




Differential Calculi on Quantum Principal Bundles Over Projective Bases

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Abstract: We propose a sheaf-theoretic approach to the theory of differential calculi on quantum principal bundles over non-affine bases. After recalling the affine case we define differential calculi on sheaves of comodule algebras as sheaves of covariant bimodules together with a morphism of sheaves -the differential- such that the Leibniz rule and surjectivity hold locally. The main class of examples is given by covariant calculi over quantum flag manifolds, which we provide via an explicit Ore extension construction. In a second step we introduce principal covariant calculi by requiring a local compatibility of the calculi on the total sheaf, base sheaf and the structure Hopf algebra in terms of exact sequences. In this case Hopf–Galois extensions of algebras lift to Hopf–Galois extensions of exterior algebras with compatible differentials. In particular, the examples of principal (covariant) calculi on the quantum principal bundles $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ and $\mathcal{O}_q(\mathrm{GL}_2(\mathbb{C}))$ over the projective space $\mathbb{P}^1(\mathbb{C})$ are discussed in detail.

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1. Introduction

While in classical differential geometry there is a canonical (functorial) construction of the differential calculus on a differentiable manifold, a main feature of noncommutative differential geometry is the non uniqueness of the calculus. Even on quantum groups and requiring (co)invariance conditions there are interesting different calculi [47]. In this perspective, the definition of a noncommutative geometry requires also a characterization of its differential calculus structure. Going beyond quantum groups we have quantum homogeneous spaces and quantum principal bundles. Their study has been very active in the last three decades when the base space is a noncommutative deformation of an affine variety. In this case, as usual in noncommutative geometry, the affine geometric objects are replaced by a deformation of the algebra of functions on them. The case of bundles on projective varieties is intrinsically more difficult since it requires patching affine opens. In noncommutative geometry, when it is necessary to go beyond the affine setting, we must take the ringed space approach as expressed by Grothendieck: *to do geometry one does not need the space itself, but only the category of sheaves on that would-be space* (see [18] and also [44] with refs. therein for more insight on this point of view).

In this paper we propose an approach to noncommutative differential geometry which employs the sheaf-theoretic language developed in [4]. This does not only cover the established affine theory but also allows to consider differential structures for quantum principal bundles over non-affine bases.

Important steps towards a sound foundation of noncommutative algebraic geometry were taken in the works [1, 27, 38, 45]. In [1], quantum projective schemes are understood as the category of modules associated with a given graded noncommutative algebra, while in [38], an affine quantum space is viewed as the spectrum of a noncommutative ring and quantum projective schemes are defined accordingly with a gluing procedure in analogy with the classical Proj construction. Indeed this is our perspective on non-affine noncommutative spaces: we shall define, following [4], a quantum space as a locally ringed space and then build quantum principal bundles over quantum spaces as sheaves with suitable (co)invariant properties with respect to a fixed quantum structure group.

While we have extensive studies on quantum principal bundles in the affine setting (see e.g. [6, 7, 12, 19]), the more general non-affine case (considered e.g. in [43]) lacks a comprehensive and exhaustive treatment. In [4] we study this more general setting combining the sheaf approach to principal bundles with affine bases of [35] (see also [11] and [3]) with the quantum projective homogeneous spaces construction of [10, 16], where a graded quantum ring is built out of the key notion of *quantum section*, associated to a quantum line bundle, with a gluing procedure similar to the classical Proj one in [38].

In particular, we study the important example of quantum principal bundles on quantum flags of algebraic groups, the total space algebra being a quantum group.

More specifically, the main construction is based on quantizations of the classical principal bundle $G \rightarrow G/P$, where G is a semisimple complex algebraic group and P a parabolic subgroup, so that G/P is a projective variety. In this case the P -invariant ring $\mathcal{O}(G/P)$, i.e., the $\mathcal{O}(P)$ -coaction invariant (for short coinvariant) ring $\mathcal{O}(G/P)$ is trivially \mathbb{C} and is then replaced by the homogeneous coordinate ring $\tilde{\mathcal{O}}(G/P)$ of G/P with respect to a chosen projective embedding, associated with a very ample line bundle \mathcal{L} . This line bundle \mathcal{L} can be recovered more algebraically via a character χ of P ; the corresponding sections are the $\mathcal{O}(P)$ -semi-coinvariant elements of $\mathcal{O}(G)$ with respect to χ and generate the homogeneous coordinate ring $\tilde{\mathcal{O}}(G/P)$ of G/P . The relation between this latter and the structure sheaf $\mathcal{O}_{G/P}$ of G/P is then as usual by considering projective localizations (zero degree subalgebras of the localizations) of $\tilde{\mathcal{O}}(G/P)$.

Let $\mathcal{O}_q(G)$ be a quantum deformation of G and $\mathcal{O}_q(P)$ a quantum deformation of a parabolic subgroup; both $\mathcal{O}_q(G)$ and $\mathcal{O}_q(P)$ are Hopf algebras (see [10] Sect. 3 and [4]). Via the datum of a quantum section $s \in \mathcal{O}_q(G)$, quantum version of the lift to $\mathcal{O}(G)$ of the character χ of P defining the line bundle \mathcal{L} , we obtain a quantization of \mathcal{L} . Indeed, through the quantum section we define a graded algebra $\tilde{\mathcal{O}}_q(G/P)$ (quantum homogeneous coordinate ring): each graded component consists of the elements of $\mathcal{O}_q(G)$, which are not coinvariant, but transform according to powers of the projection of s on $\mathcal{O}_q(P)$. Furthermore, the quantum section s allows to define two sheaves \mathcal{F}_G and \mathcal{O}_M on $M = G/P$ and in [4] we prove that \mathcal{F}_G is a sheaf of $\mathcal{O}_q(P)$ -comodule algebras on the quantum ringed space (M, \mathcal{O}_M) . Given the locally principal comodule algebra property (or faithfully flatness property) of \mathcal{F}_G , then this is a quantum principal bundle. In particular the (coproduct of the) quantum section determines an open cover $\{V_i\}$ of G . The projected open cover $\{U_i := p(V_i)\}$ of M (where $p : G \rightarrow M = G/P$) then lead to the intersections $U_I := U_{i_1} \cap \dots \cap U_{i_r}$, $I = (i_1, \dots, i_r)$ which form the basis \mathcal{B} for the topology used to define both the sheaves \mathcal{F}_G and \mathcal{O}_M .

In the present paper we develop a theory of differential calculi on sheaves which is suited but not limited to affine and non-affine examples such as quantum flags for quantum algebraic groups. In particular we construct first order differential calculi on the quantum principal bundles \mathcal{F}_G that are canonically given once a calculus on $\mathcal{O}_q(G)$ is chosen. For a different approach to quantum differential calculi on quantum flags we refer to [21, 22, 34].

We start with the discussion of first order differential calculus in the affine setting, that is, when the base and total spaces are affine. Our focus is on the construction and induction of differential structures: via tensor products, algebra homomorphisms and quotients. This also on the level of covariant and bicovariant first order differential calculi. In particular, we recall the construction of covariant calculi on smash product algebras from [36], where a calculus on a module algebra and a bicovariant calculus on the structure Hopf algebra shape the so-called smash product calculus. This affine framework has been fruitfully investigated in a series of works, including [7, 13, 25, 29, 33, 36].

We conclude our discussion of the affine setting with a treatment of principal covariant calculi. Associated with a commutative principal bundle we canonically have the exact sequence of horizontal forms into forms on the total space onto vertical forms. Even for principal comodule algebras (faithfully flat Hopf–Galois extensions) exactness of this sequence in the noncommutative case is an extra requirement. It is known that the exterior algebra Ω_H^\bullet of a bicovariant calculus on a quantum group H is a graded Hopf algebra. We here for simplicity study the quotient algebra $\Omega_H^{\leq 1} = H \oplus \Gamma_H$ where forms

of degree 2 and higher are set to zero. Similarly we can study when the (truncated) exterior algebra $\Omega_A^{\leq 1} = A \oplus \Gamma_A$ is a $\Omega_H^{\leq 1}$ -comodule algebra, and we can study the induced module of one forms Γ_B on the subalgebra $B = A^{\text{co}H} \subseteq A$ of coinvariants. It was shown in [39, 40] that exactness of the sequence of modules

$$0 \rightarrow A \otimes_B \Gamma_B \rightarrow \Gamma_A \rightarrow A \square_H \Gamma_H \rightarrow 0$$

(where the cotensor product $A \square_H \Gamma_H$ equals $A \otimes^{\text{co}H} \Gamma_H$, with ${}^{\text{co}H} \Gamma_H$ the module of left coinvariant one-forms on the quantum group H) is equivalent to principality of the graded Hopf–Galois extension $(\Omega_A^{\leq 1})^{\text{co}\Omega_H^{\leq 1}} \subseteq \Omega_A^{\leq 1}$. We revisit these results considering differential graded algebras, where the above sequence becomes that of noncommutative horizontal forms, forms on A , vertical forms. The sequence then defines a principal covariant calculus on A . We show that these calculi correspond to graded Hopf–Galois extensions where the $\Omega_H^{\leq 1}$ -coaction is differentiable (compatible with the differentials on A and on $A \otimes H$). The main examples of differential calculi on sheaves we shall encounter satisfy such an exact sequence locally. In this section dedicated to the affine setting we further clarify the relation of principal covariant calculi to quantum principal bundles as pioneered in [7].

Building on the affine case results we proceed to introduce the sheaf approach to first order differential calculi. Namely, we define a first order differential calculus (Υ, d) on a sheaf \mathcal{F} of comodule algebras as a sheaf Υ of covariant \mathcal{F} -bimodules together with a morphism $d : \mathcal{F} \rightarrow \Upsilon$ of sheaves of comodules. We demand the Leibniz rule and surjectivity of the \mathcal{F} -linear span of the differential only locally on stalks. These are the two characterizing properties of a first order differential calculus, and their holding true on stalks goes along with the property that stalks of a quantum principal bundle are principal comodule algebras.

The following result, proven in Theorem 4.16, shows that the construction of the sheaf \mathcal{F}_G from the Hopf algebra $\mathcal{O}_q(G)$ developed in [4] (with G a semisimple complex algebraic group, P a parabolic subgroup) extends to the level of first order differential calculi.

Theorem 1.1. *Let (Γ, d) be a right covariant first order differential calculus on the Hopf algebra $\mathcal{O}_q(G)$.*

(i) *The assignment*

$$\Upsilon_G : U_I \mapsto \Upsilon_G(U_I) := \mathcal{F}_G(U_I) \otimes_{\mathcal{O}_q(G)} \Gamma \otimes_{\mathcal{O}_q(G)} \mathcal{F}_G(U_I)$$

with $d : U_I \mapsto (d_I : \mathcal{F}(U_I) \rightarrow \Upsilon_G(U_I))$ induced by Ore extensions, defines a right $\mathcal{O}_q(P)$ -covariant first order differential calculus on the sheaf \mathcal{F}_G .

(ii) *The first order differential calculus (Υ_G, d) on the sheaf \mathcal{F}_G induces a first order differential calculus (Υ_M, d_M) on the sheaf $\mathcal{O}_M = \mathcal{F}_G^{\text{co}\mathcal{O}_q(P)}$.*

(iii) *If \mathcal{F}_G is a quantum principal bundle, the sheaf of base forms is isomorphic, as a sheaf of \mathcal{O}_M -bimodules, to the intersection of that of horizontal and $\mathcal{O}_q(P)$ -coinvariant forms: $\Upsilon_M \cong \Upsilon_G^{\text{co}\mathcal{O}_q(P)} \cap \Upsilon_G^{\text{hor}}$.*

For the sheaf \mathcal{F}_G , which is obtained from the Hopf algebra $\mathcal{O}_q(G)$, the definition of the calculus on $\mathcal{O}_q(G)$ uniquely determines the quotient calculus on $\mathcal{O}_q(P)$, and from Theorem 4.16, the Ore extended calculus on \mathcal{F}_G and that on the coinvariant subsheaf $\mathcal{O}_M = \mathcal{F}_G^{\text{co}\mathcal{O}_q(P)}$. Thus, despite non uniqueness of the noncommutative differential calculus, once defined the calculus on $\mathcal{O}_q(G)$ the other calculi are canonically obtained.

