ESAIM: COCV 30 (2024) 72
https://doi.org/10.1051/cocv/2024058

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

SUB-RIEMANNIAN CUT TIME AND CUT LOCUS IN REITER–HEISENBERG GROUPS

Annamaria Montanari* and Daniele Morbidelli

Abstract. We study the sub-Riemannian cut time and cut locus of a given point in a class of step-2 Carnot groups of Reiter–Heisenberg type. Following the Hamiltonian point of view, we write and analyze extremal curves, getting the cut time of any of them, and a precise description of the set of cut points.

Mathematics Subject Classification. 53C17, 53C22, 49K15.

Received December 22, 2023. Accepted August 1, 2024.

1. Introduction

In this paper we study the sub-Riemannian cut time and cut locus (of the origin) in a class of step-2 Carnot groups of Reiter–Heisenberg type. Following a Hamiltonian point of view, we will write extremal curves, and we will identify cut points as either points reached by two different minimizing geodesics, or conjugate points. In spite of the apparent similarity with the well known Heisenberg group, we will see that these more general models display several different features.

In order to state our results, let us describe briefly Reiter–Heisenberg groups. Let $V_1 = \mathbb{R}^{q\times p} \times \mathbb{R}^{p\times 1}$ and $V_2 = \mathbb{R}^{q\times 1}$. On $V_1 \times V_2$ define the operation

$$\begin{aligned}(x, y, t) \cdot (\xi, \eta, \tau) :=& \Big(x + \xi, y + \eta, t + \tau + Q((x, y), (\xi, \eta))\Big)\\ :=& \Big(x + \xi, y + \eta, t + \tau + \frac{1}{2}(x\eta - \xi y)\Big).\end{aligned} \tag{1.1}$$

It turns out that $\mathbb{G}_{qp} = (V_1 \times V_2, \cdot)$ is a step-2 nilpotent Carnot group (see [1], [2]). Following the terminology in [3, 4] and [5], we call it a Reiter–Heisenberg group. These models are a significant generalization of the familiar Heisenberg group which involves nontrivial interesting issues. For example, we will see that for any $q \geq 2$, the group $\mathbb{G}_{qp}$ contains abnormal geodesics, while for $q = 1$, $\mathbb{G}_{1p}$ is the standard Heisenberg group, where minimizers are always strictly normal.

To equip $(\mathbb{G}_{qp}, \cdot)$ with a sub-Riemannian structure, we introduce on V_1 the inner product $\langle (x, y), (\xi, \eta)\rangle := \operatorname{trace} x^T\xi + \operatorname{trace} y^T\eta = \operatorname{trace} x^T\xi + y^T\eta$. On matrix spaces, we shall always use the Hilbert–Schmidt inner product $\langle a, b\rangle := \operatorname{trace} a^T b$ for all $a, b \in \mathbb{R}^{\mu\times\nu}$ and for all $\mu, \nu \in \mathbb{N}$.

Keywords and phrases: Carnot groups, cut locus, sub-Riemannian geodesic.

Dipartimento di Matematica, Alma Mater Studiorum Università di Bologna, Italy.

* Corresponding author: annamaria.montanari@unibo.it

In order to write length-minimizing curves, we will adopt the Hamiltonian point of view (see [2], see Sect. 2.2). Denote by $H((x,y,t),(\xi,\eta,\tau))$ the quadratic Hamiltonian for normal extremals (which will be written in Sect. 2.2). Extremal curves are smooth and are parametrized by $(\xi,\eta,\tau)\in T_0^*(V_1\times V_2)\simeq V_1\times V_2$. We denote them by $\gamma(\cdot,\xi,\eta,\tau):\mathbb{R}\to V_1\times V_2$. We always assume $\gamma(0,\xi,\eta,\tau)=(0,0,0)$ (different starting points can be easily managed by group translations, see [1]). The (constant) horizontal speed of such path is by definition $|(\xi,\eta)|=\sqrt{|\xi|^2+|\eta|^2}$. We denote by $t_{\rm cut}\in\,]0,+\infty]$ the related cut time

$$t_{\rm cut}=t_{\rm cut}(\xi,\eta,\tau):=\sup\{\bar s>0:\gamma(\cdot,\xi,\eta,\tau)\text{ minimizes length on }[0,\bar s]\ \}.$$

To state our first result, for $\tau\in\mathbb{R}^q\setminus\{0\}$, introduce the notation $P_\tau=\frac{\tau\tau^T}{|\tau|^2}\in\mathbb{R}^{q\times q}$ to denote the orthogonal projection on $\operatorname{span}\{\tau\}\subset\mathbb{R}^q$. Let also $P_\tau^\perp:=I_q-P_\tau$ be the orthogonal projection along $\tau^\perp$. On a matrix $x=[x_1,\dots,x_p]\in\mathbb{R}^{q\times p}$, where $x_k\in\mathbb{R}^{q\times 1}\simeq\mathbb{R}^q$ for $k=1,\dots,p$ are the columns of x, the operators $P_\tau x$ and $P_\tau^\perp x$ project separately each column of x.

Then we have the following theorem.

Theorem 1.1 (Cut-time)**.** *Consider $(\xi,\eta)\in V_1\setminus\{0\}$ and $\tau\in\mathbb{R}^q\setminus\{0\}$. Assume also that $|\eta|+|P_\tau\xi|>0$. Then, the length-extremal $s\mapsto\gamma(s,\xi,\eta,\tau)$ such that $\gamma(0,\xi,\eta,\tau)=(0,0,0)$ minimizes length up to*

$$t_{\rm cut}(\xi,\eta,\tau)=\frac{2\pi}{|\tau|}.\tag{1.2}$$

Cut points of the origin can be consequently described as points of the form $\gamma(\frac{2\pi}{|\tau|},\xi,\eta,\tau)$ with (ξ,η,τ) satisfying the previous requirements.

If one of the assumptions $\tau\neq0$ and $|\eta|+|P_\tau\xi|>0$ is violated, then we have an extremal of the form $\gamma(s,\xi,\eta,0)=(s\xi,s\eta,0)$ which minimizes length globally (see Rem. 2.2). Some of these lines are normal minimizers, some other are abnormal (abnormal extremal curves will be identified in Prop.2.3).

To comment on Theorem 1.1, let us look at the "horizontal speed vector" $(\dot x,\dot y)$ of a given extremal $\gamma(\cdot,\xi,\eta,\tau)$. We have , $(\dot x(s),\dot y(s))=a(\xi,\eta)\cos(|\tau|s)+b(\xi,\eta)\sin(|\tau|s)+z(\xi,\eta)$, for suitable functions a,b,z of the variables ξ,η (see Sect. 2). The cut time corresponds to a complete period of the circular functions. This agrees with the standard Heisenberg group and with Heisenberg-type groups [6]. There is however a constant part $z(\xi,\tau)$ which, if nonzero, makes the "horizontal part" (x,y) of a cut point nonzero. Previous results with similar features on cut times for Carnot groups of step two were proved in [7, 8] for structures of low corank (at most two). The case of free, step-2 Carnot groups is dealt in [9–11]. Here, working in model (1.1), we have no bounds on the corank $\dim(V_2)$. The comprehensive survey paper [12] should be consulted to have a complete account on the mentioned models and on sub-Riemannian manifolds outside the setting of Carnot groups. In the recent papers [13] and [14], the authors analyze the cut-time in a rather large class of step-two Carnot groups, including the limiting case $\mathbb{G}_{q1}$ of the family of models $\mathbb{G}_{qp}$ object of the present paper.

Our second result involves the description of the cut locus as a set. It turns out from Theorem 1.1 and from the form of extremals written in Section 3, that cut points have the form

$$\begin{cases} x\Big(\dfrac{2\pi}{|\tau|},\xi,\eta,\tau\Big)=\dfrac{2\pi}{|\tau|}P_\tau^\perp\xi\\ y\Big(\dfrac{2\pi}{|\tau|},\xi,\eta,\tau\Big)=0\\ t\Big(\dfrac{2\pi}{|\tau|},\xi,\eta,\tau\Big)=\dfrac{\pi}{|\tau|^2}\Big(|P_\tau\xi|^2+|\eta|^2\Big)\dfrac{\tau}{|\tau|}-\dfrac{2\pi}{|\tau|^2}P_\tau^\perp\xi\dfrac{\xi^T\tau}{|\tau|}\end{cases}$$

where $\tau\neq0$ and $|\eta|^2+|P_\tau\xi|^2>0$. Note first that if $q=1$ and $\tau\in\mathbb{R}^1\setminus\{0\}$, trivially we have $\tau^\perp=\{0\}$ and $P_\tau^\perp=0$. Then cut points have the form $(0,0,t)$, *i.e.* they are contained in the t-axis. This is the familiar case of the

Heisenberg group $\mathbb{H}^p$ (see [2], Chap. 13). If instead $q > 1$, then $P_\tau^\perp$ is always nonzero and we have points $(x, 0, t) \in \mathrm{Cut}(\mathbb{G}_{qp})$ with $x \neq 0$. The natural question is now whether or not we may have $\mathrm{Cut}(\mathbb{G}_{qp}) = \{(x, 0, t) \in V_1 \times V_2\}$, for some choices of q and p. The answer is not, by the following theorem. To state it, given $x \in \mathbb{R}^{q\times p}$, introduce $P_{\mathrm{Im}\,x}$ and $P^\perp_{\mathrm{Im}\,x} : \mathbb{R}^q \to \mathbb{R}^q$ to denote respectively the orthogonal projection on $\mathrm{Im}\,x := \{xy : y \in \mathbb{R}^p\} \subset \mathbb{R}^q$ and on its orthogonal. The linear map $x^\dagger \in \mathbb{R}^{p\times q}$ denotes instead the Moore-Penrose inverse of x (see below).

Theorem 1.2 (Identification of the cut locus)**.** *Let $q, p \in \mathbb{N}$ and let $\mathbb{G}_{qp}$ be the associated Carnot group. Then*

1. *We have*

$$\mathrm{Cut}(\mathbb{G}_{qp}) = \big\{(x, 0, t) \in \mathbb{G}_{qp} : t \notin \mathrm{Im}\,x \ \text{ and } \ |P^\perp_{\mathrm{Im}\,x} t| \geq \pi |x^\dagger t|^2\big\}. \tag{1.3}$$

2. *If $p \geq 2$, then all cut points are conjugate points.*
3. *A cut point $(x, 0, t)$ is reached by a unique unit-speed length-minimizing curve if and only if equality $|P^\perp_{\mathrm{Im}\,x} t| = \pi |x^\dagger t|^2$ holds.*
4. *Finally, the distance from the origin of a cut point $(x, 0, t)$ has the form*

$$d(x, 0, t) = \sqrt{|x|^2 + 4\pi |P^\perp_{\mathrm{Im}\,x} t|}. \tag{1.4}$$

If $p = 1$, item 2 is drastically different. Note that for $p = 1$ we have $x^\dagger t = \frac{\langle x,t\rangle}{|x|^2}$.

Theorem 1.3. *If $p = 1$ and $q \geq 2$, a cut point $(x, 0, t)$ is conjugate if and only if*

$$|x|\big(|P^\perp_{\mathrm{Im}\,x} t| - \pi |x^\dagger t|^2\big) = 0. \tag{1.5}$$

In the statement above, for the notion of conjugate point, see Definition 2.4. The conjugate point $(x, 0, t)$ is necessarily a *first conjugate point*, see Remark 2.5. For the Moore-Penrose inverse $x^\dagger \in \mathbb{R}^{p\times q}$ of x, see *e.g.* [15]. Precisely, given $t \in \mathbb{R}^q$, $x^\dagger t$ is uniquely defined by $xx^\dagger t = P_{\mathrm{Im}\,x} t$ and $x^\dagger t \perp \ker x$. If $\ker x = \{0\}$, then $x^\dagger t$ is the unique solution $y \in \mathbb{R}^p$ of the system $xy = P_{\mathrm{Im}\,x} t$. Otherwise, $x^\dagger t$ is the smallest-norm element of the set of solutions $\{y = x^\dagger t + \eta : \eta \in \ker x\}$ of $xy = P_{\mathrm{Im}\,x} t$. In other words, $|x^\dagger t| = |x^\dagger P_{\mathrm{Im}\,x} t| = \min\{|y| : y \in \mathbb{R}^p \text{ and } xy = P_{\mathrm{Im}\,x} t\}$. Note that we do not include $(0, 0, 0)$ in our definition of cut-locus. Observe that (1.3) implies that $\mathrm{Im}\,x \neq \mathbb{R}^q$ for any point $(x, 0, t) \in \mathrm{Cut}(\mathbb{G}_{qp})$. Furthermore, since given $x = 0 \in \mathbb{R}^{q\times p}$, we have $0^\dagger = 0 \in \mathbb{R}^{p\times q}$, all points of the form $(0, 0, t)$ with $t \neq 0$ belong to the cut locus as expected. Finally, for all $r > 0$ the set in (1.3) is dilation invariant with respect to the standard Carnot homogeneous dilation $(x, y, t) \mapsto (rx, ry, r^2 t)$.

Remark 1.4. Some remarks on Theorems 1.1 and 1.2 are now in order.

1. In Theorem 1.2, we describe precisely all cut points reached by a unique length-minimizer. Such kind of minimizers are absent in p-dimensional Heisenberg groups $\mathbb{G}_{1p}$ (case $(q = 1)$), and in the free step-2, rank-three model (not included in our class $\mathbb{G}_{qp}$). They appear, but they remained unnoticed in [8]. Points of such kind appear classically on Riemannian equatorial geodesics in oblate revolution ellipsoids (see [16]). In Remark 5.1 we prove that points in $\mathrm{Cut}(\mathbb{G}_{qp})$ reached by more than one unit-speed length-minimizer are dense in $\mathrm{Cut}(\mathbb{G}_{qp})$. See [17], for the Riemannian analogous.
2. The set $\mathrm{Cut}(\mathbb{G}_{qp}) \cup \{(0, 0, 0)\}$ is not closed for any $q \geq 2$ and $p \in \mathbb{N}$. Indeed, consider a family $(x, 0, \varepsilon t)$ with $x \neq 0$, $t \neq 0$ and $t \perp \mathrm{Im}\,x$ (this forces $q \geq 2$). We have $(x, 0, \varepsilon t) \in \mathrm{Cut}(\mathbb{G}_{qp})$ for all $\varepsilon > 0$, but $(x, 0, 0)$ is not a cut point for any $x \in \mathbb{R}^{q\times p}$. In Proposition 2.3, we prove that these points $(x, 0, 0) \neq 0$ belonging to $\overline{\mathrm{Cut}(\mathbb{G}_{qp})} \setminus \mathrm{Cut}(\mathbb{G}_{qp})$ are always abnormal points. Recall that, in absence of abnormal minimizers, Rifford and Trélat proved in [18], Lemma 2.11 that in a sub-Riemannian Carnot group $\mathbb{G}$ the set $\mathrm{Cut}(\mathbb{G}) \cup \{0\}$ is closed.
3. At abnormal points appearing in item 2, the function $t_{\mathrm{cut}} : T_0^*\mathbb{G}_{qp} \to \,]0, +\infty]$, defined as $(\xi, \eta, \tau) \mapsto t_{\mathrm{cut}}(\xi, \eta, \tau)$ has a discountinuous behaviour (upper semicontinuous, unbounded), as a function of the dual

variables (ξ,η,τ). See Example 3.6. The function $(s,\xi,\eta,\tau)\mapsto\gamma(s,\xi,\eta,\tau)$ is instead smooth in all its arguments, by standard ODE theory.

