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28 October 2024

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Abstract

In this brief note we show that under a volume non-preserving scaling it is possible to recover the basics for a regularity theory regarding local weak solutions to the fully anisotropic equation

$$\partial_t u = \sum_{i=1}^{N} \partial_i(|\partial_i u|^{p_i-2}\partial_i u) \quad \text{in} \quad \Omega_T = \Omega \times (-T, T), \quad \text{with} \quad \Omega \subset\subset \mathbb{R}^N. \tag{0.1}$$

We characterize self-similar solutions regarding this particular scaling and we show that semi-continuity for solutions to this equation is a consequence of a simple property that is itself invariant under scaling.

MSC 2020: 35K65, 35K92, 35B65.

Keywords: Anisotropic p-Laplacian, Critical Mass Lemma, Intrinsic Scaling, Lower Semi-Continuity.

To celebrate Francesco Altomare's 70th genethliac

1 Introduction to the problem

Equation (0.1) is a parabolic anisotropic equation with non-standard growth. We refer to the introduction of [5], [23] and the surveys [20], [22] for a non-exhaustive introduction to the origin of the problem, and to the introduction of [6] and the book [1] for a more general account to the parabolic problem. At a first glance equation (0.1) may look similar to the equation

$$u_t - \text{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{locally weakly in } \Omega_T. \tag{1.1}$$

Literature on this topic is very developed, and even if the problem of regularity of solutions to (0.1) is old more than fifty years, still very much is unknown from the point of view of basic regularity, as local Hölder continuity or Harnack inequality. The principal motivation is that the techniques usually employed

for nonlinear equations (as p-Laplacian equations, porous medium equations, doubly nonlinear equations, and so on) are not directly applicable to it. Let us explain this point in detail.

Up to our knowledge, in the setting of evolutionary nonlinear operators of p-growth (whose prototype is (1.1) with $p \neq 2$), the main technique to prove a Harnack inequality is exploiting a parabolic continuous transformation having the general form

$$w(x,t) = e^{t/(p-2)}u(x, e^t), \qquad x \in \Omega, \, t > 0. \tag{1.2}$$

This transformation maps super-solutions to (1.1) to super-solutions to a similar equation, that has an exponential dependence on time only on the non-homogeneous terms. Along this strategy, the possibility to stretch time and control the non-homogeneous terms is crucial, in order to employ a technique originally conceived by E. DeGiorgi for solutions to elliptic partial differential equations (see, e.g., [10], [13]), based on a version of the isoperimetric inequality (cf. [12, Lemma 2.2., page 5]). This argument allows to prove an expansion of positivity for the transformed super-solutions that, if carried back to solutions to (1.1), provides the expansion of positivity necessary for an intrinsic Harnack inequality to hold true.

The main issue dealing with (0.1) is that, in general, a continuous transformation with an exponential-type dependence on time necessarily affects the space variables. Taking into account also the strong nonlinear behavior of the equation along the space variables, the control of the non-homogeneous terms in the transformed equation is encumbered. More precisely, from the energetic point of view, the new equation is no more of the same kind of (0.1), and this leads the whole machinery to fail.

On the other hand, in [8] the authors proved that an intrinsic Harnack type inequality is valid for local weak solutions to (0.1), by adapting a classic idea of E. DiBenedetto (see [11]) consisting in a comparison between the solution and a particular one, called Barenblatt solution in honor to its discoverer (see the original in [2] and [7] for an overview on the anisotropic case). However, the generalization of this inequality to a wider class of parabolic operators patterned after (0.1) is still an open and challenging problem. The purpose of the present work is to investigate a particular scaling of the equation: it would permit to free the time variable from the space ones, opening the way to an application of a transformation similar to (1.2). This homogenization seems to unveil a new insight on the anisotropic behavior of these operators. From the energetic point of view, serious difficulties appear even with the stationary counterpart of (0.1), because the competition among different directional p_i-diffusions encodes both singular and degenerate behavior. Roughly speaking, this can be illustrated within the scaling of [9], looking at the kind of degeneration that the set

$$\prod_{i=1}^{N} \left\{ |x_i| < \rho^{\frac{\bar{p}}{p_i}} M^{\frac{p_i - \bar{p}}{p_i}} \right\}, \qquad M, \rho > 0$$

exhibits as M vanishes. This is a volume-preserving set of self-similar geometry where the equation evolves, and the parameter M is usually chosen to be a multiple of the oscillation of u, in order to restore the homogeneity of the energy. The problem is that, depending on the sign of $(p_i - \bar{p})$, the set stretches or vanishes along the respective coordinates.

The different scaling that we propose in this note (see (2.2)-(2.3)) possesses the following properties: the intrinsic geometry associated with it degenerates monotonically with M, so we say that the geometry is only *degenerate*, not singular; it does not affect anyhow the time variables from the intrinsic point of view. From this perspective, this particular scaling seems promising; see for instance, the energy $\mathcal{E}_n$ in Lemma 3.2. The crucial point is that we can identify the self-similar solutions to (0.1), namely, the solutions that coincide with their scaled functions; this is done via correspondence with a Fokker-Planck equation (cf. Proposition 2.3). As a consequence, all the properties of solutions to (0.1) proved in [8] hold true, in a re-interpreted formulation, also for solutions of a 'wild' Fokker-Planck equation (see (2.11)). The existence of a Barenblatt solution is of fundamental importance to understand the behaviour of solutions.

Moreover, we show that this special scaling preserves the energy of the solutions, as well as other properties, that will be called for this reason *invariants* (standing for scale-invariants). An example is furnished by the Critical Mass Lemma, that can be regarded as a measure-theoretical maximum principle (see Lemma 3.2 for details; see also [25, p. 8] and Proposition 2 of [17] in the context of anisotropic porous medium).

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Dimensional analysis is a simple consequence of the well-known covariance principle of Physics: all physical laws can be represented in a form which is equally valid for all observers. The very idea of self-similarity is connected with the group of transformations of solutions: see, e.g., [4]. These groups are inborn in the differential equations governing the process, and are determined by the physical dimensions of the variables appearing in them; transformations of units of time, length, mass, etc. are the simplest examples. This kind of self-similarity is obtained by power laws with exponents that are simple fractions defined in an elementary way from dimensional considerations. These arguments led to an interpretation of nonlinear parabolic theory, developed by DiBenedetto ([12]), Vazquez ([24]), and many others, which is nowadays known as method of intrinsic scaling (cf. also [26]). The key feature of the argument of intrinsic scaling is that, by appropriately scaling the geometry, the energy of solutions enjoys a homogeneous form that is easier to manipulate. This idea can be used in turn to interpret the energy of solutions to anisotropic equations like (0.1) in a homogeneous fashion. This is the purpose of the present scaling, whose side-effect on energy $\mathcal{E}_n$ is here shown by the non-scaled version of Lemma 3.3. We present here a general version of this lemma, that we could not find in literature for the full parabolic anisotropic equation and that is propaedeutic to the study of further properties.