We next proceed to extend our sheaf-theoretic treatment to the notion of principal covariant calculus on a quantum principal bundle, which accounts for compatibility of the calculus (Υ, d) on the total sheaf \mathcal{F} , the calculus (Υ_M, d_M) on the base sheaf \mathcal{O}_M and a bicovariant calculus (Γ_H, d_H) on the structure Hopf algebra H . For a principal covariant calculus we demand the sequence of stalks

$$0 \rightarrow \mathcal{F}_p \otimes_{(\mathcal{O}_M)_p} (\Upsilon_M)_p \rightarrow \Upsilon_p \longrightarrow \mathcal{F}_p \square_H \Gamma_H \rightarrow 0 \tag{1.1}$$

to be exact for every $p \in M$, and the calculus on H to be bicovariant, cf. equation (4.14).

The rest of the article is devoted to providing three explicit examples of this canonical construction. We study calculi on the quantum principal bundles $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ and $\mathcal{O}_q(\mathrm{GL}_2(\mathbb{C}))$, both over $\mathbf{P}^1(\mathbb{C})$. The first example is based on a 4D bicovariant differential calculus on $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$. The associated sequence (1.1) is well-defined but not exact. Despite the base sheaf $\mathcal{O}_{\mathbf{P}^1(\mathbb{C})}$ being commutative the resulting calculus is noncommutative (the bimodule of one forms is noncommutative).

The second example is based on a 3D calculus on $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ and the associated sequence is exact but not a principal covariant sequence since the induced calculus on the quantum parabolic subgroup $\mathcal{O}_q(\mathrm{P})$ ($\mathrm{P} \subseteq \mathrm{SL}_2$ being the Borel subgroup of upper triangular matrices) is not bicovariant.

The third example has total space algebra $\mathcal{O}_q(\mathrm{GL}_2(\mathbb{C}))$ with a 4D bicovariant calculus. This induces a principal covariant calculus on the quantum principal bundle $\mathcal{F}_{\mathrm{GL}_q}$ over the base sheaf $\mathcal{O}_{\mathbf{P}^1(\mathbb{C})}$. Even though the sheaf $\mathcal{F}_{\mathrm{GL}_q}$ is locally trivial, i.e., on an open cover $\{U_i\}$ of $\mathbf{P}^1(\mathbb{C})$ the algebras $\mathcal{F}_{\mathrm{GL}_q}(U_i)$ are isomorphic to smash product algebras, the associated smash product calculi do not give a first order differential calculus on the sheaf $\mathcal{F}_{\mathrm{GL}_q}$. This is an instance where, while the sheaf defining the quantum principal bundle is locally trivial, the sheaf of one forms is not.

2. Preliminary Concepts

In this section we establish our notation and provide the key results we will need in the sequel. Though most of the material of this section is known, we include it by completeness, so that the reader has readily available the results in the form we need them later on.

2.1. Notation on modules and comodules. Let \mathbb{k} be a commutative unital ring (in later sections specialized to $\mathbb{C}[q, q^{-1}]$ and to a field) and let \otimes be the tensor product of \mathbb{k} -modules. Algebras over \mathbb{k} will be associative and unital. Let H be a Hopf algebra over \mathbb{k} . Throughout this paper we denote the coproduct of H by Δ and its counit by ϵ . Recall that those are algebra homomorphisms $\Delta : H \rightarrow H \otimes H$ and $\epsilon : H \rightarrow \mathbb{k}$ such that $(\Delta \otimes \mathrm{id}_H) \circ \Delta = (\mathrm{id}_H \otimes \Delta) \circ \Delta$ (coassociativity) and $(\epsilon \otimes \mathrm{id}_H) \circ \Delta = (\mathrm{id}_H \otimes \epsilon) \circ \Delta = \mathrm{id}_H$ (counitality) hold. We employ Sweedler’s notation $\Delta(h) = h_1 \otimes h_2$ for the coproduct of an element $h \in H$, i.e. we omit summation symbols and summation indices. By the coassociativity we write

$$h_1 \otimes h_2 \otimes h_3 := (\Delta \otimes \mathrm{id}_H)(\Delta(h)) = (\mathrm{id}_H \otimes \Delta)(\Delta(h))$$

for all $h \in H$ and similarly for higher coproducts. The antipode $S : H \rightarrow H$ of H is an anti-bialgebra homomorphism, namely $S(hh') = S(h')S(h)$, $S(1) = 1$, $\Delta(S(h)) = S(h_2) \otimes S(h_1)$, $\epsilon \circ S = \epsilon$, and furthermore we have $S(h_1)h_2 = \epsilon(h)1 = h_1S(h_2)$ for

all $h, h' \in H$. We always assume that H has *invertible antipode* and denote its inverse by $\bar{S} : H \rightarrow H$. The latter is an anti-bialgebra homomorphism such that $\bar{S}(h_2)h_1 = \epsilon(h)1 = h_2\bar{S}(h_1)$ for all $h \in H$.

A *right H -comodule* is a \mathbb{k} -module M , together with a \mathbb{k} -linear map $\Delta_R : M \rightarrow M \otimes H$ such that

$$(\Delta_R \otimes \text{id}_H) \circ \Delta_R = (\text{id}_M \otimes \Delta) \circ \Delta_R \tag{2.1}$$

and $(\text{id}_M \otimes \epsilon) \circ \Delta_R = \text{id}_M$, while a *left H -comodule* is a \mathbb{k} -module M with a \mathbb{k} -linear map $\Delta_L : M \rightarrow H \otimes M$ such that

$$(\text{id}_H \otimes \Delta_L) \circ \Delta_L = (\Delta \otimes \text{id}_M) \circ \Delta_L \tag{2.2}$$

and $(\epsilon \otimes \text{id}_M) \circ \Delta_L = \text{id}_M$. The maps Δ_R and Δ_L are called right and left H -coaction, respectively. We use the Sweedler's like notations $\Delta_R(m) = m_0 \otimes m_1$ and $\Delta_L(m) = m_{-1} \otimes m_0$, where $m \in M$. In case M is a right H -comodule, we denote the \mathbb{k} -submodule of right H -coaction invariant elements, simply called H -coinvariant elements, by

$$M^{\text{co}H} := \{m \in M \mid \Delta_R(m) = m \otimes 1\}$$

Similarly, if M is a left H -comodule, ${}^{\text{co}H}M := \{m \in M \mid \Delta_L(m) = 1 \otimes m\}$ is the \mathbb{k} -submodule of left H -coinvariant elements. Accordingly to (2.1) and (2.2), we further use the notations

$$m_0 \otimes m_1 \otimes m_2 := (\Delta_R \otimes \text{id}_H)(\Delta_R(m)) = (\text{id}_M \otimes \Delta)(\Delta_R(m)),$$

$m_{-2} \otimes m_{-1} \otimes m_0 := (\text{id}_H \otimes \Delta_L)(\Delta_L(m)) = (\Delta \otimes \text{id}_M)(\Delta_L(m))$ and similarly for higher coactions. A right H -comodule M which is also a left H -comodule is said to be an *H -bicomodule* if Δ_R and Δ_L are commuting coactions, i.e.,

$$(\Delta_L \otimes \text{id}_H) \circ \Delta_R = (\text{id}_H \otimes \Delta_R) \circ \Delta_L.$$

In Sweedler's like notation this commutativity reads $m_{-1} \otimes (m_0 \otimes m_1) = (m_{-1} \otimes m_0) \otimes m_1$, and hence we simply write $m_{-1} \otimes m_0 \otimes m_1$. If M is an H -bicomodule then $M^{\text{co}H}$ and ${}^{\text{co}H}M$ are left and right H -subcomodules, respectively. A *right H -comodule morphism* is a \mathbb{k} -linear map $\phi : M \rightarrow N$ between right H -comodules (M, Δ_M) and (N, Δ_N) such that $\Delta_N \circ \phi = (\phi \otimes \text{id}) \circ \Delta_M$, i.e.,

$$\phi(m)_0 \otimes \phi(m)_1 = \phi(m_0) \otimes m_1 \tag{2.3}$$

for all $m \in M$. A \mathbb{k} -linear map satisfying (2.3) is also called right H -colinear. Similarly, left H -colinear and H -bilinear maps are defined.

2.2. Covariant bimodules. Algebras in the category of comodules have a multiplication and unit that are compatible with the coaction. Since we are encountering several different coactions in the course of this section we utilize the convention to denote the right H -coaction on a right H -comodule algebra A by $\delta_R : A \rightarrow A \otimes H$.

Definition 2.1. A right H -comodule algebra is a right H -comodule (A, δ_R) together with an associative product $\mu : A \otimes A \rightarrow A$ and a unit $\eta : \mathbb{k} \rightarrow A$, such that

$$\delta_R \circ \mu = (\mu \otimes \mu_H) \circ (\text{id}_A \otimes \tau_{H,A} \otimes \text{id}_H) \circ (\delta_R \otimes \delta_R)$$

and $\delta_R \circ \eta = \eta \otimes \eta_H$, where $\tau_{H,A} : H \otimes A \rightarrow A \otimes H, h \otimes a \mapsto a \otimes h$ denotes the flip of \mathbb{k} -modules and (μ_H, η_H) is the algebra structure of H . Using the previously introduced short notation, the compatibility conditions read

$$\delta_R(aa') = (aa')_0 \otimes (aa')_1 = a_0a'_0 \otimes a_1a'_1 = \delta_R(a)\delta_R(a')$$

for all $a, a' \in A$ and $\delta_R(1) = 1_0 \otimes 1_1 = 1 \otimes 1_H$. A right H -comodule algebra homomorphism is a right H -comodule homomorphism which is also an algebra homomorphism. Similarly, one defines left H -comodule algebras and H -bicomodule algebras and their morphisms.

Fix a right H -comodule algebra (A, δ_R) in the following. We now introduce A -bimodules in the category of right H -comodules.

Definition 2.2. An A -bimodule M is called a right H -covariant A -bimodule if there is a right H -comodule action $\Delta_R : M \rightarrow M \otimes H$ on M , such that

$$\Delta_R(a \cdot m \cdot a') = \delta_R(a)\Delta_R(m)\delta_R(a'),$$

or equivalently, in short notation, $(a \cdot m \cdot a')_0 \otimes (a \cdot m \cdot a')_1 = a_0 \cdot m_0 \cdot a'_0 \otimes a_1m_1a'_1$ for all $a, a' \in A$ and $m \in M$. A right H -covariant right (resp. left) A -module is a right (resp. left) A -module, where we only ask compatibility of a right H -coaction with the right (resp. left) A -module structure. Similarly, left H -covariant A -modules and H -bicovariant A -bimodules are defined, if A is a left H -comodule algebra or an H -bicomodule algebra, respectively.

We are particularly interested in the case $A = H$, since H is naturally a right and a left H -comodule algebra with respect to the coactions given by the coproduct.

Definition 2.3. A bicovariant H -bimodule is an H -bimodule and an H -bicomodule M , such that

$$\Delta_R(h \cdot m \cdot h') = \Delta(h)\Delta_R(m)\Delta(h') \text{ and } \Delta_L(h \cdot m \cdot h') = \Delta(h)\Delta_L(m)\Delta(h')$$

for all $h, h' \in H$ and $m \in M$.

For H -covariant H -modules there is the following fundamental theorem (c.f. [31] Thm. 1.9.4, [41]).

Proposition 2.4. For any right H -covariant right H -module M , there is an isomorphism

$$M \rightarrow M^{\text{co}H} \otimes H, \quad m \mapsto m_0S(m_1) \otimes m_2, \tag{2.4}$$

of right H -covariant right H -modules. If M is a bicovariant H -bimodule, (2.4) extends to an isomorphism of bicovariant H -bimodules.

If M is a left H -covariant left H -module, there is an isomorphism

$$M \rightarrow H \otimes {}^{\text{co}H}M, \quad m \mapsto m_{-2} \otimes S(m_{-1})m_0, \tag{2.5}$$

of left H -covariant left H -modules. If M is a bicovariant H -bimodule, (2.5) extends to an isomorphism of bicovariant H -bimodules.

Note that in (2.4) the tensor product $M^{\text{co}H} \otimes H$ is a right H -covariant right H -module with respect to the diagonal coaction and the right H -action $(\bar{m} \otimes h) \cdot h' := \bar{m} \otimes (hh')$, where $\bar{m} \in M^{\text{co}H}$ and $h, h' \in H$. If M is furthermore a bicovariant H -bimodule we endow $M^{\text{co}H} \otimes H$ with the diagonal left H -coaction and the left H -action $h \cdot (\bar{m} \otimes h') := h_1 \bar{m} S(h_2) \otimes h_3 h'$. Similarly in (2.5), $H \otimes {}^{\text{co}H}M$ becomes a left H -covariant left H -module via the diagonal coaction and the left H -action given by $h \cdot (h' \otimes \bar{m}) := (hh') \otimes \bar{m}$, where $h, h' \in H$ and $\bar{m} \in {}^{\text{co}H}M$. If M is furthermore a bicovariant H -bimodule we endow $H \otimes {}^{\text{co}H}M$ with the diagonal right H -coaction and the right H -action $(h \otimes \bar{m}) \cdot h' := hh'_1 \otimes S(h'_2) \bar{m} h'_3$.

