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(Article begins on next page)

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The geometry of quantum computing

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In this expository paper, we present a brief introduction to the geometrical modeling of some quantum computing problems. After a brief introduction to establish the terminology, we focus on quantum information geometry and ZX-calculus, establishing a connection between quantum computing questions and quantum groups, i.e. Hopf algebras.

Keywords: Quantum computing; quantum fisher matrix; quantum metrology; quantum circuits.

1. Introduction

The idea of using the laws of quantum mechanics for a new approach to computer algorithms is due to Feyman, who introduced the concept of quantum computing in a seminal talk in 1981 [1]. Feyman himself, a few years later [2], analyzed the concept of universal quantum computer built from the simplest quantum mechanical system (a two-level system or qubit) and making use of elementary quantum gates. So Feynman, for the first time, introduced the idea of a computer, functioning according to *non-binary* i.e. *quantum* logic. Feyman also gave a graphical description of the quantum logic gates, a notation we still use today, as many of Feyman extraordinary ones. Around the same time, Manin [3] suggested a quantum approach to information theory from a mathematical point of view. Later on, Manin enhanced the theory of quantum groups with a new geometric approach [4]. In 1985, Deutsch [5] formalized the notion of quantum computer and was the first

to raise the question of "quantum advantage", which later on inspired Shor towards the formulation of a quantum algorithm for factoring large numbers more efficient than the classical ones [6, 7].

We are unable to properly account for the many discoverings in this fast developing subject, we send the reader to the exhaustive treatments by Preskill [8–10].

In this paper, we want to show how quantum computing, quantum information geometry and quantum groups, in the form of Hopf algebras, can be related, providing new models for interesting quantum computing questions. Moreover, our elementary treatment of the topics hopefully will allow experts in different fields to share their knowledge toward a more unified and sound geometric theory of quantum computing.

Our paper is organized as follows.

In Sec. 2, we establish the terminology introducing the notions of *qubit*, *density operator* in 2.1, and *quantum logic gate* in 2.2, that will be key for our subsequent treatment.

In Sec. 3, we first introduce basic concepts of information geometry, as the *Fisher matrix* in Sec. 3.1, and then we explore their quantum information counterparts in Sec. 3.2, leading to the *Quantum Geometric Tensor* in Sec. 3.3, a natural Kähler metric on the space of qubits.

Finally in Sec. 4, we discuss ZX-calculus as an effective method to describe quantum logic gates and quantum circuits, making use of the Hopf algebra language and leading to a surprising connection with the quantum group theory.

Our goal is to introduce mathematicians and physicists to some theoretical questions arising from the quantum computing world. We are convinced that geometric and theoretical physical modeling can greatly help this fast developing field.

2. An Overview on Quantum Computing

In this section we give the basics of quantum computing theory, establishing a dictionary, which will enable us to state some interesting mathematical questions that we will discuss in the next sections. Our treatment is very terse, we refer the interested read to [10] (and references therein) and to [11].

2.1. *The Qubit*

In a classical computer, the fundamental unit of information is the *bit*, which can assume either 0 or 1 value. In quantum computing, the corresponding notion is the quantum bit or, in short, *qubit*. A qubit differs from a classical bit since its state is represented by any *superposition* of two independent states that correspond to the classical 0 and 1. Mathematically, we represent the state $|\psi\rangle$ of a qubit as the linear combination of two basis states $|0\rangle$ and $|1\rangle$, the *computational basis*, that we identify with the canonical basis of $\mathbf{C}^2$:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1, \quad \alpha,\ \beta \in \mathbf{C} \tag{1}$$

together with a *phase condition* that we detail below. Hence, $|\psi\rangle$ is an element of the Hilbert space $\mathcal{H} = \mathbf{C}^2$, of unit norm. This condition follows from the probabilistic interpretation of the coefficients α and β: the result of a measurement in the computational basis finds the qubit in the state $|0\rangle$ or in the state $|1\rangle$ with probability $|\alpha|^2$ or $|\beta|^2$. Notice that this interpretation does not distinguish between the states $|\psi\rangle$ and $e^{i\gamma}|\psi\rangle$, for any phase γ.

We write the qubit $|\psi\rangle$ using the Dirac ket notation, as customary in quantum mechanics to represent a quantum mechanical state.

Since we pursue a geometrical interpretation, we also think of $|\psi\rangle$ as a complex *line* in the vector space $\mathcal{H}$. Hence in Eq. (1), we need to choose a representative of such line by looking at the intersections of the line with the sphere (the condition $|\alpha|^2 + |\beta|^2 = 1$) and then identifying all points that differ just by a phase factor $e^{i\gamma}$.

For this reason, we represent the qubit $|\psi\rangle$ parametrically as

$$|\psi\rangle = e^{i\gamma}\left[\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right], \tag{2}$$

where we can set the parameter $\gamma = 0$ (phase) to account for the identification described above. So a qubit $|\psi\rangle$ is also equivalently given by the two parameters (φ, θ), or by Eq. (2), together with the phase condition $\gamma = 0$.

We can then view a qubit as an element of a 2-dimensional (real) sphere in $\mathbf{R}^3$, called the *Bloch sphere*, as in Fig. 1. Mathematically this is the *Riemann sphere*, and it is identified with $\mathbf{P}^1(\mathbf{C})$ the projective line, consisting of lines in $\mathbf{C}^2$. Note that, conventionally, the basis vectors $|0\rangle$ and $|1\rangle$ are, respectively, the north and south poles of the Bloch sphere.

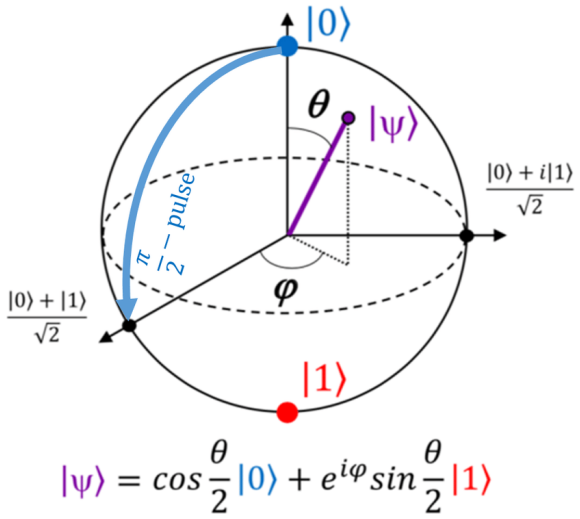


Fig. 1. The Bloch sphere.

Later on, when we introduce quantum information geometry, we will equip the Bloch sphere with a natural metric, the *Fubini-Study metric*, coming from its identification with the complex projective line (Riemann Sphere).

When we measure the qubit, we have that its state $|\psi\rangle$ collapses into one of the basis states $|0\rangle$, $|1\rangle$ with a certain probability. The problem with quantum computing, in comparison with classical computing, is that we cannot measure a state with arbitrary accurate precision, since the act of measuring will affect the state itself. As a consequence of this quantum mechanical phenomenon, the quantum algorithms need to be suitably adapted and are not simple generalizations of the classical ones. Moreover, they are more prone to errors, and the correction of errors leads to *fault tolerant algorithms* that we mention later on in our note, see also [12–14].