Indeed, as a byproduct of our analysis, by applying the ideas of [19] to the parabolic setting, we show that lower semi-continuity of super-solutions is a sole consequence of these general invariants.

The existence of a lower semi-continuous representative for local weak super-solutions has already been obtained in [14] by using an idea of [18]. The authors observe that a proper $L^r - L^\infty$ estimate for weak super-solutions suffices to obtain a lower semi-continuous representative. This technique is however linked to the particular structure of the equation, that allowed them to add a constant to the solution to generate another solution. The new approach of [19] is more general, since the existence of a lower semi-continuous representative is linked only to a more general property, that is the analogue of Lemma 3.2. In this way the authors of the aforementioned [14], [18], [19] proved that weak solutions are p-super-harmonic solutions. The latter ones are, on an appropriate setting, proper lower semi-continuous functions, that can be compared with any sufficiently regular solution to the same equation. Since the comparison principle for equations driven by monotone operators holds true, the main step consists in proving semi-continuity.

It would be an interesting subject to determine whether p-super-harmonic functions, whose derivatives a priori may be even unbounded, can satisfy a Critical Mass Lemma as Lemma 3.2.

Structure of the paper

Section 2 is devoted to set up the functional framework and propose the particular scaling. In Section 3 we show that energetic properties of the equation are invariant under this scaling. Finally, in Section 4, we furnish a new proof of semi-continuity for super-solutions to (0.1).

Notation

- Let $N \in \mathbb{N}$, $N > 1$, and let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Given $T > 0$, we set $\Omega_T = \Omega \times (-T, T)$.
  The symbol $A \subset\subset B$ means 'A is compactly contained in B'.

- For any $\varphi \in W^{1,1}(\Omega)$, we denote by $\partial_i \varphi$ the i-th directional weak derivative of φ.
  If moreover $\varphi \in W^{1,2}([s,t], L^2(\Omega))$ for some $s, t \in \mathbb{R}$, $s < t$, then $\partial_t \varphi$ stands for the weak time-derivative of φ.

- We denote the cube of side $2\rho > 0$ and center $x \in \mathbb{R}^N$ with $x + K_\rho$, while

$$\begin{cases} (x,t) + Q^-_{\rho,\tau} = (x + K_\rho) \times (-\tau, 0], \\ (x,t) + Q^+_{\rho,\tau} = (x + K_\rho) \times [0, \tau), \\ (x,t) + Q_{\rho,\tau} = (x + K_\rho) \times (-\tau, \tau], \end{cases}$$

  stand for, respectively, the backward, forward and full cylinders centered at $(x,t) \in \mathbb{R}^{N+1}$.
  When $\tau = 1$ we simply write Q^-_ρ, Q_ρ, Q^+_ρ instead of $Q^-_{\rho,1}$, $Q_{\rho,1}$, $Q^+_{\rho,1}$.

- We fix a vector of N numbers $\mathbf{p} = (p_1, \ldots, p_N)$; the index i will run through $1, \ldots, N$. We define the harmonic mean of p_is as $\bar{p} = N(\sum_{i=1}^N 1/p_i)^{-1}$, and for $\bar{p} < N$ the Sobolev exponent of the harmonic mean by $\bar{p}^* = N\bar{p}/(N - \bar{p})$. Hereafter we suppose
$$2 < p_1 \leq p_2 \leq \cdots \leq p_N < \bar{p}^*.$$

- In the sequel we will make use of the following numbers:

$$\lambda = N(\bar{p}-2) + \bar{p}, \quad \alpha = \frac{N}{\lambda}, \quad \alpha_i = \frac{N(\bar{p}-p_i) + \bar{p}}{\lambda p_i}. \tag{1.3}$$

- For any $M, \rho > 0$, the *intrinsic cube* and the *backward intrinsic cylinder* are defined respectively as

$$\mathcal{K}_\rho(M) = \prod_{i=1}^N \left\{ |x_i| < M^{\frac{p_i-2}{p_i}} \rho^{\frac{\bar{p}}{p_i}} \right\}, \qquad \mathcal{Q}_\rho^-(M) = \prod_{i=1}^N \left\{ |x_i| < M^{\frac{p_i-2}{p_i}} \rho^{\frac{\bar{p}}{p_i}} \right\} \times \left(-\rho^{\bar{p}}, 0 \right].$$

The notation of forward and full intrinsic cylinders is analogous to the one above.
- The function $\pi_i : \mathbb{R}^N \to \mathbb{R}$, $\pi_i(x) = x_i$, $i = 1, \ldots, N$, will denote the projection with respect to the i-th space variable. Moreover, $\pi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$, $\pi(x,t) = x$, stands for the projection in the space variables.
- We denote by γ a positive constant (depending only on the data, i.e., N and p_is) that may vary from line to line.

2 Preliminaries

We introduce the parabolic anisotropic spaces, which are the natural setting to work within. We define

$$W_o^{1,\mathbf{P}}(\Omega) := \{ u \in W_o^{1,1}(\Omega) | \, \partial_i u \in L^{p_i}(\Omega) \},$$

$$W_{loc}^{1,\mathbf{P}}(\Omega) := \{ u \in W_{loc}^{1,1}(\Omega) | \, \partial_i u \in L^{p_i}(\Omega) \},$$

$$L_{loc}^{\mathbf{P}}(0,T; W_o^{1,\mathbf{P}}(\Omega)) := \{ u \in L_{loc}^1(0,T; W_o^{1,1}(\Omega)) | \, \partial_i u \in L_{loc}^{p_i}(0,T; L_{loc}^{p_i}(\Omega)) \}.$$

A function

$$u \in C_{loc}^0(0,T; L_{loc}^2(\Omega)) \cap L_{loc}^{\mathbf{P}}(0,T; W_{loc}^{1,\mathbf{P}}(\Omega))$$

is called a *local weak solution* of (0.1) if, for any $0 < t_1 < t_2 < T$ and any compact set $K \subset\subset \Omega$, it satisfies

$$\int_K u\varphi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K \Big(-u \, \partial_t \varphi + \sum_{i=1}^N |\partial_i u|^{p_i-2} \partial_i u \, \partial_i \varphi \Big) \, dxdt = 0, \tag{2.1}$$

for all $\varphi \in C_{loc}^\infty(0,T; C_o^\infty(\Omega))$. By a density and approximation argument, we can consider test functions in (2.1) in the bigger space

$$\varphi \in W_{loc}^{1,2}(0,T; L_{loc}^2(\Omega)) \cap L_{loc}^{\mathbf{P}}(0,T; W_o^{1,\mathbf{P}}(\Omega)),$$

provided $\Omega \subset\subset \mathbb{R}^N$ is a rectangular domain (see [15] for an extension to more general domains).

