Vietnam Journal of Mathematics
https://doi.org/10.1007/s10013-023-00617-3

ORIGINAL ARTICLE

The Wallis Products for Fermat Curves

Alessandro Gambini[1] · **Giorgio Nicoletti**[2] · **Daniele Ritelli**[2]

Received: 25 February 2022 / Accepted: 19 November 2022

Abstract
After revisiting the properties of generalized trigonometric functions, i.e., the trigonometric function linked to the planar (Fermat) curve $x^p + y^p = 1$, using the tool of Keplerian trigonometry, introduced in (Gambini et al.: Monatsh. Math. 195, 55–72, 2021), we present the extension to this class of functions of the Wallis product, discovering connections with the representations of ordinary trigonometric functions by means of infinite products.

Keywords Generalized trigonometric functions · Keplerian maps · Eulerian functions · Infinite products · Wallis product

Mathematics Subject Classification (2010) Primary 33B10 · Secondary 40A20 · 33B15

1 Introduction

This paper studies generalized trigonometric functions, i.e., functions that parametrize the planar curve of Cartesian equation $x^p + y^p = 1$, where $p > 0$ is a positive integer. In the following we will refer to these curves as "Fermat curves".

This branch of mathematical research, in the case $p = 3$, was started by Cayley [4] and, shortly after, continued and extended by Dixon [5], who studied the cubic $x^3+y^3-3\alpha\, xy = 1$: the parametrizing functions "cm" and "sm" of that cubic, which were presented there, are mentioned by Whittaker and Watson in the classic monograph [29], and are known today as "Dixon's elliptic functions". Due to the fact that they both have a rhombic fundamental domain with 60 and 120 degree angles, Dixon's functions were employed to devise a conformal projection of the sphere on a regular hexagon by Adams [1] and on a equilateral triangle by

Work supported by RFO 2020–2021 (Panel 13) Italian grant funding.

✉ Daniele Ritelli
daniele.ritelli@unibo.it

Alessandro Gambini
alessandro.gambini@uniroma1.it

Giorgio Nicoletti
giorgio.nicoletti@gmail.com

1 Dipartimento di Matematica, Sapienza Università di Roma, Roma, Italy

2 Dipartimento di Scienze Statistiche, Università di Bologna, Bologna, Italy

Published online: 10 May 2023

Lee [16]. More recently, such functions have been deeply studied by Conrad and Flajolet [28], revealing their combinatorial nature and their link to the stochastic evolution of a Pólya urn process (see [7] and [13] for further details).

The first paper dedicated to the general case of exponent p is due to Grammel [10]. Actually trigonometric and hyperbolic functions generated by the Fermat curve have had visibility, in the entire mathematical community, thanks to the contributions of [2, 23, 25, 26, 30, 31].

Dixon himself observed that the variable of functions “cm” and “sm” when $\alpha = 0$ measures twice the area swept by the ray from the origin to the moving point on the curve; that idea was extended in [8] by the authors of this paper to the so called call “Keplerian Trigonometry”, that is a way to parametrize every curve $\mathscr{F}$ of a general family by means of the solution $(\cos_{\mathscr{F}}, \sin_{\mathscr{F}})$ of a suitable differential problem.

In this paper, we first revisit in Keplerian terms the properties of the trigonometric functions $\cos_p$ and $\sin_p$, and of their analogous $\cosh_p$ and $\sinh_p$, arising from the curve $x^p - y^p = 1$.

These preparatory lines will lead us to the main result of our article, which concerns the extension of the Wallis formula to generalized trigonometric functions, which in turn will allow us to obtain, probably by what is the most natural route, the formulas for representing trigonometric sine and cosine in terms of infinite products.

The similarities between the generalized trigonometric functions and the usual sine and cosine functions led us to retrace successfully the steps leading to the determination of the Wallis product. We feel it is important to point out that, starting with the integrals defining the inverses of generalized trigonometric functions, the fundamental tool that allowed us to carry out the integrations is due to the geometric relation between the integral of a function and that of its inverse.

In order to regain the expansions of the trigonometric sine and cosine in terms of infinite products, we point out that the properties of generalized hyperbolic functions also come into play.

Before starting our discussion, it is important to note that many authors have proposed generalizations of trigonometric functions close to those examined by us, but not coincident with them. We limit here to refer at the monograph [15], while for further indications we refer at the bibliography of [8]. Although we have highlighted this aspect in the introduction of our article [8] for the sake of completeness, we briefly explain what these non-negligible differences consist of. The generalized sine function introduced in [15], say $\sin_p(x)$, being $p \geq 1$, is the inverse of the integral

$$J_p(u) := \int_0^u (1 - t^p)^{-1/p} \mathrm{d}t.$$

The cosine $\cos_p$ is, then, defined as $\cos_p := \sin_p'$. In such a way, the identity $|\sin_p|^p + |\cos_p|^p = 1$ holds true. To appreciate the difference between this theory and ours, it is sufficient to observe that the sine function considered by us is based on the inversion of a different integral, see equation (6) below. Consequently, the sine and cosine functions obtained in [15] allow to parameterize the curve $|x|^p + |y|^p = 1$ while our treatment leads to the parametrization of $x^p + y^p = 1$.

2 Generalized Trigonometric and Hyperbolic Functions

In this section, first we briefly recall concepts and results related to “Keplerian facts”, as expounded in [8]; then, we define p-generalized trigonometric (hyperbolic) functions as the components of the unique Keplerian map that parameterizes the Fermat curve $\mathscr{F}_p$ ($\mathscr{F}_p^*$, respectively). The main properties of such functions are then shown. Lastly, equivalents of π are defined and computed.

2.1 Wedge Operator

The *wedge operator* Λ acts on a given smooth planar map $\mathbf{m}(t) := (x(t), y(t))$ to produce the signed area of the oriented parallelogram of sides $\mathbf{m}(t)$, $\mathbf{m}'(t)$, that is,

$$\Lambda\mathbf{m}(t) := x(t)y'(t) - x'(t)y(t).$$

If a map $\mathbf{m}(\kappa)$, whose domain contains 0, satisfies the axioms

$$\begin{cases} \mathbf{m}(0) = (1, 0), \\ \Lambda\mathbf{m} = 1, \end{cases}$$

then, its variable κ measures twice the signed area swept out by the the ray from the origin to $\mathbf{m}(t)$ when moving from $U_x := (1, 0)$ to $P := (x(\kappa), y(\kappa))$; this fact suggests to us to call such a map *Keplerian* (or *k-map* for short) (Fig. 1).