Using the inverse \bar{S} of the antipode S there are analogous isomorphism of right H -covariant left H -modules and left H -covariant right H -modules, respectively. The fundamental theorem allows us to write any element m of a bicovariant H -bimodule as $m = a_i m^i$, where $\{m^i\}_i \in I \subseteq {}^{\text{co}H}M$ is a basis of (left) coinvariant elements and a_i are coefficients in H . This is particularly useful when one is dealing with H -linear maps of bicovariant bimodules. Then, it is sufficient to specify properties on coinvariant elements.

2.3. Hopf–Galois extensions. In this section H denotes a Hopf algebra and A a right H -comodule algebra with right H -comodule action $\delta_R : A \rightarrow A \otimes H$. In particular H is a right H -comodule algebra with coaction Δ .

Definition 2.5. The algebra extension $B := A^{\text{co}H} \subseteq A$ is said to be

(i) a *Hopf–Galois extension*, if the \mathbb{k} -linear map

$$\begin{aligned} \chi &:= (\mu \otimes \text{id}_H) \circ (\text{id}_A \otimes_B \delta_R) : A \otimes_B A \rightarrow A \otimes H, \\ a \otimes_B a' &\mapsto \chi(a \otimes_B a') = aa'_0 \otimes a'_1 \end{aligned}$$

is a bijection, where $\mu : A \otimes_B A \rightarrow A$ is the product of A induced on the balanced tensor product $A \otimes_B A$.

- (ii) a *cleft extension*, if there is a \mathbb{k} -linear map $j : H \rightarrow A$, the *cleaving map*, such that
- (1) j is right H -colinear, i.e., $j(h)_0 \otimes j(h)_1 = j(h_1) \otimes h_2$ for all $h \in H$,
 - (2) j is convolution invertible, i.e., there is a \mathbb{k} -linear map $j^{-1} : H \rightarrow A$ such that $j(h_1) j^{-1}(h_2) = \epsilon(h)1 = j^{-1}(h_1) j(h_2)$ for all $h \in H$,
 - (3) j respects the units, i.e., $j(1_H) = 1$.
- (iii) a *trivial extension*, if there is a right H -comodule algebra homomorphism $j : H \rightarrow A$.

A trivial extension $A^{\text{co}H} \subseteq A$ is automatically a cleft extension since an H -comodule algebra map $j : H \rightarrow A$ is right H -colinear and unital by assumption and its convolution inverse is $j^{-1} = j \circ S$. This trivially implies that the composition $j^{-1} \circ \bar{S} : H \rightarrow A$ is right H -colinear. In the following lemma we prove that this H -colinearity property holds for any cleft extension.

Lemma 2.6. *The inverse of the cleaving map satisfies*

$$\delta_R \circ j^{-1} = (j^{-1} \otimes S) \circ \tau_{H,H} \circ \Delta, \tag{2.6}$$

where $\tau_{H,H}$ is the flip. In particular, $j^{-1} \circ \bar{S}$ is right H -colinear.

Proof. Equation (2.6) is proven in [31] Lem. 7.2.6, 1). Then,

$$\begin{aligned} \delta_R(j^{-1}(\bar{S}(h))) &= j^{-1}(\bar{S}(h)_2) \otimes S(\bar{S}(h)_1) = j^{-1}(\bar{S}(h_1)) \otimes S(\bar{S}(h_2)) \\ &= j^{-1}(\bar{S}(h_1)) \otimes h_2 \end{aligned}$$

for all $h \in H$, where we used that \bar{S} is an anti-bialgebra homomorphism. □

To complete the hierarchy of Definition 2.5, we note that cleft extensions are in particular Hopf–Galois extensions. We shall frequently call them cleft Hopf–Galois extensions.

Proposition 2.7. $B := A^{\text{co}H} \subseteq A$ is a cleft extension if and only if $B \subseteq A$ is a Hopf–Galois extension and there is a left B -module and right H -comodule isomorphism $B \otimes H \cong A$.

The proof of this proposition (see e.g. [31] Thm. 8.2.4) relies on the construction of the following left B -module and right H -comodule map

$$\theta : B \otimes H \rightarrow A, \quad b \otimes h \mapsto bj(h)$$

with inverse given by

$$\theta^{-1} : A \rightarrow B \otimes H, \quad a \mapsto a_0j^{-1}(a_1) \otimes a_2.$$

Note that θ^{-1} is well-defined, i.e. $a_0j^{-1}(a_1) \in B$, since

$$\begin{aligned} \delta_R(a_0j^{-1}(a_1)) &= a_0j^{-1}(a_2)_0 \otimes a_1j^{-1}(a_2)_1 = a_0j^{-1}(a_3) \otimes a_1S(a_2) \\ &= a_0j^{-1}(a_2) \otimes \epsilon(a_1)1 = a_0j^{-1}(a_1) \otimes 1, \end{aligned}$$

where we employed Lemma 2.6 (see also [31] Lem. 7.2.6, 2.).

We conclude this section by recalling the notion of principal comodule algebra from [12] and (part of) Schneider theorem for faithfully flat Hopf–Galois extensions.

Definition 2.8. A right H -comodule algebra A is said to be a principal comodule algebra if $B := A^{\text{co}H} \subseteq A$ is a Hopf–Galois extension and if A is right H -equivariantly projective as a left B -module. The latter means that there exists a left B -linear and right H -colinear map $s : A \rightarrow B \otimes A$ such that $m \circ s = \text{id}_A$, where $m : B \otimes A \rightarrow A$ denotes the restricted product of A .

Any cleft extension is a principal comodule algebra with section $s : A \rightarrow B \otimes A$, $a \mapsto s(a) = a_0j^{-1}(a_1) \otimes j(a_2)$. Let $B := A^{\text{co}H} \subseteq A$ be a Hopf–Galois extension. If \mathbb{k} is a field, and recalling that we always consider Hopf algebras H with invertible antipode, the right H -equivariant projectivity of A is equivalent to the existence of a strong connection, it is also equivalent to faithfully flatness of A as a left B -module, see e.g. [8] Part VII, Thm. 6.16, 6.19, 6.20. For faithfully flat Hopf–Galois extensions we will later use the following main result which is an immediate consequence of Schneider [42, Thm. 1] by considering that categorical equivalences are adjoint equivalences and that adjoint functors are unique up to natural isomorphism.

Theorem 2.9. Let $B := A^{\text{co}H} \subseteq A$ be a faithfully flat Hopf–Galois extension. We have the equivalence of categories via the following inverse functors:

$$\Phi : {}_B\mathcal{M} \longrightarrow {}_A\mathcal{M}^H, \quad \Phi(M) = A \otimes_B M; \quad \Psi : {}_A\mathcal{M}^H \longrightarrow {}_B\mathcal{M}, \quad \Psi(N) = N^{\text{co}H}$$

where ${}_A\mathcal{M}^H$ denotes the category of right H -covariant left A -modules and ${}_B\mathcal{M}$ the category of left B -modules.

2.3.1. Smash products A left H -module algebra (B, \triangleright) is a left H -module structure $\triangleright : H \otimes B \rightarrow B$ on an algebra B such that $h \triangleright (bb') = (h_1 \triangleright b)(h_2 \triangleright b')$ and $h \triangleright 1_B = \epsilon(h)1_B$ for all $h \in H$ and $b, b' \in B$.

Definition 2.10. For a left H -module algebra (B, \triangleright) the *smash product* $B\#H$ is defined as the \mathbb{k} -module $B \otimes H$ endowed with the multiplication

$$(b \otimes h) \cdot_{\#} (b' \otimes h') := b(h_1 \triangleright b') \otimes h_2 h'$$

for all $b, b' \in B$ and $h, h' \in H$.

In the following we write $b\#h$ instead of $b \otimes h$ if we want to view $b \otimes h$ as an element of $B\#H$. Moreover, we set $(b\#h)(b'\#h') := (b \otimes h) \cdot_{\#} (b' \otimes h')$. The algebra $B\#H$ is naturally endowed with the right H -comodule structure

$$B\#H \rightarrow B\#H \otimes H, \quad b\#h \mapsto (b\#h_1) \otimes h_2.$$

The compatibility of this coaction with the product $\cdot_{\#}$ is easily checked and thus $B\#H$ is a right H -comodule algebra (see also [31] Sect. 4.1). The smash product algebra $B\#H$ is the trivial Hopf–Galois extension $B = (B\#H)^{\text{co}H} \subseteq B\#H$ with cleaving map defined by the algebra inclusion $H \rightarrow B\#H$. Vice versa trivial Hopf–Galois extensions are isomorphic (as H -comodule algebras) to smashed products.

For a trivial Hopf Galois extension $B = A^{\text{co}H} \subseteq A$, conjugation via the cleaving map $j : H \rightarrow A$ defines the left H -module algebra action on B

$$h \triangleright b := j(h_1)bj^{-1}(h_2).$$

This is easily verified using that j is an algebra map and that its convolution inverse $j^{-1} = j \circ S$ is an anti-algebra map. Furthermore, $j(h_1)bj^{-1}(h_2) \in B$ follows from the right H -colinearity of j and Lemma 2.6. The compatibility of \triangleright with the algebra structure holds because j^{-1} is the convolution inverse of j and j is unital.

Proposition 2.11. *If $B := A^{\text{co}H} \subseteq A$ is a trivial extension the map $\theta : B\#H \rightarrow A$, $\theta(b\#h) := bj(h)$ is an isomorphism of right H -comodule algebras. The inverse is $\theta^{-1}(a) = a_0 j^{-1}(a_1)\#a_2$.*

Proof. In Proposition 2.7 it has been proven that θ is an isomorphism of right H -comodules. It remains to prove that θ is an algebra homomorphism. Trivially we have $\theta(1_B\#1_H) = 1_A$ and also

$$\begin{aligned} \theta((b\#h)(b'\#h')) &= \theta(b(h_1 \triangleright b')\#h_2 h') = b(h_1 \triangleright b')j(h_2 h') = bj(h_1)b'j^{-1}(h_2)j(h_3)j(h') \\ &= bj(h)b'j(h') = \theta(b\#h)\theta(b'\#h'), \end{aligned}$$

which concludes the proof. □

3. Covariant Differential Calculi on Hopf–Galois Extensions

We begin this section with the definition of the category of noncommutative differential calculi and discuss pullback and quotient calculi. We then specialize to covariant differential calculi and bicovariant calculi, building on previous works of [23] and [47]. Examples of (bi)covariant calculi on the quantum groups $\mathcal{O}_q(\text{SL}_2(\mathbb{C}))$ and $\mathcal{O}_q(\text{GL}_2(\mathbb{C}))$ are recalled and the induced calculi on their parabolic quantum subgroups are presented. We next revisit a result of Pflaum and Schauenburg [36], where an H -module calculus

and a bicovariant calculus merge to the *smash product calculus*: a covariant calculus on the smash product algebra. This latter is a trivial principal bundle and the smash product calculus is a FODC on it.

In Sect. 3.5 we show that for right H -covariant FODCi on principal comodule algebras base forms are the intersection of horizontal and coinvariant forms. Furthermore, when the injection of horizontal forms into total space forms is completed in an exact sequence with vertical forms, we obtain a graded Hopf–Galois extension with compatible differentials on the total space algebra and on the quantum structure group.

3.1. Noncommutative differential calculi. In this section we give the definition and some results on noncommutative differential calculi. Though this material is well established in the context of bicovariant calculi on Hopf algebras, since we take a slightly more general point of view, we prefer to review the main points. We refer the reader to [47], [26, Chpt. 12], [5, Chpt. 2] for more details.

We give the definition of a differential calculus.

Definition 3.1. A *first order differential calculus* (FODC) on a (noncommutative) algebra A is a couple (Γ, d) , where

- (i) Γ is an A -bimodule,
- (ii) $d : A \rightarrow \Gamma$ is a \mathbb{k} -linear map which satisfies the Leibniz rule $d(ab) = (da)b + adb$ for all $a, b \in A$,
- (iii) $\Gamma = AdA := \text{span}_{\mathbb{k}}\{adb \mid a, b \in A\}$.