2.1 Scaling properties of solutions

In the present subsection we show some important scaling properties of solutions to (0.1) and their correspondence with stationary solutions to a Fokker-Planck-type equation.

Proposition 2.1. *Let u be a local weak solution to the equation* (0.1) *in Ω_T. For any $M, \rho > 0$ appropriate for the inclusion $\mathcal{Q}_\rho(M) \subset \Omega_T$, we define the parametric transformation*

$$\tilde{T}_{\rho,M}(x,t) = \left(M^{\frac{p_i-2}{p_i}} \rho^{\frac{\bar{p}}{p_i}} x_i, \rho^{\bar{p}} t \right). \tag{2.2}$$

Then the transformed function

$$\mathcal{T}(u)(x,t) = M^{-1} u\left(\tilde{T}_{\rho,M}(x,t) \right) = M^{-1} u\left(M^{\frac{p_i-2}{p_i}} \rho^{\frac{\bar{p}}{p_i}} x_i, \rho^{\bar{p}} t \right) \tag{2.3}$$

is a solution to (0.1) *in $\tilde{T}_{\rho,M}^{-1}(\Omega_T)$.*

Proof. We perform some formal algebraic computations, representing change of variables in the integrals of definition (2.1).

If we generally suppose

$$\mathcal{T}(u) = M^{-1}u\Big(L_ix_i, Tt\Big)$$

for some $M, L_i, T > 0$, then

$$\partial_t\Big(\mathcal{T}u\Big) = M^{-1}T\partial_tu(L_ix_i, Tt) \quad \text{and} \quad \partial_i\Big(\mathcal{T}u\Big) = M^{-1}L_i\partial_iu(L_ix_i, Tt).$$

Thus, imposing the equation for $\mathcal{T}u$, namely,

$$\partial_t\Big(\mathcal{T}u\Big) = \sum_{i=1}^{N}\partial_i\Big(|\partial_i(\mathcal{T}u)|^{p_i-2}\partial_i(\mathcal{T}u)\Big),$$

we find

$$M^{-1}T\partial_tu(L_ix_i, Tt) = \partial_t\Big((\mathcal{T}u)(x,t)\Big) = \sum_{i=1}^{N}\partial_i\Big(|\partial_i(\mathcal{T}u)(x,t)|^{p_i-2}\partial_i(\mathcal{T}u)(x,t)\Big)$$

$$= \sum_{i=1}^{N}L_i^{p_i}M^{1-p_i}\partial_i\Big(|\partial_iu|^{p_i-2}\partial_iu\Big)(L_ix_i, Tt).$$

Furthermore, we impose

$$M^{-1}T = L_i^{p_i}M^{1-p_i} \quad \forall i = 1, \ldots, N$$

to restore the homogeneity in the equation. We find $L_i = [M^{p_i-2}T]^{\frac{1}{p_i}}$, whence

$$\mathcal{T}u = M^{-1}u\Big(\Big[M^{p_i-2}T\Big]^{\frac{1}{p_i}}x_i, Tt\Big). \tag{2.4}$$

Taking $T = \rho^{\bar{p}}$ concludes the proof. $\qquad\square$

Remark 2.1. *The peculiarity of the scaling* (2.3) *is that it does not alter the time variable, from the point of view of intrinsic geometry. Indeed, the parameter M is usually chosen to be a suitable multiple of either the oscillation or the L^∞ norm of the solution itself, therefore leading to a geometry within the equation evolves in an intrinsic fashion (see [12],[26]).*
Moreover, the proof of Proposition 2.1 reveals that (2.3) *is not the only invariant: we may consider, for instance, also the transformation*

$$\mathcal{T}_{\rho,M}u = M^{-1}u\Big(M^{\frac{p_i-\bar{p}}{p_i}}\rho^{\frac{\bar{p}}{p_i}}x_i, M^{2-\bar{p}}\rho^{\bar{p}}t\Big), \tag{2.5}$$

corresponding to $T = M^{2-\bar{p}}\rho^{\bar{p}}$ in (2.4). *This transformation has been used extensively in [8], with the aim of obtaining a Harnack inequality which intrinsically scales within the particular geometry dictated by the transformation.*

Definition 2.1. *We define the intrinsic anisotropic cube by transformation* (2.2) *on the space variables,*

$$\mathcal{K}_\rho(M) = \prod_{i=1}^{N}\Big\{|x_i| < M^{\frac{p_i-2}{p_i}}\rho^{\frac{\bar{p}}{p_i}}\Big\}, \tag{2.6}$$

and the intrinsic anisotropic cylinders

$$\mathcal{Q}_\rho^-(M) := \tilde{T}_{\rho,M}(Q_1^-) = \prod_{i=1}^{N}\Big\{|x_i| < M^{\frac{p_i-2}{p_i}}\rho^{\frac{\bar{p}}{p_i}}\Big\} \times \Big(-\rho^{\bar{p}}, 0\Big]. \tag{2.7}$$

Similarly we define forward and full intrinsic cylinders.

Remark 2.2. *Definition 2.1 is motivated by Proposition 2.1 and leads to the following consequence. If u solves (0.1) in $\mathcal{Q}_\rho^-(M)$, then $\mathcal{T}u$ solves (0.1) in Q_1^-. Vice-versa if u solves (0.1) in Q_1^- then $\mathcal{T}^{-1}(u) = Mu\left(M^{\frac{2-p_i}{p_i}}\rho^{-\frac{\bar{p}}{p_i}}x_i, \rho^{-\bar{p}}t\right)$ solves (0.1) in $\mathcal{Q}_\rho^-(M)$.*

Proposition 2.2. *Let u be a local weak solution to equation (0.1). Then the parametric transformations preserving the L^1 norm of u correspond to (2.3) for $M = \rho^{-\alpha\bar{p}}$, that is,*

$$\mathcal{T}_\rho u = \rho^{\alpha\bar{p}}u\left(\rho^{\alpha_i\bar{p}}x_i, \rho^{\bar{p}}t\right), \tag{2.8}$$

where α, α_i were defined in (1.3).