The image $\mathscr{C}$ of any smooth planar map $\mathbf{f} : I \to \mathbb{R}^2$, satisfying conditions

$$\begin{cases} \Lambda\mathbf{f}(t) \neq 0 & \text{for all } t \in I, \\ \mathbf{f}(t_0) = U_x & \text{for } t_0 \in I \end{cases}$$

is parametrized by the Keplerian map $\mathbf{m}(\kappa) := \mathbf{f}(t(\kappa))$, where $t(\kappa)$ is the inverse of the function

$$\kappa(t) := \int_{t_0}^{t} \Lambda\mathbf{f}\, \mathrm{d}u.$$

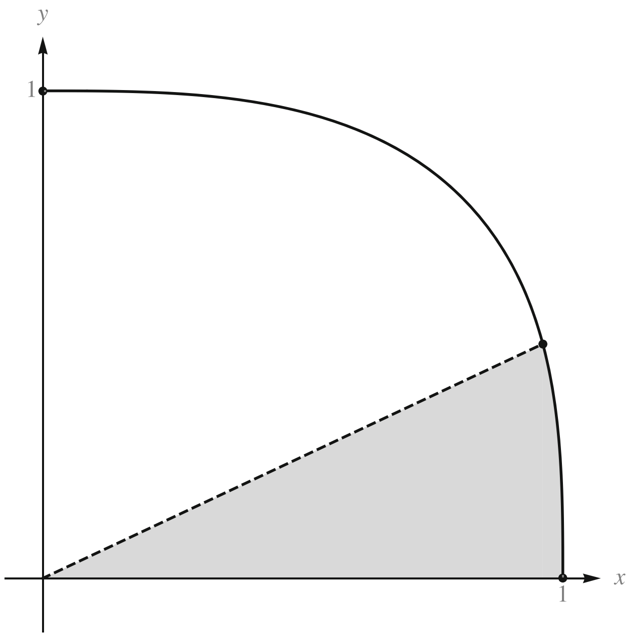

Fig. 1 Keplerian parametrization of $\mathscr{F}_p$ (here, $p = 3$)

The implicit curve $\mathscr{C} := \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$, where the function f satisfies the conditions

$$\begin{cases} f(1,0) = 0, \\ x f_x + y f_y \neq 0, \end{cases}$$

is parametrized by the Keplerian map solution of the problem

$$\begin{cases} x' = -\dfrac{f_y}{x f_x + y f_y}, & x(0) = 1, \\ y' = \dfrac{f_x}{x f_x + y f_y}, & y(0) = 0. \end{cases} \tag{1}$$

2.2 Useful Identities

The case that interests our discussion concerns the *p-Fermat curve* $\mathscr{F}_p := \{x^p + y^p = 1\}$, being p a natural number such that $p \geq 2$ for which, being $f(x, y) = x^p + y^p - 1$, we have that the differential system (1) becomes:

$$\begin{cases} x' = -y^{p-1}, & x(0) = 1, \\ y' = x^{p-1}, & y(0) = 0. \end{cases} \tag{2}$$

Once solved (2) $\mathscr{F}_p$ is the image of the keplerain map $\mathbf{t}_p(\kappa) = \big(x(\kappa), y(\kappa)\big)$, where we used the letter $\mathbf{t}$ to emphasize that the one found generates the "trigonometric system" associated to $\mathscr{F}_p$. It is therefore consistent—as done in [8, 10, 25, 28]—to call the components of $\mathbf{t}_p$ as the *p-cosine and p-sine functions* $\cos_p \kappa := x(\kappa)$, $\sin_p \kappa := y(\kappa)$, with inverses $\mathrm{arcos}_p(x)$, $\mathrm{arsin}(y)$, also setting

$$\tan_p \kappa := \frac{\sin_p \kappa}{\cos_p \kappa}, \quad \csc_p \kappa := -\frac{1}{\cos'_p \kappa}, \quad \sec_p \kappa := \frac{1}{\sin'_p \kappa}. \tag{3}$$

From identities (2) and (3), the following ones are derived[1]:

$$\begin{aligned} &\sin'_p 0 = 1, \qquad \sin''_p 0 = \sin_p 0 = 0, \\ &\tan'_p 0 = 1, \qquad \tan''_p 0 = \tan_p 0 = 0, \\ &\tan_p \kappa = \frac{\sin_p \kappa}{\cos_p \kappa} = \frac{\sin''_p \kappa}{\cos''_p \kappa}, \qquad \tan_p \kappa = \int_0^\kappa \frac{1}{\cos_p^2 u} \mathrm{d}u, \\ &\tan'_p \kappa = \frac{1}{\cos_p^2 \kappa}, \qquad \lim_{\kappa \to 0} \frac{\sin_p \kappa}{\kappa} = 1. \end{aligned} \tag{4}$$

In the following, in the case $p = 2$ of the usual trigonometric functions we will continue to write sin, cos etc. without using the subscript 2. From identities (2) we easily realize, by induction, that the nth derivatives of functions $\cos_p$ and $\sin_p$ are elements of the ring $\mathbb{Z}_{[\cos_p, \sin_p]}$. More precisely, $\mathrm{D}^i \cos_p$ and $\mathrm{D}^i \sin_p$ are polynomials in the variables $\cos_p$ and $\sin_p$ of degree $i(p-2)+1$, with integer coefficients. Still by induction, we derive the following relevant result about Maclaurin expansions of $\cos_p$ and $\sin_p$.

[1] Actually, such identities are shared by all Keplerian maps.