4. Observe that $\mathrm{Cut}(\mathbb{G}_{qp})$ can be a quite large set. Namely, if $p=1$, it turns out that $\mathrm{Cut}(\mathbb{G}_{q1})$ contains the set $\{(x,0,t)\in\mathbb{G}_{q1}: |P^{\perp}_{\operatorname{Im}x}t|\gneqq\pi|x^{\dagger}t|^2 \text{ and } \operatorname{rank}x \text{ is maximal}\}$, which is an open[1] dilation invariant subset of the codimension-one hyperplane of equation $y=0$. Note that in free step-2 Carnot groups $\mathbb{F}_k$ of rank $k=2$ and $k=3$, the cut-locus is a smooth manifold of dimension $\dim(\mathrm{Cut}(\mathbb{F}_k))=\dim(\mathbb{F}_k)-2$ (for the case $k=3$, see [9], Rem. 4.3). The conjectured dimension of $\mathrm{Cut}(\mathbb{F}_k)$ is again $\dim(\mathbb{F}_k)-2$ in rank-k, step-2 groups $\mathbb{F}_k$ with $k\geq 3$. See [11]. Observe finally that in the limiting case $p=1$ our set (1.5) agrees with the one found by [14], Section 10 with completely different methods.
5. Finally, note that in the model $\mathbb{G}_{q1}$, all points $(x,0,t)\in\mathrm{Cut}$ such that (1.5) is violated give examples of extremals whose cut time is strictly less than the first conjugate time. See [8] for previous different examples.

Since the cut locus appearing in (1.3) is not easy to visualize for general values of q,p, in Section 5.1 we discuss in some details the group $\mathbb{G}_{q1}$. In that case, the cut locus turns out to be defined as a sublevel set of an explicit scalar polynomial function. Its regularity properties are analyzed in Proposition 5.3.

Let us describe now the structure of the paper. In Section 2, after providing general notation and known facts, we write the length-extremals of our sub-Riemannian problem and we characterize abnormal ones. To prove Theorem 1.1, starting from the candidate cut time $\frac{2\pi}{|\tau|}$ appearing in (1.2), we show in Section 3 the upper estimate $t_{\mathrm{cut}}\leq\frac{2\pi}{|\tau|}$ for extremals which are not Euclidean lines. This is achieved by the analysis of conjugate points (Props. 3.1 and 3.2). We also characterize points reached by a unique minimizer (see Prop. 3.4). In Section 4, we prove the lower bound $t_{\mathrm{cut}}\geq\frac{2\pi}{|\tau|}$ by using a geometric-control argument in part inspired to the paper [8]. Finally, in Section 5, we conclude the proof of Theorem 1.2. In Section 5.1, we also explicitly describe $\mathrm{Cut}(\mathbb{G}_{q1})$ and we analyze its regularity. In Section 6, we prove Theorem 1.3.

2. Reiter–Heisenberg groups and their length-extremals

In this section we briefly recall the notion of sub-Riemannian length and distance in Reiter–Heisenberg groups. Then we write the explicit form of extremals and, among them, we characterize the abnormal ones.

2.1. General facts

Let $(\mathbb{G}_{qp},\cdot)$ be the Reiter–Heisenberg group defined in (1.1). A horizontal curve is a Lipschitz-continuous solution $\gamma=(x,y,t):[0,T]\to V_1\times V_2$ of the ODE

$$(\dot x,\dot y)=(u,v)\quad \text{and } \dot t=Q((x,y),(u,v)),\quad \text{a.e. on } [0,T], \tag{2.1}$$

where $(u,v):[0,T]\to V_1$ is an L^∞ control. The horizontal speed of γ is $|\dot\gamma|_{\mathrm{hor}}:=\sqrt{|u|^2+|v|^2}:=\sqrt{(\operatorname{trace}u^Tu)^2+|v|^2}$. The length of γ is $\int_0^T|\dot\gamma|_{\mathrm{hor}}(s)\mathrm{d}s$. Since we have Hörmander's rank condition $\operatorname{span}\{Q((x,y),(\xi,\eta)):(x,y),(\xi,\eta)\in V_1\}=V_2$, it turns out that any pairs of points can be connected by a horizontal curve (this follows from Chow–Rashevskii theorem). Minimizing such length we obtain the well known sub-Riemannian distance.

Let us introduce some notation in $\mathbb{G}_{qp}$. We sometimes identify $\mathbb{R}^p$ with $\mathbb{R}^{p\times 1}$ and the same for $\mathbb{R}^q$. As we declared, we use the Hilbert–Schmidt inner product in $\mathbb{R}^{q\times p}$, *i.e.* $\langle x,\xi\rangle:=\operatorname{trace}(\xi^Tx)$. Then, in $\mathbb{R}^{q\times p}\times\mathbb{R}^p$ we define $\langle(x,y),(\xi,\eta)\rangle:=\langle x,\xi\rangle+\langle y,\eta\rangle$, so that $e_\alpha e_k^T$ as $\alpha\in\{1,\dots,q\}$ and $k\in\{1,\dots,p\}$ is an orthonormal basis of $\mathbb{R}^{q\times p}$, where e_α and e_j, with $\alpha\in\{1,\dots,q\}$ and $j\in\{1,\dots p\}$ denote the canonical basis of $\mathbb{R}^q$ and $\mathbb{R}^p$. Introduce, for $\tau\in\mathbb{R}^q$, the linear map $A_\tau:V_1\to V_1$

$$A_\tau(\xi,\eta)=(\tau\eta^T,-\xi^T\tau).$$

[1] By [19], the map $x\mapsto x^\dagger$ is continuous on the open set $\{x\in\mathbb{R}^{q\times p}:\operatorname{rank}(x) \text{ is maximal}\}$.

Since the map is also linear in τ, we have $A_\tau = \sum_{\alpha=1}^q \tau_\alpha A_{e_\alpha}$, with $A_{e_\alpha}(\xi,\eta) = (e_\alpha \eta^T, -\xi^T e_\alpha)$ for all $(\xi,\eta) \in V_1$. Thus, we have

$$\langle (x,y), A_{e_\alpha}(\xi,\eta)\rangle = \langle (x,y), (e_\alpha\eta^T, -\xi^T e_\alpha)\rangle = \langle x, e_\alpha \eta^T\rangle - \langle y, \xi^T e_\alpha\rangle = \langle x\eta - \xi y, e_\alpha\rangle.$$

Each map A_{e_α} is skew-symmetric and we have $Q((x,y),(\xi,\eta)) = \frac{1}{2}\sum_{\alpha=1}^q \langle (x,y), A_{e_\alpha}(\xi,\eta)\rangle e_\alpha$.

2.2. Extremal curves

We are interested in writing length-minimizing curves. In order to write them, we follow [2], Section 13.1. Note that in step-2 Carnot groups it is known that all extremals are normal (see [2], Cor. 12.14). We first write a frame of left-invariant horizontal orthonormal vector fields. For $\alpha \in \{1,\dots,q\}$ and $k,j \in \{1,\dots,p\}$ we have

$$\begin{aligned} X_{\alpha k}(x,y,t) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(x,y,t)\cdot(se_\alpha e_k^T,0,0) = \Big(e_\alpha e_k^T, 0, -\frac{1}{2}y_k e_\alpha\Big) \\ Y_j(x,y,t) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(x,y,t)\cdot(0,se_j,0) = \Big(0,e_j,\frac{1}{2}xe_j\Big) = \Big(0,e_j,\frac{1}{2}x_j\Big), \end{aligned} \tag{2.2}$$

where we wrote $x = [x_1,\dots,x_p]$ with $x_j \in \mathbb{R}^q$ for all $j \in \{1,\dots,p\}$. Introducing the functions $u_{\alpha k}(x,y,t,\xi,\eta,\tau) := \langle (\xi,\eta,\tau), X_{\alpha k}(x,y,t)\rangle$ and $v_j(x,y,t,\xi,\eta,\tau) = \langle (\xi,\eta,\tau), Y_j(x,y,t)\rangle$, extremals are furnished by the Hamiltonian

$$H((x,y,t),(\xi,\eta,\tau)) = \frac{1}{2}\sum_{\alpha,k} u_{\alpha k}(x,y,t,\xi,\eta,\tau)^2 + \frac{1}{2}\sum_j v_j(x,y,t,\xi,\eta,\tau)^2.$$

Namely, to obtain all minimizers from $(0,0,0)$, one integrates the Hamiltonian system $(\dot x,\dot y,\dot t) = \nabla_{(\xi,\eta,\tau)}H$ and $(\dot\xi,\dot\eta,\dot\tau) = -\nabla_{(x,y,t)}H$ with initial data $(x(0),y(0),t(0)) = (0,0,0)$ and $(\xi(0),\eta(0),\tau(0)) = (\xi,\eta,\tau) \in T_0^*\mathbb{G}_{qp} \simeq V_1\times V_2$.[2] It turns out that extremals from the origin are horizontal curves

$$s \mapsto \gamma(s) = \gamma(s,\xi,\eta,\tau) \in V_1 \times V_2 \tag{2.3}$$

parametrized by $(\xi,\eta,\tau) \in V_1\times V_2$. Furthermore, again, by [2], Section 13.1, given (ξ,η,τ), the extremal curve $\gamma(\cdot,\xi,\eta,\tau)$ is the solution of (2.1), with

$$(u(s),v(s)) = \mathrm{e}^{-sA_\tau}(\xi,\eta) \in V_1\times V_2. \tag{2.4}$$

Next we give the form of extremal curves in terms of the three functions

$$T(\varphi) = \frac{\sin\varphi}{\varphi}, \quad U(\varphi) = \frac{\varphi - \sin\varphi\cos\varphi}{4\varphi^2} \quad \text{and} \quad V(\varphi) = \frac{\sin\varphi - \varphi\cos\varphi}{2\varphi^2}$$

defined for $\varphi > 0$.

[2] Since we are taking global coordinates $((x,y,t),(\xi,\eta,\tau)) \in (V_1\times V_2)\times(V_1\times V_2)$ on $T^*(V_1\times V_2)$, we identify covectors in $T^*_{(0,0,0)}(V_1\times V_2)$ with $(\xi,\eta,\tau)\in V_1\times V_2$.

Proposition 2.1. *For all* $(\xi,\eta,\tau)\in T^*(V_1\times V_2)\simeq V_1\times V_2$ *with* $\tau\neq 0$, *the curve* $\gamma(\cdot,\xi,\eta,\tau)=(x(\cdot,\xi,\eta,\tau),y(\cdot,\xi,\eta,\tau),t(\cdot,\xi,\eta,\tau))$ *in* (2.3) *has the form*

$$\begin{aligned}
x(s,\xi,\eta,\tau) &= sT\big(\tfrac{|\tau|s}{2}\big)\Big\{P_\tau\xi\,\cos\big(\tfrac{|\tau|s}{2}\big)-\frac{\tau\eta^T}{|\tau|}\sin\big(\tfrac{|\tau|s}{2}\big)\Big\}+sP_\tau^\perp\xi\\
y(s,\xi,\eta,\tau) &= sT\big(\tfrac{|\tau|s}{2}\big)\Big\{\eta\cos\big(\tfrac{|\tau|s}{2}\big)+\frac{\xi^T\tau}{|\tau|}\sin\big(\tfrac{|\tau|s}{2}\big)\Big\}\\
t(s,\xi,\eta,\tau) &= s^2U\big(\tfrac{|\tau|s}{2}\big)\big\{|P_\tau\xi|^2+|\eta^2|\big\}\frac{\tau}{|\tau|}\\
&\quad+s^2V\big(\tfrac{|\tau|s}{2}\big)P_\tau^\perp\xi\Big\{-\eta\sin\big(\tfrac{|\tau|s}{2}\big)+\frac{\xi^T\tau}{|\tau|}\cos\big(\tfrac{|\tau|s}{2}\big)\Big\}.
\end{aligned}\tag{2.5}$$

In formula (2.5), recall that $P_\tau:=\frac{\tau\tau^T}{|\tau|^2}\in\mathbb{R}^{q\times q}$ and $P_\tau^\perp=I_q-P_\tau$. Note also that $|P_\tau\xi|^2=\frac{|\xi^T\tau|^2}{|\tau|^2}$. Observe the known property

$$\gamma(\lambda s,\xi,\eta,\tau)=\gamma(s,\lambda\xi,\lambda\eta,\lambda\tau),\quad\text{for all }\lambda>0,\ (\xi,\eta,\tau)\in V_1\times V_2.\tag{2.6}$$

Proof. Since

$$A_\tau^2(\xi,\eta)=(-\tau\tau^T\xi,-|\tau|^2\eta)\text{ and }A_\tau^3(\xi,\eta)=-|\tau|^2(\tau\eta^T,-\xi^T\tau)=-|\tau|^2A_\tau(\xi,\eta),\tag{2.7}$$

summing up the series we get

$$(u(s),v(s))=(\xi,\eta)-\frac{\sin(|\tau|s)}{|\tau|}(\tau\eta^T,-\xi^T\tau)-\frac{1-\cos(|\tau|s)}{|\tau|^2}(\tau\tau^T\xi,|\tau|^2\eta).$$

Let now

$$a=a(\xi,\eta):=(P_\tau\xi,\eta),\quad b=b(\xi,\eta):=\Big(-\frac{\tau\eta^T}{|\tau|},\frac{\xi^T\tau}{|\tau|}\Big),\quad z=z(\xi,\eta):=(P_\tau^\perp\xi,0).$$

Then, we can write $(u(s),v(s))=a\cos(|\tau|s)+b\sin(|\tau|s)+z$. Using the function $T(\varphi):=\frac{\sin\varphi}{\varphi}$ and by trigonometry we get

$$\begin{aligned}
(x(s),y(s)) &= \frac{\sin(|\tau|s)}{|\tau|}a+\frac{1-\cos(|\tau|s)}{|\tau|}b+sz\\
&= sT\Big(\frac{|\tau|s}{2}\Big)\Big\{a\cos\Big(\frac{|\tau|s}{2}\Big)+b\sin\Big(\frac{|\tau|s}{2}\Big)\Big\}+sz\\
&= sT\Big(\frac{|\tau|s}{2}\Big)\Big\{(P_\tau\xi,\eta)\cos\Big(\frac{|\tau|s}{2}\Big)+\Big(-\frac{\tau\eta^T}{|\tau|},\frac{\xi^T\tau}{|\tau|}\Big)\sin\Big(\frac{|\tau|s}{2}\Big)\Big\}+s(P_\tau^\perp\xi,0).
\end{aligned}$$

To calculate $t(s) = \int_0^s Q((x,y),(u,v))$, integrating and by bilinearity we get

$$\begin{aligned} t(s) &= \int_0^s Q\Big(\frac{\sin(|\tau|\sigma)}{|\tau|}a + \frac{1-\cos(|\tau|\sigma)}{|\tau|}b + \sigma z, a\cos(|\tau|\sigma) + b\sin(|\tau|\sigma) + z\Big)\mathrm{d}\sigma \\ &= \frac{|\tau|s - \sin(|\tau|s)}{|\tau|^2}Q(a,b) + \frac{2(1-\cos(|\tau|s)) - |\tau|s\sin(|\tau|s)}{|\tau|^2}Q(a,z) \\ &\qquad + \frac{|\tau|s(1+\cos(|\tau|s)) - 2\sin(|\tau|s)}{|\tau|^2}Q(b,z). \end{aligned} \tag{2.8}$$

The form of t has a structure analogous to [9], equation (2.6). By the definition $Q((x,y),(\bar{x},\bar{y})) := \frac{1}{2}(x\bar{y} - \bar{x}y)$ and $\frac{|\xi^T\tau|^2}{|\tau|^2} = |P_\tau\xi|^2$, an easy calculation gives

$$Q(a,b) = \frac{1}{2}\Big\{\frac{|\xi^T\tau|^2}{|\tau|^2} + |\eta|^2\Big\}\frac{\tau}{|\tau|} = \frac{1}{2}(|P_\tau\xi|^2 + |\eta|^2)\frac{\tau}{|\tau|}$$
$$Q(a,z) = -\frac{1}{2}P_\tau^\perp\xi\,\eta \in \mathbb{R}^{q\times 1} \quad \text{and} \quad Q(b,z) = -\frac{1}{2}P_\tau^\perp\xi\frac{\xi^T\tau}{|\tau|}.$$