Proof. Performing a change of variables, besides recalling (1.3), we get

$$\int_{\pi(\tilde{T}_{\rho,M}(\mathcal{Q}_1))} \mathcal{T}u(x,t)dx = M^{-1}\prod_{i=1}^N \left(M^{\frac{p_i-2}{p_i}}\rho^{\frac{\bar{p}}{p_i}}\right)^{-1}\int_{K_1} u(y,s)dy = \left(M^{-\frac{\lambda}{\bar{p}}}\rho^{-N}\right)\int_{K_1} u(y,s)dy. \tag{2.9}$$

Hence, imposing $M^{-\frac{\lambda}{\bar{p}}}\rho^{-N} = 1$, we find $M = \rho^{-\alpha\bar{p}}$, as desired. $\qquad\square$

Remark 2.3. *It is worth noticing the following important geometric property, used also in the proof of Proposition 2.2: for any $M, \rho > 0$, the total volumes of the anisotropic cube and the anisotropic cylinder depend on p_is, i.e.,*

$$|\mathcal{K}_\rho(M)| = 2^N\rho^N M^{\frac{N(\bar{p}-2)}{\bar{p}}} = \rho^N M^{\frac{N(\bar{p}-2)}{\bar{p}}}|K_1|, \qquad |\mathcal{Q}_\rho^-(M)| = 2^N\rho^{N+\bar{p}}M^{\frac{N(\bar{p}-2)}{\bar{p}}} = \rho^{N+\bar{p}}M^{\frac{N(\bar{p}-2)}{\bar{p}}}|Q_1^-|.$$

Definition 2.2. *A solution u to (0.1) in $\mathbb{R}^{N+1}$ is said to be a self-similar solution if it satisfies $\mathcal{T}_\rho u = u$ for all $\rho > 0$, where T_ρ was defined in (2.8).*

Now we consider the continuous transformation Φ and its inverse Φ^{-1} defined as

$$\Phi(u)(x,t) = w(x,t) = e^{\alpha t}u(e^{\alpha_i t}x_i, e^t), \quad \Phi^{-1}(w)(y,s) = u(y,s) = s^{-\alpha}w(s^{-\alpha_i}y_i, \log s). \tag{2.10}$$

This map formally sends solutions to (0.1) in $\Sigma^+ := \mathbb{R}^N \times (0, +\infty)$ into solutions of the anisotropic Fokker-Planck-type equation

$$\partial_t w = \sum_{i=1}^N \partial_i[(|\partial_i w|^{p_i-2}\partial_i w) + \alpha_i y_i w] \quad \text{in} \quad \Sigma := \mathbb{R}^N \times \mathbb{R}. \tag{2.11}$$

For each fixed time $t = \log(\rho^{-\lambda})$, $\rho > 0$, Φ corresponds to a parametric transformation of type (2.8), thus preserving the L^1 norm; indeed, it is readily seen that

$$\Phi(u)(x, \log(\rho^{-\lambda})) = \mathcal{T}_{\rho^{-\lambda/\bar{p}}}u(x, 1). \tag{2.12}$$

Now we present a characterization of the self-similar solutions.

Proposition 2.3. *Self-similar solutions to (0.1) in Σ^+ correspond to stationary solutions to the Fokker-Planck equation (2.11) and vice-versa.*

Proof. Let us consider a self-similar solution u to (0.1) in Σ^+. We already know that $w = \Phi u$ is a solution to (2.11). It remains to show that w is stationary. By (2.12) and the self-similarity of u, for all $(x,t) \in \Sigma$ we get

$$w(x,t) = w(x, \log(\rho^{-\lambda})) = \mathcal{T}_{\rho^{-\lambda/\bar{p}}}u(x,1) = u(x,1) = w(x,0),$$

being $t = \log(\rho^{-\lambda})$ for some $\rho > 0$.
Vice-versa, let w be a stationary solution to (2.11). We already know that u solves (0.1) in Σ^+, so it suffices

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to show that u is self-similar. For any $\rho > 0$, we choose $t = \log(\rho^{\bar{p}} l)$, $l > 0$, in (2.10) and use the fact that w is stationary to obtain

$$l^\alpha \rho^{\alpha \bar{p}} u(\rho^{\alpha_i \bar{p}} l^{\alpha_i} x_i, \rho^{\bar{p}} l) = w(x, t) = w(x, \log l) = l^\alpha u(l^{\alpha_i} x_i, l).$$

Dividing by l^α, besides performing the change of variables $y_i = l^{\alpha_i} x_i$, leads to

$$\mathcal{T}_\rho u = u \quad \forall \rho > 0,$$

which is the self-similarity of u. $\qquad\square$

Definition 2.3. *A self-similar solution to* (0.1) *in* Σ^+ *(or, equivalently, a solution to* (0.1) *corresponding to a stationary solution to the Fokker-Planck equation* (2.11)*) is said to be a Barenblatt Fundamental solution; it is denoted by* $\mathcal{B}$*, in analogy with the literature regarding the p-Laplacian*[1].

3 Scaling invariants

Definition 2.3 is invariant under the scalings (2.3) and (2.5). In this section we show that also the energy of solutions is invariant, and the same holds for a particular energetic property of solutions, that can be regarded as a measure-theoretical maximum principle.

Lemma 3.1 (Energy Estimates)**.** *Let u be a local weak solution to equation* (0.1) *in* Ω_T*. Let* $(x_o, t_o) \in \Omega_T$ *and* $\rho, M > 0$ *be such that* $(x_o, t_o) + \mathcal{Q}_\rho^-(M) \subset \Omega_T$*. Then, for each function of the form*

$$C_o^\infty((x_o, t_o) + \mathcal{Q}_\rho^-(M)) \ni \eta = \prod_{i=1}^N \eta_i^{p_i}(x_i, t) \quad \text{with} \quad \eta_i \in C_o^\infty(\pi_i(x_o + \mathcal{K}_\rho(M)) \times (t_o - \rho^{\bar{p}}, t_o]),$$

we have the following estimates, valid for all $t_o - \rho^{\bar{p}} < s < t < t_o$ and $k \in \mathbb{R}$:

$$\int_{\mathcal{K}_\rho(M)} (u - k)_\pm^2 \eta(x, \tau) dx \Big|_{\tau=s}^{\tau=t} + \sum_{i=1}^N \int_s^t \int_{\mathcal{K}_\rho(M)} |\partial_i[\eta(u - k)_\pm]|^{p_i} dx d\tau \tag{3.1}$$
$$\leq \gamma \left\{ \int_s^t \int_{\mathcal{K}_\rho(M)} (u - k)_\pm^2 \partial_\tau \eta(x, \tau) dx d\tau + \sum_{i=1}^N \int_s^t \int_{\mathcal{K}_\rho(M)} |(u - k)_\pm|^{p_i} \hat{\eta}_i |\partial_i \eta_i|^{p_i} dx d\tau \right\},$$

where $\hat{\eta}_i := \eta/\eta_i^{p_i}$ and $\gamma > 0$ is a suitable constant (depending only on N and $p_i s$).