Theorem 2.1 *For every natural $p \geq 2$, there exist sequences of integers $C_p := ({}_p c_i)_{i\in\mathbb{N}}$ and $S_p := ({}_p s_i)_{i\in\mathbb{N}}$ such that*

$$\cos_p \kappa = \sum_{i=0}^{\infty} {}_p c_i \frac{\kappa^{ip}}{(ip)!}, \qquad \sin_p \kappa = \sum_{i=0}^{\infty} {}_p s_i \frac{\kappa^{ip+1}}{(ip+1)!}.$$

Proof First of all, we note by recursion that the hth derivative of a monomial $\cos_p^a \sin_p^b$ is a polynomial in $\mathbb{Z}_{[\cos_p, \sin_p]}$ of the form

$$\mathrm{D}^h \cos_p^a \sin_p^b = \sum_{i\geq 0}^{h} \theta_{h,i} \ \cos_p^{a_h - i(p-2)} \sin_p^{b_h + i(p-2)}.$$

Moreover, still for recurrence we get

$$b_h + h \equiv b \ (\mathrm{mod}\ p)$$

and if $b < p$

$$b_h < p$$

which, in our cases, provides

$$D^h \ \cos_p(0) = 0 \quad \text{whenever } b_h \not\equiv 0 \ (\mathrm{mod}\ p),$$
$$D^h \ \sin_p(0) = 0 \quad \text{whenever } b_h \not\equiv 1 \ (\mathrm{mod}\ p),$$

from which the statement is proved. □

Remark 2.2 It is worth noting that for $p = 3$ and $p = 4$ the relevant sequences C_p and S_p are known: they are listed in the famous repertoire [27] as entries $C_3 = A104134$, $S_3 = A104133$, $C_4 = A153300$, $S_4 = A153301$.

2.3 The Inverse Functions

Considering the identities $\cos_p' = -\sin_p^{p-1}$ and $\sin_p' = \cos_p^{p-1}$, we are able to to express the inverse functions as follows:

$$\operatorname{arcos}_p(x) = \int_1^x \frac{1}{-\sin_p^{p-1}} \mathrm{d}u = \int_x^1 (1-u^p)^{\frac{1-p}{p}} \mathrm{d}u, \tag{5}$$

$$\operatorname{arsin}_p(y) = \int_0^y \frac{1}{\cos_p^{p-1}} \mathrm{d}u = \int_0^y (1-u^p)^{\frac{1-p}{p}} \mathrm{d}u. \tag{6}$$

This enables us to state the following:

Theorem 2.3 *For every $p \in \mathbb{N}$, $p \geq 2$ the inverse of the function $\sin_p$ has Maclaurin expansion*

$$\operatorname{arsin}_p(y) = y + \sum_{i=1}^{\infty} \frac{\prod_{j=1}^{i}(jp-1)}{\prod_{j=1}^{i} jp} \frac{y^{ip+1}}{ip+1}.$$

Proof Thesis follows by a straightforward, but tedious, induction argument from (6). □

2.4 *p*th Hyperbolic Functions

The solution $\mathbf{h}_p = (x, y)$ of the problem

$$\begin{cases} x' = y^{p-1}, & x(0) = 1, \\ y' = x^{p-1}, & y(0) = 0, \end{cases}$$

is the Keplerian parametrization of the *p-Fermat* curve* $\mathscr{F}_p^* := \{x^p - y^p = 1,\ x \geq y\}$; the components of $\mathbf{h}_p$ are called the *hyperbolic p-cosine and p-sine functions*

$$\cosh_p \kappa := x(\kappa), \quad \sinh_p \kappa := y(\kappa),$$

with inverse functions $\operatorname{arcosh}_p(x)$, $\operatorname{arsinh}(y)$, and even in this case we set

$$\tanh_p \kappa := \frac{\sinh_p \kappa}{\cosh_p \kappa}, \quad \operatorname{csch}_p \kappa := -\frac{1}{\cosh'_p \kappa}, \quad \operatorname{sech}_p \kappa := \frac{1}{\sinh'_p \kappa}.$$

Identities similar to (4), (5), and (6) hold for hyperbolic functions.

2.5 π_p and π_p^*

Having generalised hyperbolic and trigonometric functions, it remains now to define the analogues of π:

- for odd values of p, π_p will denote the area of the region bounded by $\mathscr{F}_p$ and its asymptote; π_p^* is defined in a similar way, being $\pi_p = \pi_p^*$;
- for even values of p, π_p denotes the area of the region bounded by $\mathscr{F}_p$, while π_p^* measures the area of the region bounded by $\mathscr{F}_p^*$ and their asymptotes; note that $\mathscr{F}_2^*$ is unbounded.

In order to compute the values of these new $\pi's$, we will simply evaluate two additional parameters:

- the area of the region bounded by $\mathscr{F}_p$ and positive semi-axes, denoted by λ_p, and
- the area of the region bounded by the curve $\mathscr{F}_p^*$, the positive semi-axis and its asymptote, denoted by λ_p^* (Figs. 2 and 3).

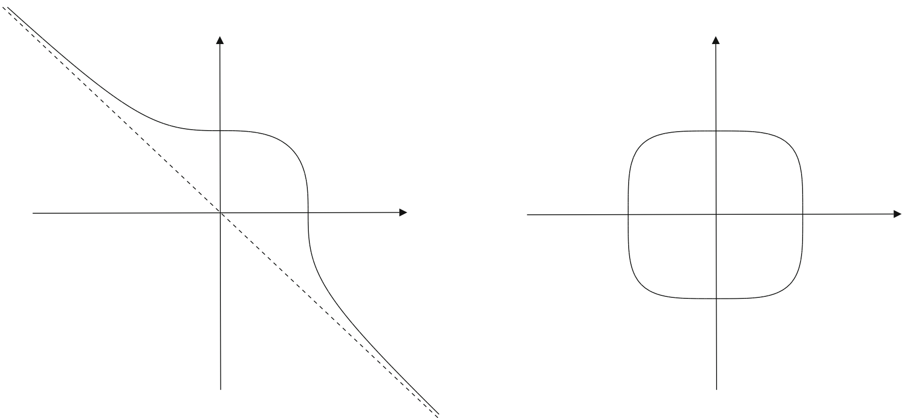

Fig. 2 $\mathscr{F}_3$ and $\mathscr{F}_4$

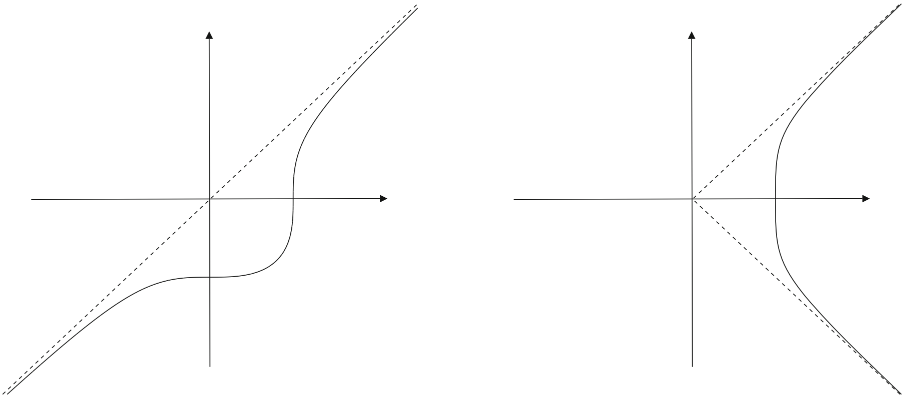

Fig. 3 $\mathscr{F}_3^*$ and $\mathscr{F}_4^*$

From the symmetries of the curves $\mathscr{F}_p$ and $\mathscr{F}_p^*$, we get the following identities:

- for every even p

$$\pi_p = 4\lambda_p, \tag{7}$$
$$\pi_p^* = 2\lambda_p^*, \tag{8}$$

- for every odd p

$$\pi_p = \pi_p^* = \lambda_p + 2\lambda_p^*. \tag{9}$$

Before computing the values of π_p and π_p^*, we point out that, following Knut [12, (2.12)] therein, in these pages the *rising factorial* is used in place of the so-called Pochhammer symbol, that is

$$x^{\overline{n}} := \prod_{i=0}^{n-1}(x+i) = \frac{\Gamma(x+n)}{\Gamma(x)}.$$

We will also use the *falling factorial*

$$x^{\underline{n}} := \prod_{i=0}^{n-1}(x-i) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}.$$

Lemma 2.4 *For every natural p, we have:*

$$\lambda_p = \frac{1}{2p}\mathrm{B}\left(\tfrac{1}{p}, \tfrac{1}{p}\right) = \frac{1}{2p}\frac{\Gamma^2\left(\frac{1}{p}\right)}{\Gamma\left(\frac{2}{p}\right)}. \tag{10}$$

Proof The computation is straightforward: if follows from the basic properties of Euler Gamma and Beta functions. □

Remark 2.5 Identity (10) provides also a geometric meaning of the Gamma function that extends the identity $\Gamma^2\left(\frac{1}{2}\right) = \pi = 4\lambda_2$ to

$$\Gamma^2\left(\tfrac{1}{p}\right) = \Gamma\left(\tfrac{2}{p}\right) 2p\,\lambda_p.$$

Remark 2.6 We notice that $\lambda_4 = \sqrt{2}L/4$, where L is the *lemniscate constant*, see [17], that is:

$$L = \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{2\pi}} = \sqrt{2}\, K\left(\tfrac{1}{\sqrt{2}}\right).$$

Remark 2.7 A result similar to Lemma 2.4 recently appeared in [22], where the L^p—measure of the disk $|x|^p + |y|^p = 1$ equals $4\lambda_p$.

Lemma 2.8 *For every $p \geq 3$ we have:*

$$\lambda_p^* = \frac{1}{2p}\mathrm{B}\left(\tfrac{1}{p}, 1-\tfrac{2}{p}\right) = \frac{1}{2p}\frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{2}{p}\right)}{\Gamma\left(1-\frac{1}{p}\right)}. \tag{11}$$

Proof In this case, we propose a detailed proof that is based on integration by series, using properties of Gauss hypergeometric functions. We have:

$$\begin{aligned}
\lambda_p^* &= \frac{1}{2} + \int_1^{+\infty}\left(x - (x^p-1)^{\frac{1}{p}}\right)\mathrm{d}x && \text{(12a)}\\
&= \frac{1}{2} + \int_0^1 \frac{1-(1-t^p)^{\frac{1}{p}}}{t^3}\mathrm{d}t\\
&= \frac{1}{2} + \int_0^1 t^{-3}\left(1 - \sum_{n=0}^{+\infty}\frac{\left(-\frac{1}{p}\right)^{\overline{n}}}{n!}t^{np}\right)\mathrm{d}t && \text{(12b)}\\
&= \frac{1}{2} - \int_0^1 \sum_{n=1}^{+\infty}\frac{\left(-\frac{1}{p}\right)^{\overline{n}}}{n!}t^{np-3}\,\mathrm{d}t && \text{(12c)}\\
&= -\sum_{n=0}^{+\infty}\frac{\left(-\frac{1}{p}\right)^{\overline{n}}}{(np-2)\,n!} = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{-\frac{2}{p}\left(-\frac{1}{p}\right)^{\overline{n}}}{\left(n-\frac{2}{p}\right)}\frac{1}{n!} && \text{(12d)}\\
&= \frac{1}{2}\sum_{n=0}^{+\infty}\frac{\left(-\frac{2}{p}\right)^{\overline{n}}\left(-\frac{1}{p}\right)^{\overline{n}}}{\left(1-\frac{2}{p}\right)^{\overline{n}}}\frac{1}{n!} = \frac{1}{2}{}_2\mathrm{F}_1\left(\left.\begin{matrix}-\frac{2}{p}, -\frac{1}{p}\\ 1-\frac{2}{p}\end{matrix}\right| 1\right) && \text{(12e)}\\
&= \frac{1}{2}\frac{\Gamma\left(1-\frac{2}{p}\right)\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1-\frac{1}{p}\right)} = \frac{1}{2p}\frac{\Gamma\left(1-\frac{2}{p}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(1-\frac{1}{p}\right)} && \text{(12f)}\\
&= \frac{1}{2p}\mathrm{B}\left(\tfrac{1}{p}, 1-\tfrac{2}{p}\right).
\end{aligned}$$

We give, for the reader's convenience, a brief explanation of the steps: in (12a) we change variable setting $t := x^{-1}$, in (12b) we use the binomial series expansion, in (12c) we eliminated the first term of the series, in (12d) we integrate term by term, in (12e) we use the identity

$$\left(-\frac{2}{p}\right)^{\overline{n}} = -\frac{2}{np-2}\left(1-\frac{2}{p}\right)^{\overline{n}}$$

and finally (12f) follows from a well-known Gauss result on the hypergeometric function, see [6, pp. 352–353]. Then the proof of (11) is completed. $\square$

Remark 2.9 Unlike Lemma 2.4 this result has no analogy with what is described in [22] because since we have considered the Fermat curve $x^p + y^p = 1$, the part of the plane described by λ_p^* is unbounded when p is odd, and this cannot happen in the case considered by [22] which is related to the curve $|x|^p + |y|^p = 1$. Nevertheless, it allows us to widen the geometric interpretation of the Gamma function.