We say that Γ is generated (as a left A -module) by exact forms and refer to (iii) as the *surjectivity property* of the FODC. A morphism between a FODC (Γ, d) on A and a FODC (Γ', d') on another algebra A' is a couple (ϕ, Φ) , where $\phi : A \rightarrow A'$ is an algebra homomorphism and $\Phi : \Gamma \rightarrow \Gamma'$ is a \mathbb{k} -linear map such that

$$\Phi(a \cdot \omega \cdot b) = \phi(a) \cdot' \Phi(\omega) \cdot' \phi(b)$$

and

$$\Phi \circ d = d' \circ \phi$$

for all $a, b \in A, \omega \in \Gamma$. Two FODCi are called *equivalent* if there is an isomorphism of FODCi between them.

The classical example of a FODC comes from differential geometry. Given a smooth manifold M , the $C^\infty(M)$ -bimodule $\Omega^1(M)$ of differential 1-forms on M together with the de Rham differential is a FODC on $C^\infty(M)$. On any algebra A there is the so-called *universal FODC* (Γ_u, d_u) , where $\Gamma_u := \ker \mu_A \subseteq A \otimes A$ is the kernel of the multiplication $\mu_A : A \otimes A \rightarrow A$ and $d_u a := 1 \otimes a - a \otimes 1$ for all $a \in A$. The left and right A -module action on Γ_u is given by the multiplication on the first and the second tensor factor, respectively. This FODC is universal in the sense that every FODC on A is isomorphic to a quotient of (Γ_u, d_u) . (c.f. e.g. [47]).

It is well-known that the tensor product $A \otimes A'$ of two algebras A and A' is an algebra with associative product $(a \otimes a')(b \otimes b') = ab \otimes a'b'$ and unit $1_A \otimes 1_{A'}$. This construction extends to the level of FODCi (see also [36] Thm. 2.2).

Proposition 3.2. *Given a FODC (Γ, d) on A and a FODC (Γ', d') on A' , there is a FODC $(\Gamma_{A \otimes A'}, d_{A \otimes A'})$ on $A \otimes A'$, where $\Gamma_{A \otimes A'} = \Gamma \otimes A' \oplus A \otimes \Gamma'$ and*

$$d_{A \otimes A'} : A \otimes A' \rightarrow \Gamma_{A \otimes A'}, \quad a \otimes a' \mapsto da \otimes a' + a \otimes d'a'. \quad (3.1)$$

The $A \otimes A'$ -bimodule structure on $\Gamma_{A \otimes A'}$ is

$$(a \otimes a') \cdot (\omega \otimes b' + b \otimes \omega') \cdot (c \otimes c') = a\omega c \otimes a'b'c' + abc \otimes a'\omega'c', \quad (3.2)$$

where $a, b, c \in A$, $a', b', c' \in A'$, $\omega \in \Gamma$ and $\omega' \in \Gamma'$. This construction is associative, i.e. $(\Gamma_{(A \otimes A') \otimes A''}, d_{(A \otimes A') \otimes A''}) = (\Gamma_{A \otimes (A' \otimes A'')}, d_{A \otimes (A' \otimes A'')})$ for another algebra A'' .

Proof. Clearly, (3.2) defines an $A \otimes A'$ -bimodule structure and (3.1) a \mathbb{k} -linear map. This latter satisfies the Leibniz rule

$$\begin{aligned} d_{A \otimes A'}((a \otimes a')(b \otimes b')) &= d_{A \otimes A'}(ab \otimes a'b') \\ &= d(ab) \otimes a'b' + ab \otimes d'(a'b') \\ &= (da)b \otimes a'b' + adb \otimes a'b' + ab \otimes (d'a')b' + ab \otimes a'd'b' \\ &= (da \otimes a' + a \otimes d'a') \cdot (b \otimes b') + (a \otimes a') \cdot (db \otimes b' + b \otimes d'b') \\ &= d_{A \otimes A'}(a \otimes a') \cdot (b \otimes b') + (a \otimes a') \cdot d_{A \otimes A'}(b \otimes b') \end{aligned}$$

for all $a, b \in A$ and $a', b' \in A'$. Furthermore, $\Gamma_{A \otimes A'}$ is generated by $A \otimes A'$ and $d_{A \otimes A'}$, since

$$\begin{aligned} (A \otimes A') \cdot d_{A \otimes A'}(A \otimes A') &= (A \otimes A') \cdot (dA \otimes A' \oplus A \otimes d'A') \\ &= AdA \otimes A' \oplus A \otimes A'd'A' \\ &= \Gamma \otimes A' \oplus A \otimes \Gamma' \\ &= \Gamma_{A \otimes A'} \end{aligned}$$

as an equality of sets. Finally, associativity follows from the equality of $A \otimes A' \otimes A''$ -bimodules

$$\begin{aligned} \Gamma_{(A \otimes A') \otimes A''} &= \Gamma_{A \otimes A'} \otimes A'' \oplus (A \otimes A') \otimes \Gamma'' \\ &= (\Gamma \otimes A' \oplus A \otimes \Gamma') \otimes A'' \oplus A \otimes A' \otimes \Gamma'' \\ &= \Gamma \otimes A' \otimes A'' \oplus A \otimes \Gamma' \otimes A'' \oplus A \otimes A' \otimes \Gamma'' \\ &= \Gamma \otimes (A' \otimes A'') \oplus A \otimes \Gamma_{A' \otimes A''} \\ &= \Gamma_{A \otimes (A' \otimes A'')} \end{aligned}$$

and from $d_{(A \oplus A') \oplus A''} = d_{A \oplus (A' \oplus A'')}$. \square

Remark 3.3. In other words, Proposition 3.2 proves that FODCi form a monoidal category. The monoidal product is

$$(\Gamma, d, A) \otimes (\Gamma', d', A') := (\Gamma_{A \otimes A'}, d_{A \otimes A'}, A \otimes A')$$

and the monoidal unit is the trivial calculus $(\Gamma = \{0\}, d = 0)$ on the algebra $A = \mathbb{k}$.

Given a FODC on A we can induce a FODC on any subalgebra and quotient algebra of A . For a surjective algebra map $\pi : A \rightarrow A'$, we identify A' with A/I where $I := \ker \pi$ and denote by $[a] := a + I$ the class of $a \in A$.

Proposition 3.4. *Let (Γ, d) be a FODC on A .*

(i) An algebra map $\iota : A' \rightarrow A$ induces a FODC (Γ_ι, d_ι) on A' , where

$$\Gamma_\iota := \iota(A')d\iota(A') \subseteq \Gamma \quad (3.3)$$

and $d_\iota := d \circ \iota : A' \rightarrow \Gamma_\iota, a' \mapsto d\iota(a')$.

(ii) A surjective algebra map $\pi : A \rightarrow A'$ induces a FODC (Γ_π, d_π) on A' , where the A' -bimodule

$$\Gamma_\pi := \Gamma / \Gamma_I$$

is the quotient with $\Gamma_I := IdA + AdI$, where $I := \ker \pi \subseteq A$ and $d_\pi : A' \rightarrow \Gamma_\pi, \pi(a) \mapsto [da]$.

(iii) If the exact sequence of algebras $0 \rightarrow \ker \pi \rightarrow A \xrightarrow{\pi} A' \rightarrow 0$ splits via $\iota : A' \rightarrow A$, so that $\pi \circ \iota = \text{id}_{A'}$, then (Γ_ι, d_ι) is equivalent to (Γ_π, d_π) with isomorphism (ϕ, Φ) given by

$$\phi = \text{id}_{A'}, \quad \Phi : \Gamma_\iota \rightarrow \Gamma_\pi, \omega \mapsto [\omega]. \quad (3.4)$$

Proof. (i) The \mathbb{k} -submodule Γ_ι in (3.3) is structured as an A' -bimodule via

$$a' \cdot \omega \cdot b' := \iota(a')\omega\iota(b') \quad (3.5)$$

for all $a', b' \in A'$ and $\omega \in \Gamma_\iota \subseteq \Gamma$. To see this, we first verify that the maps defined in (3.5) close in Γ_ι . By definition, any $\omega \in \Gamma_\iota$ is a finite sum $\omega = \iota(a^i)d\iota(b'_i)$, for $a^i, b'_i \in A'$. Then, for all $a', b' \in A'$,

$$\begin{aligned} a' \cdot \omega \cdot b' &= \iota(a')\iota(a^i)(d\iota(b'_i))\iota(b') \\ &= \iota(a'a^i)(d(\iota(b'_i)\iota(b'))) - \iota(b'_i)d\iota(b') \\ &= \iota(a'a^i)d\iota(b'_i b') - \iota(a'a^i b'_i)d\iota(b') \in \Gamma_\iota, \end{aligned}$$

where we used that ι is an algebra map. The maps in (3.5) are left and right A' -actions because ι is an algebra map. They are trivially commuting. Next, the \mathbb{k} -linear map d_ι satisfies the Leibniz rule

$$d_\iota(a'b') = d\iota(a'b') = d(\iota(a')\iota(b')) = (d\iota(a'))\iota(b') + \iota(a')d\iota(b') = (d_\iota a') \cdot b' + a' \cdot d_\iota b'$$

for all $a', b' \in A'$. Since by definition of Γ_ι any $\omega \in \Gamma_\iota$ is of the form $\omega = \iota(a^i)d\iota(b'_i)$ for some $a^i, b'_i \in A'$ (finite sum understood), we have $\omega = \iota(a^i)d\iota(b'_i) = a^i \cdot d_\iota b'_i$, proving that (Γ_ι, d_ι) is a FODC on A' .

(ii) Since $I = \ker \pi \subseteq A$ is an ideal, by the Leibniz rule of d it immediately follows that $\Gamma_I = IdA + AdI \subseteq \Gamma$ is an A -subbimodule. Hence, Γ/Γ_I is an A -bimodule. Since $I \cdot \Gamma \subseteq \Gamma_I, \Gamma \cdot I \subseteq \Gamma_I$, the A -actions on Γ descend to $A' = A/I$ -actions on Γ/Γ_I ,

$$\pi(a) \cdot [\omega] \cdot \pi(b) := [a\omega b]$$

for all $a, b \in A$ and $\omega \in \Gamma$. It follows that the map d induced on the quotient, $d_\pi : A' \rightarrow \Gamma_\pi, \pi(a) \mapsto [da]$, is \mathbb{k} -linear and satisfies the Leibniz rule:

$$\begin{aligned} d_\pi(\pi(a)\pi(b)) &= d_\pi(\pi(ab)) = [d(ab)] = [(da)b + adb] \\ &= [da] \cdot \pi(b) + \pi(a) \cdot [db] = (d_\pi \pi(a)) \cdot \pi(b) + \pi(a) \cdot d_\pi(\pi(b)) \end{aligned}$$

for all $a, b \in A$. Finally, since any $\omega \in \Gamma$ is of the form $\omega = a^i db_i$ (sum understood) we have $[\omega] = [a^i db_i] = \pi(a^i) \cdot d_\pi(\pi(b_i))$ so that $\Gamma_\pi = A' d_\pi A'$. We conclude that (Γ_π, d_π) is a FODC on A' .

(iii) According to (i) and (ii) (Γ_ι, d_ι) and (Γ_π, d_π) are FODCi on A' . The splitting $\iota : A' \rightarrow A$ of the exact sequence $0 \rightarrow \ker \pi \rightarrow A \xrightarrow{\pi} A' \rightarrow 0$ implies the direct sum of A' -bimodules $A = \iota(A') \oplus I$ where the A' -action is via the algebra embedding $A' \rightarrow \iota(A') \subseteq A$. This induces the direct sum of A' -bimodules

$$\Gamma = AdA = \iota(A')d_\iota(A') \oplus (IdA + AdI) = \Gamma_\iota \oplus \Gamma_I.$$

Considering the quotient with respect to Γ_I gives the isomorphism $\Gamma_\iota \rightarrow \Gamma/\Gamma_I$, $\omega \mapsto \pi(\omega)$, which equals the Φ morphism defined in (3.4). We are left to prove the compatibility of $(\phi = \text{id}_{A'}, \Phi)$ with the exterior derivatives, for all $a' \in A'$

$$\Phi(d_\iota a') = \Phi(d_\iota(a')) = [d_\iota(a')] = d_\pi \pi(\iota(a')) = d_\pi a' = d_\pi \phi(a').$$

□

We call (Γ_ι, d_ι) the *pullback calculus*, while we call (Γ_π, d_π) the *quotient calculus*. The above proposition provides a useful tool to produce examples of noncommutative differential calculi. In the context of Drinfel'd twist deformation quantization it has been employed to describe noncommutative differential geometry on submanifolds of \mathbb{R}^n given by generators and relations [15]. More abstractly, the braided Cartan calculus of a braided commutative algebra with triangular Hopf algebra symmetry is related to the braided Cartan calculus on a submanifold algebra in the above sense [46].