Therefore, starting from (2.8), we obtain

$$\begin{aligned} t(s) &= \frac{|\tau|s - \sin(|\tau|s)}{2|\tau|^2}(|P_\tau\xi|^2 + |\eta|^2)\frac{\tau}{|\tau|} - \frac{2(1-\cos(|\tau|s)) - |\tau|s\sin(|\tau|s)}{2|\tau|^2}P_\tau^\perp\xi\,\eta \\ &\quad - \frac{|\tau|s(1+\cos(|\tau|s)) - 2\sin(|\tau|s)}{2|\tau|^2}P_\tau^\perp\xi\frac{\xi^T\tau}{|\tau|}. \end{aligned} \tag{2.9}$$

Writing trigonometric functions in terms of $\frac{|\tau|s}{2}$ instead of $|\tau|s$, we get

$$\begin{aligned} t(s,\xi,\eta,\tau) &= s^2U(\tfrac{|\tau|s}{2})(|P_\tau\xi|^2 + |\eta|^2)\frac{\tau}{|\tau|} \\ &\quad + s^2V(\tfrac{|\tau|s}{2})\Big(-\sin(\tfrac{|\tau|s}{2})P_\tau^\perp\xi\,\eta + \cos(\tfrac{|\tau|s}{2})P_\tau^\perp\xi\frac{\xi^T\tau}{|\tau|}\Big), \end{aligned} \tag{2.10}$$

as desired. The proof is finished. ☐

Remark 2.2. It is easy to check that constant-speed Euclidean lines of the form $(x(s),y(s),t(s)) = (su,sv,0)$ for some $(u,v) \in V_1 \setminus \{(0,0)\}$ are always globally minimizing. Furthermore, an extremal of the form $(u(s),v(s)) = \mathrm{e}^{-sA_\tau}(\xi,\eta)$ gives rise to an Euclidean line of that form if and only if

$$\begin{cases} \tau = 0 \\ |\xi|^2 + |\eta|^2 > 0 \end{cases} \quad \text{or} \quad \begin{cases} |\tau|\,|P_\tau^\perp\xi| > 0 \\ |\eta|^2 + |P_\tau\xi|^2 = 0. \end{cases}$$

In the first case, $\dot{\gamma}(s) = (\xi,\eta,0)$ and we have $\gamma(s) = (s\xi,s\eta,0)$. In the second case we have instead $\xi^T\tau = 0$ and (2.5) implies that $\gamma(s) = (sP_\tau^\perp\xi,0,0) = (s[P_\tau^\perp\xi_1,\dots,P_\tau^\perp\xi_p],0,0)$. Note that in both cases we have $A_\tau(\xi,\eta) = (\tau\eta^T, -\xi^T\tau) = 0$, getting $\mathrm{e}^{-sA_\tau}(\xi,\eta) = (\xi,\eta)$ trivially.

We conclude this section with a characterization of abnormal extremal curves. Recall that, by definition, given a L^2 control $(u,v) : (0,T) \to V_1$, the curve $\gamma_{(u,v)} : [0,T] \to \mathbb{G}_{qp}$ is abnormal if $d_{(u,v)}E : L^2((0,T) \to \mathbb{G} \sim T_{E(u,v)}\mathbb{G})$ is not onto. Here $E : L^2(0,T) \to \mathbb{G}_{qp}$ is the end point map, *i.e.* $E(u,v) := \gamma_{(u,v)}(T)$, where $\gamma_{(u,v)}$ is the curve corresponding to (u,v) and with $\gamma(0)=(0,0,0)$. We say instead that $\gamma_{(u,v)}$ is *strictly normal* when it is not abnormal in any subsegment of $[0,T]$.

Proposition 2.3. *Let $q\geq 2$ and let $(\xi,\eta,\tau)\in\mathbb{G}_{qp}\simeq T^*_{(0,0,0)}\mathbb{G}_{qp}$ and assume that $|\xi|^2+|\eta|^2>0$. The curve $\gamma(\cdot,\xi,\eta,\tau)$ is abnormal if and only if it has the form $\gamma(s)=(s\xi,0,0)$ with $\operatorname{Im}\xi\neq\mathbb{R}^q$.*

If $q=1$, all extremals are strictly normal. Recall also that it is known from second-order analysis that in step-2 Carnot groups all abnormal curves are also normal. See [2], Corollary 12.14 and [20], Section 20.5.

Proof. By [21], Section 3, we know that an extremal curve $\gamma_{(\xi,\eta,\tau)}=(x_{(\xi,\eta,\tau)},y_{(\xi,\eta,\tau)},t_{(\xi,\eta,\tau)})$ corresponding to a given control $(u(s),v(s))=\mathrm{e}^{-sA_\tau}(\xi,\eta)$ is abnormal if and only of there is $\sigma\in\mathbb{R}^q\setminus\{0\}$ such that

$$A_\sigma\mathrm{e}^{-sA_\tau}(\xi,\eta)=(0,0)\quad\text{ for all } s\in\mathbb{R}. \tag{2.11}$$

In order to show the "if" part, consider $(u(s),v(s))=(\xi,0)$ for all $s\in[0,T]$, where $\operatorname{Im}\xi\neq\mathbb{R}^q$. Take $\sigma\in(\operatorname{Im}\xi)^\perp\setminus\{0\}$. We have $A_\sigma(\xi,0)=(0,-\xi^T\sigma)=(0,0)$ and (2.11) follows.

To show the "only if" part, let us look first at case $\tau=0$. In this case $(u(s),v(s))=(\xi,\eta)$ and condition (2.11) furnishes $A_\sigma(\xi,\eta)=(\sigma\eta^T,-\xi^T\sigma)=(0,0)$. Since $\sigma\in\mathbb{R}^q$ must be nontrivial, first coordinate gives $\eta=0$. The second is equivalent to $P_\sigma\xi=0$. Thus, it must be $P_\sigma^\perp\xi\neq 0$. Existence of such a $\sigma\neq0$ is equivalent to condition $\operatorname{Im}\xi\neq\mathbb{R}^q$.

Let finally $(u(s),v(s))=\mathrm{e}^{-sA_\tau}(\xi,\eta)$ be abnormal with $\tau\neq0$. From the discussion above, given $\sigma\in\mathbb{R}^q\setminus\{0\}$, we have $\ker A_\sigma=\{(\xi,0):\operatorname{Im}\xi\subset\sigma^\perp\}$. Evaluating (2.11) and its s-derivative at $s=0$, we get $A_\sigma(\xi,\eta)=(0,0)$ and $A_\sigma A_\tau(\xi,\eta)=A_\sigma(\tau\eta^T,-\xi^T\tau)=(0,0)$. Therefore, we get

$$\eta=0,\quad \operatorname{Im}\xi\subset\sigma^\perp,\quad \xi^T\tau=0,\quad \operatorname{Im}(\tau\eta^T)\subset\sigma^\perp.$$

Requirements $\eta=0$ and $\xi^T\tau=0$ already imply that $A_\tau(\xi,0)=(0,0)$ and $\operatorname{Im}\xi\subset\tau^\perp$, which is a nontrivial subspace. Therefore $\mathrm{e}^{-sA_\tau}(\xi,0)=(\xi,0)$ for all s, and $\operatorname{Im}\xi\neq\mathbb{R}^q$, as we wished. ☐

We briefly recall the notion of conjugate point.

Definition 2.4. Recall that, given an extremal $\gamma(\cdot,\bar\xi,\bar\eta,\bar\tau)$ with $|(\bar\xi,\bar\eta)|>0$ and a time $\bar s>0$, we say that $\bar s$ is a conjugate time for $\gamma(\cdot,\bar\xi,\bar\eta,\bar\tau)$ if the differential of the map $(\xi,\eta,\tau)\in\mathbb{G}_{qp}\mapsto\gamma(\bar s,\xi,\eta,\tau)\in\mathbb{G}_{qp}$ is singular at $(\bar\xi,\bar\eta,\bar\tau)$.

Remark 2.5. If $\gamma(\cdot,\xi,\eta,\tau)$ is abnormal, then all times $\bar s>0$ are trivially conjugate times (see [2], Rem. 8.46). If instead $\gamma(\cdot,\xi,\eta,\tau)$ is not abnormal in any subsegment $[0,s]$, it is known that there is a strictly positive smallest conjugate time t_{conj}. Furthermore, we have $t_{\mathrm{conj}}\geq t_{\mathrm{cut}}$. The time t_{conj} is usually called *first conjugate time*. The corresponding point $\gamma(t_{conj},\bar\xi,\bar\eta,\bar\tau)$ is called *first conjugate point*. All these facts are proved in [2], Section 8.8.

3. Upper estimate $t_{\mathrm{cut}}\leq 2\pi/|\tau|$

In this section we consider extremal curves $\gamma(\cdot,\xi,\eta,\tau)$ which are not Euclidean lines (in particular strictly normal, see Prop. 2.3 and Rem. 2.5). We show that if $p\geq2$, then all points of the form $\gamma(\frac{2\pi}{|\tau|},\xi,\eta,\tau)$ with $\tau\neq0$ and $|\eta|+|P_\tau\xi|>0$ are conjugate points. See Definition 2.4. Among them, we characterize those that are reached by at least two different (unit-speed) minimizing geodesics exiting from the origin (see Prop. 3.4). The remaining points are reached by a unique unit-speed minimizing curve. In $\mathbb{G}_{q1}$, i.e. $p=1$ and $q\geq2$, it is not true that all points $\gamma(\frac{2\pi}{|\tau|},\xi,\eta,\tau)$ are conjugate points. Conjugate points in this limiting case are described in Proposition 3.2 for the sufficient part and in the separate Section 6, Theorem 6.1 for the necessary part, which is slightly more technical.

Starting from the form of extremals established in Proposition 2.1, we get

$$\begin{aligned} x\Big(\frac{2\pi}{|\tau|},\xi,\eta,\tau\Big) &= \frac{2\pi}{|\tau|}P_\tau^\perp\xi \\ y\Big(\frac{2\pi}{|\tau|},\xi,\eta,\tau\Big) &= 0 \\ t\Big(\frac{2\pi}{|\tau|},\xi,\eta,\tau\Big) &= \frac{\pi}{|\tau|^2}(|P_\tau\xi|^2+|\eta|^2)\frac{\tau}{|\tau|}-\frac{2\pi}{|\tau|^2}P_\tau^\perp\xi\frac{\xi^T\tau}{|\tau|}. \end{aligned} \tag{3.1}$$

Proposition 3.1 (Conjugate points for $p\geq 2$)**.** *Let $q\in\mathbb{N}$ and $p\geq 2$. Given $(\xi,\eta,\tau)\in V_1\times V_2$, assume that $\tau\neq 0$ and $|\eta|^2+|P_\tau\xi|^2>0$ and consider the extremal $\gamma(\cdot,\xi,\eta,\tau)$. Then, the time $\bar{s}:=\frac{2\pi}{|\tau|}$ is conjugate.*

As we already observed, if $\tau=0$ or $\tau\neq 0$ and $|\eta|+|P_\tau\xi|=0$, then γ has the form $\gamma(s)=(s\xi,0,0)$. If $\operatorname{Im}\xi=\mathbb{R}^q$, then it must be $\tau=0$, because there is no $\tau\neq 0$ such that $P_\tau\xi=0$. In this case, γ is strictly normal and it has no conjugate points. If $\operatorname{Im}\xi\neq\mathbb{R}^q$, then γ is abnormal and all its points are trivially conjugate (see Prop. 2.3).

Proof. Let $(\bar\xi,\bar\eta,\bar\tau)\in V_1\times V_2\simeq T_0^*\mathbb{G}_{qp}$ with $|\bar\tau|(|P_{\bar\tau}\bar\xi|+|\bar\eta|)>0$. Consider the cylinder $\Lambda=\{(\xi,\eta,\tau)\in V_1\times V_2: |\xi|^2+|\eta|^2=|\bar\xi|^2+|\bar\eta|^2\}\subset\mathbb{G}_{qp}\simeq T_0^*\mathbb{G}_{qp}$. Consider the map $(s,\xi,\eta,\tau)\in\mathbb{R}\times\Lambda\mapsto\gamma(s,\xi,\eta,\tau)\in V_1\times V_2$. To prove the proposition, it suffices to show that $\det[\partial_s\gamma,d_\Lambda\gamma]|_{(s,\xi,\eta,\tau)=(\frac{2\pi}{|\bar\tau|},\bar\xi,\bar\eta,\bar\tau)}=0$. Here $d_\Lambda\gamma\in\mathbb{R}^{N\times(N-1)}$ denotes any family of independent derivatives in $T_{(\bar\xi,\bar\eta,\bar\tau)}\Lambda$ with $N:=\dim\mathbb{G}_{qp}=qp+p+q$. If $\bar\eta\neq 0$, choose $v\in\mathbb{R}^p\setminus\{0\}$ such that $\langle v,\bar\eta\rangle=0$ and consider the derivative $D_v:=v\cdot\nabla_\eta$ (note that existence of such a direction v needs $p\geq 2$). Since (3.1) is radial in η, it turns out easily that $D_v\gamma(\frac{2\pi}{|\bar\tau|},\bar\xi,\bar\eta,\bar\tau)=0$. This implies that the determinant above vanishes. If instead $\eta=0$, we have

$$\gamma\Big(\frac{2\pi}{|\bar\tau|},\bar\xi,0,\bar\tau\Big)=\lim_{\varepsilon\to 0}\gamma\Big(\frac{2\pi}{|\bar\tau|},\bar\xi,\varepsilon\eta_0,\bar\tau\Big)$$

where $\eta_0\in\mathbb{R}^p$ is a fixed nonzero vector. Thus, the point with $\bar\eta=0$ is conjugate, being a limit of a family of conjugate points (recall that γ is a smooth map). ☐

Proposition 3.2 (Conjugate points for $q\geq 2$ and $p=1$, sufficient condition)**.** *Let $q\geq 2$ and consider the model $\mathbb{G}_{q1}$. Given $(\xi,\eta,\tau)\in V_1\times V_2$, assume that $\tau\neq 0$ and $|\eta|^2+|P_\tau\xi|^2>0$ and consider the extremal $\gamma(\cdot,\xi,\eta,\tau)$. If*

$$\eta P_\tau^\perp\xi=0, \tag{3.2}$$

then the time $\bar{s}:=\frac{2\pi}{|\tau|}$ is conjugate.

Note that assumption (3.2) does not appear in Proposition 3.1, where $p\geq 2$. We shall prove in Theorem 6.1 that condition (3.2) is also necessary to have $\frac{2\pi}{|\tau|}$ as a conjugate time. We do not discuss the case $p=q=1$: this is the familiar Heisenberg group $\mathbb{H}^1$, and in this case, using rotation invariance around t-axis, it is easy to see that all cut points are conjugate. Extremals in the Heisenberg group $\mathbb{G}_{11}$ can be read in plenty of papers, *e.g.* [22–24]. See also the general discussions in Section 13.2, 13.3 of [2].