Besides the above-mentioned single or 1 qubit (1), we may have a 2-qubit system, whose states are elements of $\mathbf{C}^2 \otimes \mathbf{C}^2$. We represent a 2-qubit system in the basis:

$$|00\rangle, \quad |01\rangle, \quad |10\rangle, \quad |11\rangle$$

called again the *computational basis*, where we use the notation:

$$|ij\rangle := |i\rangle \otimes |j\rangle. \tag{3}$$

More in general, we can speak of an n qubit system in a similar fashion: states are expressed as linear superposition (combination) of the computational basis states:

$$|00\ldots00\rangle, \quad |00\ldots01\rangle \quad |00\ldots10\rangle, \quad \ldots \quad |11\ldots11\rangle. \tag{4}$$

Note that the 2^n states of the computational basis are ordered from $00\ldots0$ to $11\ldots1$ according to the binary representation of the integer i, $1 = 0, 1, \ldots, 2^n - 1$ enumerating them. The space of n qubits is $\mathcal{H} := \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$ (n tensor product). Hence, a state of n qubits is an element of the Hilbert space $\mathcal{H} \cong \mathbf{C}^N$, for $N = 2^n$

$$|\psi\rangle = \alpha_0|00\ldots0\rangle + \alpha_1|00\ldots1\rangle + \cdots + \alpha_{2^n-1}|11\ldots1\rangle, \quad \sum_{i=0}^{2^n-1} |\alpha_i|^2 = 1.$$

The last condition takes into account the above mentioned fact about measurements in quantum mechanics: after a measurement, the state $|\psi\rangle$ is found in the ith state of the computational basis with probability $|\alpha_i|^2$. We should also take into account the phase condition, but we do not write such condition explicitly here.

A vector $\psi \in \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$ describing the state of n qubits is said to be *entangled* if it is not an indecomposable tensor, that is if we cannot write it as $|\psi\rangle = u_1 \otimes \cdots \otimes u_n$, with $u_i \in \mathbf{C}^2$.

The states considered up to this points are called *pure* states, to be distinguished from the case in which a system of n qubits can be described by a *mixed* state, i.e. by a statistical mixture defined by an ensemble of states $\{|\psi_j\rangle\}_{j=1}^m$ that can occur with a probability p_j, with $0 \leq p_j \leq 1$ and $\sum_{j=1}^m p_j = 1$. Clearly, a pure state is a particular example of a mixed state with $p_1 = 1,\ p_2 = \cdots = p_m = 0$.

Both pure and mixed states can be represented by means of the so-called *density operator*

$$\rho = \sum_s p_s |\psi_s\rangle\langle\psi_s|, \tag{5}$$

where $\langle\psi_s| \in \mathcal{H}^*$ is the dual vector of $|\psi_s\rangle$.

It is immediate to verify that ρ is a self-adjoint, (semi)positive definite operator with unit trace. Also, one can prove that it is idempotent (hence it is a projection operator) if and only if the state is pure.

Once an ON (orthonormal) basis is fixed, we can express ρ as a matrix and speak of *density matrix*. With an abuse of notation, we shall use ρ_ψ to refer to both density matrix and operator.

For example, for a 1 qubit system in the pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we have in the computational basis:

$$\rho_\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (\overline{\alpha} \quad \overline{\beta}) = \begin{pmatrix} |\alpha|^2 & \alpha\overline{\beta} \\ \overline{\alpha}\beta & |\beta|^2 \end{pmatrix}$$

while the mixed state defined by the ensemble $\{|0\rangle, |1\rangle\}$ with $p_0 = |\alpha|^2, p_1 = |\beta|^2$ we have:

$$\rho_\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (\overline{\alpha} \quad \overline{\beta}) = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}.$$

2.2. *Quantum logic gates*

In classical computing, we have logic gates operating on bits. For example, the classical logic gate "NOT" operates as follows:

$$\begin{aligned} 0 &\mapsto 1, \\ 1 &\mapsto 0. \end{aligned} \tag{6}$$

Quantum logic gates operate on states which are linear combinations of the computational basis elements, as in (3) for 1 qubit and in (4) for n qubits. In order to preserve probabilities, quantum logic gates are represented by unitary matrices U, i.e. $U^\dagger U = I$.

Let us see examples of logic gates for single qubits; these examples will be especially relevant for the ZX calculus that we will treat more in detail in the last section. We can take advantage of the Pauli matrices X, Y, Z and the operator H called the *Hadamard operator*:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The Hadamard operator transforms the computational basis $|0\rangle$, $|1\rangle$ (which are the eigenvectors of the Z operator) into the *Hadamard basis* corresponding to the

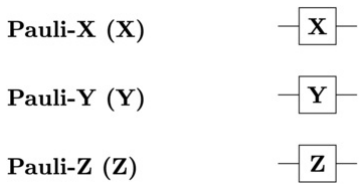


Fig. 2. Quantum logic gates for 1 qubit.

eigenvectors of the X operator. The Hadamard basis is denoted by

$$|+\rangle := H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle := H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

These operators are also given graphically in Fig. 2.

We now want to discuss an example of a quantum logic gate for a 2 qubit system: the CNOT, *control not*. The unitary transformation expressing the CNOT is given in the computational basis by the matrix:

$$U_{\text{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} |00\rangle \mapsto |00\rangle \\ |01\rangle \mapsto |01\rangle \\ |10\rangle \mapsto |11\rangle \\ |11\rangle \mapsto |10\rangle \end{matrix}$$

and is graphically represented as in Fig. 3.

This transformation is called "control not", since it reverses the computational basis state in Fig. 3 represented by $|y\rangle$ (i.e. $y = 0, 1$) if only if the *control state* $|x\rangle = |1\rangle$. Note that the sum $x \oplus y$ means sum modulo 2.

Despite the specificity of the quantum gates described above, we have a beautiful theoretical result, [15], showing that actually single qubit gates and CNOT are all we need to construct the most general quantum logic gates.

Theorem 2.1. *The set of gates that consists of all* 1 *qubit quantum gates* (*namely* $U(2)$ *operators*) *and the* 2 *qubit gate CNOT is universal in the sense that all unitary operations on* n *qubits* (*namely* $U(2^n)$) *can be expressed as compositions of these gates.*

This theorem is very significant, since it shows the universality of the set consisting of unitary single qubit operators and the CNOT: from this set we can then

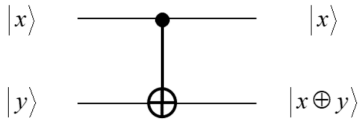


Fig. 3. CNOT logic gate.

get the operators to build any quantum algorithm. Thus, this is the counterpart of the classical observation that with classical NAND gates, we can realize all logic functions: AND, OR, NOT and therefore any classical algorithm.

Still, it is not clear how (and if!) a quantum computer can solve any classical computability question; essentially the obstruction comes from two main reasons. First, because the classical NAND is a non-reversible gate, while quantum gates are necessarily unitary and hence invertible. Then, we have the so-called *No-cloning theorem*, that forbids the existence of a quantum gate that allows to copy a generic state of a qubit.

Theorem 2.2 (No-cloning theorem). *There is no unitary operator U on $\mathcal{H}\otimes\mathcal{H}$ such that for all pure states $|\phi\rangle$ and $|B\rangle$ in $\mathcal{H}$, we have*

$$U(|\phi\rangle \otimes |B\rangle) = e^{i\alpha(\phi,B)}|\phi\rangle \otimes |\phi\rangle$$

for some real number α depending on ϕ and B.

