\Psi_{M-1}\rangle = \sum_{k=0}^{N} d_k |\Psi_{M-1+k}\rangle, \tag{10}$$

where we used $B_M|\Psi_{M-1}\rangle = 0$, so that the number or excitations cannot decrease. Now, we measure the number of excitations, using the circuit described in Result 2, neglecting for simplicity exponentially small errors in the circuit depth. In case we obtain $k = 1$ we have succeeded. If $k = 0$, we have not changed anything so that we can repeat the procedure. If $k \geq 2$, then we have failed. The probability of failing and obtaining $M = 1$ are, respectively, $O(\theta^4)$ and $O(\theta^2)$. We can iterate this procedure to prepare $|\Psi_M\rangle$ starting from $|0\rangle^{\otimes N}$. It is easy to show that the preparation time scales as $O(M/\theta^2)$, while the success probability is $O(e^{-M\theta^2})$, independent of N.

If we do not neglect the term $O(M/N)$ in (9), then the above construction introduces additional errors. While this is not relevant for the probabilities, the state (10) contains corrections for each M. The latter can be estimated as follows. If we have to repeat the procedure r times (on average), the error is rM/N for each step. Accordingly, the total error will be $\varepsilon = rM^2/N$. Since $r = 1/\theta^2$, and the probability of not detecting $M = 0{,}1$ in any procedure scales as $p_{\text{fail}} = rM\theta^4$, by setting $p_{\text{fail}} = 1/2$ we have $\varepsilon = M^3/N$. Thus, this allows us to create $M = O(N^{1/3})$ excitations if we take N large.

Outlook—We have introduced protocols to prepare many-body quantum states using QCs and LOCCs. We have shown how we can save resources by relaxing the condition of preparing the states exactly and deterministically but allowing for controlled infidelities and probabilities of failure. Our results are expected to be relevant for quantum-state preparation in present-day quantum devices, also in light of recent experiments operating QCs assisted by feed-forward operations [26,73–75]. Our work also raises several theoretical questions. For instance, it would be interesting to explore the possibilities of this approach to prepare eigenstates of more general interacting Hamiltonians. In addition, it would be important to understand how the classification of phases of matter via quantum circuits and LOCC introduced in Ref. [16] is modified by allowing for finite infidelities. We leave these questions for future work.

Acknowledgments—We thank Harry Buhrman and Marten Folkertsma for useful discussions. The research is part of the Munich Quantum Valley, which is supported by the Bavarian state government with funds from the Hightech Agenda Bayern Plus. We acknowledge funding from the German Federal Ministry of Education and Research (BMBF) through EQUAHUMO (Grant No. 13N16066) within the funding program Quantum Technologies—From Basic Research to Market. This work was funded by the European Union (ERC, QUANTHEM, 101114881). Views and opinions expressed are, however, those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

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