Proof. Let $(\bar\xi,\bar\eta,\bar\tau)\in V_1\times V_2\simeq T_0^*\mathbb{G}_{qp}$ with $|\bar\tau|(|P_{\bar\tau}\bar\xi|+|\bar\eta|)>0$. As in the proof of Proposition 3.1 above, consider the cylinder $\Lambda=\{(\xi,\eta,\tau)\in V_1\times V_2:|\xi|^2+\eta^2=|\bar\xi|^2+\bar\eta^2\}\subset\mathbb{G}_{q1}\simeq T_0^*\mathbb{G}_{q1}$. Consider the map $(s,\xi,\eta,\tau)\in\mathbb{R}\times\Lambda\mapsto\gamma(s,\xi,\eta,\tau)\in V_1\times V_2$. To prove the proposition, it suffices to show that $\det[\partial_s\gamma,d_\Lambda\gamma]|_{(s,\xi,\eta,\tau)=(2\pi/|\bar\tau|,\bar\xi,\bar\eta,\bar\tau)}=0$. Here $d_\Lambda\gamma\in\mathbb{R}^{N\times(N-1)}$ denotes any family of independent derivatives in $T_{(\bar\xi,\bar\eta,\bar\tau)}\Lambda$ with $N:=\dim\mathbb{G}_{q1}=2q+1$. Consider the vector field $Z:=-\langle\xi,\tau\rangle\partial_\eta+\eta D_\tau$, where $D_\tau=\langle\tau,\nabla_\xi\rangle$. Note that Z is tangent to Λ and $Z\neq 0$, because $\eta^2+|P_\tau\xi|^2\neq 0$. Taking the form (3.1) of $\gamma(\frac{2\pi}{|\tau|},\xi,\eta,\tau)$ into account, after a computation we get

$$Zx\Big(\frac{2\pi}{|\bar\tau|},\bar\xi,\bar\eta,\bar\tau\Big)=(-\langle\bar\xi,\bar\tau\rangle\partial_\eta+\bar\eta D_{\bar\tau})\frac{2\pi}{|\bar\tau|}P_{\bar\tau}^{\perp}\bar\xi=\frac{2\pi}{|\bar\tau|}P_{\bar\tau}^{\perp}\bar\tau=0 \tag{3.3}$$

and $Zy(\frac{2\pi}{|\bar\tau|},\bar\xi,\bar\eta,\bar\tau)=0$. Also,

$$\begin{aligned}Zt\Big(\frac{2\pi}{|\bar\tau|},\bar\xi,\bar\eta,\bar\tau\Big)&=\frac{\pi}{|\bar\tau|^3}(-\langle\bar\xi,\bar\tau\rangle\partial_\eta+\bar\eta D_{\bar\tau})\Big((|P_{\bar\tau}\bar\xi|^2+\bar\eta^2)\bar\tau-2P_{\bar\tau}^{\perp}\bar\xi\bar\xi^T\bar\tau\Big)\\&=\frac{\pi}{|\bar\tau|^3}\Big\{-2\bar\eta\langle\bar\xi,\bar\tau\rangle\bar\tau+2\bar\eta\langle P_{\bar\tau}\bar\xi,D_{\bar\tau}P_{\bar\tau}\bar\xi\rangle\bar\tau-2\bar\eta P_{\bar\tau}^{\perp}\bar\xi D_{\bar\tau}\langle\bar\xi,\bar\tau\rangle\Big\}\\&=-\frac{2\pi\bar\eta}{|\bar\tau|}P_{\bar\tau}^{\perp}\bar\xi,\end{aligned} \tag{3.4}$$

because $D_{\bar\tau}P_{\bar\tau}^{\perp}\bar\xi=P_{\bar\tau}^{\perp}\bar\tau=0$ and terms along τ cancel. From these computation, we see that assumption $\bar\eta P_{\bar\tau}^{\perp}\bar\xi=0$ implies that $\det[\partial_s\gamma,d_\Lambda\gamma]|_{(s,\xi,\eta,\tau)=(\frac{2\pi}{|\bar\tau|},\bar\xi,\bar\eta,\bar\tau)}=0$. □

Propositions 3.1 and 3.2 imply estimate $t_{\rm cut}\le 2\pi/|\tau|$ if $p\ge 2$. If $q\ge 2$ and $p=1$, the upper estimate follows only if $\eta P_\tau^\perp\xi=0$.

Remark 3.3. The upper estimate in case $q\ge 2$, $p=1$ and $\eta P_\tau^\perp\xi\neq 0$ can be obtained easily from the fact that if $\eta\neq 0$, then $\gamma(\frac{2\pi}{|\tau|},\xi,\eta,\tau)=\gamma(\frac{2\pi}{|\tau|},\xi,-\eta,\tau)$. Thus there are two different extremals reaching the point $\gamma(\frac{2\pi}{|\tau|},\xi,\eta,\tau)$ with length $\frac{2\pi}{|\tau|}|(\xi,\eta)|$.

The remark above concludes the proof of the upper estimate in all cases. In Section 4 we shall prove the opposite estimate $t_{\rm cut}\ge 2\pi/|\tau|$.

In next proposition, we find for all p,q, all points $\gamma(\frac{2\pi}{|\tau|},\xi,\eta,\tau)\in\mathbb{G}_{qp}$ reached by more than one extremal of length $\frac{2\pi}{|\tau|}|(\xi,\eta)|$.

In the statement of the following proposition, to have a clean exposition, we take for granted equality $t_{\rm cut}=2\pi/|\tau|$ for all extremals different from Euclidean lines. (Note that arguments in Prop. 3.4 and in Sect. 4 are independent of each other.)

It is easy to show that for an extremal of the form $\gamma(s,\xi,\eta,\tau)$ with $\tau\neq 0$ and $\eta\neq 0$, we have $\gamma(\frac{2\pi}{|\tau|},\xi,\eta,\tau)=\gamma(\frac{2\pi}{|\tau|},\xi,R\eta,\tau)$ for all $R\in O(p)$. Therefore the point is reached by more than one minimizer of equal speed. There are also cut points with $\eta=0$ reached by at least two minimizers of equal speed, and we now characterize them. All other cut points are reached by a unique minimizer.

Proposition 3.4 (Cut points reached by at least two different arclength minimizers)**.** *Let $\tau\in\mathbb{R}^q\setminus\{0\}$. Let $|\xi|^2+|\eta|^2=1$ and assume also that $|\eta|^2+|P_\tau\xi|^2>0$. Then there is $(\xi',\eta',\tau')\neq(\xi,\eta,\tau)$ such that*

$$\begin{cases}|\xi'|^2+|\eta'|^2=|\xi|^2+|\eta|^2=1, \quad |\tau'|=|\tau| \quad \textit{and}\\ \gamma(\frac{2\pi}{|\tau'|},\xi',\eta',\tau')=\gamma(\frac{2\pi}{|\tau|},\xi,\eta,\tau)\end{cases} \tag{3.5}$$

if and only if either $\eta\neq 0$, or

$$\eta=0 \quad \textit{and} \quad \xi^T\tau \ \textit{ is not orthogonal to } \ker P_\tau^\perp\xi. \tag{3.6}$$

Remark 3.5. 1. Assumption $\tau\neq 0$ and $|\eta|^2+|P_\tau\xi|^2>0$ ensure that the curve $\gamma(s,\xi,\eta,\tau)$ is not an Euclidean line contained in the plane $t=0$. See Remark 2.2.

2. The second assumption in (3.6) implicitely implies that the columns $P_\tau^\perp \xi_1, \dots, P_\tau^\perp \xi_p$ of $P_\tau^\perp \xi$ are linearly dependent in $\mathbb{R}^q$ (*i.e.*, $\operatorname{rank} P_\tau^\perp \xi < p$) and that $\xi^T\tau \neq 0 \in \mathbb{R}^p$. If $p = 1$, this means $P_\tau^\perp \xi = 0$ and $P_\tau \xi \neq 0$.
3. In terms of coordinates (x, y, t) on $V_1 \times V_2$, points reached by a unique unit-speed length-minimizer will be characterized in Section 5.
4. If $q = 1$, the group $\mathbb{G}_{1p} = \mathbb{H}^p$ is the familiar p-dimensional Heisenberg group, and it is easy to see that if $\tau \neq 0$ and $\eta = 0 \in \mathbb{R}^{p\times 1}$, condition (3.6) is satisfied by any $\xi \in \mathbb{R}^{1\times p} \setminus \{0\}$. In case $q = 1$, $P_\tau^\perp \xi \in \mathbb{R}^1$ vanishes trivially.

Proof of Proposition 3.4. Let $(\xi, \eta, \tau) \in T_0^*(V_1 \times V_2) \simeq V_1 \times V_2$ be such that $|\xi|^2 + |\eta|^2 = 1$. We first characterize all (ξ', η', τ') satisfying (3.5). By (3.1), condition $x(\frac{2\pi}{|\tau'|}, \xi', \eta', \tau') = x(\frac{2\pi}{|\tau|}, \xi, \eta, \tau)$ and $|\tau| = |\tau'|$ give

$$P_\tau^\perp \xi = P_{\tau'}^\perp \xi', \tag{3.7}$$

while assumption $t(\frac{2\pi}{|\tau'|}, \xi', \eta', \tau') = t(\frac{2\pi}{|\tau|}, \xi, \eta, \tau)$ gives

$$(|P_\tau \xi|^2 + |\eta|^2)\tau - 2P_\tau^\perp \xi\, \xi^T \tau = \Big(|P_{\tau'}\xi'|^2 + |\eta'|^2\Big)\tau' - 2P_{\tau'}^\perp \xi' \xi'^T \tau'. \tag{3.8}$$

Now we write $P_\tau^\perp \xi = P_{\tau'}^\perp \xi' =: v = [v_1, \dots, v_p]$, where the column space of v satisfies $\operatorname{Im} v \subset \operatorname{span}\{\tau, \tau'\}^\perp$. We have then

$$|\xi|^2 + |\eta|^2 = 1 = |\eta|^2 + |P_\tau \xi|^2 + |v|^2 \quad \text{and} \quad |\xi'|^2 + |\eta'|^2 = 1 = |\eta'|^2 + |P_{\tau'}\xi'|^2 + |v|^2.$$

Thus $|\eta'|^2 + |P_{\tau'}^\perp \xi'|^2 = |\eta|^2 + |P_\tau \xi|^2$. Projecting orthogonally (3.8) along $\operatorname{span}\{\tau, \tau'\}$, we get then $(|\eta|^2 + |P_\tau\xi|^2)(\tau - \tau') = 0$. The parenthesis can not vanish by assumption. This forces $\tau' = \tau$. Thus, (3.8) becomes $P_\tau^\perp \xi\, \xi^T \tau = P_\tau^\perp \xi'\, \xi'^T \tau$. Passing to a shorter notation write $\alpha := \frac{\xi^T\tau}{|\tau|}$ and $\alpha' := \frac{\xi'^T\tau}{|\tau|} \in \mathbb{R}^p$ (keep in mind that $|P_\tau \xi|^2 = \frac{|\xi^T\tau|^2}{|\tau|^2}$). Thus, (ξ', η', τ') satisfies all conditions (3.5) if and only if $\tau' = \tau$, $P_\tau^\perp \xi' = P_\tau^\perp \xi =: v$ and there are α' and $\eta' \in \mathbb{R}^p$ such that

$$\begin{cases} v\alpha' = v\alpha \\ |\eta'|^2 + |\alpha'|^2 = |\eta|^2 + |\alpha|^2 = 1 - |v|^2, \end{cases} \tag{3.9}$$

where $v = [v_1 \dots, v_p] \in \mathbb{R}^{q\times p}$. In order to conclude the proof, we need to understand for which given $\eta, \alpha \in \mathbb{R}^p$ the pair $\eta', \alpha' \in \mathbb{R}^p$ satisfying (3.9) must be chosen uniquely in the form $\eta' = \eta$ and $\alpha' = \alpha$. Note first that if $\eta \neq 0$, any choice $\eta' = R\eta$ with $R \in O(p)$, $R \neq I_p$ gives a solution $\eta' = R\eta$ and $\alpha' = \alpha$ different from η, α (if $p = 1$, just choose $\eta' = -\eta$). Therefore, to have uniqueness it must be $\eta = 0$. If $\eta = 0$, then (3.9) becomes

$$v\alpha' = v\alpha \text{ and } |\eta'|^2 + |\alpha'|^2 = |\alpha|^2 = 1 - |v|^2.$$

If $\ker v = 0$, then it must be $\alpha' = \alpha$, $\eta' = \eta$, and we have uniqueness (if $p = 1$, this occurs when $P_\tau^\perp \xi \neq 0 \in \mathbb{R}^q$). Let now $\ker v$ be nontrivial. If $\alpha \in (\ker v)^\perp$, then $\alpha = \alpha_{\mathrm{LS}}$, the least-squares solution of the system $v\beta = v\alpha$ with unknown $\beta \in \mathbb{R}^p$. Then, $|\alpha'| \gneqq |\alpha|$ for all $\alpha' \neq \alpha$ solving $v\alpha' = v\alpha$. Then, it must be $\alpha' = \alpha = \alpha_{\mathrm{LS}}$ and we have again uniqueness in the choice of η', α'. Finally, if $\alpha \notin (\ker v)^\perp$, we can choose $\eta' = 0$ and $\alpha' = 2\alpha_{\mathrm{LS}} - \alpha \neq 0$. In this case $\alpha' \neq \alpha$, $|\alpha'| = |\alpha|$ and $v\alpha' = v\alpha$ as required. If $p = 1$ and $q \geq 2$, $\ker v \neq 0$ means $v = 0$. Then, $\alpha_{\mathrm{LS}} = 0$ and the non uniqueness choice becomes $\alpha' = -\alpha$.

To resume, if $\eta \neq 0$, we can choose $\eta' = -\eta$, $\alpha' = \alpha$ and we have found a choice of $(\xi', \eta', \tau') \neq (\xi, \eta, \tau)$. If $\eta = 0$, by assumption (3.6), given $\eta = 0$, $v = [v_1, \dots, v_p] \in \mathbb{R}^{q\times p}$ and $\alpha \neq 0$, there is a second solution $\eta' = 0$, $\alpha' = 2\alpha_{\mathrm{LS}} - \alpha \neq \alpha$, where α_{LS} solves $v\alpha_{\mathrm{LS}} = v\alpha$ and $\alpha_{\mathrm{LS}} \perp \ker v$.

We have ultimately proved that there is nonuniqueness if and only if either $\eta \neq 0$, or $\eta = 0$ and $\alpha \notin (\ker v)^{\perp}$. The proof is concluded. □

A careful inspection of the proof above shows that, if $p \geq 2$, when the choice of (ξ', η', τ') is not unique, then we have an infinite, continuous family of choices. If instead $p = 1$ and $q \geq 2$, then in case of non uniqueness there are either two or infinitely many minimizers.

We conclude this section with an example showing that the function $(\xi, \eta, \tau) \in \mathbb{G}_{qp} \mapsto t_{\mathrm{cut}}(\xi, \eta, \tau)$ is discontinuous for $q \geq 2$.

Example 3.6. Let $q \geq 2$, let $\xi \in \mathbb{R}^{q\times p} \setminus \{0\}$ with $\operatorname{Im}\xi \subsetneq \mathbb{R}^q$ and $\tau \in (\operatorname{Im}\xi)^{\perp} \setminus \{0\}$ (*i.e.* $\xi = P_\tau^{\perp}\xi$). Assume that $|\tau| = 1$ and consider $\eta \in \mathbb{R}^p \setminus \{0\}$. Then for any $\varepsilon > 0$ we have the family of cut points

$$\gamma(2\pi, \xi, \varepsilon\eta, \tau) = (2\pi\xi, 0,\ \mathrm{e}^2\pi|\eta|^2\tau) \to (2\pi\xi, 0, 0) = \gamma(2\pi, \xi, 0, \tau).$$

The corresponding cut-times are $t_{\mathrm{cut}}(\xi, \varepsilon\eta, \tau) = 2\pi$ for all $\varepsilon > 0$. However, $t_{\mathrm{cut}}(\xi, 0, \tau) = +\infty$. Note that the point $(2\pi\xi, 0, 0)$ in this example is abnormal.

4. Lower bound $t_{\mathrm{cut}} \geq \frac{2\pi}{|\tau|}$

Here we prove the following lower bound for the cut-time.