Corollary 2.10 *The following identity holds:*

$$\frac{\lambda_p^*}{\lambda_p} = \frac{1}{2} \sec \frac{\pi}{p}.$$

Proof Comparing identities (10) and (11) we have:

$$\frac{\lambda_p^*}{\lambda_p} = \frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{2}{p}\right)}{\Gamma\left(1-\frac{1}{p}\right)} \frac{\Gamma\left(\frac{2}{p}\right)}{\Gamma^2\left(\frac{1}{p}\right)} = \frac{\Gamma\left(1-\frac{2}{p}\right)}{\Gamma\left(1-\frac{1}{p}\right)} \frac{\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}.$$

Now, the Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ gives

$$\frac{\lambda_p^*}{\lambda_p} = \frac{\sin\frac{\pi}{p}}{\sin\frac{2\pi}{p}} = \frac{1}{2} \sec \frac{\pi}{p}.$$

□

By identities (7), (8), and (9), we can conclude:

Theorem 2.11 (The value of π and π^*) *For every even $p \geq 2$, we have*

$$\pi_p = 4\lambda_p = \frac{2}{p} \frac{\Gamma^2\left(\frac{1}{p}\right)}{\Gamma\left(\frac{2}{p}\right)},$$

and for every even $p \geq 2$

$$\pi_p^* = \lambda_p \sec \frac{\pi}{p} = \frac{1}{2p} \frac{\Gamma^2\left(\frac{1}{p}\right)}{\Gamma\left(\frac{2}{p}\right)} \sec \frac{\pi}{p}.$$

For every odd $p \geq 3$

$$\pi_p = \pi_p^* = \lambda_p \left(1 + \sec \frac{\pi}{p}\right) = \frac{1}{2p} \frac{\Gamma^2\left(\frac{1}{p}\right)}{\Gamma\left(\frac{2}{p}\right)} \left(1 + \sec \frac{\pi}{p}\right).$$

Proof The proof is straightforward, indeed it follows by direct integration and gamma function well-known properties. □

Remark 2.12 It should be noted that in case $p = 2$, the relation for π_p^* returns, correctly, $+\infty$.

3 The Wallis Product for λ_p

In this section, we treat for a generic exponent p, the relationships between Wallis integrals and generalized trigonometric functions. The Wallis product, which dates back to 1655[2], comes from $p = 2$:

$$\frac{\pi}{2} = \prod_{i=1}^{\infty} \left(\frac{2i}{2i-1} \cdot \frac{2i}{2i+1} \right). \tag{13}$$

Wallis's original demonstration starts with the observation that $\frac{\pi}{2}$, which is our $2\lambda_2$, is the area intercepted by $\mathscr{F}_2$ and the first quadrant[3], and then, the proof is based on the calculation of the integrals of integer powers of the trigonometric functions sine and cosine.

The original scheme can be successfully replicated in the case of generalized trigonometric functions. First, using the Euler Beta function, we will find formulas for integrating powers of binomials of the type $\cos_p^m \sin_p^n$ in the interval $[0, 2\lambda_p]$.

This fact will allow us to express the Wallis integrals in terms of infinite products, being intimately related to a classical property of the Gamma function, which below and for the benefit of the reader, we will report in Rainville's formulation[4].

Theorem 3.1 *If* $\sum_{i=1}^{s} a_1 = \sum_{i=1}^{s} b_i$ *and none of the* a_i, b_i *is a negative integer, then*

$$\prod_{n=1}^{\infty} \frac{(n+a_1)(n+a_2)\ldots(n+a_s)}{(n+b_1)(n+b_2)\ldots(n+b_s)} = \frac{\Gamma(1+b_1)\Gamma(1+b_2)\ldots\Gamma(1+b_s)}{\Gamma(1+a_1)\Gamma(1+a_2)\ldots\Gamma(1+a_s)}.$$

3.1 Wallis p-Integrals

The Wallis integral, rewritten in the notation of this paper, reads as

$$w_2^{(m)} := \int_0^{\pi/2} \cos^m \kappa \, \mathrm{d}\kappa = \int_0^{\pi/2} \sin^m \kappa \, \mathrm{d}\kappa = \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}.$$

The upper boundary $\pi/2$ of the integral verifies identities $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$, and it should be read as twice the area of the plane region intercepted by the first quadrant and $x^2 + y^2 \leq 1$.

Therefore, to generalize this notion to the $p > 2$ case, it is natural to define the *Wallis p-integrals* by setting

$$w_p^{(m)} := \int_0^{2\lambda_p} \cos_p^m \kappa \mathrm{d}\kappa = \int_0^{2\lambda_p} \sin_p^m \kappa \mathrm{d}\kappa.$$

Observe that we have, as in the case of usual trigonometry, i.e., $p = 2$, that for every $m \in \mathbb{N}$, $w_p^{(m+1)} < w_p^{(m)}$. As in case $p = 2$, the integrals $w_p^{(m)}$ can be expressed in terms of the gamma function, in fact, we can state the following theorem.

Theorem 3.2 *For every* $p \geq 2$ *and* $m \geq 0$, *we have*

$$w_p^{(m)} = \frac{1}{p} \frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1+m}{p}\right)}{\Gamma\left(\frac{2+m}{p}\right)}. \tag{14}$$

[2] For a quick, but accurate historical overview see [9].

[3] See [11, pp. 65–66].

[4] See [24, Theorem 5 of Section 14]

Moreover,

$$\lim_{m\to\infty} w_p^{(m)} = 0, \tag{15a}$$

$$\lim_{m\to\infty} \frac{w_p^{(m+1)}}{w_p^{(m)}} = 1. \tag{15b}$$

Proof Given that the functions $\cos_p$ and $\sin_p$ map bijectively the interval $[0, 2\lambda]$ to $[0, 1]$, we are driven to calculate such integrals through the inverse function integration rule,

$$\int_{f(0)}^{f(1)} f^{\mathrm{inv}}(x)\mathrm{d}x = \mathrm{sgn}(f') \int_0^1 t f'(t)\mathrm{d}t,$$

and its generalization to the powers of the inverse function:

$$\int_{f(0)}^{f(1)} \left(f^{\mathrm{inv}}(x)\right)^m \mathrm{d}x = \mathrm{sgn}(f') \int_0^1 t^m f'(t)\mathrm{d}t.$$