Proof. Assume such U exists, and assume for simplicity $\alpha = 0$. Given the two states $|\phi\rangle$, $|\psi\rangle$ in $\mathcal{H}$, we write

$$U : \mathcal{H}\otimes\mathcal{H} \longrightarrow \mathcal{H}\otimes\mathcal{H}, \quad U(|\phi\rangle \otimes |B\rangle) = |\phi\rangle \otimes |\phi\rangle, \quad U(|\psi\rangle \otimes |B\rangle) = |\psi\rangle \otimes |\psi\rangle.$$

Since U preserves the hermitian product, we have that

$$|\phi\rangle \otimes |B\rangle \cdot |\psi\rangle \otimes |B\rangle = |\phi\rangle \otimes |\phi\rangle \cdot |\psi\rangle \otimes |\psi\rangle$$

which gives the equality:

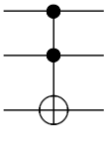
$$\langle\psi|\phi\rangle\langle B|B\rangle = \langle\psi|\phi\rangle\langle\psi|\phi\rangle \implies \langle\psi|\phi\rangle = \langle\psi|\phi\rangle^2$$

yielding either $\langle\psi|\phi\rangle = 0$, i.e. ψ, ϕ orthogonal or $\langle\psi|\phi\rangle = 1$, i.e. $\phi = \psi$. □

We overcome this difficulty by introducing a suitable 3 qubit gate (e.g. the Toffoli gate or CCNOT, see Fig. 4).

Such gate allows to realize both a copy of the state of a qubit and the NAND operation between two qubits, by suitably choosing the entry of the third qubit.

When performing a quantum algorithm on a real physical platform, the interaction of the system with the environment, due to either measurements or noise, can



$$\text{CCNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Fig. 4. CCNOT logic gate.

degrade the information. This differs from what happens in a real classical device because a quantum channel that describes either a measurement or a noisy evolution does not simply flip the state of a qubit in a random way, but can actually transform a pure state into a mixed one, thus resulting in a loss of information. Also, error correction schemes are more difficult to implement because of the No-cloning theorem.

Still, we can develop fault tolerant algorithms to minimize the damage, and we have an important theoretical result, called the *threshold theorem*. This is the analog of the classical Von Neumann theorem, and states that a quantum computer with a physical error rate below a certain threshold can, through application of quantum error correction methods, reduce the error rate to arbitrarily low levels. Hence, we have the hope to create fault-tolerant algorithms and feasible quantum computing in general. We invite the reader to look at the works [12–14] for topological approaches to such algorithms.

3. Classical and Quantum Information Geometry

Information geometry is a new field at the intersection of statistics, probability and geometry, that takes advantage of techniques from differential and Riemannian geometry to study *statistical manifolds*, that is manifolds whose points represent families of probability distributions. We focus on the most important notion of classical Fisher Information Matrix, its quantum counterpart, the Quantum Fisher Matrix, arriving to define the Quantum Geometric Tensor, providing a natural hermitian metric on the space of qubits.

3.1. *The Fisher information matrix*

A central object in Information Geometry is the Fisher information matrix (see [16] for the notation and the references therein).

Definition 3.1. *Let $p(\theta)$ be a discrete empirical probability distribution, depending on parameters $\theta \in \mathbf{R}^n$ (or to a statistical manifold embedded in $\mathbf{R}^n$). We define F the Fisher information matrix or Fisher matrix for short, as*

$$F_{ij}(\theta) = \mathbf{E}_p[\partial_{\theta_i} \log p(\theta) \partial_{\theta_j} \log p(\theta)], \tag{7}$$

where $\mathbf{E}_p$ represents the expected value with respect to the probability distribution p.

We can also express more concisely the Fisher matrix as

$$F(\theta) = \mathbf{E}_p[\nabla \log p(\theta) (\nabla \log p(\theta))^t],$$

where we understand $\nabla \log p(\theta)$ as a column vector and t denotes the transpose.

In his pioneering work [17], Amari gave a geometric interpretation of the Fisher matrix, as a meaningful metric for the statistical manifold and provided a new method of gradient descent based on his considerations. Later on, this treatment inspired the quantum counterparts [18].

It is therefore important to relate, in the spirit of Amari, the Fisher matrix to the information loss of a probability distribution:

$$I(\theta) = -\log(p(\theta))$$

which is also linked to *Shannon entropy* $H(\theta)$ as follows:

$$H(\theta) = -\mathbf{E}_p[I(\theta)] = -\sum p_i(\theta)\log(p_i(\theta))$$

H measures how "diffused" the information in our probability distribution is; if $p(\theta)$ is a probability mass distribution (i.e. $p_i(\theta) = \delta_{ij}$ for a fixed i and $j = 1, \ldots, n$), then $H(\theta) = 0$, while $H(\theta)$ is maximal if $p_j(\theta) = 1/n$.

We now express some well-known facts on the Fisher matrix.

Proposition 3.2. *The Fisher matrix is the covariance matrix of the gradient of the information loss.*

Proof. The gradient of the information loss is

$$\nabla_\theta I(\theta) = -\frac{\nabla_\theta p(\theta)}{p(\theta)}.$$

Note that

$$\mathbf{E}_p(\nabla_\theta I) = \sum p_i \frac{\nabla_\theta p_i}{p_i} = \sum_i \nabla_\theta p_i = \nabla_\theta(\sum_i p_i) = 0.$$

The covariance matrix of $\nabla_\theta I(\theta)$ is (by definition)

$$\begin{aligned}\mathrm{Cov}(I) &= \mathbf{E}_p[(\nabla_\theta I - \mathbf{E}_p(\nabla_\theta I))^t(\nabla_\theta I - \mathbf{E}_p(\nabla_\theta I))] \\ &= \mathbf{E}_p[(\nabla_\theta I)^t(\nabla_\theta I)] = F(\theta).\end{aligned}$$

□

We conclude our brief treatment of Information Geometry by an observation regarding the metric on the parameter space.

Observation 3.3. We observe that the Fisher metric is related to the Kullback–Leibler divergence (see [16] and references therein) that gives the statistical distance between two probability distributions. Indeed, if we consider $\mathrm{KL}(p(\theta + \delta w)||p(\theta))$ that measures how $p(\theta + \delta w)$ and $p(\theta)$ differ, for a small variation of the parameters δw, we have:

$$\mathrm{KL}(p(\theta + \delta w)||p(\theta)) \cong \frac{1}{2}(\delta w)^t F(\theta)(\delta w) + \mathcal{O}(\|\delta w\|^3).$$

This clearly shows how the Fisher can be interpreted as a metric on the statistical manifold.

3.2. *The quantum information matrix*

In quantum metrology, which is the study of physical measurements via quantum theory, the *quantum information matrix* or *quantum Fisher matrix* is the analog of the Fisher matrix, introduced in the previous section. We now give its definition and main properties; for the details, we refer the reader to [19, 20] and references therein.

We start with the definition of *symmetric logarithmic derivative*. This is a generalization of the usual notion of logarithmic derivative and it is necessary in the quantum setting, since the scalar expressing the probability distribution p appearing in the definition Fisher matrix (see (7)) is replaced by the density operator ρ as in (5).

We assume that ρ is depending on a parameter θ; when ρ is defined for pure states (a special, but important case), this parameter comes from a parametrization of the vectors in the Hilbert space.

Definition 3.4. *We define, implicitly, the symmetric logarithmic derivative L of a given density operator ρ as*

$$\partial_\theta \rho = \frac{L\rho + \rho L}{2}. \tag{8}$$

We notice immediately that, whenever L and ρ are scalars, L is the gradient of the logarithm of ρ as it appears in (7), that is $L = \partial_\theta \rho/\rho = \partial_\theta(\log(\rho))$.

We can derive the explicit expression of L, when we fix an ON (orthonormal) basis $\{|\psi_j\rangle\}$ for qubits.

Observation 3.5. Let $\{|\psi_j\rangle\}$ be an ON basis for $\mathcal{H}$. Thanks to the properties of the density matrix ρ, we can consider the ON basis $\{|\psi_j\rangle\}$ of its eigenvectors, with $\rho|\psi_j\rangle = p_j|\psi_j\rangle$, $j = 1, \ldots, \dim(\mathcal{H}) = N$. Here $0 \leq p_j \leq 1, \sum p_j = 1$ and indeed we can assume that only the first s values are different form zero, s being the rank of ρ.