Proposition 4.1. *Let $\bar u = u(\cdot, \bar\xi, \bar\eta, \bar\tau)$ be a given extremal control with $\bar\tau \neq 0$ and $|\bar\eta| + |P_{\bar\tau}\bar\xi| > 0$. Then the cut time of $\gamma_{\bar u}$ satisfies $t_{\mathrm{cut}} \geq \frac{2\pi}{|\bar\tau|}$*

Proof. We work following the argument of [8], Section 2.3.2, which is in turn based on [20], Chapter 12.4. Let $\bar\xi, \bar\eta, \bar\tau$ be given and let

$$(\bar u(s), \bar v(s)) = (u, v)(s, \bar\xi, \bar\eta, \bar\tau) = \mathrm{e}^{-sA_{\bar\tau}}(\bar\xi, \bar\eta) \tag{4.1}$$

and $\bar\gamma(s) = \gamma(s, \bar\xi, \bar\eta, \bar\tau)$ be a given extremal path which we assume to be parametrized by arclength, *i.e.* $|\bar\xi|^2 + |\bar\eta|^2 = 1$. Fix a positive time $\hat s < \frac{2\pi}{|\bar\tau|}$. We want to show that $\bar\gamma$ minimizes length between $(0, 0, 0)$ and $(\bar x, \bar y, \bar t) := \bar\gamma(\hat s)$.

Consider a control $(u, v) \in L^2((0, \hat s), \mathbb{R}^{q\times p} \times \mathbb{R}^p)$ and the corresponding path $\gamma_{(u,v)}$ as defined in (2.1). Denote $\gamma_{(u,v)}(s) = (x_{(u,v)}(s), y_{(u,v)}(s), t_{(u,v)}(s))$ for all s. On the control (u, v) we require the following three properties.

1. $|(u, v)| = 1$ a.e. on $[0, \hat s]$ (*i.e.* $\gamma_{(u,v)}$ is arclength).
2. We have $(x(\hat s), y(\hat s)) = (\bar x, \bar y)$.
3. (u, v) maximizes the cost $J(u, v) := \langle \bar\tau, t_{(u,v)}(\hat s)\rangle =: \int_0^{\hat s} \varphi((x, y, t),(u, v))$.

We claim first that the three statements (i), (ii), and (iii) of Lemma 21 in [8] hold in our setting too.

Statement (i) claims that there is $(u, v) \in L^2$ such that (1), (2), and (3) hold. This is a standard compactness argument. Roughly speaking, it suffices to take a minimizing sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}} \in L^\infty([0, \hat s], V_1)$. Using (1) and the ODE (2.1) one can easily check that the sequence $\{(x_n, y_n, t_n)\}$ is equicontinuous and uniformly bounded. Passing to a subsequence, we may assume that (x_n, y_n, t_n) converges uniformly to a Lipschitz function (x, y, t) on $[0, \hat s]$. This ensures (2). To check that the limit (x, y, t) satisfies (2.1), we use the weak compactness of the sequence $\{(u_n, v_n)\}$ which by (1) has a subsequence converging to a limit (u, v) satisfying $|(u(s), v(s))| \leq 1$ for a. e. $s \in [0, \hat s]$. Observe that the set $\{(u, v) \in V_1 : |(u, v)| \leq 1\}$ is convex.

Statement (ii) claims that $\gamma_{(u,v)}$ is a length-minimizer on $[0, \hat s]$. To show this property, assume by contradiction that there is $\varepsilon > 0$ and an arclength control (u', v') on $[0, \hat s - \varepsilon]$ such that $\gamma_{(u',v')}(0) = (0, 0, 0)$ and $\gamma_{(u',v')}(\hat s - \varepsilon) = \gamma_{(u,v)}(\hat s) = (\bar x, \bar y, t_{(u,v)}(\hat s))$. Since the sub-Riemannian ε-ball is open, we can extend (u', v') on $[\hat s - \varepsilon, \hat s]$ to achieve a final point $(\bar x, \bar y, t_{(u',v')}(\hat s))$ such that $\langle \bar\tau, t_{(u',v')}(\hat s)\rangle > \langle \bar\tau, t_{(u,v)}(\hat s)\rangle$ contradicting (3).

Claim (iii) asserts that the solution $\gamma_{(u,v)}$ discussed in (i) and (ii) has the precise form $\gamma_{(u,v)} = \gamma(\cdot,\xi,\eta,\lambda\bar\tau)$ for a suitable (ξ,η) of unit norm and $\lambda > 0$. To accomplish this step, observe that the cost function $\langle\bar\tau, t_{(u,v)}\hat s\rangle$ is the integral on $[0,\hat s]$ of

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\langle\bar\tau,t\rangle &= \langle\bar\tau,\dot t\rangle = \frac12\langle\bar\tau, xv-uy\rangle\\ &= \frac12\big(\langle x,\bar\tau v^T\rangle - \langle y, u^T\bar\tau\rangle\big) = \frac12\langle(x,y),A_{\bar\tau}(u,v)\rangle =: \varphi((x,y,t),(u,v)).\end{aligned}$$

Therefore, by [20], Theorem 12.13, the Hamiltonian to study problem (1), (2) and (3) is

$$\begin{aligned}H_{(u,v)}((x,y,t),(\xi,\eta,\tau)) &= \sum_{\substack{\alpha=1,\dots,q\\ k=1,\dots,p}} u_{\alpha k}\langle X_{\alpha k}(x,y,t),(\xi,\eta,\tau)\rangle + \sum_{j=1,\dots,p} v_j\langle Y_j(x,y,t),(\xi,\eta,\tau)\rangle\\ &\qquad + 2\nu\varphi((x,y,t),(u,v))\\ &= \langle u,\xi\rangle + \langle v,\eta\rangle - \frac12\langle u,\tau y^T\rangle + \frac12\langle v, x^T\tau\rangle + \nu\big(\langle x,\bar\tau v^T\rangle - \langle y,u^T\bar\tau\rangle\big)\end{aligned}$$

where we used (2.2). Since we are maximizing $\int_0^{\widehat s}\varphi$, we have $\nu \ge 0$. An optimal control for our problem should satisfy the related Hamilton equations for suitable ν. Furthermore it must satisfy the *transversality condition*, with target manifold $N_1 := \{(x,y,t)\in V_1\times V_2 : (x,y) = (\bar x,\bar y)\}$. Thus, $(\xi,\eta,\tau)(\widehat s) \perp N_1$, which becomes $\tau(\widehat s) = 0$. Since $\dot\tau = -\nabla_t H = 0$, we have $\tau(s) = 0$ on $[0,\hat s]$ for the requested solution. Therefore (along an optimal control) we can write the Hamiltonian in the form

$$\begin{aligned}H_{(u,v)}((x,y,t),(\xi,\eta,\tau)) &= \langle u,\xi\rangle + \langle v,\eta\rangle + \nu\big(\langle x,\bar\tau v^T\rangle - \langle y,u^T\bar\tau\rangle\big)\\ &= \big\langle (u,v),(\xi,\eta) - \nu A_{\bar\tau}(x,y)\big\rangle.\end{aligned}$$

The remaining Hamilton equations are

$$\dot\xi = -\nabla_x H = -\nu\bar\tau v^T \quad \text{and } \dot\eta = -\nabla_y H = \nu u^T\bar\tau, \quad \textit{i.e. } (\dot\xi,\dot\eta) = -\nu A_{\bar\tau}(u,v). \tag{4.2}$$

The maximality condition [20], Theorem 12.13, equation (12.31) and Remark 12.2 states that along the optimal control $(u(s),v(s))$ we have for almost all $s\in[0,\hat s]$

$$\begin{aligned}&\Big\langle (u(s),v(s)),(\xi(s),\eta(s)) - \nu A_{\bar\tau}(x(s),y(s))\Big\rangle\\ &\qquad = \max_{|(u,v)|=1}\Big\langle (u,v),(\xi(s),\eta(s)) - \nu A_{\bar\tau}(x(s),y(s))\Big\rangle\\ &\qquad = \Big\|(\xi(s),\eta(s)) - \nu A_{\bar\tau}(x(s),y(s))\Big\| = 1.\end{aligned}$$

Therefore, the unit-norm control $(u(s),v(s))$ has the form $(u(s),v(s)) = (\xi(s),\eta(s)) - \nu A_{\bar\tau}(x(s),y(s))$. This shows first that $(u(0),v(0)) = (\xi(0),\eta(0))$. Furthermore, since $(\dot x,\dot y) = (u,v)$, differentiating the latter formula we get $(\dot u,\dot v) = (\dot\xi,\dot\eta) - \nu A_{\bar\tau}(u,v) = -2\nu A_{\bar\tau}(u,v) = -A_{2\nu\bar\tau}(u,v)$, where we also used (4.2). Thus the solution has the form

$$(u(s),v(s)) = \mathrm{e}^{-sA_{2\nu\bar\tau}}(\xi,\eta) \tag{4.3}$$

where $\nu > 0$ and (ξ,η) is a unit vector. This shows (iii) and completes the analogous of [8], Lemma 21.

We are left with the proof that the control $(\bar u,\bar v)$ in (4.1) and (u,v) in (4.3) are the same, *i.e.* that $\nu=\frac12$. Precisely, we prove the following claim (see again [8]).

Claim. Let $(\bar u,\bar v):\mathbb{R}\to\mathbb{R}^{q\times p}\times\mathbb{R}^p$ be the control in (4.1). Let $(\xi,\eta)\in V_1$ be of unit norm, let $\hat s\in\left]0,\frac{2\pi}{|\bar\tau|}\right[$ and let $\nu>0$. Let $(u,v)(s)=\mathrm{e}^{-sA_{2\nu\bar\tau}}(\xi,\eta)$ and put $(x,y)(s)=\int_0^s(u,v)$. Then, if $(x(\hat s),y(\hat s))=(\bar x,\bar y)$ and $\gamma_{(u,v)}$ minimizes length at least until $\hat s$, we have $\nu=\frac12$ and $(\xi,\eta)=(\bar\xi,\bar\eta)$.

To prove the claim note first that, since $\gamma_{(u,v)}$ minimizes length at least until $\hat s$ it must be $\hat s\le\frac{2\pi}{2\nu|\bar\tau|}$, which gives the upper bound on $\nu\le\frac{\pi}{\hat s|\bar\tau|}$. Write now $\xi=P_{\bar\tau}\xi+P_{\bar\tau}^\perp\xi=\frac{\bar\tau}{|\bar\tau|}\lambda^T+v$ where $v=P_{\bar\tau}^\perp\xi$ and $\lambda=\frac{\xi^T\bar\tau}{|\bar\tau|}$. In view of (2.5), assumption $(x(\hat s),y(\hat s))=(\bar x,\bar y)$ reads

$$\begin{aligned}&\hat sT(\hat s\nu|\bar\tau|)\frac{\bar\tau}{|\bar\tau|}\Big\{\lambda^T\cos(\hat s\nu|\bar\tau|)-\eta^T\sin(\hat s\nu|\bar\tau|)\Big\}+\hat sP_{\bar\tau}^\perp\xi=\bar x\\&\hat sT(\hat s\nu|\bar\tau|)\Big\{\eta\cos(\hat s\nu|\bar\tau|)+\lambda\sin(\hat s\nu|\bar\tau|)\Big\}=\bar y\end{aligned}$$

Recall that $\bar\tau$ is given. Then we can project the system along $\bar\tau^\perp$ and $\bar\tau$, obtaining $P_{\bar\tau}^\perp\xi=\frac1{\hat s}P_{\bar\tau}^\perp\bar x$ and

$$\begin{cases}\hat sT(\hat s\nu|\bar\tau|)\Big\{\lambda\cos(\hat s\nu|\bar\tau|)-\eta\sin(\hat s\nu|\bar\tau|)\Big\}=\dfrac{\bar x^T\bar\tau}{|\bar\tau|}\\ \hat sT(\hat s\nu|\bar\tau|)\Big\{\eta\cos(\hat s\nu|\bar\tau|)+\lambda\sin(\hat s\nu|\bar\tau|)\Big\}=\bar y.\end{cases}\tag{4.4}$$

Taking the norm and summing up we find $\hat s^2T(\hat s\nu|\bar\tau|)^2\big(|\lambda|^2+|\eta|^2\big)=\frac{|\bar x^T\bar\tau|^2}{|\bar\tau|^2}+|\bar y|^2$. Since $|\lambda|^2+|\eta|^2=|P_{\bar\tau}\xi|^2+|\eta|^2=1-|P_{\bar\tau}^\perp\xi|^2=1-\frac{|P_{\bar\tau}^\perp\bar x|^2}{\hat s^2}$, we find

$$\hat s^2\Big(\frac{\sin(\nu\hat s|\bar\tau|)}{\nu\hat s|\bar\tau|}\Big)^2\Big(1-\frac{|P_{\bar\tau}^\perp\bar x|^2}{\hat s^2}\Big)=\frac{|\bar x^T\bar\tau|^2}{|\bar\tau|^2}+|\bar y|^2.\tag{4.5}$$

Since $\hat s,\bar x,\bar y$ and $\bar\tau$ are known, the only unknown here is $\nu\in\left]0,\pi/(\hat s|\bar\tau|)\right]$. We already know that $\nu=\frac12$ belongs to that interval and is a solution of the equation (4.5). Since the function $\nu\mapsto(\sin(\nu\hat s|\bar\tau|)/(\nu\hat s|\bar\tau|))^2$ is strictly decreasing on $\left]0,\pi/(\hat s|\bar\tau|)\right]$, the solution $\nu=\frac12$ is unique. Letting then $\nu=\frac12$ we go to the Cramer system (4.4) and we find uniquely $\eta=\bar\eta$ and $\lambda=\frac{\bar\xi^T\bar\tau}{|\bar\tau|}$. The proof is finished. □

Remark 4.2. We give an idea of our choice of the cost (3) in the proof of Proposition 4.1. Our choice is suggested by the form of the cost in problem (P) at page 570 of [8], which is actually a control problem in a corank-1 quotient of the general corank-2 group the authors are working with. In our case, to get corank-1 quotients, we can take any unit vector $\omega\in\mathbb{R}^{q\times1}$ and we can consider the quotient $\mathbb{G}_\omega$ defined as follows. $\mathbb{G}_\omega=\{(x,y,\omega\omega^Tt)\in\mathbb{G}_{qp}:(x,y,t)\in\mathbb{G}_{qp}\}$. As a set, $\mathbb{G}_\omega=V_1\times\operatorname{span}\{\omega\}$. It can be equipped with the operation $(x,y,\lambda\omega)\cdot(\xi,\eta,\mu\omega)=\Big(x+\xi,y+\eta,\omega(\lambda+\mu+\frac12\omega^T(x\eta-\xi y))\Big)$. This turns out to be a sub-Riemannian Carnot group of corank-1. Now, taking the extremal control $(\bar u,\bar v)$ in (4.1) and the corresponding extremal $\bar\gamma(\cdot,\bar\xi,\bar\eta,\bar\tau)=(x,y,t)(\cdot,\bar\xi,\bar\eta,\bar\tau)$, if we project such extremal on a quotient $\mathbb{G}_\omega$, it can be checked that this gives an extremal control in $\mathbb{G}_\omega$ if and only if $\omega=\frac{\bar\tau}{|\bar\tau|}\in\operatorname{span}\{\bar\tau\}$. In such case, the vertical coordinate in $\mathbb{G}_\omega=\mathbb{G}_{\bar\tau/|\bar\tau|}$ is proportional to the cost appearing in (3).

5. Description of the cut locus

In this section we identify precisely the cut-locus and in a significant example we discuss some of its regularity properties. Some of the results of this section will be used in the proof of Theorem 6.1 in Section 6.

Item 2 of Theorem 1.2 has been proved in Proposition 3.1. Next we prove the remaining ones.

Proof of Theorem 1.2, items 1, and 3 and 4. Let $\gamma(\cdot,\xi,\eta,\tau)$ be the extremal appearing in Proposition 2.1. We know that

$$\begin{aligned}\mathrm{Cut}(\mathbb{G}_{qp}) &= \Big\{\gamma\Big(\frac{2\pi}{|\tau|},\xi,\eta,\tau\Big) : (\xi,\eta,\tau)\in V_1\times V_2,\ \tau\neq 0 \text{ and } |\eta|^2+|P_\tau\xi|^2>0\Big\}\\ &= \{\gamma(2\pi,\xi,\eta,\tau) : (\xi,\eta,\tau)\in V_1\times V_2,\ \ |\tau|=1 \text{ and } |\eta|^2+|P_\tau\xi|^2>0\},\end{aligned}\tag{5.1}$$

by (2.6).