Recalling that the inverse of $\cos_p$ is

$$\mathrm{arcos}_p(x) = \int_x^1 \frac{\mathrm{d}\xi}{(1-\xi^p)^{1-\frac{1}{p}}},$$

we infer that

$$\begin{aligned} w_p^{(m)} &= \int_0^{2\lambda_p} \cos_p^m u \mathrm{d}u = -\int_0^1 x^m \mathrm{arcos}_p'(x)\,\mathrm{d}x \\ &= \int_0^1 x^m (1-x^p)^{\frac{1}{p}-1}\,\mathrm{d}x = \frac{1}{p}\,\frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1+m}{p}\right)}{\Gamma\left(\frac{2+m}{p}\right)}. \end{aligned}$$

showing (14). Limits (15a) and (15b) follow, recalling Stirling formula

$$\frac{e^x\sqrt{x}}{\sqrt{2\pi}\,x^x}\,\Gamma(x) = 1 + \frac{1}{12\,x} + \mathrm{O}\left(\frac{1}{x^2}\right) \quad \text{as } x \to +\infty.$$

We have

$$w_p^{(m)} = m^{-1/p}\left(p^{\frac{1}{p}-1}\Gamma\left(\frac{1}{p}\right) + \mathrm{O}\left(\frac{1}{m}\right)\right)$$

proving (15a), and

$$\frac{w_p^{(m+1)}}{w_p^{(m)}} = 1 - \frac{1}{m\,p}$$

proving (15b). As it is easy to guess, computer algebra has allowed us to manage the calculations just presented with great speed and precision. □

Remark 3.3 We would like to point out that the technique we used provides the primitive of $\sin_p^m$ on the first quadrant; indeed, fixing $0 < x \le 1$, we get

$$\begin{aligned} \int_0^{\mathrm{arsin}_p(x)} \sin_p^m(s)\,\mathrm{d}s &= \int_0^x \frac{u^m}{(1-u^p)^{1-\frac{1}{p}}}\,\mathrm{d}u = \int_0^1 \frac{x^{m+1}v^n}{(1-x^p v^p)^{1-\frac{1}{p}}}\,\mathrm{d}v \\ &= \frac{x^{m+1}}{m+1}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}\frac{m+1}{p}, \frac{p-1}{p} \\ \frac{m+p+1}{p}\end{matrix}\,\right|\, x^p\right). \end{aligned}$$

Remark 3.4 In analogy to the case of usual trigonometric functions, we can put $t = \cos_p^p(\kappa)$ in the definition of Beta function $\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$, $x, y > 0$, thus obtaining

$$\mathrm{B}(x, y) = p \int_0^{2\lambda_p} \cos_p^{px-1}(\kappa) \sin_p^{py-1}(\kappa)\, \mathrm{d}\kappa$$

from which we get

$$\int_0^{2\lambda_p} \cos_p^m(\kappa) \sin_p^n(\kappa)\, \mathrm{d}\kappa = \frac{1}{p}\, \mathrm{B}\left(\frac{m+1}{p}, \frac{n+1}{p}\right).$$

This formula reduces to (14) for $n = 0$. Therefore, it is a generalisation, but equation (14) is of some interest because of the direct method by which we obtained it.

From identity (14) we easily evaluate the following special cases:

$$w_p^{(0)} = 2\lambda_p, \quad w_p^{(p-1)} = 1, \quad w_p^{(p)} = \lambda_p, \tag{16}$$

and thanks the Euler's reflection formula

$$w_p^{(p-2)} = \frac{\pi}{p} \csc \frac{\pi}{p}, \tag{17}$$

$$w_p^{(p-3)} = 2\lambda_p^* = \lambda_p \sec \frac{\pi}{p}. \tag{18}$$

Also (14) allows to write down the table of values of the Wallis p integrals, which we report to order $p = 5$ and for powers from $m = 0$ to $m = 4$ (Table 1).

3.2 The mod p Lowering

To better understand the nature of the generalization we are going to present, let us start with the classic formulation in terms of semifactorials of Wallis integrals $w_2^{(m)}$

$$w_2^{(2m)} = \int_0^{\frac{\pi}{2}} \cos^{2m} u\mathrm{d}u = \frac{2m-1}{2m}\frac{2m-3}{2m-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} = \frac{(2m-1)!!}{(2m)!!}\cdot\frac{\pi}{2}$$

$$w_2^{(2m+1)} = \int_0^{\frac{\pi}{2}} \cos^{2m+1} u\mathrm{d}u = \frac{2m}{2m+1}\frac{2m-2}{2m-1}\cdots\frac{2}{3}\cdot 1 = \frac{(2m)!!}{(2m+1)!!}.$$

Table 1 Table of values of the Wallis p integrals, for $1 \le p \le 5$ and $0 \le m \le 4$

	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$p=1$	1				
$p=2$	$\frac{\pi}{2}$	1			
$p=3$	$\frac{\Gamma^3(1/3)}{2\sqrt{3}\pi}$	$\frac{2\pi}{3\sqrt{3}}$	1		
$p=4$	$\frac{\Gamma^2(1/4)}{4\sqrt{\pi}}$	$\frac{\sqrt{\pi}\Gamma(1/4)}{4\Gamma(3/4)}$	$\frac{\pi}{2\sqrt{2}}$	1	
$p=5$	$\frac{2\sqrt{\pi}\Gamma(1/5)}{2^{1/5}5\Gamma(7/10)}$	$\frac{2^{1/5}\Gamma(1/10)\Gamma(2/5)}{10\sqrt{\pi}}$	$\frac{\Gamma(1/5)\Gamma(3/10)}{2^{2/5}5\sqrt{\pi}}$	$\sqrt{2+\frac{2}{\sqrt{5}}}\frac{\pi}{5}$	1

Ascending factorials are also linked to double factorials, see [3, p. 20] so that

$$w_2^{(2m)} = \frac{\pi}{2}\frac{\left(\frac{1}{2}\right)^{\overline{m}}}{(1)^{\overline{m}}} = w_2^{(0)} \prod_{i=0}^{m-1} \frac{2i+1}{2i+2},$$

$$w_2^{(2m+1)} = \frac{(1)^{\overline{m}}}{\left(\frac{3}{2}\right)^{\overline{m}}} = w_2^{(1)} \prod_{i=0}^{m-1} \frac{2i+2}{2i+1}.$$

In a very similar way, the Wallis integrals $w_p^{(m)}$ fall into p families, according to the residual classes mod p.