Proof. The function u solves equation (0.1) in $(x_o, t_o) + \mathcal{Q}_\rho^-(M)$, so $\mathcal{T}(u)$ (defined in (2.3)) solves (0.1) in Q_1^-, according to Remark 2.2. Now, Lemma 3.1 of [14] on unitary cylinders ensures that for each function of the form

$$C_o^\infty(Q_1) \ni \eta = \prod_{i=1}^N \eta_i^{p_i}(x_i, t) \quad \text{with} \quad \eta_i \in C_o^\infty(\pi_i(K_1) \times (-1, 0]), \tag{3.2}$$

we have, for all $-1 < s_1 < s_2 < 0$ and $\bar{k} \in \mathbb{R}$,

$$\int_{K_1} (\mathcal{T}u - \bar{k})_\pm^2 \eta \, dy \Big|_{s_1}^{s_2} + \sum_{i=1}^N \int_{s_1}^{s_2} \int_{K_1} |\partial_i[\eta(\mathcal{T}u - \bar{k})_\pm]|^{p_i} \, dy ds \tag{3.3}$$
$$\leq \gamma \int_{s_1}^{s_2} \int_{K_1} |(\mathcal{T}u - \bar{k})_\pm|^2 \partial_s \eta \, dy ds + \sum_{i=1}^N \int_{s_1}^{s_2} \int_{K_1} |(\mathcal{T}u - \bar{k})_\pm|^{p_i} \hat{\eta}_i |\partial_i \eta_i|^{p_i} \, dy ds.$$

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[1]Indeed, the epithet *Fundamental* does not mean that solutions are represented by an integral convolution with kernel $\mathcal{B}$, but that the classic $\mathcal{B}$ function approaches to the heat kernel as $p \to 2$. The Barenblatt solution for the p-Laplacian equation can be found in [2].

Now we show that (3.1) comes from (3.3) by performing the change of variables (2.2)-(2.3), besides observing that $[\mathcal{T}u > \bar{k}] \cap K_1 = [u > k] \cap \mathcal{K}_\rho(M)$ provided $\bar{k} = k/M$. Indeed, let us consider the change of variables

$$u(x,t) = u(M^{(p_i-2)/p_i}\rho^{\bar{p}/p_i}y_i, \rho^{\bar{p}}s) = M\mathcal{T}(u)(y,s),$$

with the stipulations

$$\begin{cases} x_i = M^{\frac{p_i-2}{p_i}}\rho^{\frac{\bar{p}}{p_i}}y_i, \\ t = \rho^{\bar{p}}s. \end{cases}$$

We observe that

$$\begin{cases} \partial_{x_i}u(x,t) = M^{2/p_i}\rho^{-\bar{p}/p_i}\partial_{y_i}\mathcal{T}u(y,s), \\ \partial_t u(x,t) = M\rho^{-\bar{p}}\partial_s\mathcal{T}u(y,s), \end{cases}$$

and $dx(y) = \left(\prod_i |dx_i/dy_i|\right)dy = \rho^N M^{\frac{N(\bar{p}-2)}{\bar{p}}}dy$. Hence the first integral in (3.3) becomes

$$\int_{K_1}(\mathcal{T}u - \bar{k})_\pm^2\eta\,dy\Big|_{s_1}^{s_2} = \int_{K_\rho(M)}(M^{-1}(u(x,t) - k))_\pm^2\eta(y(x),s(t))\left(\rho^{-N}M^{\frac{-N(\bar{p}-2)}{\bar{p}}}dx\right)\Big|_{t_1}^{t_2}$$

$$= \rho^{-N}M^{-\frac{[N(\bar{p}-2)+2\bar{p}]}{\bar{p}}}\int_{K_\rho(M)}(u(y,t) - k)_\pm^2\eta\,dx\Big|_{t_1}^{t_2},$$

being $t_1 := \rho^{\bar{p}}s_1 < \rho^{\bar{p}}s_2 =: t_2$. Similarly we evaluate the other integrals of (3.3), obtaining

$$\int_{s_1}^{s_2}\int_{K_1}|\partial_{y_i}[\eta(\mathcal{T}u - \bar{k})_\pm]|^{p_i}\,dyds = \rho^{-N}M^{-\frac{[N(\bar{p}-2)+2\bar{p}]}{\bar{p}}}\int_{t_1}^{t_2}\int_{K_\rho(M)}|\partial_{x_i}[\eta(u - k)_\pm]|^{p_i}\,dxdt.$$

$$\int_{s_1}^{s_2}\int_{K_1}|(\mathcal{T}u - \bar{k})_\pm|^2\partial_s\eta\,dyds = \rho^{-N}M^{-\frac{[N(\bar{p}-2)+2\bar{p}]}{\bar{p}}}\int_{t_1}^{t_2}\int_{K_\rho(M)}|(u(y,t) - k)_\pm^2\partial_t\eta\,dxdt.$$

$$\int_{s_1}^{s_2}\int_{K_1}|(\mathcal{T}u - \bar{k})_\pm|^{p_i}\hat{\eta}_i|\partial_{y_i}\eta_i|^{p_i}\,dyds = \rho^{-N}M^{-\frac{[N(\bar{p}-2)+2\bar{p}]}{\bar{p}}}\int_{t_1}^{t_2}\int_{K_\rho(M)}|u(x,t) - k)_\pm|^{p_i}\hat{\eta}_i|\partial_{x_i}\eta_i|^{p_i}\,dxdt.$$

Collecting the terms $\rho^{-N}M^{-\frac{[N(\bar{p}-2)+2\bar{p}]}{\bar{p}}}$ we get (3.1). Hence, energy estimates are invariant under the scaling transformation (2.3). $\square$

Remark 3.1. *Clearly, the energy estimates above are valid also in forward and full cylinders $(x_o, t_o) + \mathcal{Q}_\rho^+(M)$, $(x_o, t_o) + \mathcal{Q}_\rho(M)$, provided they are contained in Ω_T.*

The next Lemma is a sort of measure-theoretical maximum principle, popular amongst nonlinear analysts as Critical Mass Lemma (following Caffarelli), or De Giorgi-type Lemma (following DiBenedetto). It may be proven at ease for unitary cylinders, and then re-interpreted in the intrinsic geometry dictated by the scaling (2.3). To show the convenience of using (2.3), first we prove the lemma in its general form, and then we discuss its invariance with respect to the scaling.