Step 1. We show first that the set (5.1) is contained in the set (1.3). By (3.1), a point $(x,0,t)$ in (5.1) has the form

$$\begin{cases} x = 2\pi\, P_\tau^\perp\xi\\ t = \pi\Big(|\eta|^2+|P_\tau\xi|^2\Big)\tau - 2\pi P_\tau^\perp\xi\,\xi^T\tau.\end{cases}$$

The first line tells immediately that $\tau\perp\operatorname{Im}x$. Furthermore, it must be $|\eta|^2+|P_\tau\xi|^2>0$, see (5.1). Eliminating $P_\tau^\perp\xi$ we get $t=\pi\Big(|\eta|^2+|P_\tau\xi|^2\Big)\tau - x\,\xi^T\tau$. Thus

$$\pi\Big(|\eta|^2+|P_\tau\xi|^2\Big)\tau = P^\perp_{\operatorname{Im}x}t \quad\text{and}\quad -\,x\,\xi^T\tau = P_{\operatorname{Im}x}t.$$

Letting $\beta=\xi^T\tau\in\mathbb{R}^{p\times 1}$, we get $\pi\Big(|\eta|^2+|\beta|^2\Big)\tau = P^\perp_{\operatorname{Im}x}t$ and $-x\beta = P_{\operatorname{Im}x}t$. From last formula we get immediately

$$\begin{aligned}|P^\perp_{\operatorname{Im}x}t| = \pi(|\eta|^2+|\beta|^2) \geq \pi|\beta|^2 &\geq \pi\min\{|\beta'|^2 : -x\beta' = P_{\operatorname{Im}x}t\}\\ &=: \pi|x^\dagger P_{\operatorname{Im}x}t|^2 = \pi|x^\dagger t|^2.\end{aligned}\tag{5.2}$$

The inclusion is proved.

Step 2. Now we show the opposite inclusion and we characterize which cut points are reached by a unique minimizer.

Let $(x,0,t)$ be in the set in the right-hand side of (1.3). We must find $(\xi,\eta)\in V_1$ and $\tau\in V_2$ such that

$$\begin{cases} 2\pi\, P_\tau^\perp\xi = x\\ \pi\Big(|\eta|^2+|P_\tau\xi|^2\Big)\tau - 2\pi P_\tau^\perp\xi\,\xi^T\tau = t\\ |\tau|=1 \quad\text{ and }\quad |\eta|^2+|P_\tau\xi|^2>0.\end{cases}\tag{5.3}$$

Since τ is a unit vector and from the first line, we can write $\xi = P_\tau\xi + P_\tau^\perp\xi = \tau\tau^T\xi + \frac{x}{2\pi} =: -\tau\lambda^T + \frac{x}{2\pi}$, where we put $\lambda = -\xi^T\tau\in\mathbb{R}^{p\times 1}$. Thus, to find ξ, it suffices to know the vector $\lambda=-\xi^T\tau\in\mathbb{R}^p$. Concerning the vector τ we are looking for, it must be orthogonal to $\operatorname{Im}x$. Then, projecting the second line of (5.3) along $(\operatorname{Im}x)^\perp$, we get

$$\pi\Big(|\eta|^2+|P_\tau\xi|^2\Big)\tau = P^\perp_{\operatorname{Im}x}t\neq 0.\tag{5.4}$$

Note that $P^\perp_{\operatorname{Im}x}t\neq 0$ by assumption. Therefore, since τ is unit-norm, it must be first $|\eta|^2+|P_\tau\xi|^2>0$ and furthermore $\tau = \frac{P^\perp_{\operatorname{Im}x}t}{|P^\perp_{\operatorname{Im}x}t|}$. Next project the second line of (5.3) along $\operatorname{Im}x$. We get the equation $x\lambda = P_{\operatorname{Im}x}t$. If the columns $x_1, \dots, x_p$ of x are independent, then $x^T x$ is nonsingular and we find a unique $\lambda = (x^T x)^{-1} x^T t = x^\dagger t$ solving the problem. If instead $x_1, \dots, x_p$ are dependent, the solutions $\lambda \in \mathbb{R}^p$ of $x\lambda = P_{\operatorname{Im} x} t$ form an affine space of the form $\{x^\dagger t + \mu : \mu \in \ker x\}$, where $x^\dagger t$ satisfies $x x^\dagger t = P_{\operatorname{Im} x} t$ and has the further property $x^\dagger t \perp \ker x$, *i.e.* $x^\dagger t$ it is the minimal-norm solution: $|x^\dagger t| \le |x^\dagger t + \mu|$ for all $\mu \in \ker x$. More precisely, since $x^\dagger t \perp \mu$, we have $|x^\dagger t + \mu|^2 = |x^\dagger t|^2 + |\mu|^2$. Fix now any $\mu \in \ker x$ and choose then $\lambda = x^\dagger t + \mu$. Multiply the second line of (5.3) scalarly by τ (taking into account that $-\xi^T \tau = \lambda = x^\dagger t + \mu$). This gives

$$\pi(|\eta|^2 + |x^\dagger t + \mu|^2) = \langle t, \tau \rangle = \Big\langle t, \frac{P^\perp_{\operatorname{Im} x} t}{|P^\perp_{\operatorname{Im} x} t|} \Big\rangle = |P^\perp_{\operatorname{Im} x} t|.$$

Again by orthogonality $\mu \perp x^\dagger t$ we get

$$\pi(|\eta|^2 + |\mu|^2) = |P^\perp_{\operatorname{Im} x} t| - \pi |x^\dagger t|^2. \tag{5.5}$$

The right-hand side is nonnegative by assumption. Therefore, any choice of $\eta \in \mathbb{R}^p$ and $\mu \in \ker x$ such that (5.5) is fulfilled, will provide a solution fulfilling (5.3). This finishes the proof of the inclusion.

To prove item 3 of Theorem 1.2, observe that, given $(x, 0, t) \in \operatorname{Cut}(\mathbb{G}_{qp})$, uniqueness of the choice of (ξ, η, τ) holds if and only if the choice of η and $\mu \in \mathbb{R}^p$ satisfying (5.5) is unique. This happens if and only if the right-hand side of (5.5) vanishes, *i.e.* $|P^\perp_{\operatorname{Im} x} t| = \pi |x^\dagger t|^2$.

Step 3. We finally check formula (1.4). Let $(x, 0, t) = \gamma(2\pi, \xi, \eta, \tau) \in \operatorname{Cut}(\mathbb{G}_{qp})$, where $|\tau| = 1$. The horizontal speed of the minimizer $\gamma(\cdot, \xi, \eta, \tau) : [0, 2\pi] \mapsto V_1 \times V_2$ is $\sqrt{|\eta|^2 + |\xi|^2}$. We have $|\eta|^2 + |\xi|^2 = |\eta|^2 + |P_\tau \xi|^2 + |P_\tau^\perp \xi|^2$. From the first line of (5.3) we have $|P_\tau^\perp \xi|^2 = \frac{|x|^2}{4\pi^2}$. From (5.4) we find $|\eta|^2 + |P_\tau \xi|^2 = \frac{|P^\perp_{\operatorname{Im} x} t|}{\pi}$. Collecting formulas, we conclude that

$$d(x, 0, t)^2 = 4\pi^2 (|\xi|^2 + |\eta|^2) = |x|^2 + 4\pi |P^\perp_{\operatorname{Im} x} t|,$$

as required. Note that the component $P_{\operatorname{Im} x} t$ does not appear in the distance. □

Remark 5.1. Observe that the set of points $(x, 0, t) \in \operatorname{Cut}(\mathbb{G}_{qp})$ reached by more than one unit-speed length-minimizer is dense in $\operatorname{Cut}(\mathbb{G}_{qp})$. To see that, it suffices to take a point $(x, 0, t) = (x, 0, P_{\operatorname{Im} x} t + P^\perp_{\operatorname{Im} x} t)$ such that $P^\perp_{\operatorname{Im} x} t \neq 0$ and satisfying $|P^\perp_{\operatorname{Im} x} t| = \pi |x^\dagger t|^2$. This point can be reached by a unique unit-speed minimizer. Next consider the family of approximating points $(x_\varepsilon, 0, t_\varepsilon) := (x, 0, P_{\operatorname{Im} x} t + (1 + \varepsilon) P^\perp_{\operatorname{Im} x} t)$, where $\varepsilon > 0$. We have easily $|P^\perp_{\operatorname{Im} x_\varepsilon} t_\varepsilon| = (1 + \varepsilon) |P^\perp_{\operatorname{Im} x} t|$. Furthermore, note that $x_\varepsilon^\dagger t_\varepsilon$ only depends on $P_{\operatorname{Im} x_\varepsilon} t_\varepsilon = P_{\operatorname{Im} x} t$ and not on $P^\perp_{\operatorname{Im} x_\varepsilon} t_\varepsilon$ which changes with ε. Thus $x_\varepsilon^\dagger t_\varepsilon = x^\dagger t$ for all $\varepsilon > 0$. Thus the approximating point satisfies the strict inequality $|P_{\operatorname{Im} x_\varepsilon} t_\varepsilon| > \pi |x_\varepsilon^\dagger t_\varepsilon|^2$. Consequently, it can be reached by more than one length-minimizer.

Remark 5.2. As a byproduct of Step 2 of the proof above, and for future reference, observe that, if $(x, 0, t) \in \operatorname{Cut}(\mathbb{G}_{qp})$ and if $\ker x$ is trivial, then we can write $(x, 0, t) = \gamma(2\pi, \xi, \eta, \tau)$, where $\tau = \frac{P^\perp_{\operatorname{Im} x} t}{|P^\perp_{\operatorname{Im} x} t|}$, $\xi = \tau (x^\dagger t)^T + \frac{x}{2\pi}$ and η satisfies

$$\pi |\eta|^2 = |P^\perp_{\operatorname{Im} x} t| - \pi |x^\dagger t|^2. \tag{5.6}$$

5.1. The group $\mathbb{G}_{q1}$

Here $(x, 0, t) \in \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}^q$. All points of the form $(0, 0, t) \in \mathbb{G}_{q1}$ with $t \neq 0$ belong to $\operatorname{Cut}(\mathbb{G}_{q1})$. Let $(x, 0, t) \in \mathbb{G}_{q1}$ be such that $x \neq 0$ and $t \neq 0$. The equation $x\beta = P_{\operatorname{Im} x} t$ has a unique real solution $x^\dagger (P_{\operatorname{Im} x} t) = \frac{\langle x, t \rangle}{|x|^2}$. Then

$|x^\dagger t|^2 = |x^\dagger P_{\mathrm{Im}\,x} t|^2 = \frac{\langle x,t\rangle^2}{|x|^4} = \frac{|P_{\mathrm{Im}\,x} t|^2}{|x|^2}$. Thus we have

$$\begin{aligned}\mathrm{Cut}(\mathbb{G}_{q1}) &= \Big\{(x,0,t)\in\mathbb{G}_{q1} : t\neq 0 \text{ and } |x|^2\,|P^\perp_{\mathrm{Im}\,x} t| \geq \pi |P_{\mathrm{Im}\,x} t|^2\Big\}\\ &= \{(x,0,t) : t\neq 0 \ \textit{and}\ |x|^8|t|^2 - |x|^6\langle t,x\rangle^2 \geq \pi^2\langle t,x\rangle^4\},\end{aligned}$$

where, passing from the first to the second line, we used the Pythagorean theorem $|P^\perp_{\mathrm{Im}\,x} t|^2 = |t|^2 - |P_{\mathrm{Im}\,x} t|^2 = |t|^2 - \frac{\langle t,x\rangle^2}{|x|^2}$.

In the next proposition, we analyze the regularity of the set of cut points where equality holds. Namely, of

$$\Sigma := \{(x,0,t) : t\neq 0 \ \textit{and}\ |x|^8|t|^2 - |x|^6\langle t,x\rangle^2 = \pi^2\langle t,x\rangle^4\}. \tag{5.7}$$

Proposition 5.3. *Let $\Sigma \subset \mathrm{Cut}(\mathbb{G}_{q1})$ be the set in (5.7). Let $\Sigma_0 := \{(x,0,t)\in\Sigma : x\neq 0\}\subset\Sigma$. Then Σ_0 is a smooth codimension-one embedded submanifold of $\{(x,0,t) : (x,t)\in\mathbb{R}^q\times\mathbb{R}^q\}$. Furthermore, the whole Σ is not a manifold.*

Observe that Σ_0 has codimension 2 in $\mathbb{G}_{q1}$. The surface Σ_0 contains all points $(x,0,t)\in\mathrm{Cut}(\mathbb{G}_{qp})$ which are reached by a unique length- minimizing unit-speed curve.

Proof. To check this statement, it suffices to observe that Σ_0 is the zero-level set of the function $\psi : (\mathbb{R}^q\setminus\{0\})\times(\mathbb{R}^q\setminus\{0\})\to\mathbb{R}$

$$\psi(x,t) = |t|^2 - \frac{\langle x,t\rangle^2}{|x|^2} - \pi^2\frac{\langle x,t\rangle^4}{|x|^8}. \tag{5.8}$$

A short computation gives

$$\begin{cases}\nabla_t\psi = 2t - \dfrac{2}{|x|^2}\langle x,t\rangle\Big\{1+2\pi^2\dfrac{\langle t,x\rangle^2}{|x|^6}\Big\}x\\ \nabla_x\psi = -\dfrac{2}{|x|^2}\langle x,t\rangle\Big\{1+2\pi^2\dfrac{\langle t,x\rangle^2}{|x|^6}\Big\}t + \dfrac{2}{|x|^4}\langle x,t\rangle^2\Big\{1+\dfrac{4\pi^2}{|x|^6}\langle x,t\rangle^2\Big\}x.\end{cases}$$

Observe now that at all point of Σ_0, we have $\langle x,t\rangle\neq 0$ (see (5.8)). Furthermore, x and t are independent (otherwise, again (5.8) fails). Denoting $\nabla_t\psi =: at+bx$ and $\nabla_x\psi =: bt+cx$, we see that the Jacobian of ψ has full rank if and only if $ac-b^2\neq 0$. A computation gives

$$ac-b^2 = -\frac{16\pi^4}{|x|^{16}}\langle x,t\rangle^6 \neq 0$$

for any $(x,t)\in\Sigma_0$. Thus, Σ_0 is a smooth manifold.

To show that the whole Σ is not a manifold, we argue by contradiction. Without loss of generality, consider $q=2$ and take the point $(x,t)=(0,e_2)=(0,0,0,1)$. Assume that Σ is a smooth embedded manifold in a neighborhood of $(0,e_2)$.

Step 1. If Σ is a smooth surface containing, $(0,e_2)$, a tangent vector $(\xi,\tau)\in T_{(0,e_2)}\Sigma$, would have the form $(\xi,\tau)=(x'(0),t'(0))$, where $s\mapsto(x(s),t(s))$ is a curve belonging to Σ for $s\in(-1,1)$, $(x(0),t(0))=(0,e_2)$ and $(x'(0),t'(0))=(\xi,\tau)$. We claim that all such vectors satisfy $\xi\perp e_2$. This implies that $T_{(0,e_2)}\Sigma = \{(\xi_1,0,\tau_1,\tau_2) : \xi_1,\tau_1,\tau_2\in\mathbb{R}\}$. To prove the claim, take a curve $(x(s),t(s))$ as above and expand at the first order $x(s) = x's+o(s)$ and $t(s)=e_2+t's+o(s)=e_2+o(1)$, where $(x',t')=(x'(0),t'(0))$, $o(1)\to 0$ and $\frac{o(s)}{s}\to 0$ as $s\to 0$.