3.2. *Covariant calculi on comodule algebras.* In this section we discuss the theory for covariant calculi on comodule algebras, following the perspectives of [23,47] and [36]. Let H be a Hopf algebra.

Definition 3.5. (i) A FODC (Γ, d) on a right H -comodule algebra (A, δ_R) is said to be *right H -covariant* if

$$adb \mapsto a_0 db_0 \otimes a_1 b_1, \tag{3.6}$$

for all $a, b \in A$, extends to a well-defined \mathbb{k} -linear map $\Gamma \rightarrow \Gamma \otimes H$.

(ii) A FODC (Γ, d) on a left H -comodule algebra (A, δ_L) is said to be *left H -covariant* if

$$adb \mapsto a_{-1} ab_{-1} \otimes a_0 db_0, \tag{3.7}$$

for all $a, b \in A$, extends to a well-defined \mathbb{k} -linear map $\Gamma \rightarrow H \otimes \Gamma$.

(iii) A FODC (Γ, d) on an H -bicomodule algebra (A, δ_R, δ_L) is said to be *H -bicovariant* if it is right and left H -covariant.

If $A = H$ is an H -comodule algebra with respect to the coproduct we call an H -covariant FODC simply covariant. We recall the following proposition.

Proposition 3.6. ([47])

(i) A FODC (Γ, d) on a right H -comodule algebra (A, δ_R) is right H -covariant if and only if (Γ, Δ_R) is a right H -covariant A -bimodule and d is right H -colinear. In this case Δ_R is determined by (3.6).

- (ii) A FODC (Γ, d) on a left H -comodule algebra (A, δ_L) is left H -covariant if and only if (Γ, Δ_L) is a left H -covariant A -bimodule and d is left H -colinear. In this case Δ_L is determined by (3.7).
- (iii) A FODC (Γ, d) on an H -bicomodule algebra (A, δ_R, δ_L) is an H -bicovariant FODC if and only if $(\Gamma, \Delta_R, \Delta_L)$ is an H -bicovariant A -bimodule and d is H -bicolinear.

We define a morphism of right/left/bi H -covariant FODCi (Γ, d) and (Γ', d') on right/left/bi H -comodule algebras A, A' to be a morphism (ϕ, Φ) between FODCi such that ϕ and Φ are right/left/bi H -colinear maps. Since Φ , if it exists, is determined by ϕ via $\Phi(a^i db_i) = \phi(a^i) d\phi(b_i)$ for $a^i, b_i \in A$ (finite sum understood), we see that the FODC morphism (ϕ, Φ) is right/left/bi H -colinear if and only if $\phi : A \rightarrow A'$ is right/left/bi H -colinear, indeed, for all $a \in A$,

$$\begin{aligned} \Delta_R(\Phi(da)) &= \Delta_R d\phi(a) = d\phi(a)_0 \otimes \phi(a)_1 \\ &= d\phi(a_0) \otimes a_1 = \Phi(da_0) \otimes a_1 = (\Phi \otimes \text{id}_H)\Delta_R(da). \end{aligned}$$

The natural question arises if Proposition 3.4 generalizes to the H -covariant setting.

Proposition 3.7. *Let (Γ, d) be a right H -covariant FODC on a right H -comodule algebra A .*

- (i) *If $\iota : A' \rightarrow A$ is a right H -comodule algebra homomorphism the pullback calculus (Γ_ι, d_ι) is a right H -covariant FODC on A' .*
- (ii) *If $\pi : A \rightarrow A'$ is a surjective right H -comodule algebra homomorphism the quotient calculus (Γ_π, d_π) is a right H -covariant FODC on A' .*
- (iii) *If the exact sequence of H -comodule algebras $0 \rightarrow \ker \pi \rightarrow A \xrightarrow{\pi} A' \rightarrow 0$ splits via $\iota : A' \rightarrow A$, then (Γ_ι, d_ι) is equivalent to (Γ_π, d_π) as a right H -covariant FODC.*

Analogous statements hold for left H -covariant and H -bicovariant FODCi on left H -comodule algebras and H -bicomodule algebras, respectively.

Proof. From Proposition 3.4 it follows that (Γ_ι, d_ι) and (Γ_π, d_π) are FODCi on A' . We prove (i) by observing that for all $a^i, b^i \in A$ (and understanding finite sums on the indices i)

$$\iota(a^i)_0 d(\iota(b^i)_0) \otimes \iota(a^i)_1 \iota(b^i)_1 = \iota(a^i_0) d(\iota(b^i_0)) \otimes a^i_1 b^i_1 \in \Gamma_\iota \otimes H$$

by the right H -colinearity of ι . Thus, the right H -coaction $\Delta_R : \Gamma \rightarrow \Gamma \otimes H$ restricts to a \mathbb{k} -linear map $\Delta_R|_{\Gamma_\iota} : \Gamma_\iota \rightarrow \Gamma_\iota \otimes H$ and consequently (Γ_ι, d_ι) is a right H -covariant FODC. For (ii) we first note that $I = \ker \pi \subseteq A$ satisfies $\delta_R(I) \subseteq I \otimes H$ since π is right H -colinear. Then, by the right H -covariance of (Γ, d) we have that $\Gamma_I \subseteq \Gamma$ is a right H -subcomodule: $\Delta_R(\Gamma_I) \subseteq \Gamma_I \otimes H$, so that $\Delta_R : \Gamma \rightarrow \Gamma \otimes H$ induces a well-defined right H -coaction on Γ_π , which reads

$$\Delta_R([adb]) = [a_0 db_0] \otimes a_1 ab_1 = \pi(a_0) \cdot d_\pi \pi(b_0) \otimes a_1 b_1$$

for all $a, b \in A$. This shows that (Γ_π, d_π) is right H -covariant. The third statement trivially holds recalling that for the morphism (ϕ, Φ) of FODCi Φ is H -colinear if ϕ is H -colinear, which is the case since $\phi = \text{id}_{A'}$. □

We have the following corollary of Proposition (3.7) (ii).

Corollary 3.8. *Let $\pi : A \rightarrow H$ be a Hopf algebra epimorphism. Any right/left/bicovariant FODC (Γ, d) on the Hopf algebra A gives a right/left H -covariant or H -bicovariant FODC on A and projects to a right/left/bicovariant FODC (Γ_π, d_π) on the Hopf algebra H .*

Proof. Here we prove the case of a bicovariant FODC (Γ, d) on A . The others are similarly proven. The left and right H -coactions on A defined by

$$\begin{aligned} \delta_L &:= (\pi \otimes \text{id}_A) \circ \Delta : A \rightarrow H \otimes A, \quad a \mapsto a_{-1} \otimes a_0 := [a_1] \otimes a_2 \\ \delta_R &:= (\text{id}_A \otimes \pi) \circ \Delta : A \rightarrow A \otimes H, \quad a \mapsto a_0 \otimes a_1 := a_1 \otimes [a_2] \end{aligned}$$

and those on Γ defined by

$$\begin{aligned} \Delta_L^H &:= (\pi \otimes \text{id}_A) \circ \Delta_L : \Gamma \rightarrow H \otimes \Gamma, \quad adb \mapsto a_{-1}b_{-1} \otimes a_0db_0 \\ \Delta_R^H &:= (\text{id}_A \otimes \pi) \circ \Delta_R : \Gamma \rightarrow \Gamma \otimes H, \quad adb \mapsto a_0db_0 \otimes a_1b_1, \end{aligned}$$

where $a, b \in A$, structure (Γ, d_π) as an H -bicovariant FODC on A . The projection $\pi : A \rightarrow H$ is left and right H -colinear so that from Proposition (3.7) the induced FODC (Γ_π, d_π) on H is H -bicovariant with H -coaction given by the H -coproduct. \square

3.3. Examples of covariant calculi. In this section we describe covariant calculi on the quantum groups $\mathcal{O}_q(\text{GL}_2(\mathbb{C}))$, $\mathcal{O}_q(\text{SL}_2(\mathbb{C}))$ and on their parabolic subalgebras.

Example 3.9. (4D calculus on $\mathcal{O}_q(\text{GL}_2(\mathbb{C}))$) Let $q \in \mathbb{C}$ be a non-zero complex number which is not a root of unity. The free \mathbb{C} -algebra $\mathbb{C}\langle \alpha, \beta, \gamma, \delta \rangle$ generated by indeterminates $\alpha, \beta, \gamma, \delta$ modulo the relations

$$\begin{aligned} \alpha\beta &= q^{-1}\beta\alpha, \quad \alpha\gamma = q^{-1}\gamma\alpha, \quad \beta\delta = q^{-1}\delta\beta, \quad \gamma\delta = q^{-1}\delta\gamma, \\ \beta\gamma &= \gamma\beta, \quad \alpha\delta - \delta\alpha = (q^{-1} - q)\beta\gamma \end{aligned} \tag{3.8}$$

is denoted by $\mathcal{O}_q(\text{M}_2(\mathbb{C})) := \mathbb{C}\langle \alpha, \beta, \gamma, \delta \rangle / I_M$, where I_M is the ideal in $\mathbb{C}\langle \alpha, \beta, \gamma, \delta \rangle$ generated by the (3.8). The quotient algebra $\mathcal{O}_q(\text{M}_2(\mathbb{C}))$ is a bialgebra with coalgebra structure

$$\Delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \epsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.9}$$

where \otimes denotes the tensor product and matrix multiplication. Furthermore, there is a central element $\det_q := \alpha\delta - q^{-1}\beta\gamma \in \mathcal{O}_q(\text{M}_2(\mathbb{C}))$, satisfying $\Delta(\det_q) = \det_q \otimes \det_q$ and $\epsilon(\det_q) = 1$. On the quotient algebra $\mathcal{O}_q(\text{GL}_2(\mathbb{C})) := \mathcal{O}_q(\text{M}_2(\mathbb{C}))\langle r \rangle / \langle r \det_q - 1 \rangle$ we induce a Hopf algebra structure by (3.9) and $\Delta(r) = r \otimes r$, $\epsilon(r) = 1$, together with the antipode

$$S \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = r \begin{pmatrix} \delta & -q\beta \\ -q^{-1}\gamma & \alpha \end{pmatrix}, \quad S(r) = \det_q. \tag{3.10}$$

On the quantum group $A := \mathcal{O}_q(\text{GL}_2(\mathbb{C}))$ there is a 4-dimensional bicovariant FODC $(\Gamma_{\text{GL}}, d_{\text{GL}})$ which we are going to describe following the explicit exposition of [2].¹ Γ

¹ Note that in our convention we used q^{-1} instead of q and we rescaled ω^2, ω^3 by q^{-1} for later use in Example 3.10

is a free left A -module generated by a basis $\{\omega^1, \omega^2, \omega^3, \omega^4\}$ of left coinvariant 1-forms obeying the commutation relations

$$\begin{aligned}
 \omega^1\alpha &= q\alpha\omega^1, & \omega^1\beta &= q^{-1}\beta\omega^1, \\
 \omega^2\alpha &= \alpha\omega^2, & \omega^2\beta &= -q^{-1}\lambda\alpha\omega^1 + \beta\omega^2, \\
 \omega^3\alpha &= -q^{-1}\lambda\beta\omega^1 + \alpha\omega^3, & \omega^3\beta &= \beta\omega^3, \\
 \omega^4\alpha &= -\lambda\beta\omega^2 + q^{-1}\alpha\omega^4, & \omega^4\beta &= q^{-1}\lambda^2\beta\omega^1 - \lambda\alpha\omega^3 + q\beta\omega^4, \\
 \omega^1\gamma &= q\gamma\omega^1, & \omega^1\delta &= q^{-1}\delta\omega^1, \\
 \omega^2\gamma &= \gamma\omega^2, & \omega^2\delta &= \delta\omega^2 - q^{-1}\lambda\gamma\omega^1, \\
 \omega^3\gamma &= \gamma\omega^3 - q^{-1}\lambda\delta\omega^1, & \omega^3\delta &= \delta\omega^3, \\
 \omega^4\gamma &= -\lambda\delta\omega^2 + q^{-1}\gamma\omega^4, & \omega^4\delta &= q^{-1}\lambda^2\delta\omega^1 - \lambda\gamma\omega^3 + q\delta\omega^4
 \end{aligned}
 \tag{3.11}$$

with $\lambda := q^{-1} - q$. The differential is given in terms of this basis by

