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On the propagation of flatness for second order hypoelliptic operators

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Abstract

For a class of hypoelliptic operators with real-analytic coefficients, we provide a criterion ensuring a partial analyticity result. As a consequence, even when the "elliptic" strong unique continuation (i.e. a solution of the homogeneous equation which vanishes of infinite order at a point is zero near such a point) fails, a weaker form of "propagation" of zeroes still holds.

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1. Introduction and statement of the results

Let $\Omega \subset \mathbb{R}^d$ be an open convex set and let $P = P(x, \partial)$ be a C^∞ hypoelliptic second order partial differential operator (i.e. $u \in C^\infty$ if $Pu \in C^\infty$). We consider an arbitrary solution of the equation

$$P(x, \partial)u = 0 \qquad \text{in} \qquad \Omega \tag{1.1}$$

and we set[2]

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[2] We assume that $0 \in \mathbb{N}$.

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$$Z_\infty(u) = \{x \in \Omega : \partial^\alpha u(x) = 0, \quad \forall \alpha \in \mathbb{N}^d\},$$

i.e. $Z_\infty(u)$ is the set of all the points where u vanishes of infinite order.

We study the structure of the set $Z_\infty(u)$. More precisely, we want to find conditions on P guaranteeing that $Z_\infty(u)$ contains no isolated points.

As a partial explanation of the hypoellipticity assumption, we point out that, as shown in the next example, $Z_\infty(u)$ may be a singleton if P is hyperbolic.

Example 1. Consider the function

$$\varphi(s) = \begin{cases} 0 & \text{for } s \le 0, \\ e^{-\frac{1}{s}} & \text{for } s > 0, \end{cases}$$

and, for $x, t \in \mathbb{R}$, set

$$u(x,t) = \varphi(x-t) + \varphi(x+t).$$

Then, $(\partial_t^2 - \partial_x^2)u = 0$ and u is flat only at the origin, i.e. $Z_\infty(u) = \{(0,0)\}$.

In order to gain some intuition on the problem, let us begin by recalling some basic facts in the case of P with constant coefficients. The following (well-known) characterization holds

Theorem 1.1. *Let P be a linear partial differential operator with constant coefficients. Then the following assertions are equivalent*

(A) *P is elliptic;*
(B) *P satisfies the strong unique continuation property (i.e. zero is the only solution of* (1.1) *flat at least at one point);*
(C) *P is analytic hypoelliptic (i.e. Pu is real-analytic implies that u is real-analytic).*

Remark 1.1. Let us provide an explanation of the equivalence above. The implication $(A) \Longrightarrow (B)$ can be understood from the point of view of the hypoellipticity. If P is an elliptic operator then, since it has constant coefficients, it is analytic hypoelliptic[3] then, all the (distributional) solutions of $Pu = 0$ are real-analytic functions. Then, the strong unique continuation property is a consequence of the fact that a real-analytic function vanishing of infinite order at a given point is identically zero (on the connected component of its domain containing such a point). The opposite implication, $(B) \Longrightarrow (A)$, can be proved by showing that non-$(A) \Longrightarrow$ non-(B) and using the fact that the characteristic Cauchy Problem is not well-posed. More precisely, if P is not elliptic, then there exists a characteristic hyperplane passing through the origin of $\mathbb{R}^d$ and a smooth solution of $Pu = 0$ vanishing on one side of the hyperplane but with $0 \in \operatorname{supp} u$ (see [15, Theorem 5.2.2]), i.e. the strong unique continuation property fails.

[3] For instance, in [15, Corollary 4.4.1], it is shown that all the solutions of $Pu = 0$ are real-analytic if and only if P is elliptic.

We observe that if P (possibly with variable coefficients) satisfies the strong unique continuation property, then for every solution u of (1.1) either $Z_\infty(u) = \emptyset$ or $Z_\infty(u) = \Omega$. Furthermore, if P is analytic hypoelliptic, then it satisfies the strong unique continuation property.