Then $|x(s)|^2 = |x'|^2s^2 + o(s^2)$, $|t(s)|^2 = 1 + o(1)$, while $\langle x(s), t(s)\rangle = \langle x', e_2\rangle s + o(s) = x_2's + o(s)$. Inserting into (5.7), we get

$$(|x'|^8s^8 + o(s^8))(1 + o(1)) - (|x'|^6s^6 + o(s^6))((x_2')^2s^2 + o(s^2)) - \pi^2((x_2')^4s^4 + o(s^4)) = 0.$$

Comparing powers of s, we get $x_2' = \langle x'(0), e_2\rangle = 0$, as desired.

Step 2. Being $T_{(0,e_2)}\Sigma = \{(\xi_1, 0, \tau_1, \tau_2) : \xi_1, \tau_1, \tau_2 \in \mathbb{R}\}$, the manifold Σ can be written as a graph of a function F of the variables x_1, t_1, t_2. Namely, there are neighborhoods $V := \{(x_1, t_1, t_2) \in \mathbb{R}^3 : |(x_1, t_1, t_2) - (0, 0, 1)| < \varepsilon\}$ and $W = \{x_2 \in \mathbb{R} : |x_2| < \delta\}$ such that we have

$$(V \times W) \cap \Sigma = \{(x_1, F(x_1, t_1, t_2), t_1, t_2) : (x_1, t_1, t_2) \in V\}.$$

Precisely, for all $(x_1, t_1, t_2) \in V$ there is a unique $F = F(x_1, t_1, t_2) \in \left]-\delta, \delta\right[$ such that $\psi(x_1, F, t_1, t_2) = 0$. Let us test such property on points of the form $(x_1, 0, t_2) \in V$. Letting $F = F(x_1, 0, t_2)$, we have after some simpifications

$$\psi(x_1, F, 0, t_2) = t_2^2 - \frac{(t_2F)^2}{x_1^2 + F^2} - \pi^2\frac{(t_2F)^4}{(x_1^2 + F^2)^4} = 0. \tag{5.9}$$

Since F appears quadratically, its uniqueness gives $F(x_1, 0, t_2) = 0$ as soon as $x_1t_2 \neq 0$. However, inserting $F = 0$ into (5.9), we get $t_2^2 = 0$ for all $(x_1, 0, t_2) \in V$, which gives a contradiction. □

6. Lower estimate of conjugate time for $p = 1$

In this section we show that for $q \geq 2$ and $p = 1$, in the model $\mathbb{G}_{q1}$ there are minimizers such that the t_{cut} is strictly smaller than the first conjugate time. The following statement completes Proposition 3.2. Such phenomenon has already been encountered in [8].

Theorem 6.1 (Conjugate points, $q \geq 2$, $p = 1$, necessary condition)**.** *Let $q \geq 2$ and consider the model $\mathbb{G}_{q1}$. Given $(\xi, \eta, \tau) \in V_1 \times V_2$, assume that $\tau \neq 0$ and $|\eta|^2 + |P_\tau\xi|^2 > 0$ and consider the extremal $\gamma(\cdot, \xi, \eta, \tau)$. Then, if $\bar{s} := \frac{2\pi}{|\tau|}$ is a conjugate time of $\gamma(\cdot, \xi, \eta, \tau)$, it must be*

$$\eta P_\tau^\perp\xi = 0. \tag{6.1}$$

We do not discuss the case $p = q = 1$, the lower dimensional Heisenberg group, where it is known that cut time and first conjugate times are always the same.

Proof. Consider $\mathbb{G}_{q1}$ and a point $(\xi, \eta, \tau) \in T^*_{(0,0,0)}\mathbb{G}_{q1}$. Define $\Lambda = \{(\xi', \eta', \tau') \in T^*_{(0,0,0)}\mathbb{G}_{q1} : |\xi'|^2 + {\eta'}^2 = |\xi|^2 + \eta^2\} \subset T^*_{(0,0,0)}\mathbb{G}_{q1}$. It suffices to show that $\eta|P_\tau^\perp\xi| \neq 0$ implies that $s = \frac{2\pi}{|\tau|}$ is not conjugate. Fix an orthogonal frame in $T_0^*\mathbb{G}_{q1} \simeq \mathbb{R}^q$ choosing

$$\tau, \quad \omega_{q-1} := P_\tau^\perp\xi, \quad \text{and} \quad \omega_1, \dots, \omega_{q-2} \in \operatorname{span}\{\tau, P_\tau^\perp\xi\}^\perp. \tag{6.2}$$

Note that if $q = 2$, the frame will be formed by $\omega_1 = P_\tau^\perp\xi$ and τ only and the discussion will be easier (see below). Fix also the frame of vector fields

$$Z = -\langle\xi, \tau\rangle\partial_\eta + \eta D_\tau \qquad V_k = -\langle\xi, \omega_k\rangle\partial_\eta + \eta D_{\omega_k} \quad \text{where} \quad k \leq q - 1$$

and $D_{\omega_k} := \langle \omega_k, \nabla_\xi \rangle$ and $D_\tau = \langle \tau, \nabla_\xi \rangle$. Observe that the frame is tangent to Λ.[3] Furthermore, the frame in independent, because $\eta \neq 0$ and $\omega_1, \ldots, \omega_{q-1}, \tau$ are independent. We need to show that the following determinant is nonzero.

$$\det M := \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_q \wedge \mathrm{d}y \wedge \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_q(\partial_s\gamma, Z\gamma, V_1\gamma, \ldots, V_{q-1}\gamma, d_\tau\gamma) \neq 0$$

Let $\ell := \xi \cdot \mathrm{d}x + \eta \mathrm{d}y + \tau \cdot \mathrm{d}t$ be the Liouville form. Since $|\tau| \neq 0$, up to a nonzero factor we change $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t_1 \wedge \cdot \wedge \mathrm{d}t_q$ with $\mathrm{d}x \wedge \mathrm{d}y \wedge \ell \wedge \mu_1 \wedge \cdots \wedge \mu_{q-1}$, where $\mu_k := \omega_k \cdot \mathrm{d}t$. By Lemma A.1, we have $\ell(Z\gamma) = \ell(V_k\gamma) = 0$, for all $k = 1, \ldots, q-1$, while $\ell(\partial_s\gamma) = 1$.

$$\det M \sim \ell(\partial_s\gamma)(\mathrm{d}x \wedge \mathrm{d}y \wedge \mu_1 \wedge \cdots \wedge \mu_{q-1})(Z\gamma, W\gamma, V_{q-1}\gamma, d_\tau\gamma)$$
$$\sim \det \begin{bmatrix} Zx & Wx & V_{q-1}x & d_\tau x \\ Zy & Wy & V_{q-1}y & d_\tau y \\ \mu_1(Zt) & \mu_1(Wt) & \mu_1(V_{q-1}t) & \mu_1(d_\tau t) \\ \vdots & \vdots & \vdots & \\ \mu_{q-1}(Zt) & \mu_{q-1}(Wt) & \mu_{q-1}(V_{q-1}t) & \mu_{q-1}(d_\tau t) \end{bmatrix}.$$

where $\sim$ means that the left-hand side vanishes if and only if the right-hand side vanishes. Here $x = [x_1, \ldots, x_q]^T$, while W stands for $[V_1, \ldots, V_{q-2}]$ and $d_\tau = [\partial_{\tau_1}, \ldots, \partial_{\tau_q}]$.

We already calculated the Z-derivatives in (3.3) and (3.4). Namely $Zx(\frac{2\pi}{|\tau|}, \xi, \eta, \tau) = 0$, $Zy(\frac{2\pi}{|\tau|}, \xi, \eta, \tau) = 0$ and $Zt(\frac{2\pi}{|\tau|}, \xi, \eta, \tau) = -\frac{2\pi\eta}{|\tau|} P_\tau^\perp \xi$. By (6.2) we have then $\mu_j(Zt) = 0$, for $j \leq q-2$ and $\mu_{q-1}(Zt) = (P_\tau^\perp \xi)^T(-\frac{2\pi\eta}{|\tau|} P_\tau^\perp \xi) = -\frac{2\pi\eta}{|\tau|} |P_\tau^\perp \xi|^2 \neq 0$, by our assumptions. Therefore, if $q \geq 3$,

$$\det M \sim \det \begin{bmatrix} Wx & V_{q-1}x & d_\tau x \\ Wy & V_{q-1}y & d_\tau y \\ \mu_1(Wt) & \mu_1(V_{q-1}t) & \mu_1 d_\tau t \\ \vdots & \vdots & \vdots \\ \mu_{q-2}(Wt) & \mu_{q-2}(V_{q-1}t) & \mu_{q-2}(d_\tau t) \end{bmatrix}. \tag{6.3}$$

If $q = 2$, we have instead the simpler form

$$\det M \overset{q=2}{\sim} = \det \begin{bmatrix} V_1 x & d_\tau x \\ V_1 y & d_\tau y \end{bmatrix}.$$

Next, for $k \leq q-1$ we have $V_k x = (-\langle \omega_k, \xi \rangle + \eta D_{\omega_k}) \frac{2\pi}{|\tau|} P_\tau^\perp \xi = \frac{2\pi\eta}{|\tau|} P_\tau^\perp \omega_k = \frac{2\pi\eta}{|\tau|} \omega_k$, because $\omega_k \perp \tau$ for all $k \leq q-1$. Easily, $V_k y = 0$ and finally

$$\begin{aligned} V_k t &= \frac{\pi}{|\tau|^3}(-\langle \xi, \omega_k \rangle \partial_\eta + \eta D_{\omega_k})\Big((|P_\tau \xi|^2 + \eta^2)\tau - 2P_\tau^\perp \xi \xi^T \tau\Big) \\ &= \frac{\pi}{|\tau|^3}\Big(- 2\eta\langle \xi, \omega_k \rangle \tau + 2\eta \langle P_\tau \xi, P_\tau \omega_k \rangle \tau - 2\eta P_\tau^\perp \omega_k \xi^T \tau - 2\eta P_\tau^\perp \xi \omega_k^T \tau\Big) \\ &= -\frac{2\pi\eta}{|\tau|^3}\Big\{ \langle \xi, \omega_k \rangle \tau + \langle \xi, \tau \rangle \omega_k \Big\} \end{aligned}$$

[3]This comes from the fact the set Λ is defined by the equation $F(\xi, \eta, \tau) := |\xi|^2 + \eta^2 = $ constant, and we have $ZF = V_kF = 0$ identically.

where we used $\omega_k \in \tau^\perp$. Therefore, for $j,k \le q-2$ we have $\mu_j(V_k t) = \omega_j^T(V_k t) = -\frac{2\pi\eta}{|\tau|^3}\langle \xi, \tau\rangle \omega_j^T \omega_k$. Introducing the matrix $\Omega = [\omega_1, \dots, \omega_{q-2}] \in \mathbb{R}^{q\times(q-2)}$, if $q \ge 3$, we can write the first $q-2$ columns of the matrix in (6.3) as

$$\begin{bmatrix} \frac{2\pi\eta}{|\tau|}\Omega \\ 0 \\ -\frac{2\pi\eta}{|\tau|^3}\langle\xi,\tau\rangle\Omega^T\Omega \end{bmatrix} \in \mathbb{R}^{(2q-1)\times(q-2)}.$$

Passing to the $(q-1)$-th column, since we let $\omega_{q-1} = P_\tau^\perp \xi$, computations above give $V_{q-1}x = \frac{2\pi\eta}{|\tau|}P_\tau^\perp \omega_{q-1} = \frac{2\pi\eta}{|\tau|}P_\tau^\perp \xi$. Furthermore, $V_{q-1}y = 0$ and

$$V_{q-1}t = -\frac{2\pi\eta}{|\tau|^3}\big(\langle \xi, P_\tau^\perp\xi\rangle\tau + \langle\xi,\tau\rangle P_\tau^\perp \xi\big) \quad\Rightarrow\quad \mu_j(V_{q-1}t) = 0 \quad \forall\, j \le q-2.$$

Ultimately, the $(q-1)$-th column becomes $\begin{bmatrix} |\tau|^{-1}2\pi\eta P_\tau^\perp\xi \\ 0 \\ 0\end{bmatrix} \in \mathbb{R}^{2q-1}$.

Let us pass to the columns involving ∂_{τ_α}. In all differentiations below, we omit all terms that vanish when $s = 2\pi/|\tau|$. The calculations of $\partial_{\tau_\alpha}\gamma(s,\xi,\eta,\tau)$ for $s \ne \frac{2\pi}{|\tau|}$ would be much longer.

$$\begin{aligned}
\partial_{\tau_\alpha} x\Big|_{s=\frac{2\pi}{|\tau|}} &= \partial_{\tau_\alpha}\bigg(\tfrac{2}{|\tau|}\sin\big(\tfrac{|\tau|s}{2}\big)\Big\{P_\tau\xi\,\cos\big(\tfrac{|\tau|s}{2}\big) - \frac{\tau\eta^T}{|\tau|}\sin\big(\tfrac{|\tau|s}{2}\big)\Big\} + sP_\tau^\perp\xi\bigg)\Big|_{s=\frac{2\pi}{|\tau|}} \\
&= \frac{2}{|\tau|}\cos(\tfrac{|\tau|s}{2})\frac{s}{2}\frac{\tau_\alpha}{|\tau|}(-P_\tau\xi) + \frac{2\pi}{|\tau|}\partial_{\tau_\alpha}\Big(\xi - \frac{\tau\tau^T}{|\tau|^2}\xi\Big)\Big|_{s=\frac{2\pi}{|\tau|}} \\
&= \frac{2\pi}{|\tau|^3}\tau_\alpha P_\tau\xi + \frac{2\pi}{|\tau|}\Big(\frac{2}{|\tau|^3}\frac{\tau_\alpha}{|\tau|}\tau\tau^T\xi - \frac{e_\alpha\tau^T}{|\tau|^2}\xi - \frac{\tau e_\alpha^T}{|\tau|^2}\xi\Big) \\
&= \frac{2\pi}{|\tau|^3}\big(3\tau_\alpha P_\tau\xi - e_\alpha\langle\xi,\tau\rangle - \tau\xi_\alpha\big).
\end{aligned}$$

So, the $q\times q$ north east block is $d_\tau x|_{s=\frac{2\pi}{|\tau|}} = \frac{2\pi}{|\tau|^3}\big(3\langle\xi,\tau\rangle P_\tau - \langle\xi,\tau\rangle I_q - \tau\xi^T\big) \in \mathbb{R}^{q\times q}$. An analogous computation furnishes $\partial_{\tau_\alpha} y|_{s=2\pi/|\tau|} = \frac{2\pi}{|\tau|^3}\eta\tau_\alpha$ for $\alpha = 1,\dots,q$. To conclude, for $q \ge 3$, we have to calculate derivatives $\partial_{\tau_\alpha} t|_{s=2\pi/|\tau|}$. Keeping into account that we need only to know $\mu_j(\partial_{\tau_\alpha} t) = \omega_j^T\partial_{\tau_\alpha} t$ with $j \le q-2$, since $\operatorname{span}\{\omega_1,\dots,\omega_{q-2}\} = \{\tau, P_\tau^\perp\xi\}^\perp$, below we work ignoring terms in $\operatorname{span}\{\tau, P_\tau^\perp\xi\}$ writing below $u \simeq u'$ when $u - u' \in \operatorname{span}\{\tau, P_\tau^\perp\xi\} \subset \mathbb{R}^q$.