We recall that local weak sub-solutions (resp, super-solutions) to (0.1) are locally bounded from above (resp., below) in Ω_T (see, e.g., [14], [21]), provided an additional condition constraining the spareness of p_is is ensured. Let us fix a cylinder $(y,s) + \mathcal{Q}_{2\rho}(\theta) \subset\subset \Omega_T$, being $(y,s) \in \Omega_T$ and $\rho, \theta > 0$ appropriate. Let μ^+, μ^- be such that

$$\mu^- \leq \operatorname*{ess\,inf}_{(y,s)+\mathcal{Q}_{2\rho}(\theta)} u \leq \operatorname*{ess\,sup}_{(y,s)+\mathcal{Q}_{2\rho}(\theta)} u \leq \mu^+.$$

We also fix $\omega > 0$, $\xi \in (0,1]$, and $a \in (0,1)$.

Lemma 3.2 (De Giorgi-type/Critical Mass)**.** *Let u be a local weak super-solution to (0.1) in Ω_T locally bounded from below, and let $\rho, \theta, \mu^\pm, \omega, \xi, a$ be defined as above. Then there exists $\nu^- \in (0,1)$, depending on the data N, p_is and on the parameters θ, ω, ξ, a but not on the radius ρ, such that if*

$$|[u \leq \mu^- + \xi\omega] \cap [(y,s) + \mathcal{Q}_{2\rho}^-(\theta)]| \leq \nu^-|\mathcal{Q}_{2\rho}^-(\theta)| \tag{3.4}$$

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then

$$u \geq \mu^- + a\xi\omega \quad \text{a.e. in } \mathcal{Q}_\rho^-(\theta). \tag{3.5}$$

Likewise, if u is a local weak sub-solution to (0.1) *in Ω_T which is locally bounded from above, then there exists $\nu^+ \in (0,1)$, depending on the data $N, p_i s$ and on the parameters θ, ω, ξ, a but not on the radius ρ, such that if*

$$|[u \geq \mu^+ - \xi\omega] \cap [(y,s) + \mathcal{Q}_{2\rho}^-(\theta)]| \leq \nu^+ |\mathcal{Q}_{2\rho}^-(\theta)| \tag{3.6}$$

then

$$u \leq \mu^+ - a\xi\omega \quad \text{a.e. in } \mathcal{Q}_\rho^-(\theta). \tag{3.7}$$

Proof. We prove (3.5), since the proof of (3.7) is analogous. Without loss of generality we assume $(y,s) = (0,0)$, just to ease the notation. Let us set, for any $n, \bar{n} \in \mathbb{N} \cup \{0\}$,

$$\rho_n = \rho + \frac{\rho}{2^n}, \qquad \mathcal{K}_n = \prod_{i=1}^{N} \left\{ |x_i| < \theta^{\frac{p_i-2}{p_i}} \rho^{\frac{\bar{p}}{p_i}} \left(1 + \frac{1}{2^{n+\bar{n}}} \right) \right\}, \qquad \mathcal{Q}_n = \mathcal{K}_n \times (-\rho_n^{\bar{p}}, 0].$$

Since $K_0 \to \mathcal{K}_\rho(\theta)$ as $\bar{n} \to \infty$, we fix $\bar{n}$ such that $\mathcal{K}_0 \subset \mathcal{K}_{2\rho}(\theta)$. Notice that $\bar{n}$ can be chosen in such a way that it depends only on N and p_is. We apply energy estimates (3.1) over $\mathcal{K}_n, \mathcal{Q}_n$ to the truncations $(u - k_n)_-$ at the levels

$$k_n = \mu^- + \xi_n \omega, \quad \text{where} \quad \xi_n = a\xi + \frac{(1-a)\xi}{2^n}.$$

Incidentally, notice that

$$|(u - k_n)_-| \leq \xi_n \omega \leq \xi\omega.$$

For any n, we pick a cut-off function η_n of the form $\eta_n = \bar{\eta}(t) \prod_{i=1}^{N} \eta_i^{p_i}(x_i)$, where

$$\eta_i(x_i) = \begin{cases} 1, & \text{in} \quad \pi_i(\mathcal{K}_{n+1}), \\ 0, & \text{in} \quad \mathbb{R} \setminus \pi_i(\mathcal{K}_n), \end{cases} \qquad |\partial_i \eta_i| \leq \frac{\gamma 2^n}{\theta^{\frac{p_i-2}{p_i}} \rho^{\frac{\bar{p}}{p_i}}}.$$

$$\bar{\eta}(t) = \begin{cases} 1, & \text{when} \quad t \geq -\rho_{n+1}^{\bar{p}}, \\ 0, & \text{when} \quad t < -\rho_n^{\bar{p}}, \end{cases} \qquad |\partial_t \bar{\eta}| \leq \frac{\gamma}{(\rho_n^{\bar{p}} - \rho_{n+1}^{\bar{p}})} \leq \frac{\gamma 2^{(n+1)\bar{p}}}{\rho^{\bar{p}}},$$

The energy estimate (3.1), applied with these choices, yields

$$\mathcal{E}_n := \sup_{(-\rho_n^{\bar{p}}, 0]} \int_{\mathcal{K}_n} (u - k_n)_-^2 \eta_n \, dx + \sum_{i=1}^{N} \int\!\!\int_{\mathcal{Q}_n} |\partial_i[\eta_n (u - k_n)_-]|^{p_i} \, dx dt$$

$$\leq \gamma \int\!\!\int_{\mathcal{Q}_n} \left\{ (u - k_n)_-^2 |\partial_t \bar{\eta}| + \sum_{i=1}^{N} |(u - k_n)_-|^{p_i} |\partial_i \eta_i|^{p_i} \right\} dx dt$$

$$\leq \frac{\gamma 2^{\bar{p}(n+1)}}{\rho^{\bar{p}}} (\xi\omega)^2 \left(1 + \sum_{i=1}^{N} \left(\frac{\xi\omega}{\theta} \right)^{p_i-2} \right) |[u < k_n] \cap \mathcal{Q}_n|.$$

Now we combine these estimates of the energies $\mathcal{E}_n$ with the embedding inequality (see [14])

$$\int\!\!\int_{\mathcal{Q}_n} |(u - k_n)_- \eta_n|^{\bar{p}(\frac{N+2}{N})} dx dt$$

$$\leq \gamma \left(\sup_{(-\rho_n^{\bar{p}}, 0]} \int_{\mathcal{K}_n} (u - k_n)_-^2 \eta_n^2 \, dx + \sum_{i=1}^{N} \int\!\!\int_{\mathcal{Q}_n} |\partial_i[\eta_n[(u - k_n)_-]|^{p_i} \, dx dt \right)^{\frac{N+\bar{p}}{N}} = \gamma \mathcal{E}_n^{\frac{N+\bar{p}}{N}}.$$