Remark 1.2. We point out that it is a difficult (open) problem to find a characterization of the operators P that are analytic-hypoelliptic (see e.g. [21,22], [19], [13], [2], [4], [6,7]). Furthermore, we observe that the equivalence contained in Theorem 1.1 fails in the case of variables coefficients, e.g., in dimension two, the operator $P = \partial_x^2 + x^{2k}\partial_y^2$ is analytic hypoelliptic (see, e.g., [1,2]) but it is not elliptic.

The study of the heat operator suggests that, even in the absence of the strong unique continuation property, some weaker form of unique continuation may still hold true. Indeed, $P = \partial_t - \Delta$ does not satisfy the strong unique continuation property (this is a consequence of the fact that P is not elliptic and by Theorem 1.1) but it satisfies a unique continuation along time slice, i.e. a solution of the equation

$$(\partial_t - \Delta)u = 0 \qquad \text{in} \qquad \Omega,$$

which is flat[4] at $(t_0, x_0) \in \Omega$, it is zero in $\Omega \cap \{t = t_0\}$. This result can be proved, e.g., using either Carleman estimates with a singular weight (see, e.g., [20]) or arguing as in the proof of Theorem 1.2 (ii) below.

We consider the following question:

- how we can describe "geometrically" the set where the points of flatness propagate?

We will see that the answer to the previous question suggests a condition ensuring that $Z_\infty(u)$ contains no isolated points.

The symbol of the heat operator is

$$i\tau + |\xi|^2, \qquad \text{its principal part is} \qquad |\xi|^2,$$

and the characteristic manifold (i.e. the set where the principal part of the operator vanishes) is given by

$$\mathrm{Char}(P) = \{(t, x; \tau, 0) | \tau \neq 0\}.$$

We observe that, since P has constant coefficients, every point (t_0, x_0) can be considered as the projection into the base of a point $(t_0, x_0; \tau, 0) \in \mathrm{Char}(P)$. The characteristic manifold admits an Hamiltonian foliation. More precisely, at every $\rho \in \mathrm{Char}(P)$ one associates the vector space

$$T_\rho \mathrm{Char}(P) \cap (T_\rho \mathrm{Char}(P))^\sigma,$$

[4] $P = \partial_t - \Delta$ is C^∞ hypoelliptic, then, in particular, all the distributional solutions of the homogeneous equation are smooth.

where $T_\rho \operatorname{Char}(P)$ is the tangent space to $\operatorname{Char}(P)$ at the point ρ (i.e. $T_\rho \operatorname{Char}(P) = \{(\delta t, \delta x; \delta\tau, 0)\}$), while $(T_\rho \operatorname{Char}(P))^\sigma$ is the orthogonal, w.r.t. the symplectic form $\sigma = \sum d\xi_j \wedge dx_j$, to $T_\rho \operatorname{Char}(P)$ (i.e. $(T_\rho \operatorname{Char}(P))^\sigma = \{(0, \delta x; 0, 0)\}$).

We observe that the Hamiltonian leaf passing through the point $\rho := (t_0, x_0; \tau_0, 0)$ (i.e. the integral manifold of the distribution, in the sense of the differential geometry, $T_\rho \operatorname{Char}(P) \cap (T_\rho \operatorname{Char}(P))^\sigma = \{(0, \delta x; 0, 0)\}$ passing through ρ) is

$$F_\rho = \{(t_0, x; \tau_0, 0) \, : \, (t_0, x) \in \Omega\},$$

and its projection on the base is given by the manifold

$$\pi(F_\rho) = \{(t_0, x) \, : \, (t_0, x) \in \Omega\}.$$

In this perspective, the (parabolic) unique continuation can be recast as a sort of "propagation" result as follows: if a solution of the heat equation vanishes of infinite order at a point then it vanishes of infinite order along the projection into the base of the Hamiltonian leaf through (a lifting of) such a point.

Motivated by the above considerations, one can try to show that, for an hypoelliptic operator with real–analytic coefficients, the vanishing of infinite order at a point propagates along the projection, into the base, of the Hamiltonian leaf through a lifting of the given point.