Now, for any $N \times N$ matrix A, we can express the (i, j) entry of A as $A_{ij} = \langle\psi_i|A|\psi_j\rangle$. Applying this observation to (8) and using the definition (5) for ρ, we get

$$\begin{aligned}(\partial_\theta \rho)_{ij} &= \langle\psi_i|\partial_\theta \rho|\psi_j\rangle = \frac{1}{2}\langle\psi_i|(L\rho + \rho L)|\psi_j\rangle \\ &= \frac{1}{2}(\langle\psi_i|Lp_j|\psi_j\rangle + \langle\psi_i|p_i L|\psi_j\rangle) = \frac{p_i + p_j}{2}L_{ij}.\end{aligned}$$

Note that $p_i + p_j = 0$ if both $i, j > s$ so that we can assume

$$L_{ij} = 0, \quad i, j = s, \ldots, N$$

while the other matrix elements of L are given by

$$L_{ij} = \frac{2(\partial_\theta \rho)_{ij}}{p_i + p_j}. \tag{9}$$

Now we can define the quantum information matrix.

Definition 3.6. *Let ρ be a density operator as in (5). We define quantum information matrix as*

$$F_Q = \mathrm{tr}[\rho L^2], \tag{10}$$

where L is the symmetric logarithmic derivative of ρ.

We notice that F_Q is actually an operator, however we adhere to the most common terminology *quantum information matrix*. With an abuse of notation, we write F_Q also for the matrix representing F_Q in a fixed ON basis.

There is a strict analogy between F_Q and the Fisher information matrix F as in (7).

Observation 3.7. Let the notation be as above and let us fix the ON basis of eigenvectors of ρ. Then, we have

$$F_Q = \sum_{k=1}^{s}\sum_{l=1}^{N} p_k L_{kl} L_{lk}. \tag{11}$$

In fact

$$F_Q = \mathrm{tr}[\rho L^2] = \sum_k \langle\psi_k|\rho L^2|\psi_k\rangle = \sum_{k,l}\langle\psi_k|\rho|\psi_l\rangle\langle\psi_l|L^2|\psi_k\rangle$$

$$= \sum_k p_k\langle\psi_k|L^2|\psi_k\rangle = \sum_{k,l} p_k\langle\psi_k|L|\psi_l\rangle\langle\psi_l|L|\psi_k\rangle$$

which is the expression (11). Note that the sum over k is in practice restricted to $k = 1, \ldots, N$ since $p_k = 0$ if $k > s$. As one readily sees, this is the analog of the expression (7) for F.

We now give another useful and interesting expression of F_Q.

Proposition 3.8. *Let the notation be as above. Then*

$$F_Q = \sum_{i=1}^{s}\frac{1}{p_i}(\partial_\theta p_i)^2 + \sum_{i=1}^{s}\sum_{j=1}^{N}\frac{4p_i(p_i-p_j)^2}{(p_i+p_j)^2}|\langle\psi_i|\partial_\theta\psi_j\rangle|^2, \tag{12}$$

where s is the rank of ρ.

Proof. We substitute the expression (9) for L_{ij} in (11)

$$F_Q = \sum_{i=1}^{s}\sum_{j=1}^{N} 4p_i\frac{(\partial_\theta\rho)_{ij}(\partial_\theta\rho)_{ji}}{(p_i+p_j)^2}. \tag{13}$$

From the very definition of ρ, we get

$$(\partial_\theta\rho)_{ij} = \delta_{ij}\partial p_i + (p_j - p_i)\langle\psi_i|\partial_\theta\psi_j\rangle,$$

where we used that

$$0 = \partial_\theta(\langle\psi_i|\psi_j\rangle) = \langle\partial_\theta\psi_i|\psi_j\rangle + \langle\psi_i|\partial_\theta\psi_j\rangle.$$

Note that if $i \neq j$, $(\partial_\theta \rho)_{ij} = -(\partial_\theta \rho)_{ji}$, so that

$$F_Q = \sum_{i=1}^{s} \frac{1}{p_i} (\partial_\theta p_i)^2 + \sum_{i=1}^{s} \sum_{i \neq j, j=1}^{N} \frac{|(\partial_\theta \rho)_{ij}|^2}{(p_i + p_j)^2}.$$

Now, substituting the expression for $(\partial_\theta \rho)_{ij}$ we obtain the result. □

The next observation compares, once again, the classical Fisher information matrix with the quantum F_Q.

Observation 3.9. We notice that the first term of the Quantum Fisher (12) coincides with the Classical Fisher Matrix (7): indeed it is given by changes in the eigenvalues only, i.e. of the probabilities p_k, of the density matrix. Instead, the second term of Eq. (12) arises from changes in the eigenvectors and it is therefore quantum in nature.

3.3. *Quantum geometric tensor*

The quantum geometric tensor is a key object in quantum information geometry and it is strictly related to the quantum information matrix introduced in the previous section.

As we remarked in Sec. 2.1, the space of qubits is identified with the projective space $\mathbf{P}^N(\mathbf{C})$, that is the space of *rays* or lines in $\mathcal{H} = \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$, $N = 2^n$. The projective space is a Kähler manifold, hence it has a natural Kähler metric, called the *Fubini-Study metric*, which is a 2-tensor on the tangent space of $\mathbf{P}^N(\mathbf{C})$. It is a well-known Hermitian metric; if we fix an holomorphic frame, its expression is given by

$$ds^2 := g_{i\bar{j}}\, dz^i\, d\bar{z}^j = \frac{(1 + z_i \bar{z}^i)\, dz_j\, d\bar{z}^j - \bar{z}^j z_i\, dz_j\, d\bar{z}^i}{(1 + z_i \bar{z}^i)^2}, \tag{14}$$

where as customary, we sum over repeated indices.

The Hermitian matrix of the Fubini-–Study metric $(g_{i\bar{j}})$ in this frame is explicitly given by

$$(g_{i\bar{j}}) = \frac{1}{(1 + |\mathbf{z}|^2)^2} \begin{pmatrix} 1 + |\mathbf{z}|^2 - |z_1|^2 & -\bar{z}_1 z_2 & \cdots & -\bar{z}_1 z_n \\ -\bar{z}_2 z_1 & 1 + |\mathbf{z}|^2 - |z_2|^2 & \cdots & -\bar{z}_2 z_n \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{z}_n z_1 & -\bar{z}_n z_2 & \cdots & 1 + |\mathbf{z}|^2 - |z_n|^2 \end{pmatrix}$$

with $|\mathbf{z}| = |z_1|^2 + |z_2|^2 + \cdots + |z^n|^2$. The metric expressed by $(g_{i\bar{j}})$ is clearly invariant under unitary transformations.

Furthermore, we notice that this metric can be derived from a *Kähler potential* K, that is, we can write it as

$$g_{i\bar{j}} = \frac{\partial^2}{\partial z^i\, \partial \bar{z}^j} K, \quad K = \log(1 + z_i \bar{z}^i).$$

We briefly recall some well-known facts regarding Kähler manifolds. First, recall that a real manifold X is Kähler if it is a symplectic manifold (X, ω) with an integrable almost-complex structure J compatible with the symplectic form ω, that is

$$g(u, v) := \omega(u, Jv)$$

is a Riemannian metric on X [21]. Equivalently, a complex manifold X is Kähler if it has an hermitian metric h and a closed 2-form ω given by

$$\omega(u, v) := \operatorname{Im} h(u, v) \,(= \operatorname{Re} h(iu, v)).$$

On a Kähler manifold, we then have the following structures, compatible with each other:

- A complex structure J, i.e. $J^2 = -I$ on the tangent bundle.
- A symplectic (real) structure ω, i.e. a non degenerate closed 2-form.
- A Riemannian metric g, $(g(X, Y) = \omega(X, JY))$.
- An hermitian metric h such that (real and imaginary parts):

$$h = g - i\omega.$$

We are ready for the key definition of Quantum Geometric Tensor.