Observing that $(u - k_n)_- \geq (k_n - k_{n+1}) = (1-a)\xi\omega/2^{n+1}$ in $[u < k_{n+1}] \cap \mathcal{Q}_{n+1}$, we get the chain

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$$\left(\frac{(1-a)\xi\omega}{2^{n+1}}\right)^{\bar{p}(\frac{N+2}{N})}|[u < k_{n+1}] \cap \mathcal{Q}_{n+1}| \leq \int\int_{\mathcal{Q}_{n+1}} |(u-k_n)_-|^{\bar{p}(\frac{N+2}{N})}dxdt$$

$$\leq \int\int_{\mathcal{Q}_n} |(u-k_n)_-\eta_n|^{\bar{p}(\frac{N+2}{N})}dxdt \leq \gamma\mathcal{E}_n^{\frac{N+\bar{p}}{N}} \tag{3.8}$$

$$\leq \gamma\left[\frac{2^{\bar{p}(n+1)}}{\rho^{\bar{p}}}(\xi\omega)^2\left(1 + \sum_{i=1}^N \left(\frac{\xi\omega}{\theta}\right)^{p_i-2}\right)\right]^{\frac{N+\bar{p}}{N}}|[u < k_n] \cap \mathcal{Q}_n|^{\frac{N+\bar{p}}{N}}.$$

We set $Y_n = \frac{|[u<k_n]\cap\mathcal{Q}_n|}{|\mathcal{Q}_n|}$ and recall that up to a constant we have $|\mathcal{Q}_n|^{\frac{\bar{p}}{N}} \leq \theta^{\bar{p}-2}\rho^{\frac{\bar{p}}{N}(\bar{p}+N)}$. Dividing (3.8) for $|\mathcal{Q}_{n+1}|$, as well as noticing that $|\mathcal{Q}_{n+1}| \geq 2^{-(N+1)}|\mathcal{Q}_n|$, we obtain

$$Y_{n+1} \leq \gamma\left(\frac{2^{n+1}}{(1-a)\xi\omega}\right)^{\bar{p}(\frac{N+2}{N})}\left[\frac{2^{\bar{p}(n+1)}}{\rho^{\bar{p}}}(\xi\omega)^2\left(1 + \sum_{i=1}^N \left(\frac{\xi\omega}{\theta}\right)^{p_i-2}\right)\right]^{\frac{N+\bar{p}}{N}}Y_n^{1+\frac{\bar{p}}{N}}|\mathcal{Q}_n|^{\frac{\bar{p}}{N}}$$

$$\leq 2^{n(2N+\bar{p}+2)\frac{\bar{p}}{N}}\left[\frac{\gamma(\xi\omega/\theta)^{2-\bar{p}}}{(1-a)^{\frac{\bar{p}}{N}(N+2)}}\left(1 + \sum_{i=1}^N (\xi\omega/\theta)^{p_i-2}\right)^{\frac{N+\bar{p}}{N}}\right]Y_n^{1+\frac{\bar{p}}{N}}. \tag{3.9}$$

According to [12, Lemma 4.1, page 12], (3.9) produces $Y_\infty := \lim_{n\to\infty} Y_n = 0$ provided

$$Y_0 \leq 2^{-\frac{N}{\bar{p}}(2N+\bar{p}+2)}\left[\frac{\gamma(\xi\omega/\theta)^{2-\bar{p}}}{(1-a)^{\frac{\bar{p}}{N}(N+2)}}\left(1 + \sum_{i=1}^N (\xi\omega/\theta)^{p_i-2}\right)^{\frac{N+\bar{p}}{N}}\right]^{-\frac{N}{\bar{p}}} := \nu^*. \tag{3.10}$$

Then $Y_\infty = 0$ implies (3.5), concluding the proof. It remains to ensure (3.10).
We set $\nu^- := \nu^*/\gamma$, where ν^* stems from (3.10) and γ is such that $|\mathcal{Q}_{2\rho}^-(\theta)| \leq \gamma|\mathcal{Q}_0|$. Therefore

$$|[u < \mu^- + \xi\omega] \cap \mathcal{Q}_0| \leq |[u < \mu^- + \xi\omega] \cap \mathcal{Q}_{2\rho}^-(\theta)| \leq \nu^-\gamma|\mathcal{Q}_0| = \nu^*|\mathcal{Q}_0|,$$

which guarantees (3.10).
In order to prove (3.7), we may proceed as above, considering energy estimates (3.1) in the same iterative geometry, but this time using the truncations $(u-k_n)_+$, being $k_n = \mu^+ - \xi_n\omega$. □

Remark 3.2. *We observe that setting $\theta = \xi\omega$ frees ν^- from any dependence on θ, ω, ξ. When $\omega \geq \mu^+ - \mu^-$, this choice of θ represents the intrinsic geometry, since ω is related to the oscillation of the solution u in $\mathcal{Q}_{2\rho}^-(\theta)$.*

Simple adjustments on $\mathcal{Q}_n$ guarantee the validity of Lemma 3.2 also for forward cylinders and full cylinders.

Lemma 3.2 is invariant under the particular scalings (2.3) and (2.5). To show this, one can use the change of variables employed in the proof of Lemma 3.1 to prove the following lemma, and then one can go back to Lemma 3.2. Accordingly, the argument sketched here shows the equivalence of Lemmas 3.2 and 3.3.

Lemma 3.3. *Let u be a local weak super-solution to (0.1) in Q_1 such that $\text{ess}\inf_{Q_1^-} u \geq 0$. Then for any $a \in (0,1)$ there exists $\nu_a^+ > 0$, depending on a, p_i, N but neither on u nor on ρ, such that*

$$|[u \leq 1] \cap \mathcal{Q}_1^-(1)| \leq \nu_a^+|\mathcal{Q}_1^-(1)| \quad \Rightarrow \quad \text{ess}\inf_{\mathcal{Q}_{1/2}^-(1)} u \geq a. \tag{3.11}$$

Let u be a local weak sub-solution to (0.1) in Q_1 such that $\text{ess}\sup_{Q_1^-} u \leq 1$. Then for any $a \in (0,1)$ there exists $\nu_a^- > 0$, depending on a, p_i, N but neither on u nor on ρ, such that

$$|[u \geq 1/2] \cap \mathcal{Q}_1^-(1/2)| \leq \nu_a^-|\mathcal{Q}_1^-(1/2)| \quad \Rightarrow \quad \text{ess}\sup_{\mathcal{Q}_{1/2}^-(1/2)} u \leq (1-a/2). \tag{3.12}$$