We consider a class of second order operators of the form sums of squares of real-analytic coefficients satisfying the Hörmander's brackets generating condition.

Notice that, for this class of operators, the solutions of (1.1) are smooth with a suitable control on the growth of their derivatives and, in particular, the solutions of $Pu = 0$ may be flat at a point without being identically zero. More precisely, it is well-known that the analytic regularity assumption on the vector fields and Hörmander's condition do not imply that P be analytic-hypoelliptic (see [5]). On the other hand, let us also recall that, as a consequence of Hörmander's condition, there exists $s_0 > 1$ such that P is at least hypoelliptic in Gevrey spaces of index $s \geq s_0$ (see [11], [3], and [8]).

We assume

(H) the vector fields

$$X_j = \sum_{k=1}^{n} a_{jk}(y) \frac{\partial}{\partial y_k}, \qquad (j = 0, 1, \ldots, N)$$

are real-analytic (w.r.t. the variables $y \in \mathbb{R}^n$) and satisfy Hörmander's brackets generating condition in $\mathbb{R}^n$.

Set

$$P = X_0 + \sum_{j=1}^{N} X_j^2 + \Delta, \tag{1.2}$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_\ell^2}.$$

Remark 1.3. We notice that, due to [14], the operator given in (1.2) is C^∞ hypoelliptic. We recall that, as shown in [10, Théorème 2.2.], for an operator of the form $P = X_0 + \sum X_j^2$ with real-analytic coefficients, if at every point at least one vector field is non-singular then Hörmander's bracket generating condition is equivalent to the C^∞ hypoellipticity of P.

We have the following

Theorem 1.2. *Under Assumption (H), let P be as in (1.2), and let u be a solution of $Pu = 0$. Then,*

(i) *u is real-analytic with respect to the variables x;*
(ii) *let $(x_0, y_0) \in Z_\infty(u)$, then $(x, y_0) \in Z_\infty(u)$, for every $(x, y_0) \in \Omega$.*

Example 2. The heat operator associated with the Baouendi-Goulaouic operator,

$$P = \partial_t - \partial_x^2 - \partial_y^2 - y^{2(q-1)}\partial_z^2 \tag{1.3}$$

where $q > 1$ is a positive integer, is hypoelliptic in Gevrey spaces of index $s \ge q$ (see [11] and [8]). In this case, if a solution of $Pu = 0$ in Ω vanishes of infinite order at (t_0, x_0, y_0, z_0), then $(t_0, x, y_0, z_0) \in Z_\infty(u)$, for every $(t_0, x, y_0, z_0) \in \Omega$.

Remark 1.4. Notice that Theorem 1.2 does not depend on the presence of the vector fields X_0: the same result holds assuming that $X_1, \ldots, X_N$ satisfy the Hörmander condition and taking $P = \sum_{j=1}^N X_j^2 + \Delta$. Furthermore, we point out that, in the absence of additional information, the conclusion of Theorem 1.2 is optimal. Indeed, in [17] it is constructed a solution of the homogeneous heat equation (i.e. $N = 0$, $X_0 = \partial_{y_1}$) supported in $y_1 \in [0, \varepsilon]$, for every $\varepsilon > 0$. In other words, in general, knowing that u vanishes of infinite order at a point (t_0, x_0), the only prediction that can be done is that u vanishes along a time slice, (t_0, x).

Let us clarify how Theorem 1.2 (ii) is connected with the "rule of thumb" introduced for the heat equation. Due to the generality of the vector fields X_j, $j = 0, \ldots, N$, in Theorem 1.2, it is unclear if Char(P) is a manifold and if it is defined on it an Hamiltonian foliation. On the other hand, if we assume that there exists an Hamiltonian leaf F_{ρ_0}, then

$$\{(x_0 + x, y_0) \in \Omega\} \subset \pi(F_{\rho_0}),$$

where $\pi(\rho_0) = (x_0, y_0)$.

2. Proofs

From a technical point of view our proof of Theorem 1.2 is based on energy estimates.