Definition 3.10. *We define the Quantum Geometric Tensor (QGT) on* $\mathbf{P}^N(\mathbf{C})$ *as the (Kähler) metric*:

$$ds^2 = \frac{\langle d\psi | d\psi \rangle}{\langle \psi | \psi \rangle} - \frac{\langle d\psi | \psi \rangle \, \langle \psi | d\psi \rangle}{\langle \psi | \psi \rangle^2}, \tag{15}$$

where here $|\psi\rangle$ *is not assumed to have length* 1.

We have the following result, which is just a calculation that can be found in Appendix A of Ref. [22].

Proposition 3.11. *The QGT is equivalent to the Fubini-Study hermitian metric.*

We now show that the QGT is strictly related with F_Q the quantum Fisher (see [23] for more details).

Proposition 3.12. *The real part of the Quantum Geometric Tensor is equivalent to the metric defined by the quantum Fisher* F_Q *symmetric operator on rays in* $\mathcal{H}$.

Proof. For pure states $\rho = |\psi\rangle\langle\psi|$, it is easy to check that $L = 2d\rho = 2(|d\psi\rangle\langle\psi| + |\psi\rangle\langle d\psi|)$. From a straightforward calculation it follows that $F_Q = 4Tr[\rho(d\rho)^2] = 4(|d\psi\rangle\langle\psi| + |\psi\rangle\langle d\psi|)$, which reproduces Eq. (15) when the vector has unit norm. □

4. *ZX*-Calculus and Hopf Algebras

In Sec. 2.2, we introduced the quantum logic gates. They are used for *quantum circuits*, which consist in a sequence of quantum logic gates and measurements,

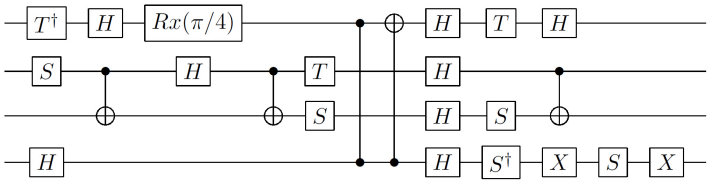


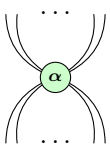
Fig. 5. X, Y, Z are the Pauli matrices gates, H the Hadamard gate. The S gate is $e^{i\pi/2}Z$, the T gate is $e^{i\pi/4}Z$, † as usual indicates the adjoint.

though in the present treatment we shall ignore the latter. Figure 5 represents a quantum circuit.

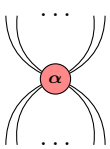
In this section, we want to show how to represent quantum circuits via ZX-diagrams, and how ZX-calculus, originally developed in [24, 25], can be effectively used to understand and optimize simple quantum circuits. Our main references are [26] and the expository treatment [27] (see also the complete list of references in [27]). We shall also explore the intriguing connection between the language of ZX-calculus and quantum groups, as expressed in [26].

4.1. *Quantum circuits and ZX diagrams*

We first establish the conventions that allow us to translate a quantum circuit into a *ZX-diagram*. The main components of a ZX-diagram are green (or Z) and red (or X) *spiders* operating on qubits as unitary operators as follows:



$$: \mathcal{H}_n \to \mathcal{H}_m, \quad \begin{cases} |0\ldots0\rangle & \mapsto |0\ldots0\rangle, \\ |1\ldots1\rangle & \mapsto e^{\mathrm{i}\alpha}|1\ldots1\rangle, \\ \text{others} & \mapsto 0, \end{cases} \tag{16}$$



$$: \mathcal{H}_n \to \mathcal{H}_m, \quad \begin{cases} |+\ldots+\rangle & \mapsto |+\ldots+\rangle, \\ |-\ldots-\rangle & \mapsto e^{\mathrm{i}\alpha}|-\ldots-\rangle, \\ \text{others} & \mapsto 0, \end{cases} \tag{17}$$

where $\mathcal{H}_n = \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$ (n times) and also $n = 0$ or $m = 0$ are allowed. If $\alpha = 0$ we omit the symbol of α.

So, for example, we can establish the correspondence between elementary operations on 1 qubits and ZX-diagrams as in Fig. 6. We will detail the rules involved in the identifications in the next section. Not that customarily the quantum circuits are read from left to right (see Fig. 5), while the ZX-diagrams are depicted from top to bottom. We will follow these conventions.

We further consider the *Hadamard gate*

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

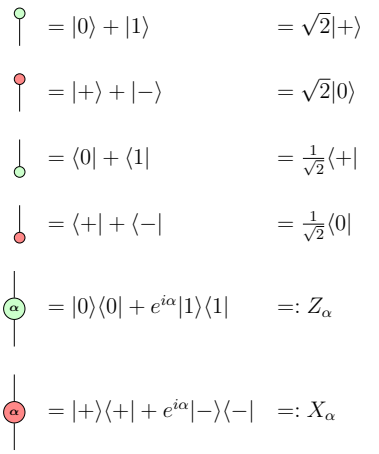


Fig. 6. Examples of elementary 1 qubit operations.

which we represent as a yellow square in the circuit and the *normalization constant* $\Diamond = \sqrt{2}$. Given a quantum circuit, we can then immediately draw the corresponding ZX-diagram.

Vice-versa, given a ZX-diagram, one can write the quantum circuit associated with it (see [28] for a full discussion of such reconstruction). This means that the rules of the ZX-calculus are complete and that we can use them to study fundamental problems such as quantum circuit optimization and or quantum error correction codes.

There are some mathematically obvious rules regarding quantum circuits, that can be translated into operations on the spiders that model them. This originates the ZX-calculus, that we describe below. Such calculus allows to simplify a ZX-diagram and then translate such simpler version to a quantum circuit *equivalent* to the starting one. This allows for a more effective and efficient realization of simple quantum circuits.

4.2. *The ZX-calculus*

The ZX-calculus is a graphical language to describe linear maps $\mathcal{H}_n \to \mathcal{H}_m$ on qubits. For example, the identity $\mathrm{id}_{\mathcal{H}} \colon \mathcal{H} \to \mathcal{H}$, $(\mathcal{H} = \mathbf{C}^2)$, is written as a single string, while the identity $\mathrm{id}_{\mathcal{H}_2} \colon \mathcal{H}_2 \to \mathcal{H}_2$ on $\mathcal{H}_2 = \mathcal{H} \otimes \mathcal{H}$ corresponds to two strings

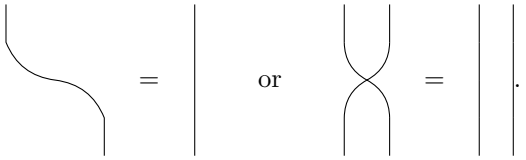
$$\mathrm{id}_{\mathcal{H}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \quad \Big| \quad , \quad \mathrm{id}_{\mathcal{H}^2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \quad \Big| \quad \Big| \quad ,$$

flowing from top to bottom. Some non-trivial operations are given by the "spiders" with an angle $\alpha \in \mathbb{R}$.