Theorem 3.5 (The mod p lowering) *For every $m, k \in \mathbb{N}$, $m > 1$, the Wallis integral $w_p^{(mp+k)}$ can be lowered to $w_p(k)$ as follows*

$$w_p^{(mp+k)} = w_p^{(k)} \frac{\left(\frac{1+k}{p}\right)^{\overline{m}}}{\left(\frac{2+k}{p}\right)^{\overline{m}}} = w_p^{(k)} \prod_{i=0}^{m-1} \frac{ip+1+k}{ip+2+k}. \tag{19}$$

Proof From identity (14) we get, for every $k \in \mathbb{N}$,

$$w_p^{(p+k)} = \frac{1}{p} \frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1+k+p}{p}\right)}{\Gamma\left(\frac{2+k+p}{p}\right)} = \frac{1}{p} \frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1+k}{p}\right)\frac{1+k}{p}}{\Gamma\left(\frac{2+k}{p}\right)\frac{2+k}{p}} = w_p^{(k)} \frac{k+1}{k+2},$$

and the identity (19) follows by induction. □

3.3 The Wallis p-Product

From identities (19) and (16), we get the identities

$$w_p^{(mp+p)} = \lambda_p \prod_{i=1}^{m} \frac{ip+1}{ip+2}, \quad w_p^{(mp+p-1)} = \prod_{i=1}^{m} \frac{ip}{ip+1},$$

which yield

$$\frac{w_p^{(mp+p)}}{w_p^{(mp+p-1)}} = \lambda_p \prod_{i=1}^{m} \frac{ip+1}{ip+2}\frac{ip+1}{ip},$$

and, by (15b)

$$1 = \lim_{m\to\infty} \frac{w_p^{(mp+p)}}{w_p^{(mp+p-1)}} = \lambda_p \prod_{i=1}^{\infty} \frac{ip+1}{ip+2}\frac{ip+1}{ip},$$

from which we obtain the expression of λ_p as an infinite product.

Theorem 3.6 (The Wallis p-product) *For every integer $p \geq 2$, the area λ_p of the region bounded by $\mathscr{F}$ positive semi axes, can be expressed as an infinite product as follows*

$$\lambda_p = w_p^{(p)} = \prod_{i=1}^{\infty} \frac{ip}{ip+1}\frac{ip+2}{ip+1}. \tag{20}$$

Proof The previous discussion motivates the structure of the formula (20), it remains to verify its effective convergence, which follows, see for example [19, Chapter 7] or [18, Chapter 2], in fact the infinite product

$$\prod_{i=1}^{\infty}(1+a_i)$$

converges if and only if the series

$$\sum_{i=1}^{\infty} a_i$$

converges. In the case of (20) we have

$$\frac{ip}{ip+1}\frac{ip+2}{ip+1} = 1 - \frac{1}{(ip+1)^2},$$

hence the convergence of (20) follows front the convergence of the harmonic series. $\square$

Remark 3.7 Equation (20) yields for $p = 2$ the usual Wallis product for π, in fact taking $p = 2$ we obtain

$$\lambda_2 = \frac{\pi}{4} = \prod_{i=1}^{\infty} \frac{2i}{2i+1}\frac{2i+2}{2i+1}. \tag{21}$$

Then, multiplying by 2 (21) we get, after an index rescaling which allows to incorporate the factor 2 into the infinite product:

$$2\lambda_2 = \frac{\pi}{2} = \prod_{i=0}^{\infty} \frac{(2i+2)^2}{(2i+1)(2i+3)}. \tag{22}$$

Now is it evident how and why the classical formulation of Wallis product (13) fits with our findings (21) and (22). Finally, we observe that the original formulation of the Wallis product for $\frac{\pi}{2}$ emerges by multiplying both sides of (20) by 2.

The convergence rate of infinite products (20) is the same as in the classical case of $p = 2$, for example, cases $p = 3, 4, 5$ need 100 factors to obtain an accuracy of three decimal figures, in accordance with what has been illustrated, of course for the case of $p = 2$, in the introduction of [20]. For the classic case of the Wallis product, an interesting piece of research is devoted to methods of speeding up the convergence of the infinite product. By the nature of our result, we are confident that the methods presented in [14, 20, 21] and many others will adapt to this new situation.

3.4 Other p-Products

The procedure we used to express λ_p as an infinite product in $\mathbb{Q}$ allows us to find an analogous result for every other mth Wallis integral.

Theorem 3.8 *For every $p \geq 2$, the mth Wallis integral equals*

$$w_p^{(m)} = \frac{m+2}{m+1}\prod_{i=1}^{\infty}\frac{ip}{ip+1}\frac{ip+m+2}{ip+m+1}. \tag{23}$$

Proof By induction. For $m = 0$ the statement is true (see (16) and (20)). Suppose now the identity holds for $m \geq 0$, then

$$1 = \lim_{h\to\infty} \frac{w_p^{(hp+m+1)}}{w_p^{(hp+m)}} = \frac{w_p^{(m+1)}}{w_p^{(m)}} \prod_{i=0}^{\infty} \frac{ip+1+m+1}{ip+2+m+1} \frac{ip+2+m}{ip+1+m},$$

from which

$$\begin{aligned} w_p^{(m+1)} &= w_p^{(m)} \prod_{i=0}^{\infty} \tfrac{ip+m+3}{ip+m+2} \tfrac{ip+m+1}{ip+m+2} = \tfrac{m+2}{m+1} \prod_{i=1}^{\infty} \tfrac{ip}{ip+1} \tfrac{ip+m+2}{ip+m+1} \prod_{i=0}^{\infty} \tfrac{ip+m+3}{ip+m+2} \tfrac{ip+m+1}{ip+m+2} \\ &= \tfrac{m+2}{m+1} \tfrac{m+3}{m+2} \tfrac{m+1}{m+2} \prod_{i=1}^{\infty} \tfrac{ip}{ip+1} \tfrac{ip+m+2}{ip+m+1} \prod_{i=1}^{\infty} \tfrac{ip+m+3}{ip+m+2} \tfrac{ip+m+1}{ip+m+2} = \tfrac{m+3}{m+2} \prod_{i=1}^{\infty} \tfrac{ip}{ip+1} \tfrac{ip+m+3}{ip+m+2}. \end{aligned}$$

□

Remark 3.9 Comparing identities (18) and (10), we re-discover easily the well-known formula

$$\cos \tfrac{\pi}{p} = \prod_{i=1}^{\infty} \tfrac{4-i^2p^2}{1-i^2p^2}. \tag{24}$$

Analogously, we get the representation of the sine as an infinite product.