$$\begin{aligned}
\partial_{\tau_\alpha} t|_{s=2\pi/|\tau|} &= \partial_{\tau_\alpha}\bigg(s^2 U\big(\tfrac{|\tau|s}{2}\big)\big\{|P_\tau\xi|^2 + |\eta^2|\big\}\frac{\tau}{|\tau|} \\
&\quad + s^2 V\big(\tfrac{|\tau|s}{2}\big)\Big\{-P_\tau^\perp\xi\,\eta\sin\big(\tfrac{|\tau|s}{2}\big) + P_\tau^\perp\xi\frac{\xi^T\tau}{|\tau|}\,\cos\big(\tfrac{|\tau|s}{2}\big)\Big\}\bigg) \\
&\simeq \Big(\frac{2\pi}{|\tau|}\Big)^2 U(\pi)(|P_\tau\xi|^2 + \eta^2)\frac{1}{|\tau|}\partial_{\tau_\alpha}\tau + \Big(\frac{2\pi}{|\tau|}\Big)^2 V(\pi)\partial_{\tau_\alpha}P_\tau^\perp\xi\Big(\frac{\xi^T\tau}{|\tau|}\cos\pi\Big) \\
&= \frac{\pi}{|\tau|^2}(|P_\tau\xi|^2 + \eta^2)\frac{e_\alpha}{|\tau|} + \frac{2\pi}{|t|^2}\frac{e_\alpha\tau^T\xi}{|\tau|^2}\frac{\xi^T\tau}{|\tau|} = \frac{\pi}{|\tau|^3}\big(\eta^2 + 3|P_\tau\xi|^2\big)e_\alpha.
\end{aligned}$$

The south east $(q-2)\times q$-block has elements $(\mu_j(\partial_{\tau_\alpha}t)) = \omega_j^T(d_{\tau_\alpha}t))$, with $j\leq q-2$ and $\alpha=1,\dots,q$. Ultimately, the matrix in (6.3) takes the form

$$M=\begin{bmatrix} \frac{2\pi\eta}{|\tau|}\Omega & \frac{2\pi\eta}{|\tau|}P_\tau^\perp\xi & \frac{2\pi}{|\tau|^3}\Big(3\langle\xi,\tau\rangle P_\tau-\langle\xi,\tau\rangle I_q-\tau\xi^T\Big)\\ 0 & 0 & \frac{2\pi\eta}{|\tau|^3}\tau^T\\ -\frac{2\pi\eta}{|\tau|^3}\langle\xi,\tau\rangle\Omega^T\Omega & 0 & \frac{\pi}{|\tau|^3}\big(\eta^2+3|P_\tau\xi|^2\big)\Omega^T\end{bmatrix}$$

where $\Omega=[\omega_1,\dots,\omega_{q-2}]\in\mathbb{R}^{q\times(q-2)}$. By linear algebra,[4] M has full rank if and only if

$$\widehat{M}:=\begin{bmatrix}\Omega & P_\tau^\perp\xi & -2\Big(\langle\xi,\tau\rangle I_q+\tau\xi^T\Big)\\ 0 & 0 & \tau^T\\ -\frac{\langle\xi,\tau\rangle}{|\tau|^2}\Omega^T\Omega & 0 & \big(\eta^2+3|P_\tau\xi|^2\big)\Omega^T\end{bmatrix}\in\mathbb{R}^{(2q-1)\times(2q-1)}$$

has full rank. If $q=2$, we have the simpler matrix

$$\widehat{M}\overset{q=2}{=}\begin{bmatrix}P_\tau^\perp\xi & -2\Big(\langle\xi,\tau\rangle I_2+\tau\xi^T\Big)\\ 0 & \tau^T\end{bmatrix}\in\mathbb{R}^{3\times3}. \tag{6.4}$$

To conclude the proof we prove the following claim.

Claim. The matrix $\widehat{M}$ has trivial kernel.

Proof of the claim for $q\geq3$ and $\langle\xi,\tau\rangle=0$. We show that $\widetilde{M}$ has trivial kernel. Let $a\in\mathbb{R}^{q-2}$, $b\in\mathbb{R}$ and $c\in\mathbb{R}^q$. If $\langle\xi,\tau\rangle=0$, the system $\widehat{M}\begin{bmatrix}a\\b\\c\end{bmatrix}=0$ becomes

$$\begin{cases}\Omega a+P_\tau^\perp\xi b-2\tau\langle\xi,c\rangle=0\\ \langle\tau,c\rangle=0\\ \big(\eta^2+3|P_\tau\xi|^2\big)\Omega^Tc=0.\end{cases} \tag{6.5}$$

By (6.2), $\Omega a\in\operatorname{span}\{\tau,P_\tau^\perp\xi\}$. Then, the first line gives three separate conditions and we get

$$\begin{cases}\Omega a=0,\quad b=0,\quad \langle\xi,c\rangle=0\\ \langle\tau,c\rangle=0\\ \big(\eta^2+3|P_\tau\xi|^2\big)\Omega^Tc=0\end{cases} \tag{6.6}$$

Furthermore, since $\Omega=[\omega_1,\dots,\omega_{q-2}]$ has no kernel, we get $a=0$. Concerning c, we see that it is orthogonal both to τ and to $\operatorname{span}\{\omega_1,\dots,\omega_{q-2}\}$, by the third line of (6.6). Then, $c=\lambda P_\tau^\perp\xi$ for suitable $\lambda\in\mathbb{R}$. Again from the first line, we get $0=\langle\xi,c\rangle=\langle\xi,\lambda P_\tau^\perp\xi\rangle=\lambda|P_\tau^\perp\xi|^2$, which implies $c=0$ as we wished.

Proof of the claim, $q\geq3$ and $\langle\xi,\tau\rangle\neq0$. The system becomes

$$\begin{cases}\Omega a+P_\tau^\perp\xi b-2\langle\xi,\tau\rangle c-2\tau\langle\xi,c\rangle=0\\ \langle\tau,c\rangle=0\\ -\dfrac{\langle\xi,\tau\rangle}{|\tau|^2}\Omega^T\Omega a+\big(\eta^2+3|P_\tau\xi|^2\big)\Omega^Tc=0.\end{cases}$$

[4] Note that the north east-term containing the factor $P_\tau=\frac{\tau}{|\tau|^2}\tau^T$ can be eliminated by subtracting to each of its rows a suitable multiple of the $(q+1)$-th row. All other simplifications are multiplications of some row/columns for nonzero scalars.

Multiply from the left the first line of (6.5) by $\frac{\langle\xi,\tau\rangle}{|\tau|^2}\Omega^T$ and add to the third. After elementary simplifications based on property $\Omega^T\tau = \Omega^T P_\tau^\perp\xi = 0$, this gives $(\eta^2 + |P_\tau\xi|^2)\Omega^T c = 0$. Thus, $\Omega^T c = 0$ which implies $c \in \operatorname{span}\{\tau, P_\tau^\perp\xi\}$. Arguing as above, we write $c = \lambda P_\tau^\perp\xi$ and projecting orthogonally the first line along τ, we get $c = 0$. The third line gives also $\Omega^T\Omega a = 0$, which implies $a = 0$ (independence of $\omega_1, \dots, \omega_{q-2}$ in $\mathbb{R}^q$ implies that $\Omega^T\Omega$ is nonsingular). The first line gives now $b = 0$ and the proof is finished.

Proof of the claim, $q = 2$. The system $\widehat{M}[\begin{smallmatrix} b \\ c\end{smallmatrix}]$ with $b \in \mathbb{R}$ and $c \in \mathbb{R}^2$ becomes

$$\begin{cases} P_\tau^\perp\xi b - 2\langle\xi,\tau\rangle c - 2\tau\langle\xi, c\rangle = 0 \\ \tau^T c = 0. \end{cases}$$

Here $\mathbb{R}^2 = \operatorname{span}\{\tau, P_\tau^\perp\xi\}$ and the second line gives $c = \lambda P_\tau^\perp\xi$ for some λ. Then the first becomes $P_\tau^\perp\xi(b - 2\langle\xi,\tau\rangle c) - 2\tau\lambda|P_\tau^\perp\xi|^2 = 0$, which by independence of τ and $P_\tau^\perp\xi$ provides $\lambda = b = 0$. This concludes the proof of the Theorem 6.1. □

We are ready to prove Theorem 1.3.

Proof. Let $q \geq 2$ and $p = 1$ and let $(x, 0, t) \in \operatorname{Cut}(\mathbb{G}_{q1})$. Write $(x, 0, t) = \gamma(2\pi, \xi, \eta, \tau)$ with $|\tau| = 1$ and $\eta^2 + |P_\tau^\perp\xi|^2 \neq 0$. We know by Proposition 3.2 and Theorem 6.1 that this point is conjugate if and only if $\eta|P_\tau^\perp\xi| = 0$. Condition $P_\tau^\perp\xi = 0$ is equivalent to $x = 0$, by the first line of (3.1). If $x \neq 0$, then $\ker x$ is trivial, and Remark 5.2 holds. In particular, by (5.6), condition $\eta = 0$ holds if and only if $|P_{\operatorname{Im} x}^\perp t| - \pi|x^\dagger t|^2 = 0$, as required. Note that here we have $|x^\dagger t|^2 = \frac{\langle x,t\rangle^2}{|x|^4}$. □

Appendix A.

The following lemma on quadratic Hamiltonian systems has been used in the proof of Theorem 6.1. We include it for completeness. This lemma is general and we use then standard Hamiltonian notation.

Lemma A.1. *Let $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N \simeq T^*(\mathbb{R}^N)$ and consider the Hamiltonian*

$$H(x, p) = \frac{1}{2}\langle M(x)p, p\rangle \tag{A.1}$$

where $M(x) \in \mathbb{R}^{N\times N}$ is symmetric, positive semidefinite and depends smoothly on x. Denote by $\Lambda = \{p \in \mathbb{R}^n = T_0^*\mathbb{R}^N : H(0, p) = \frac{1}{2}\}$. Given $p \in \Lambda$, denote by $t \in I \mapsto (X(t, p), P(t, p))$ the solution of the Hamiltonian system with initial data $(X(0, p), P(0, p)) = (0, p)$, where $I \subset \mathbb{R}$ is an interval containing 0. Let $\sigma \in \mathbb{R} \mapsto p(\sigma) \in \Lambda$ be a smooth path. Then we have

$$\sum_j P_j(t, p(\sigma))\frac{\partial}{\partial\sigma}X_j(t, p(\sigma)) = \sum_j P_j(0, p(\sigma))\frac{\partial}{\partial\sigma}X_j(0, p(\sigma)) = 0 \quad \forall\, t, \sigma.$$

In other words, the Liouville form $\ell := \sum_j p_j \mathrm{d}x_j$ satisfies $\ell(\frac{\partial}{\partial\sigma}X(t, p_\sigma)) = 0$, for all t, σ.

Proof. Write briefly $X_j = X_j(t, p(\sigma))$ and $P_j = P_j(t, p(\sigma))$ and omit summation on index $j \in \{1, \dots, N\}$. We must see that the following is zero:

$$(*) := \partial_t(P_j\partial_\sigma X_j) = \partial_t P_j\partial_\sigma X_j + P_j\partial_t\partial_\sigma X_j$$

Since the Hamiltonian is constant along flow, we get $H(X,P) = H(0,p(\sigma)) = \frac{1}{2}$ identically in t, σ. Differentiating with respect to σ gives

$$0 = \partial_{x_j} H(X,P)\partial_\sigma X_j + \partial_{p_j} H(X,P)\partial_\sigma P_j = -\partial_t P_j \partial_\sigma X_j + \partial_t X_j \partial_\sigma P_j.$$

Thus we may rewrite $(*)$ as

$$\begin{aligned}(*) &= \partial_t X_j \partial_\sigma P_j + P_j \partial_t \partial_\sigma X_j = \partial_t X_j \partial_\sigma P_j + P_j \partial_\sigma \partial_t X_j = \partial_\sigma (P_j \partial_t X_j) = \partial_\sigma (P_j \partial_{p_j} H) \\ &\overset{(\dagger)}{=} \partial_\sigma H(X,P) = \partial_\sigma H(0,p(\sigma)) = 0.\end{aligned}$$

In $\overset{(\dagger)}{=}$ we used the form (A.1) of the Hamiltonian. □

Acknowledgements

The authors are supported by PRIN 2022 F4F2LH – CUP J53D23003760006 "Regularity problems in sub-Riemannian structures". The authors are also members of the *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM). The auhors thank Ye Zhang, for a careful reading of the first version of the manuscript and for suggesting some improvements.

References

[1] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, Stratified Lie groups and potential theory for their sub-Laplacians. Springer Monographs in Mathematics. Springer, Berlin (2007).

[2] A. Agrachev, D. Barilari and U. Boscain, A comprehensive introduction to sub-Riemannian geometry, Cambridge Studies in Advanced Mathematics, Vol. 181. Cambridge University Press, Cambridge (2020), from the Hamiltonian viewpoint, With an appendix by Igor Zelenko.

[3] H. Reiter, Über den Satz von Wiener und lokalkompakte Gruppen. *Comment. Math. Helv.* **49** (1974) 333–364.

[4] J.F. Torres Lopera, Geodesics and conformal transformations of Heisenberg–Reiter spaces. *Trans. Am. Math. Soc.* **306** (1988) 489–498.

[5] A. Martini, Spectral multipliers on Heisenberg–Reiter and related groups. *Ann. Mat. Pura Appl.* **194** (2015) 1135–1155.

[6] C. Autenried and M.G. Molina, The sub-Riemannian cut locus of H-type groups. *Math. Nachr.* **289** (2016) 4–12.

[7] A. Agrachev, D. Barilari and U. Boscain, On the Hausdorff volume in sub-Riemannian geometry. *Calc. Var. Partial Diff. Equ.* **43** (2012) 355–388.

[8] D. Barilari, U. Boscain and J.-P. Gauthier, On 2-step, corank 2, nilpotent sub-Riemannian metrics. *SIAM J. Control Optim.* **50** (2012) 559–582.

[9] A. Montanari and D. Morbidelli, On the subRiemannian cut locus in a model of free two-step Carnot group. *Calc. Var. Partial Diff. Equ.* **56** (2017) Paper No. 36, 26.

[10] O. Myasnichenko, Nilpotent $(3,6)$ sub-Riemannian problem. *J. Dynam. Control Syst.* **8** (2002) 573–597.

[11] L. Rizzi and U. Serres, On the cut locus of free, step two Carnot groups. *Proc. Am. Math. Soc.* **145** (2017) 5341–5357.

[12] Yu.L. Sachkov, Left-invariant optimal control problems on Lie groups: classification and problems integrable by elementary functions. *Uspekhi Mat. Nauk* **77** (2022) 109–176. MR 4461360

[13] H.-Q. Li, The Carnot–Carathéodory distance on 2-step groups. arXiv:2112.07822 (2021).

[14] H.-Q. Li and Y. Zhang, Sub-Riemannian geometry on some step-two Carnot groups, arXiv:2102.09860 (2021).

[15] C. Meyer, Matrix Analysis and Applied Linear Algebra. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (2000), with 1 CD-ROM (Windows, Macintosh and UNIX) and a solutions manual (iv+171 pp.).

[16] W.P.A. Klingenberg, Riemannian Geometry, 2nd edn. De Gruyter Studies in Mathematics, Vol. 1. Walter de Gruyter & Co., Berlin (1995).

[17] F.W. Warner, The conjugate locus of a Riemannian manifold. *Am. J. Math.* **87** (1965) 575–604.

[18] L. Rifford and E. Trélat, On the stabilization problem for nonholonomic distributions. *J. Eur. Math. Soc.* **11** (2009) 223–255.

[19] G.W. Stewart, On the continuity of the generalized inverse. *SIAM J. Appl. Math.* **17** (1969) 33–45. MR 245583

[20] A.A. Agrachev and Y.L. Sachkov, Control theory from the geometric viewpoint. *Encyclopaedia of Mathematical Sciences*, Vol. 87. Springer-Verlag, Berlin (2004), Control Theory and Optimization, II.

[21] A. Montanari and D. Morbidelli, On the lack of semiconcavity of the subRiemannian distance in a class of Carnot groups. *J. Math. Anal. Appl.* **444** (2016) 1652–1674.

[22] L. Ambrosio and S. Rigot, Optimal mass transportation in the Heisenberg group. *J. Funct. Anal.* **208** (2004) 261–301.

[23] P. Hajłasz and S. Zimmerman, Geodesics in the Heisenberg group. *Anal. Geom. Metr. Spaces* **3** (2015) 325–337. MR 3417082

[24] R. Monti, Some properties of Carnot–Carathéodory balls in the Heisenberg group. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **11** (2000) 155–167.