2.1. *Proof of Theorem 1.2* (i)

Our proof is more than inspired by the one given in [9] for the Oleinik operator. Without loss of generality, we may suppose that $\ell = 1$, i.e. that x is a single real variable. We want to estimate the growth of the x-derivatives of u. We observe that, in [9], the (optimal) subelliptic estimate given in [18] is used. In the present situation, we take advantage of the special structure of the operator and we use the following

Lemma 2.1 *(Energy estimate). We have that*

$$\frac{1}{2}\left(\|\partial_x v\|^2 + \sum_{j=1}^{N}\|X_j v\|^2\right) \le |\langle Pv, v\rangle| + C\|v\|^2, \tag{2.4}$$

for every $v \in C_0^\infty$.

(Here $\|\cdot\|$ stands for the L^2 norm, while $\langle\cdot,\cdot\rangle$ is the L^2 scalar product.)

Proof of Lemma 2.1. For every $v \in C_0^\infty$, we have that

$$|\langle Pv, v\rangle| = \left|\left\langle\left(-X_0 - \sum_{j=1}^{N} X_j^2 - \partial_x^2\right)v, v\right\rangle\right|.$$

Integrating by parts we find that

$$\begin{aligned}
|\langle Pv, v\rangle| &= \left|-\int X_0\frac{v^2}{2} - \sum_{j=1}^{N}\langle X_j v, X_j^* v\rangle + \|\partial_x v\|^2\right| \\
&= \left|\int (\operatorname{div} X_0)\frac{v^2}{2} + \sum_{j=1}^{N}\|X_j v\|^2 + \sum_{j=1}^{N}\langle X_j v, (-X_j - {}^t X_j)v\rangle + \|\partial_x v\|^2\right| \\
&\ge \sum_{j=1}^{N}\|X_j v\|^2 + \|\partial_x v\|^2 - \frac{1}{2}\sum_{j=1}^{N}\left(\|X_j v\|^2 + \|(-X_j - {}^t X_j)v\|^2\right) \\
&\qquad - \left|\int (\operatorname{div} X_0)\frac{v^2}{2}\right| \\
&\ge \frac{1}{2}\sum_{j=1}^{N}\|X_j v\|^2 + \|\partial_x v\|^2 - C\|v\|^2,
\end{aligned}$$

for a suitable $C > 0$. (Here we used the fact that ${}^t X_j + X_j$ is a zero order term.) □

We estimate the localization of high order derivatives of u using the estimate (2.4). These localizations are of the form $\chi(x, y)\partial_x^s u$. We observe that, since we are looking for an analytic regularity w.r.t. the variables x we have to use, as localizing functions, some special cut-off functions introduced by Ehrenpreis in [12] (see [16]). For any pair of open sets ω, Ω, with ω compactly contained in Ω, there exists a constant C_0, and, for any m, $\chi_m \in C_0^\infty(\Omega)$ such that

(A) $\chi_m \equiv 1$ on ω;
(B) $|\partial^\alpha \chi_m| \le C_0(C_0 m)^{|\alpha|}$, for $|\alpha| \le 2md$.

Remark 2.5. For $|\alpha|$ close to m, the bounds given in (B) above resemble the analytic growth.[5]

In the sequel, for the sake of simplicity, we write χ instead of χ_m.

We plug $v = \chi \partial_x^r u$ into estimate (2.4), we have

$$\frac{1}{2}\left(\|\partial_x \chi \partial_x^r u\|^2 + \sum_j \|X_j \chi \partial_x^r u\|^2\right) \leq |\langle P\chi \partial_x^r u, \chi \partial_x^r u\rangle| + C\|\chi \partial_x^r u\|^2 \underbrace{=}_{\text{since } Pu=0} |\langle [P, \chi \partial_x^r]u, \chi \partial_x^r u\rangle| + C\|\chi \partial_x^r u\|^2. \tag{2.5}$$