$$\begin{aligned}
 d_{\text{GL}}\alpha &= \frac{q-1}{\lambda}\alpha\omega^1 + \frac{q^{-1}-1}{\lambda}\alpha\omega^4 - \beta\omega^2 \\
 d_{\text{GL}}\beta &= Q\beta\omega^1 + \frac{q-1}{\lambda}\beta\omega^4 - \alpha\omega^3 \\
 d_{\text{GL}}\gamma &= \frac{q-1}{\lambda}\gamma\omega^1 + \frac{q^{-1}-1}{\lambda}\gamma\omega^4 - \delta\omega^2 \\
 d_{\text{GL}}\delta &= Q\delta\omega^1 + \frac{q-1}{\lambda}\delta\omega^4 - \gamma\omega^3,
 \end{aligned}
 \tag{3.12}$$

where $Q := \frac{q^{-1}(\lambda^2+1)-1}{\lambda} = \frac{q^2+q+1}{q^2(q+1)} - 1$. The basis 1-forms are then expressed explicitly as

$$\begin{aligned}
 \omega^1 &= Q' \left((q^{-2} - q^{-1})(S(\alpha)d_{\text{GL}}\alpha + S(\beta)d_{\text{GL}}\gamma) + q^{-2}(q^{-1} - 1)(S(\gamma)d_{\text{GL}}\delta + S(\delta)d_{\text{GL}}\delta) \right) \\
 \omega^2 &= -q^2(S(\gamma)d_{\text{GL}}\alpha + S(\delta)d_{\text{GL}}\gamma) \\
 \omega^3 &= -q^2(S(\alpha)d_{\text{GL}}\beta + S(\beta)d_{\text{GL}}\delta) \\
 \omega^4 &= Q' \left((q^{-5} - q^{-3} - q^{-2} + q^{-1})(S(\alpha)d_{\text{GL}}\alpha + S(\beta)d_{\text{GL}}\gamma) \right. \\
 &\quad \left. + (q^{-2} - q^{-1})(S(\gamma)d_{\text{GL}}\delta + S(\delta)d_{\text{GL}}\delta) \right),
 \end{aligned}
 \tag{3.13}$$

where $Q' := \frac{1}{q^{-5}-q^{-4}-q^{-2}+q^{-1}}$.

We now construct a bicovariant FODC $(\Gamma_{P_{\text{GL}}}, d_{P_{\text{GL}}})$ on $\mathcal{O}_q(P_{\text{GL}}) := A/\langle \gamma \rangle$. This is done via the Hopf algebra quotient $\pi_{P_{\text{GL}}} : A \rightarrow \mathcal{O}_q(P_{\text{GL}})$, $a \mapsto \pi_{P_{\text{GL}}}(a) := [a]$, using Corollary 3.8. We recall that $\Gamma_{P_{\text{GL}}} := \Gamma_{\text{GL}}/\Gamma_{I_\gamma}$, where $\Gamma_{I_\gamma} := A \cdot d_{\text{GL}}\langle \gamma \rangle + \langle \gamma \rangle d_{\text{GL}}A \subseteq \Gamma_{\text{GL}}$. Writing $[\omega]$ for the equivalence class of $\omega \in \Gamma_{\text{GL}}$ on the quotient $\pi_{P_{\text{GL}}}^\Gamma : \Gamma_{\text{GL}} \rightarrow \Gamma_{P_{\text{GL}}}$, the differential on $\Gamma_{P_{\text{GL}}}$ is defined by $d_{P_{\text{GL}}} : \mathcal{O}_q(P_{\text{GL}}) \rightarrow \Gamma_{P_{\text{GL}}}$, $d_{P_{\text{GL}}}[a] := [d_{\text{GL}}a]$ for all $a \in \mathcal{O}_q(\text{GL}_2(\mathbb{C}))$. As a free left $H := \mathcal{O}_q(P_{\text{GL}})$ -module $\Gamma_{P_{\text{GL}}}$ is 3-dimensional with basis $\{[\omega^1], [\omega^3], [\omega^4]\}$.² Denoting the projected generators by

$$\pi_{P_{\text{GL}}} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} =: \begin{pmatrix} t & p \\ 0 & s \end{pmatrix}, \quad \pi_{\text{GL}}(r) =: r',$$

² Classically, the dimension of $\Gamma_{P_{\text{GL}}}$ is 3, so its q -deformation is at least 3-dimensional. From (3.13) it immediately follows that $[\omega^2] = 0$, which proves the claim.

the right H -action and differential are explicitly given by

$$\begin{aligned}
 [\omega^1]t &= qt[\omega^1], & [\omega^1]p &= q^{-1}p[\omega^1], & [\omega^1]s &= q^{-1}s[\omega^1], \\
 [\omega^3]t &= -q^{-1}\lambda p[\omega^1] + t[\omega^3], & [\omega^3]p &= p[\omega^3], & [\omega^3]s &= s[\omega^3], \\
 [\omega^4]t &= q^{-1}t[\omega^4], & [\omega^4]p &= q^{-1}\lambda^2 p[\omega^1] - \lambda t[\omega^3] + qp[\omega^4], & [\omega^4]s &= q^{-1}\lambda^2 s[\omega^1] + qs[\omega^4]
 \end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
 d_{P_{GL}}t &= \frac{q-1}{\lambda}t[\omega^1] + \frac{q^{-1}-1}{\lambda}t[\omega^4], \\
 d_{P_{GL}}p &= Qp[\omega^1] + \frac{q-1}{\lambda}p[\omega^4] - t[\omega^3], \quad d_{P_{GL}}s = Qs[\omega^1] + \frac{q-1}{\lambda}s[\omega^4].
 \end{aligned} \tag{3.15}$$

Example 3.10. (Bicovariant FODC on $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ and its parabolic quotient) Consider the quotient Hopf algebra $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C})) := \mathcal{O}_q(\mathrm{GL}_2(\mathbb{C}))/\langle \det_q - 1 \rangle$ with coalgebra structure and antipode induced from (3.9) and (3.10). We denote the associated Hopf algebra map projection by $\mathrm{pr}_{\mathrm{SL}} : \mathcal{O}_q(\mathrm{GL}_2(\mathbb{C})) \rightarrow \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$. The quantum group $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ is the Manin deformation (see [30]) of the ring of algebraic functions on the complex special linear group $\mathrm{SL}_2(\mathbb{C})$. It is the deformed algebra of regular functions on the complex special linear group $\mathrm{SL}_2(\mathbb{C})$. Let $\mathcal{O}_q(P)$ be the deformed algebra of functions on the Borel subgroup $P \subset \mathrm{SL}_2(\mathbb{C})$. We identify it with the quotient $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))/I_P$ where $I_P \subseteq \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ is the Hopf ideal generated by the element γ . On generators the Hopf algebra projection is given by

$$\pi_P : \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C})) \rightarrow \mathcal{O}_q(P), \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix} \tag{3.16}$$

or, in other words, $\mathcal{O}_q(P) = \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))/I_P = \mathbb{C}\langle t, t^{-1}, p \rangle / \langle tp - q^{-1}pt \rangle$. Note that $\mathcal{O}_q(P) = \mathcal{O}_q(P_{GL})/\langle \det_q - 1 \rangle \cong \mathcal{O}_q(\mathrm{GL}_2(\mathbb{C}))/\langle \gamma, \det_q \rangle$ and the corresponding projection $\mathrm{pr}_P : \mathcal{O}_q(P_{GL}) \rightarrow \mathcal{O}_q(P)$ makes the diagram

$$\begin{array}{ccc}
 \mathcal{O}_q(\mathrm{GL}_2(\mathbb{C})) & \xrightarrow{\mathrm{pr}_{\mathrm{SL}}} & \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C})) \\
 \pi_{P_{GL}} \downarrow & & \downarrow \pi_P \\
 \mathcal{O}_q(P_{GL}) & \xrightarrow{\mathrm{pr}_P} & \mathcal{O}_q(P)
 \end{array} \tag{3.17}$$

commute.

There are bicovariant FODCi $(\Gamma_{\mathrm{SL}}^+, d_{\mathrm{SL}}^+)$ on $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ and (Γ_P^+, d_P^+) on $\mathcal{O}_q(P)$ induced from Example 3.9 as the quotient calculi

$$\begin{array}{ccc}
 \Gamma_{\mathrm{GL}} & \xrightarrow{\mathrm{pr}_{\mathrm{SL}}^\Gamma} & \Gamma_{\mathrm{SL}}^+ \\
 d_{\mathrm{GL}} \uparrow & & \uparrow d_{\mathrm{SL}}^+ \\
 \mathcal{O}_q(\mathrm{GL}_2(\mathbb{C})) & \xrightarrow{\mathrm{pr}_{\mathrm{SL}}} & \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))
 \end{array} \qquad
 \begin{array}{ccc}
 \Gamma_{P_{GL}} & \xrightarrow{\mathrm{pr}_P^\Gamma} & \Gamma_P^+ \\
 d_{P_{GL}} \uparrow & & \uparrow d_P^+ \\
 \mathcal{O}_q(P_{GL}) & \xrightarrow{\mathrm{pr}_P} & \mathcal{O}_q(P)
 \end{array} .$$

By the commutativity of the square diagram in (3.17) and recalling that the modules of 1-forms are generated by the algebras and differentials we obtain the commutative cube

$$\begin{array}{ccccc}
 & & \Gamma_{P_{GL}} & \xrightarrow{\text{pr}_P^\Gamma} & \Gamma_P^+ \\
 & \nearrow \pi_{P_{GL}}^\Gamma & \uparrow \text{pr}_{SL}^\Gamma & & \nearrow \pi_P^\Gamma \\
 \Gamma_{GL} & \xrightarrow{\quad} & \Gamma_{SL}^+ & \xrightarrow{\quad} & \Gamma_P^+ \\
 \uparrow d_{GL} & & \uparrow \text{pr}_P & & \uparrow d_P^+ \\
 \mathcal{O}_q(\text{GL}_2(\mathbb{C})) & \xrightarrow{\pi_{P_{GL}}} & \mathcal{O}_q(P_{GL}) & \xrightarrow{\text{pr}_P} & \mathcal{O}_q(P) \\
 & \searrow \pi_{P_{GL}} & \downarrow \text{pr}_{SL} & & \searrow \pi_P \\
 & & \mathcal{O}_q(\text{SL}_2(\mathbb{C})) & \xrightarrow{\quad} & \mathcal{O}_q(P)
 \end{array}$$

To be more explicit, the bicovariant FODC $(\Gamma_{SL}^+, d_{SL}^+)$ on $\mathcal{O}_q(\text{SL}_2(\mathbb{C}))$ is 4-dimensional with basis of left coinvariant 1-forms $\{\omega^1, \omega^2, \omega^3, \omega^4\}$. Up to the identification $\det_q = 1$ the commutation relations and differentials coincide with (3.11) and (3.12). The bicovariant calculus Γ_P^+ is only 2-dimensional: one of the three basis vectors $[\omega^1], [\omega^3], [\omega^4]$ of $\Gamma_{P_{GL}}$ vanishes on Γ_P^+ , namely $[\omega^1]$. Indeed, denoting equivalence classes on Γ_P^+ under $\text{pr}_P^\Gamma : \Gamma_{P_{GL}} \rightarrow \Gamma_P^+$ by $[\cdot]'$ and using (3.8) and (3.12) we obtain

$$0 = [d(\alpha\delta)]' = [d(\alpha)\delta]' + [\alpha d\delta]' = \left[\left(\frac{q-1}{\lambda} + qQ \right) \omega^1 \right]',$$

which implies $[\omega^1]' = 0$. Then, the commutation relations (3.14) and the differentials (3.15) project to

$$\begin{aligned}
 [\omega^3]'t &= t[\omega^3]', \quad [\omega^3]'p = p[\omega^3]', \quad [\omega^4]'t = q^{-1}t[\omega^4]', \quad [\omega^4]'p = qp[\omega^4]' - \lambda t[\omega^3]', \\
 d_P^+ t &= \frac{q^{-1}-1}{\lambda} t[\omega^4]', \quad d_P^+ p = -t[\omega^3]' + \frac{q-1}{\lambda} p[\omega^4]'
 \end{aligned}$$

on $\mathcal{O}_q(P)$, where we identified t, p as generators in $\mathcal{O}_q(P)$.