Remark 3.10

$$\sin \tfrac{\pi}{p} = \prod_{i=0}^{\infty} \tfrac{4(ip+1)(ip+p-1)}{(2ip+p)^2}. \tag{25}$$

Proof We start from identities (17) observing

$$w_p^{(p-2)} = \frac{\pi}{p} \frac{1}{\sin \frac{\pi}{p}} = \frac{4w_2^{(2)}}{p} \frac{1}{\sin \frac{\pi}{p}}$$

then use (21) and (23) to arrive at

$$\sin \frac{\pi}{p} = \frac{4}{p} \frac{w_2^{(2)}}{w_p(p-2)} = \frac{4}{p} \prod_{i=1}^{\infty} \frac{2i}{2i+1} \frac{2i+2}{2i+1} \frac{p-1}{p} \frac{(1+ip)(p(1+i))(p(1+i)-1)}{ip(p+pi)}.$$

Eventually restarting the index from $i = 0$ we obtain (25). □

Of course (24) and (25) are consistent with the classic infinite product representations of the sine and the cosine, indeed taking $x = \frac{\pi}{p}$ we find

$$\cos x = \prod_{i=1}^{\infty} \frac{\pi^2 i^2 - 4x^2}{\pi^2 i^2 - x^2}, \quad \sin x = \prod_{i=1}^{\infty} \frac{4(\pi i - x)(\pi(i-1) + x)}{(\pi - 2\pi i)^2}.$$

Acknowledgements The authors thank the anonymous referees for their valuable suggestions which greatly improved our article

References

1. Adams, O.S.: Elliptic Functions Applied to Conformal World Maps. Washington Government Printing Office, Washington (1925)
2. Burgoyne, F.D.: Generalized trigonometric functions. Math. Comput. **18**, 314–316 (1964)
3. Carlson, B.C.: Special Functions of Applied Mathematics. Academic Press, New York (1977)
4. Cayley, A.: On the elliptic function solution of the equation $x^3 + y^3 - 1 = 0$. In: Proceedings of the Cambridge Philosophical Society, vol. 4, pp. 106–109 (1883)
5. Dixon, A.C.: On the doubly periodic functions arising out of the curve $x^3 + y^3 - 3\alpha xy = 1$. Q. J. Pure Appl. Math. **24**, 168–233 (1890)
6. Duren, P.: Invitation to Classical Analysis. American Mathematical Society, Providence, RI (2012)
7. Flajolet, P., Dumas, P., Puyhaubert, V.: Some exactly solvable models of urn process theory. In: Fourth Colloquium on Mathematics and Computer Science Algorithms Trees Combinatorics and Probabilities, pp. 59–118. Discrete Mathematics and Theoretical Computer Science (2006)
8. Gambini, A., Nicoletti, G., Ritelli, D.: Keplerian trigonometry. Monatsh. Math. **195**, 55–72 (2021)
9. Giaquinta, M., Modica, M.: Mathematical Analysis: Approximation and Discrete Processes. Springer, New York (2004)
10. Grammel, R.: Eine Verallgemeinerung der Kreis-und Hyperbelfunktionen. Arch. Appl. Mech. **16**, 188–200 (1948)
11. Hairer, E., Wanner, G.: Analysis by its History, vol. 2. Springer, New York (2006)
12. Knuth, D.E.: Two notes on notation. Am. Math. Mon. **99**, 403–422 (1992)
13. Kotz, S., Johnston, J.A.: Urn Models and Their Application: An Approach to Modern Discrete Probability Theory. John Wiley & Sons Inc, Hoboken (1977)
14. Lamperet, V.: Simple accurate balanced asymptotic approximation of Wallis' ratio using Euler-Boole alternating summation. Math. Inequal. Appl. **24**, 887–896 (2021)
15. Lang, J., Edmunds, D.: Eigenvalues, Embeddings and Generalised Trigonometric Eigenvalues. Embeddings and Generalised Trigonometric Functions. Springer, Berlin (2011)
16. Lee, L.P.: Some conformal projections based on elliptic functions. Geogr. Rev. **55**, 563–580 (1965)
17. Levin, A.: A geometric interpretation of an infinite product for the lemniscate constant. Am. Math. Mon. **113**, 519–520 (2006)
18. Little, C.H., Teo, K.L., van Brunt, B.: An Introduction to Infinite Products. Springer, New York (2022)
19. Loya, P.: Amazing and Aesthetic Aspects of Analysis. Springer, New York (2017)
20. Mortici, C.: Optimizing the convergence rate of the Wallis sequence. Hacet. J. Math. Stat. **44**, 101–109 (2015)
21. Mortici, C., Cristea, V.G.: Estimates for Wallis' ratio and related functions. Indian J. Pure Appl. Math. **47**, 437–447 (2016)
22. Pearson, J.: Viète Formula, Knar's Formula, and the Geometry of the Gamma Function. Am. Math. Mon. **125**, 704–714 (2018)
23. Poodiack, R.D.: Squigonometry, hyperellipses, and supereggs. Math. Mag. **89**, 92–102 (2016)
24. Rainville, E.D.: Special Functions. MacMillian Company, New York (1960)
25. Robinson, P.L.: The Dixonian elliptic function. arXiv:1901.04296 (2019)
26. Robinson, P.L.: Higher trigonometry: A class of nonlinear systems. arXiv:1907.05240 (2019)
27. Sloane, N.J.A.: The on-line encyclopedia of integer sequences. https://oeis.org. Accessed 31 Aug. 2022
28. van Fossen Conrad, E., Flajolet, P.: The Fermat cubic, elliptic functions, continued fractions, and a combinatorial excursion. Sémin. Lothar. Comb **B54g**, 1–44 (2006)
29. Whittaker, E.T., Watson, G.N.: A Course of Modern Analysis. Cambridge University Press, Cambridge (1902)
30. Wood, W.E., Poodiack, R.D.: Squigonometry: Trigonometry in the p-norm. In: Harris, P.E., Insko, E., Wootton, A. (eds.) A Project-Based Guide to Undergraduate Research in Mathematics Starting and Sustaining Accessible Undergraduate Research, pp. 263–286. Birkhäuser, Basel (2020)
31. Wood, W.E.: Squigonometry. Math. Mag. **84**, 257–265 (2011)