The first term in the line above lead to the bracket

$$\langle [X_j^2, \chi \partial_x^r]u, \chi \partial_x^r u\rangle = \langle [X_j^2, \chi]\partial_x^r u, \chi \partial_x^r u\rangle,$$

for $j = 1, \ldots, N$. We have that

$$\begin{aligned}\langle [X_j^2, \chi]\partial_x^r u, \chi \partial_x^r u\rangle &= \langle X_j[X_j, \chi]\partial_x^r u, \chi \partial_x^r u\rangle + \langle [X_j, \chi]X_j \partial_x^r u, \chi \partial_x^r u\rangle \\ &= \langle X_j(\chi)\partial_x^r u, {}^t X_j \chi \partial_x^r u\rangle + \langle \chi X_j \partial_x^r u, \chi X_j(\chi)\partial_x^r u\rangle\end{aligned}$$

since $X_j^* = -X_j + a$ (a zero order term)

$$\begin{aligned}&= \langle X_j(\chi)\partial_x^r u, (-X_j + a)\chi \partial_x^r u\rangle + \langle ([\chi, X_j] + X_j\chi)\partial_x^r u, X_j(\chi)\partial_x^r u\rangle \\ &\leq \frac{1}{\varepsilon}\|X_j(\chi)\partial_x^r u\|^2 + 2\varepsilon\|X_j\chi \partial_x^r u\|^2 + 2\varepsilon C_1\|\chi \partial_x^r u\|^2 \\ &\quad - \|X_j(\chi)\partial_x^r u\|^2 + \varepsilon\|X_j\chi \partial_x^r u\|^2 + \frac{1}{\varepsilon}\|X_j(\chi)\partial_x^r u\|^2 \\ &\leq \frac{2}{\varepsilon}\|X_j(\chi)\partial_x^r u\|^2 + 3\varepsilon\|X_j\chi \partial_x^r u\|^2 + 2\varepsilon C_1\|\chi \partial_x^r u\|^2,\end{aligned}$$

where $\varepsilon > 0$ and $C_1 > 0$ is a suitable constant. We observe that, for $\varepsilon < 1/6$, the second term in the sum above can be reabsorbed in the LHS of (2.5). Furthermore, we have

$$|\langle [X_0, \chi \partial_x^r]u, \chi \partial_x^r u\rangle| = |\langle X_0(\chi)\partial_x^r u, \chi \partial_x^r u\rangle| \leq \|X_0(\chi)\partial_x^r u\|^2 + \|\chi \partial_x^r u\|^2$$

Then, we deduce that

[5] Due to the Stirling formula $m^m \leq C^m m!$, for a suitable C.

$$\frac{1}{2}\|\partial_x \chi \partial_x^r u\|^2 \le \frac{2}{\varepsilon}\sum_{j=0}^{N}\|X_j(\chi)\partial_x^r u\|^2 + \left(C+1+N\frac{C_1}{3}\right)\|\chi\partial_x^r u\|^2,$$

i.e.

$$\frac{1}{4}\|\chi\partial_x^{r+1}u\|^2 \le \frac{1}{2}\|(\partial_x\chi)\partial_x^r u\|^2 + \frac{2}{\varepsilon}\sum_{j=0}^{N}\|X_j(\chi)\partial_x^r u\|^2 + \left(C+1+N\frac{C_1}{3}\right)\|\chi\partial_x^r u\|^2.$$

We observe that the latter inequality can be rewritten as

$$\|\chi\partial_x^{r+1}u\|^2 \le C_2\left\{\|(\partial_x\chi)\partial_x^r u\|^2 + \sum_{j=0}^{N}\|X_j(\chi)\partial_x^r u\|^2 + \|\chi\partial_x^r u\|^2\right\}, \quad (2.6)$$

for a suitable positive constant C_2.

Now, estimate (2.6) can be use in an iterative way: each term in the RHS gives $N+3$ terms with one x-derivative less on the function u. After r iterations, we find $(N+3)^{r+1}$ terms without any x-derivative on u and $r+1$ derivatives either on the cut-off function χ or on the coefficients of the vector fields X_j (there will be also a term with no derivatives at all but we consider only the "worst case scenario"). We claim that, for a suitable $C>0$, each one of these terms can be bounded by $CC^{r+1}(r+1)!$.