Example 3.11. (3-dim calculus on $\mathcal{O}_q(\text{SL}_2(\mathbb{C}))$ and its parabolic quotient $\mathcal{O}_q(P)$) Let $A := \mathcal{O}_q(\text{SL}_2(\mathbb{C}))$ and $H := \mathcal{O}_q(P)$ be the Hopf algebras from Example 3.10. There is a 3-dimensional left covariant FODC (Γ_{SL}, d_{SL}) on A described in [26] Sect. 14.1.4. Γ_{SL} is the free left A -module generated by the basis $\{\omega^0, \omega^1, \omega^2\}$ of left coinvariant 1-forms with commutation relations

$$\begin{aligned}
 \omega^j \alpha &= q^3 \alpha \omega^j, & \omega^j \beta &= q^{-3} \beta \omega^j, \\
 \omega^j \gamma &= q^3 \gamma \omega^j, & \omega^j \delta &= q^{-3} \delta \omega^j,
 \end{aligned} \tag{3.18}$$

for $j = 0, 2$ and

$$\begin{aligned}
 \omega^1 \alpha &= q^2 \alpha \omega^1 + (q^2 - 1) \beta \omega^2, & \omega^1 \beta &= q^{-2} \beta \omega^1 + (q^2 - 1) \alpha \omega^0, \\
 \omega^1 \gamma &= q^2 \gamma \omega^1 + (q^2 - 1) \delta \omega^2, & \omega^1 \delta &= q^{-2} \delta \omega^1 + (q^2 - 1) \gamma \omega^0.
 \end{aligned} \tag{3.19}$$

The differential $d_{SL} : A \rightarrow \Gamma_{SL}$ is determined by

$$\begin{aligned}
 d_{SL} \alpha &= \alpha \omega^1 + \beta \omega^2, & d_{SL} \beta &= \alpha \omega^0 - q^{-2} \beta \omega^1, \\
 d_{SL} \gamma &= \gamma \omega^1 + \delta \omega^2, & d_{SL} \delta &= \gamma \omega^0 - q^{-2} \delta \omega^1
 \end{aligned}$$

and thus

$$\omega^0 = \delta d_{\text{SL}}\beta - q\beta d_{\text{SL}}\delta, \quad \omega^1 = \delta d_{\text{SL}}\alpha - q\beta d_{\text{SL}}\gamma, \quad \omega^2 = -q^{-1}\gamma d_{\text{SL}}\alpha + \alpha d_{\text{SL}}\gamma.$$

As in Example 3.10 elements in the equivalence class H are denoted by $[\omega]$ with a representative $\omega \in \Gamma_{\text{SL}}$. The induced quotient calculus (Γ_P, d_P) on H is the 2-dimensional left covariant FODC with Γ_P being the free left H -module generated by the basis $\{[\omega^0], [\omega^1]\}$ of left coinvariant elements (notice that $[\omega^2] = 0$, while $[\omega^0], [\omega^1]$ are linearly independent). The resulting commutation relations are

$$\begin{aligned} [\omega^0]t &= q^3t[\omega^0], & [\omega^0]p &= q^{-3}p[\omega^0], \\ [\omega^1]t &= q^2t[\omega^1], & [\omega^1]p &= q^{-2}p[\omega^1] + (q^2 - 1)t[\omega^0] \end{aligned}$$

and the differential reads

$$d_P t = t[\omega^1], \quad d_P p = t[\omega^0] - q^{-2}p[\omega^1], \quad d_P t^{-1} = -q^{-2}t^{-1}[\omega^1].$$

3.4. The smash product calculus. In this section we recall the construction of a covariant differential calculus on the smash product algebra $B\#H$ from an H -module calculus on an H -module algebra B and a bicovariant calculus on the Hopf algebra H , given in [36]. If the H -action on B is trivial we recover the tensor product differential calculus on $B \otimes H$ described in Proposition 3.2.

Let B be a left H -module algebra, with action $\triangleright : H \otimes B \rightarrow B$. Let M be a B -bimodule, with actions $\cdot : B \otimes M \rightarrow M$, $\cdot : M \otimes B \rightarrow M$, and a left H -module with action that with slight abuse we denote $\triangleright : H \otimes M \rightarrow M$. We say that M is a *relative H -module B -bimodule* if the H and B actions have the compatibility, for all $h \in H, b, b' \in B, m \in M$,

$$h \triangleright (b \cdot m \cdot b') = (h_1 \triangleright b) \cdot (h_2 \triangleright m) \cdot (h_3 \triangleright b').$$

Similar definitions are given if M is just a left or a right B -module.

Definition 3.12. Let B be a left H -module algebra with action $\triangleright : H \otimes B \rightarrow B$. A FODC (Γ_B, d_B) on B is said to be an *H -module FODC* if for any $b^i, b_i \in B, i = 1, 2, \dots, n$, ($n \in \mathbb{N}$) and $h \in H$ we have

$$\sum_i b^i d_B b_i = 0 \Rightarrow \sum_i (h_1 \triangleright b^i) d_B (h_2 \triangleright b_i) = 0. \tag{3.20}$$

The rationale of this definition is in the following proposition.

Proposition 3.13. (Γ_B, d_B) is an H -module FODC if and only if Γ_B is a relative H -module B -bimodule and $d_B : B \rightarrow \Gamma_B$ is an H -module map: for all $h \in H, b \in B, h \triangleright d_B b = d_B (h \triangleright b)$.

Proof. Since (3.20) holds we have a well-defined H -action given by,

$$H \otimes \Gamma_B \rightarrow \Gamma_B, \quad h \triangleright \sum_i (b^i d_B b_i) := \sum_i (h_1 \triangleright b^i) d_B (h_2 \triangleright b_i)$$

where we used that $\Gamma_B = B d_B B$, so that $\sum_i b^i d_B b_i$ is a generic element of Γ_B . Trivially $d_B : B \rightarrow \Gamma_B$ is an H -module map. Compatibility of this action with the left B -action

is immediate. Compatibility with the right B -action: $h \triangleright ((\sum_i b^i d_B b_i)b) = (h_1 \triangleright \sum_i b^i d_B b_i)(h_2 \triangleright b)$, follows writing $(\sum_i b^i d_B b_i)b = \sum_i b^i d_B(b_i b) - \sum_i b^i b_i d_B b$.

Vice versa, if Γ_B is a relative H -module B -bimodule and $d_B : B \rightarrow \Gamma_B$ is an H -module map, the implication (3.20) follows from its equivalence with $\sum_i b^i d_B b_i = 0 \Rightarrow h \triangleright \sum_i (b^i d_B b_i) = 0$. \square

Given an H -module algebra B and a FODC (Γ_B, d_B) and a left covariant FODC (Γ_H, d_H) , we consider the \mathbb{k} -module

$$\Gamma_{\#} := \Gamma_B \otimes H \oplus B \otimes \Gamma_H \tag{3.21}$$

and study when there is a FODC $(\Gamma_{\#}, d_{\#})$ on $B\#H$. The \mathbb{k} -module $\Gamma_{\#}$ in (3.21) is a direct sum of tensor products of left H -modules hence it carries a left H -action canonically induced from the H -actions on the H -modules Γ_B, H, B, Γ_H : for all $h \in H, \omega_B \otimes h' + b' \otimes \omega_H \in \Gamma_{\#}$,

$$h \cdot (\omega_B \otimes h' + b' \otimes \omega_H) = h_1 \triangleright \omega_B \otimes h_2 h' + h_1 \triangleright b' \otimes h_2 \omega_H,$$

extended linearly to all $\Gamma_{\#}$. Defining the left B -action on $\Gamma_{\#}$ as the B -action on the first factors in the tensor products $\Gamma_B \otimes H$ and $B \otimes \Gamma_H$ we obtain the left $B\#H$ -action on $\Gamma_{\#}$ using that $b\#h = (b\#1_H)(1_B\#h)$:

$$(b\#h) \cdot (\omega_B \otimes h' + b' \otimes \omega_H) := b(h_1 \triangleright \omega_B) \otimes h_2 h' + b(h_1 \triangleright b') \otimes h_2 \omega_H. \tag{3.22}$$

The proof that this indeed defines an action of the algebra $B\#H$ on $\Gamma_{\#}$, actually a $B\#H$ -action on the submodules $\Gamma_B \otimes H$ and $B \otimes \Gamma_H$, uses the same steps of the proof of associativity of the multiplication in $B\#H$. We define

$$(\omega_B \otimes h' + b' \otimes \omega_H) \cdot b := \omega_B(h'_1 \triangleright b) \otimes h'_2 + b'((\omega_H)_{-1} \triangleright b) \otimes (\omega_H)_0$$

and extend it linearly to all $\Gamma_{\#}$. It is easy to prove that this is a right B -action on $\Gamma_{\#}$. Defining the right H -action on $\Gamma_{\#}$ as the right H -action on the second factors in the tensor products $\Gamma_B \otimes H$ and $B \otimes \Gamma_H$ we obtain the right $B\#H$ -action on $\Gamma_{\#}$

$$(\omega_B \otimes h' + b' \otimes \omega_H) \cdot (b\#h) := \omega_B(h'_1 \triangleright b) \otimes h'_2 h + b'((\omega_H)_{-1} \triangleright b) \otimes (\omega_H)_0 h. \tag{3.23}$$

We prove commutativity of the left and right $B\#H$ -actions on $\Gamma_B \otimes H$:

$$\begin{aligned} (b\#h) \cdot ((\omega_B \otimes h') \cdot (\tilde{b}\tilde{h})) &= (b\#h) \cdot (\omega_B(h'_1 \triangleright \tilde{b}) \otimes h'_2 \tilde{h}) \\ &= b \cdot (h_1 \triangleright (\omega_B(h'_1 \triangleright \tilde{b})) \otimes h_2 h'_2 \tilde{h}) \\ &= b(h_1 \triangleright \omega_B)(h_2 h'_1 \triangleright \tilde{b}) \otimes h_3 h'_2 \tilde{h} \\ &= (b(h_1 \triangleright \omega_B) \otimes h_2 h') \cdot (\tilde{b}\tilde{h}) \\ &= ((b\#h) \cdot (\omega_B \otimes h')) \cdot (\tilde{b}\tilde{h}). \end{aligned}$$

Commutativity of the left and right $B\#H$ -actions on $B \otimes \Gamma_H$ is similarly proven using the left H -comodule structure of Γ_H . This shows that $\Gamma_{\#}$ is a $B\#H$ -bimodule. We sometimes write $\Gamma_B\#H$ and $B\#\Gamma_H$ in order to stress that we consider them as bimodules over $B\#H$ instead of $B \otimes H$. As in [36] Thm. 2.7 we have a FODC on $B\#H$ with $B\#H$ -bimodule $\Gamma_{\#}$.

Theorem 3.14. *Let H be a Hopf algebra and B a left H -module algebra. Given an H -module FODC (Γ_B, d_B) on B and a left covariant FODC (Γ_H, d_H) on H there is a FODC $(\Gamma_{\#}, d_{\#})$ on $B\#H$, where the \mathbb{k} -module $\Gamma_{\#} := \Gamma_B \otimes H \oplus B \otimes \Gamma_H$ is endowed with the $B\#H$ -bimodule actions (3.22), (3.23) and the exterior derivative $d_{\#} : B\#H \rightarrow \Gamma_{\#}$ is defined by*

$$d_{\#}(b\#h) := d_B b \otimes h + b \otimes d_H h$$

for all $b \in B$ and $h \in H$.

Proof. We show that $d_{\#} : B\#H \rightarrow \Gamma_{\#}$ satisfies the Leibniz rule:

$$\begin{aligned} d_{\#}((b\#h)(b'\#h')) &= d_{\#}(b(h_1 \triangleright b')\#h_2h') \\ &= d_B(b(h_1 \triangleright b')) \otimes h_2h' + b(h_1 \triangleright b') \otimes d_H(h_2h') \\ &= (d_B b)(h_1 \triangleright b') \otimes h_2h' + b d_B(h_1 \triangleright b') \otimes h_2h' \\ &\quad + b(h_1 \triangleright b') \otimes (d_H h_2)h' + b(h_1 \triangleright b') \otimes h_2 d_H h' \\ &= d_{\#}(b\#h) \cdot (b'\#h') + (b\#h) \cdot d_{\#}(b'\#h') \end{aligned}$$

for all $b, b' \in B$ and $h, h' \in H$. We are left to prove that $\Gamma_{\#} = (B\#H) \cdot d_{\#}(B\#H)$. Let $b, b' \in B$ and $h, h' \in H$. Then

$$\begin{aligned} b d_B b' \otimes h &= b d_B b' \otimes h + b b' \otimes d_H h - b b' \otimes d_H h \\ &= (b\#1) \cdot d_{\#}(b'\#h) - (b b'\#1) \cdot d_{\#}(1\#h) \end{aligned}$$

and $b \otimes h d_H h' = (b\#h) \cdot d_{\#}(1\#h')$ establish the equality in question. □

The smash product construction of differential calculi is compatible with right H -coactions.