We now list some of such operation, forming the backbone of ZX-calculus rules. For this we follow [25, Sec. 2.2].

The T-rule, or, "only topology matters"

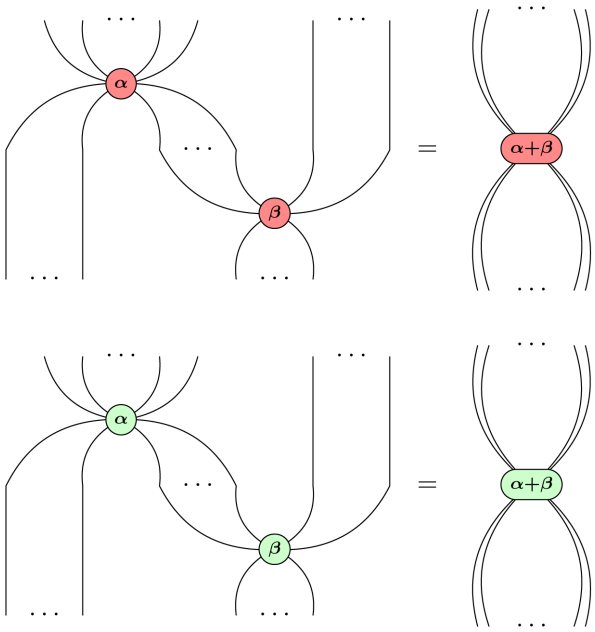
This rule postulates that we can straighten, stretch, bend or twist wires without changing the linear operation represented by the ZX-diagram. For example



It is crucial that the wires are not detached from their entry or exit points, while their position can be changed, also in relation to other wires. This also applies to points of connection with green and red vertices or Hadamard gates. In particular, the number of incoming and outgoing wires are invariants under the **T**-rule.

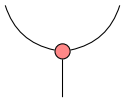
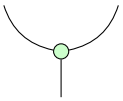
The S-rules

In a nutshell, the **S**-rules entail that all operations involving only red or green vertices correspond to red or green spiders, where only the incoming and outgoing wires matter and we sum the phases.



As a consequence of the **T** and **S** rules it follows that $\mathcal{H}$ inherits two algebra and two coalgebra structures.

Corollary 4.1. *The* 1*-qubit space* $\mathcal{H}$ *is an algebra, where*

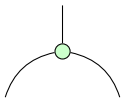
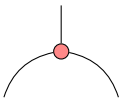
$m_\bullet =$  *and* $m_\bullet =$ 

are associative multiplication operations with unit operations

$\eta_\bullet =$ *and* $\eta_\bullet =$,

respectively.

Moreover, $\mathcal{H}$ *is a coalgebra, where*

$\Delta_\bullet :=$  *and* $\Delta_\bullet :=$ 

are coassociative comultiplication operations with counit operations

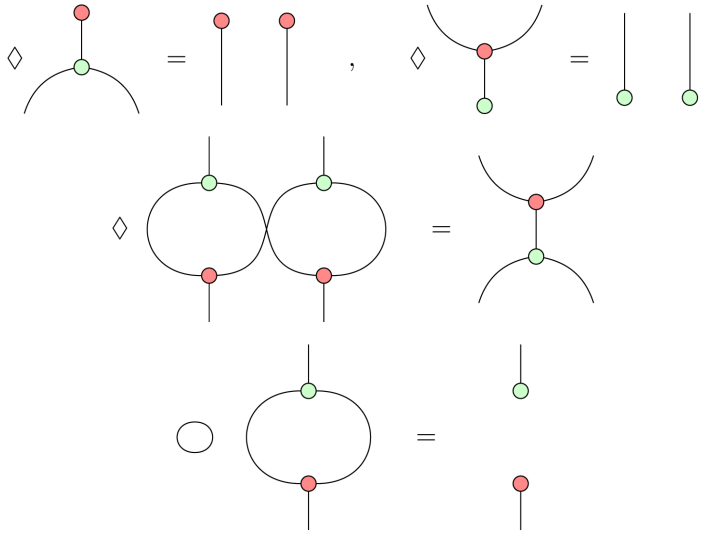
$\varepsilon_\bullet :=$ *and* $\varepsilon_\bullet :=$,

respectively.

The corresponding (co)algebra axioms, detailed in Appendix A.1, are in fact special cases of the **S**-rules.

The B-rules

As a consequence of the previous **S**-rules, we have constructed an associative unital algebra structure $(m_\bullet, \eta_\bullet)$ and a coassociative counital coalgebra structure $(\Delta_\bullet, \varepsilon_\bullet)$ on $\mathcal{H}$ (see Corollary 4.1). The **B**-rules will relate such structures with each other to realize an unnormalized Hopf algebra (see Appendix A.1). Let us first state the rules, that are manifest, once the significance of diagrams is translated into quantum circuits.



Proposition 4.2 ([25, Sec. 2.2.3]). *Let the notation be as above. Then* $(\mathcal{H}, m_\bullet, \eta_\bullet, \Delta_\bullet, \varepsilon_\bullet)$ *is an unnormalized Hopf algebra with trivial antipode.*

Proof. The first diagram above states that $\Delta_\bullet$ is unital with respect to $\eta_\bullet$ up to the normalization constant $\Diamond$, while the second diagram states that $m_\bullet$ is counital with respect to $\varepsilon_\bullet$ up to the same normalization constant $\Diamond$. The third diagram states that $\Delta_\bullet$ respects the product $m_\bullet$ (up to the constant $\Diamond$). In this situation, the antipode axiom (A.1) with respect to the trivial antipode is automatically satisfied, yielding an unnormalized Hopf algebra. □

Thus, the previous proposition gives an unexpected connection between quantum computing and Hopf algebra (quantum group) theory. We continue to describe the algebraic structure in more detail. One can show that also $\Delta_\bullet$ respects the product $m_\bullet$ (up to the same normalization constant $\Diamond$), which implies that also $(m_\bullet, \eta_\bullet, \Delta_\bullet, \varepsilon_\bullet)$ is an unnormalized Hopf algebra. A natural question is whether $(m_\bullet, \eta_\bullet, \Delta_\bullet, \varepsilon_\bullet)$ or $(m_\bullet, \eta_\bullet, \Delta_\bullet, \varepsilon_\bullet)$ are unnormalized Hopf algebras. This is not the case in general (in fact, only in trivial situations). Instead, the green (or red) algebra and coalgebra together form an *F-algebra*, with compatibility condition discussed in Definition A.2. Together with Proposition 4.2, this is the data of an F-Hopf algebra, see Appendix A.2.

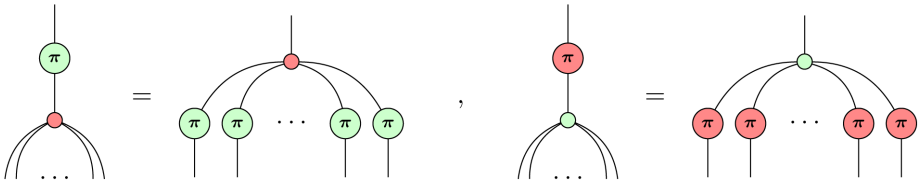
Proposition 4.3 ([24]). *The pair*

$$(A, m_\bullet, \eta_\bullet, \Delta_\bullet, \varepsilon_\bullet), \quad (A, m_\bullet, \eta_\bullet, \Delta_\bullet, \varepsilon_\bullet)$$

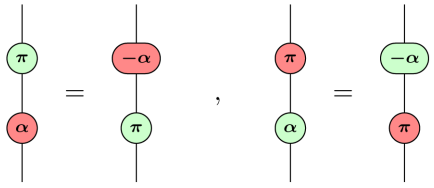
of F-algebras forms an F-Hopf algebra.