We set

$$Y_j = X_j, \quad (j=0,\dots,N), \qquad Y_{N+1} = \partial_x,$$

and, for $I=(i_1,\dots,i_{r+1})$, $i_j\in\{0,\dots,N+1\}$, we define

$$Y_I(\chi) := Y_{i_1}(Y_{i_2}(\cdots(Y_{i_{r+1}}(\chi))\cdots)).$$

We may also suppose that each coefficients and, taking $m\sim r+1$, the cut-off function and the coefficients of the vector fields satisfy estimates with the same constant. We point out that in these computations it is irrelevant that Y_{N+1} has constant coefficients and that it commutes with the other vector fields. We have

$$\begin{aligned}
&(1) \qquad |Y_{i_{r+1}}(\chi)| \le C^2Cm = C^3m,\\
&(2) \qquad |Y_{i_r}(Y_{i_{r+1}}(\chi))| \le C^3(C^2m + C^2m^2) \le C^5(m+1)^2,\\
&(3) \qquad |Y_{i_{r-1}}(Y_{i_r}(Y_{i_{r+1}}(\chi)))| \le C^4(Cm(C+Cm)(2C+Cm)) \le C^7(m+1)^3\\
&\qquad\qquad \dots\\
&(r+1) \qquad |Y_I(\chi)| \le CC^{2|I|}|I|! = CC^{2(r+1)}(m+1)^{r+1}.
\end{aligned}$$

Then, taking $C_3=C^2$ (we may always assume that $C>1$), we obtain

$$\|\chi\partial_x^{r+1}u\|^2 \le (N+3)^{r+1}C_3C_3^{r+1}(m+1)^{r+1},$$

and

$$\|\chi \partial_x^\alpha u\|^2 \leq (N+3)^{|\alpha|} C_3 C_3^{|\alpha|} (m+1)^{|\alpha|}$$

with $x \in \mathbb{R}^\ell$, $|\alpha| = \alpha_1 + \ldots + \alpha_\ell$, and $\alpha! = \alpha_1! \ldots \alpha_\ell!$. Hence, due to the Sobolev embeddings, first, we obtain the analytic estimate in the L^∞ norm, and then we deduce the pointwise estimate:

$$|\partial_x^\alpha u(x, y)| \leq C C^\alpha \alpha!, \qquad (x, y) \in \omega,$$

for a suitable positive constant which we denoted once more by C, i.e. u is real-analytic w.r.t. the variable x. Finally, replacing u with $\partial_y^\beta u$, we conclude that

$$|\partial_x^\alpha \partial_y^\beta u(x, y)| \leq C_\beta C_\beta^{|\alpha|} \alpha!, \qquad (x, y) \in \omega,$$

for a suitable C_β. This completes our proof.

2.2. *Proof of Theorem 1.2 (ii)*

Since P is C^∞ hypoelliptic u is smooth, furthermore, due to Theorem 1.2 (i), one can find a neighborhood of (x_0, y_0), W, such that u is real-analytic w.r.t. the variables x. For the sake of simplicity we can take as W a ball with center at (x_0, y_0) and radius r, for a suitably small $r > 0$. Then, for every $\alpha \in \mathbb{N}^n$, the map

$$\{x \in \mathbb{R}^\ell \mid |x - x_0| < r\} \ni x \mapsto \partial_y^\alpha u(x, y_0)$$

is real-analytic and, by assumption, it vanishes of infinite order (w.r.t. t) at $x = x_0$. Then, we find that

$$\partial_y^\alpha u(x, y_0) \equiv 0 \qquad \forall |x - x_0| < r,$$

and for all $\alpha \in \mathbb{N}^n$, i.e.

$$\partial_y^\alpha \partial_x^\beta u(x, y_0) \equiv 0 \qquad \forall |x - x_0| < r$$

for every $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^\ell$. By iteration, we find that

$$\{(x, y_0) \in \Omega\} \subset Z_\infty(u).$$

This completes the proof of Theorem 1.2.

Data availability

No data was used for the research described in the article.

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