Proof. It suffices to apply Lemma 3.2 to the function $(u-\mu^-)/(\xi\omega)$ (resp., $(\mu^+ - u)/(\xi\omega)$) with the choices $\mu^- = 0$, $\theta = \xi\omega = 1$ (resp., $\mu^+ = 1$, $\theta = \xi\omega = 1/2$), and $\rho = 1$ and in the first (resp., second) case. □

4 A topological consequence of energy invariants: lower semi-continuity of super-solutions

Theorem 4.1. *Let u be a weak local super-solution to* (0.1) *in Ω_T locally bounded from below. Then u is lower semi-continuous.*

Proof. We proceed in a way reminiscent of [19]. Set $\mathcal{Q}_\rho := \mathcal{Q}_\rho(1)$ for all $\rho > 0$, and consider the lower semi-continuous regularization of u, defined as

$$u_*(x,t) = \lim_{\rho \to 0^+} \operatorname*{ess\,inf}_{(x,t)+\mathcal{Q}_\rho} u \quad \forall (x,t) \in \Omega_T. \tag{4.1}$$

We observe that this function is well defined, since $(x,t) + \mathcal{Q}_\rho \subset \Omega_T$ for small values of ρ. It is a well-known fact that u_* is lower semi-continuous. Accordingly, proving that $u_* = u$ almost everywhere in Ω_T furnishes the lower semi-continuity of u. In order to show this equality, we also define the set

$$\mathcal{L} = \left\{ (x,t) \in \Omega_T : \ |u(x,t)| < \infty, \text{ and } \lim_{\rho \to 0^+} \fint_{(x,t)+\mathcal{Q}_\rho} |u(x,t) - u(y,t)| \, dy dt = 0 \right\}. \tag{4.2}$$

This set is well defined, since $u \in L^1_{loc}(0,T; L^1_{loc}(\Omega))$. Moreover,

$$|\mathcal{L}| = |\Omega_T|. \tag{4.3}$$

As we will see, this is a consequence of the fact that $X := (\Omega_T, \mathscr{L}^{N+1}, d)$, being $\mathscr{L}^{N+1}$ the $(N+1)$-Lebesgue measure and d a particular distance to be introduced, is a doubling space. We consider the following distance d: for any $(x,t), (y,s) \in \Omega_T$ we define

$$d((x,t),(y,s)) := \max\{|x_i - y_i|^{\frac{p_i}{\bar{p}}}, |t-s|^{\frac{1}{\bar{p}}}\},$$

and we denote by $\mathbb{B}_\rho(x,t)$ the balls with respect to distance d. It turns out that $\mathbb{B}_\rho(x,t) = (x,t) + \mathcal{Q}_\rho$. The doubling property follows from

$$\mathscr{L}^{N+1}(\mathbb{B}_{2\rho}) = \mathscr{L}^{N+1}(\mathcal{Q}_{2\rho}) = (2\rho)^{N+\bar{p}} = 2^{N+\bar{p}} \mathscr{L}^{N+1}(\mathcal{Q}_\rho) = 2^{N+\bar{p}} \mathscr{L}^{N+1}(\mathbb{B}_\rho).$$

Accordingly, [16, p. 12] provides (4.3).

Taking (4.3) into account, it is sufficient to prove $u_* = u$ in $\mathcal{L}$. For all $(x,t) \in \mathcal{L}$ we have

$$u_*(x,t) = \lim_{\rho \to 0} \operatorname*{ess\,inf}_{(x,t)+\mathcal{Q}_\rho} u \leq \lim_{\rho \to 0} \fint_{(x,t)+\mathcal{Q}_\rho} u \, dy dt = u(x,t).$$

To show the opposite inequality, let us pick $(x_o, t_o) \in \mathcal{L}$ and suppose by contradiction that $u_*(x_o, t_o) < u(x_o, t_o)$. Let $r, b > 0$ be small enough such that $(x_o, t_o) + \mathcal{Q}_r \subset \Omega_T$ and

$$\operatorname*{ess\,inf}_{(x_o,t_o)+\mathcal{Q}_r} u := \mu_- \leq u_*(x_o,t_o) < \mu_- + b < u(x_o,t_o).$$

This choice is possible, since u_* is close to μ^- for small values of ρ, as well as $\mathcal{Q}_\rho$ shrinks to (x_o, t_o) as $\rho \to 0^+$. Let us introduce $a \in (0,1)$ such that

$$\mu_- + ab > u_*(x_o,t_o), \qquad \text{i.e.,} \qquad \frac{u_*(x_o,t_o) - \mu_-}{b} < a < 1.$$

Then there exists $\nu_a^- > 0$, depending only on a, b, p_i, N, such that for some $\rho \in (0,r)$ we have

$$|[u \leq \mu_- + b] \cap (x_o,t_o) + \mathcal{Q}_\rho| \leq \nu_a^- |\mathcal{Q}_\rho|,$$

since otherwise we have, for all $\rho \in (0,r)$,

$$\int_{(x_o,t_o)+\mathcal{Q}_\rho} |u(x_o,t_o) - u(x,t)| \, dx dt \geq \int_{[u \leq \mu_-+b] \cap (x_o,t_o)+\mathcal{Q}_\rho} [u(x_o,t_o) - (\mu_- + b)] \, dx dt$$

$$\geq \nu_a^- [u(x_o,t_o) - (\mu_- + b)]|\mathcal{Q}_\rho|,$$

contradicting $(x_o, t_o) \in \mathcal{L}$. Now we are in the position to apply Lemma 3.2 and reach

$$u(x, t) \geq \mu_- + ab > u_*(x_o, t_o), \quad \text{for a.a. } (x, t) \in (x_o, t_o) + \mathcal{Q}_{\rho/2}.$$

This contradicts the definition of $u_*(x_o, t_o)$, since

$$u_*(x_o, t_o) < \operatorname*{ess\,inf}_{(x_o, t_o)+Q_{\rho/2}} u \leq \lim_{\rho \to 0^+} \operatorname*{ess\,inf}_{(x_o, t_o)+Q_\rho} u = u_*(x_o, t_o).$$

Therefore, we obtain $u_* = u$ in $\mathcal{L}$, concluding the proof. $\qquad\square$

Remark 4.1. *Semi-continuity of solutions to* (0.1), *proved in Theorem 4.1, jointly with the structure of the equation, ensures that the estimates* (3.5) *and* (3.7) *of Lemma 3.2 hold true everywhere (and not merely 'almost everywhere') in* $\mathcal{Q}_\rho^-(\theta)$.

Acknowledgements

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