The K-rules

The vertices with phase π are "classical", in the sense that they can be copied according to the following rules:



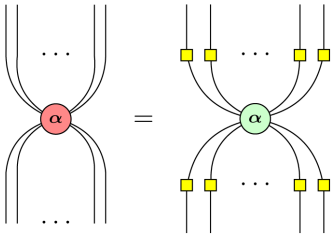
Furthermore, the vertices of phase π can be used to "invert" the phase of the other color, namely



These are known as the **K**-rules, where "K" stands for classical (*klassisch* in German).

The C-rules

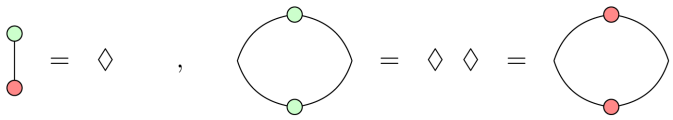
The **C**-rules regard *color*. We can use the Hadamard gate to transform a green vertex into a red one. Explicitly,



In particular, the composition of two Hadamard gates yields the identity operation.

The D-rules

Finally, the **D**-rules determine the normalization constant as the outcome of the following closed circuits:



We show how to apply the rules of the ZX calculus in order to simplify quantum circuits via the two examples in Figs. 7 and 8. Many more can be found in [24, 25].

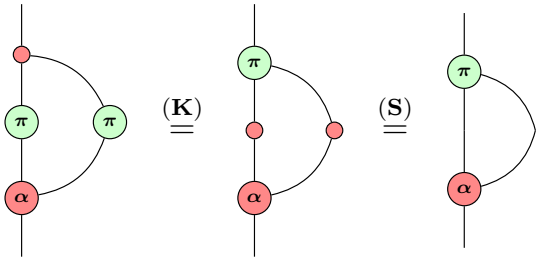


Fig. 7. Simplifying a ZX diagram.

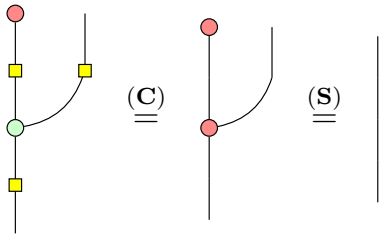


Fig. 8. Simplifying another ZX diagram.

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Appendix A. Unnormalized and F-Hopf Algebras

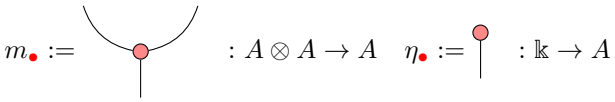
In this appendix, we recall some formal notion regarding unnormalized Hopf algebras and F-algebras. For more details on Hopf algebras, see [29–31].

A.1. *Unnormalized Hopf algebras*

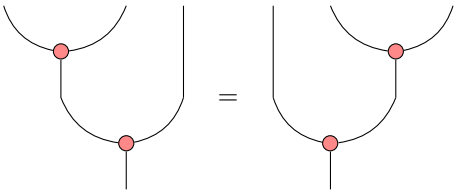
We recall the notion of an unnormalized Hopf algebra over a field $\Bbbk$. The axioms are presented in a graphical language, with string diagrams to be read from top to bottom.

An *unnormalized Hopf algebra* is a $\Bbbk$-vector space A, together with

(i) a multiplication and unit

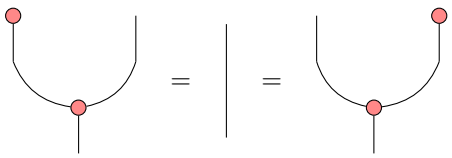


satisfying



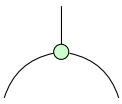
Associativity

and

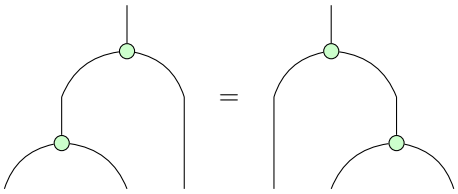


Unitality

(ii) a comultiplication and counit

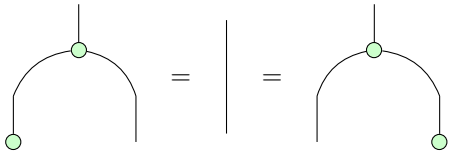
$\Delta_\bullet :=$  $: A \to A \otimes A$ $\qquad$ $\varepsilon_\bullet :=$ $: A \to \Bbbk$

satisfying



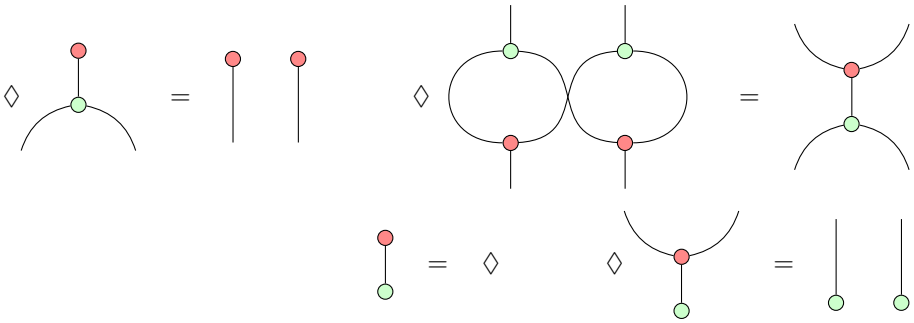
Coassociativity

and

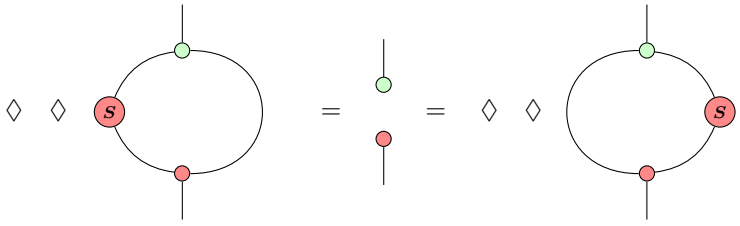


Counitality

(iii) a normalization constant $\Diamond \in \Bbbk \backslash \{0\}$ such that Δ and ε are unnormalized algebra morphisms



(iv) and a linear map $S :=$ $: A \to A$ satisfying the antipode axioms



(A.1)

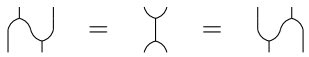
The usual notion of (normalized) Hopf algebra is recovered by setting $\Diamond = 1$. It is easy to see that, given an unnormalized Hopf algebra $(A, m_\bullet, \eta_\bullet, \Delta_\bullet, \varepsilon_\bullet, S)$, we

obtain a normalized Hopf algebra $(A, m_\bullet, \eta_\bullet, \Delta'_\bullet, \varepsilon'_\bullet, S')$ by setting $\Delta'_\bullet := \Diamond\Delta_\bullet$, $\varepsilon'_\bullet := \Diamond^{-1}\varepsilon_\bullet$ and $S' := \Diamond^{-2}S$.

Example A.1. Let G be a group with neutral element $e \in G$. Then the group algebra $\Bbbk G$ becomes a (normalized) Hopf algebra with algebra structure $m_\bullet(g \otimes h) := gh$, $\eta_\bullet(1_\Bbbk) := e$, coalgebra structure $\Delta_\bullet(g) = g \otimes g$, $\varepsilon_\bullet(g) = 1$ and antipode $S(g) = g^{-1}$ defined for all $g, h \in G$ and extended $\Bbbk$-linearly to $\Bbbk G$.

A.2. *F-Hopf algebras*

Definition A.2. *A vector space A is called F-algebra if there is an associative unital algebra structure $(m = \curlyvee, \eta)$ on A and a coassociative counital coalgebra structure $(\Delta = \curlywedge, \varepsilon)$ on A such that*



hold.

Definition A.3. *A pair*

$$(A, m_\bullet, \eta_\bullet, \Delta_\bullet, \varepsilon_\bullet), \quad (A, m_\bullet, \eta_\bullet, \Delta_\bullet, \varepsilon_\bullet)$$

of F-algebras on the same vector space A is called an F-bialgebra if

$$(A, m_\bullet, \eta_\bullet, \Delta_\bullet, \varepsilon_\bullet), \quad (A, m_\bullet, \eta_\bullet, \Delta_\bullet, \varepsilon_\bullet)$$

are bialgebras. The pair is called an F-Hopf algebra if the latter are Hopf algebras.

Example A.4. Let G be a finite group with neutral element $e \in G$. Then $A = \Bbbk G$ is an F-Hopf algebra with

$$m_\bullet(g \otimes h) = gh, \quad \eta_\bullet(1_\Bbbk) = e, \quad \Delta_\bullet(g) = \sum_{h,l \in G, hl = g} h \otimes l, \quad \varepsilon_\bullet(g) = \delta_{e,g}$$

and

$$m_\bullet(g \otimes h) = \delta_{g,h} g, \quad \eta_\bullet(1_\Bbbk) = \sum_{g \in G} g, \quad \Delta_\bullet(g) = g \otimes g, \quad \varepsilon_\bullet(g) = 1.$$

Namely, $(A, m_\bullet, \eta_\bullet, \Delta_\bullet, \varepsilon_\bullet)$ is the Hopf algebra from Example A.1 with antipode $S(g) = g^{-1}$, while $(A, m_\bullet, \eta_\bullet, \Delta_\bullet, \varepsilon_\bullet)$ is a bialgebra because, using the notation $m_\bullet(g \otimes h) =: g \bullet h$,

$$\begin{aligned} \Delta_\bullet(g) \bullet \Delta_\bullet(h) &= \sum_{k,k',n,n' \in G, kn = g, k'n' = h} k \bullet k' \otimes l \bullet l' \\ &= \sum_{k,l \in G, kl = g} k \otimes l \\ &= \Delta_\bullet(g \bullet h) \end{aligned}$$

and

$$\varepsilon_\bullet(g)\varepsilon_\bullet(h) = \delta_{e,g}\delta_{e,h} = \delta_{e,g}\delta_{g,h} = \varepsilon_\bullet(g \bullet h)$$

follow. The latter is endowed with the antipode $S(g) = g^{-1}$.

The interesting example of quantum $sl(2)$ can be found in [26, Example 2.11].

References

[1] R. P. Feynman, Simulating physics with computers, *Int. J. Theor. Phys.* **21** (1982) 467–488.

[2] R. P. Feynman, Quantum mechanical computers, *Found. Phys.* **16**(6) (1986) 507–531.

[3] Y. Manin, *Computable and Uncomputable* (Sovetskoye Radio, Moscow, 1980).

[4] Y. Manin, *Topics in Noncommutative Geometry*, M. B. Porter Lectures, (Princeton University Press, Princeton, NJ, 1991).

[5] D. Deutsch, Quantum theory, the Church–Turing principle and the universal quantum computer, *Proc. R. Soc. Lond. A Math. Phys. Sci.* **400**(1818) (1985) 97–117.

[6] P. W. Shor, Algorithms for quantum computation, discrete logarithms and factorizing, in *Proc. 35th Annual Symp. Foundations of Computer Science* (IEEE, 1994).

[7] P. W. Shor, Scheme for reducing decoherence in quantum computer memory, *Phys. Rev. A* **52**(4) (1995) R2493.

[8] J. Preskill, Reliable quantum computers, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **454**(1969) (1998) 385–410.

[9] J. Preskill, Fault-tolerant quantum computation, in *Introduction to Quantum Computation and Information* (World Scientific, 1998), pp. 213–269.

[10] J. Preskill, *Lecture Notes for Physics 229*: *Quantum Information and Computation* (Barron, 2015).

[11] A. Ashtekar and T. A. Schilling, Geometrical formulation of quantum mechanics, in *On Einstein's Path*, ed. A. Harvey (Springer, New York, 1999).

[12] A. Y. Kitaev, Fault-tolerant quantum computation by anyons, *Ann. Phys.* **303**(1) (2003) 2–30.

[13] A. Y. Kitaev, Periodic table for topological insulators and superconductors, *AIP Conf. Proc.* **1134** (2009) 22.

[14] E. Dennis, A. Kitaev, A. Landahl and J. Preskill, Topological quantum memory, *J. Math. Phys.* **43**(9) (2002) 4452–4505.

[15] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. Smolin and H. Weinfurther, Elementary gates for quantum computation, *Phys. Rev. A* **52**(5) (1995) 3457–3467.

[16] R. Fioresi and F. Zanchetta, Deep learning and geometric deep learning: An introduction for mathematicians and physicists, *Int. J. Geom. Methods Mod. Phys.* **20**(12) (2023) 2330006.

[17] S.-I. Amari, Natural gradient works efficiently in learning, *Neural Comput.* **10**(2) (1998) 251–276.

[18] J. Stokes, J. Izaac, N. Killoran and G. Carleo, Quantum natural gradient, *Quantum* **4** (2020) 269.

[19] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, 1976).

[20] J. Liu, X. Jing, W. Zhong and X. Wang, Quantum Fisher information for density matrices with arbitrary ranks, *Commun. Theor. Phys.* **61**(1) (2014) 45–50.

[21] D. Huybrechts, *Complex Geometry: An Introduction* (Springer GTM, 2005).

[22] X.-Y. Hou, Z. Zhou, X. Wang, H. Guo and C.-C. Chien, Local geometry and quantum geometric tensor of mixed states, Preprint arXiv: 2305.07597.

[23] P. Facchi, R. Kulkarni, V. I. Manko, G. Marmo, E. C. G. Sudarshan and F. Ventriglia, Classical and quantum Fisher information in the geometrical formulation of quantum mechanics, *Phys. Lett. A* **374** (2010) 4801–4803.

[24] B. Coecke and R. Duncan, Interacting quantum observables, in *ICALP 2008*: *Automata*, *Languages and Programming*, Lecture Notes in Computer Science, Vol. 5126 (Springer, Berlin, Heidelberg. 2008), pp. 298–310.
[25] B. Coecke and R. Duncan, Interacting quantum observables: categorical algebra and diagrammatics, *New J. Phys.* **13** (2011) 043016.
[26] S. Majid, Quantum and braided ZX calculus, *J. Phys. A: Math. Theor.* **55** (2022) 254007.
[27] J. van de Wetering, ZX-calculus for the working quantum computer scientist, (2020), https://api.semanticscholar.org/CorpusID:229680014.
[28] N. de Beaudrap, A. Kissinger and J. van de Wetering, Circuit extraction for ZX-diagrams can be # P-hard, in *49th Int. Colloquium on Automata, Languages, and Programming* (ICALP 2022), Leibniz International Proceedings in Informatics, Vol. 229 (Schloss Dagstuhl - Leibniz-Zentrum fr Informatik, 2022), pp. 119:1–119:19.
[29] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Lecture Notes, Vol. 82 (American Math Society, Providence, RI, 1993).
[30] C. Kassel, *Quantum Groups* (Springer-Verlag, GTM, 1995).
[31] S. Majid, *Foundations of Quantum Groups Theory* (Cambridge University Press, 1995).