Corollary 3.15. *Let H be a Hopf algebra, B a left H -module algebra, (Γ_B, d_B) an H -module FODC on B and (Γ_H, d_H) a bicovariant FODC on H . The FODC $(\Gamma_{\#}, d_{\#})$ of Theorem 3.14 is then right H -covariant.*

Proof. Define a right H -coaction on $\Gamma_{\#}$ via

$$\begin{aligned} \Gamma_{\#} &\xrightarrow{\Delta_{\#}} \Gamma_{\#} \otimes H \\ \omega_B \otimes h + b \otimes \omega_H &\longmapsto \omega_B \otimes h_1 \otimes h_2 + b \otimes (\omega_H)_0 \otimes (\omega_H)_1. \end{aligned}$$

We prove that the calculus is right H -covariant by showing right H -colinearity of the differential, cf. Proposition 3.6 (i). For all $b\#h \in B\#H$,

$$\begin{aligned} \Delta_{\#}(d_{\#}(b\#h)) &= \Delta_{\#}(d_B b \otimes h + b \otimes d_H h) \\ &= d_B b \otimes h_1 \otimes h_2 + b \otimes d_H h_1 \otimes h_2 \\ &= d_{\#}(b\#h_1) \otimes h_2 \\ &= (d_{\#} \otimes \text{id}_A) \Delta_{\#}(b\#h). \end{aligned}$$

□

3.5. Base forms, horizontal forms and principal covariant calculi. In this section we study differential calculi on noncommutative principal bundles over affine bases. We assume the ground ring \mathbb{k} to be a field and recall that the Hopf algebra (quantum structure group) H is assumed to have invertible antipode. We have seen that in this setting a principal comodule algebra $B = A^{\text{co}H} \subseteq A$ is equivalently a faithfully flat Hopf Galois extension $B = A^{\text{co}H} \subseteq A$.

Definition 3.16. Let (Γ_A, d_A) be a FODC on a right H -comodule algebra A . We call the pullback calculus $(\Gamma_B, d_B) := (Bd_A|_B B, d_A|_B)$ on $B := A^{\text{co}H} \subseteq A$ the FODC of *base forms*. We further call $\Gamma^{\text{hor}} := A\Gamma_B$, the (A, B) -bimodule of *horizontal forms*.

If (Γ_A, d_A) is a right H -covariant FODC, base and horizontal forms can be further characterized.

Theorem 3.17. *Let A be a principal comodule algebra, $B := A^{\text{co}H}$ and (Γ_A, d_A) a right H -covariant FODC on A . The natural map $A \otimes_B \Gamma_B \rightarrow \Gamma_A$, $a \otimes \omega \mapsto a\omega$ is injective and gives the left A -module isomorphism*

$$A \otimes_B \Gamma_B \cong A\Gamma_B.$$

The B -bimodule of base forms is the intersection of those of horizontal and coinvariant forms

$$\Gamma_B = \Gamma_A^{\text{hor}} \cap \Gamma_A^{\text{co}H}.$$

Proof. The inclusion $A\Gamma_B \subseteq \Gamma_A$ implies $(A\Gamma_B)^{\text{co}H} \subseteq (\Gamma_A)^{\text{co}H}$ and, since A is a flat B -module, we have the inclusion

$$A \otimes_B (A\Gamma_B)^{\text{co}H} \longrightarrow A \otimes_B (\Gamma_A)^{\text{co}H} \cong \Gamma_A \tag{3.24}$$

where for the isomorphism $A \otimes_B (\Gamma_A)^{\text{co}H} \cong \Gamma_A$ we used Theorem 2.9. The inclusion $\Gamma_B \subseteq (A\Gamma_B)^{\text{co}H}$ and flatness of A over B imply the inclusion

$$A \otimes_B \Gamma_B \longrightarrow A \otimes_B (A\Gamma_B)^{\text{co}H} \tag{3.25}$$

that composed with (3.24) gives injectivity of the natural map $A \otimes_B \Gamma_B \rightarrow \Gamma_A$ and hence the isomorphism $A \otimes_B \Gamma_B \cong A\Gamma_B$.

From Theorem 2.9 we then have $\Gamma_B \cong (A \otimes_B \Gamma_B)^{\text{co}H} \cong (A\Gamma_B)^{\text{co}H}$, that is, $\Gamma_B = \Gamma_A^{\text{hor}} \cap \Gamma_A^{\text{co}H}$. □

We now consider a FODC (Γ_H, d_H) on H , with (Γ_A, d_A) that is not necessarily right H -covariant.

Definition 3.18. A FODC (Γ_A, d_A) on a principal comodule algebra A together with a left covariant FODC (Γ_H, d_H) on H , is called a *principal calculus* on A if we have the short exact sequence

$$0 \rightarrow A \otimes_B \Gamma_B \rightarrow \Gamma_A \xrightarrow{\text{ver}} A\Box_H\Gamma_H \rightarrow 0 \tag{3.26}$$

where $A\Box_H\Gamma_H := \{a_\ell \otimes \omega_\ell^H \in A \otimes \Gamma_H \mid \delta_A(a_\ell) \otimes \omega_\ell^H = a_\ell \otimes \Delta_L^{\Gamma_H}(\omega_\ell^H)\}$ (sum over ℓ understood) is the *cotensor product* of A and Γ_H , and the *vertical map* $\text{ver} : \Gamma_A \rightarrow A\Box_H\Gamma_H$ is well-defined as a \mathbb{k} -linear map by

$$\text{ver}(ad_A a') := a_0 a'_0 \otimes a_1 d_H a'_1. \tag{3.27}$$

If, in addition, the FODC (Γ_A, d_A) is right H -covariant and the FODC (Γ_H, d_H) is bicovariant, we say we have a *principal covariant calculus* on the principal comodule algebra A .

We can easily check that ver , if it is well-defined, it satisfies

$$\text{ver}(a \cdot \omega \cdot a') = \delta_A(a)\text{ver}(\omega)\delta_A(a') \tag{3.28}$$

for all $a, a' \in A$ and $\omega \in \Gamma_A$. The sequence is therefore a sequence of left A -modules right B -modules.

The following lemma provides a sufficient condition for the existence of the vertical map.

Lemma 3.19. *Let $\pi : A \rightarrow H$ be a Hopf algebra quotient and A be a principal comodule algebra. For any left covariant calculus (Γ_A, d_A) on A and induced left covariant quotient calculus (Γ_H, d_H) on H the vertical map is well-defined.*

Proof. Let us denote the left A -coaction on Γ_A by $\Delta_L^{\Gamma_A} : \Gamma_A \rightarrow A \otimes \Gamma_A$, $\omega^A \mapsto \omega_{-1}^A \otimes \omega_0^A$, and the quotient map of forms by $\pi_\Gamma : \Gamma_A \rightarrow \Gamma_H$. Then

$$\begin{aligned} (\text{id}_A \otimes \pi_\Gamma) \circ \Delta_L^{\Gamma_A} : \Gamma_A &\rightarrow A \otimes \Gamma_H \\ \omega^A &\mapsto \omega_{-1}^A \otimes [\omega_0^A] \end{aligned}$$

is a well-defined map. We prove that this map coincides with ver . Recall that the induced right H -coaction on A is given by $\delta_A := (\text{id}_A \otimes \pi) \circ \Delta : A \rightarrow A \otimes H$, $a \mapsto a_0 \otimes a_1 := a_1 \otimes [a_2]$. For $a, a' \in A$ we obtain by the left covariance of (Γ_A, d_A)

$$\begin{aligned} (\text{id}_A \otimes \pi_\Gamma)(\Delta_L^{\Gamma_A}(ad_A a')) &= (ad_A a')_{-1} \otimes [(ad_A a')_0] = a_1 a'_1 \otimes [a_2 d_A a'_2] \\ &= a_1 a'_1 \otimes [a_2] d_H [a'_2] = a_0 a'_0 \otimes a_1 d_H a'_1, \end{aligned}$$

i.e. $(\text{id}_A \otimes \pi_\Gamma) \circ \Delta_L^{\Gamma_A} = \text{ver}$, as claimed. □

For a right H -covariant calculus on a principal comodule algebra the second arrow in (3.26) is injective by Theorem 3.17. In this context of principal comodule algebras the notion of principal covariant calculus of Definition 3.18 is then equivalent to that of strong quantum principal bundle in [7] Def. 4.9 and [5] Sect. 5.4. This follows from the canonical identifications $A \otimes_B \Gamma_B \cong A\Gamma_B$, cf. Theorem 3.17, and $A \square_H \Gamma_H \cong A \otimes^{\text{co}H} \Gamma_H$, where ${}^{\text{co}H} \Gamma_H$ denotes the module of left coinvariant one forms characterizing the covariant FODC (Γ_H, d_H) (see [47]). In this light Theorem 3.17, which does not assume an exact sequence, can be read as generalizing the conditions of [5] Cor. 5.53 for the equality $\Gamma_B = \Gamma_A^{\text{hor}} \cap \Gamma_A^{\text{co}H}$ to hold true.

We present two examples of principal covariant calculus, further examples including a principal calculus are in Sect. 4.5.

Example 3.20. (Smash product calculus) Let (Γ_H, d_H) be a bicovariant FODC on a Hopf algebra H and (Γ_B, d_B) an H -module FODC on a left H -module algebra B . Then, the smash product calculus $(\Gamma_\#, d_\#)$ of Sect. 3.4 is a principal covariant calculus on $B\#H$.

Proof. Recall from Sect. 2.3.1 that $B = (B\#H)^{\text{co}H} \subseteq B\#H$ is a trivial Hopf–Galois extension. We show that the sequence in (3.26) is equivalent to the exact sequence

$$0 \rightarrow \Gamma_{B\#H} \rightarrow \Gamma_B \# \Gamma_H \oplus B \# \Gamma_H \xrightarrow{\text{pr}} B \# \Gamma_H \rightarrow 0.$$

From [36] Thm. 4.1 and Lem. 4.2 we have the isomorphisms of right H -covariant $B\#H$ -bimodules

$$\begin{aligned} (B\#H) \otimes_B \Gamma_B &\cong \Gamma_B \# H, & (b\#h) \otimes_B \omega^B &\mapsto b(h_1 \triangleright \omega^B) \# h_2, \\ (B\#H) \square_H \Gamma_H &\cong B \# \Gamma_H, & (b\#h) \otimes \omega^H &\mapsto b \# \epsilon(h) \omega^H. \end{aligned} \tag{3.29}$$

Their inverses are given by $\omega^B \# h \mapsto (1 \# h_2) \otimes_B (\overline{S}(h_1) \triangleright \omega^B)$ and $b \# \omega^H \mapsto (b \# \omega_{-1}^H) \otimes \omega_0^H$, respectively. Under this first isomorphism the second arrow in (3.26) becomes the inclusion $\Gamma_B \# H \rightarrow \Gamma_B \# H \oplus B \# \Gamma_H$. Finally, the vertical map equals the projection $\text{pr} : \Gamma_B \# H \oplus B \# \Gamma_H \rightarrow B \# \Gamma_H$ up to the second isomorphism in (3.29) (cf. e.g. [36] Thm. 2.9). \square

Example 3.21. (3D calculus on the q -monopole fibration) Consider $A = \mathcal{O}_q(\text{SL}_2(\mathbb{C}))$ (q real, $q \neq \pm 1$) with the Hopf algebra quotient

$$\pi : A \rightarrow H, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{pmatrix}.$$

The projection π induces a right H -coaction:

$$\delta_A := (\text{id} \otimes \pi) \circ \Delta : A \rightarrow A \otimes H, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \dot{\otimes} \begin{pmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{pmatrix}.$$

The subalgebra $B := A^{\text{co}H}$ of coinvariants is given by the (complex) Podleś sphere $\mathcal{O}_q(\mathbb{S}^2)$ with generators $B_- = \alpha\beta$, $B_+ = \gamma\delta$, $B_0 = \beta\gamma$ and commutation relations

$$B_{\pm} B_0 = q^{\pm 2} B_0 B_{\pm}, \quad B_- B_+ = q^{-2} B_+ B_- + q^{-2}(q^{-1} - q) B_0^2.$$

(see [37] and refs. therein). It is known that $B \subseteq A$ is a faithfully flat Hopf–Galois extension, i.e. that A is a principal comodule algebra [32], (cf. also [8] Ex. 6.26). The 3-dimensional left covariant FODC on A introduced in [48] is H -bicovariant. It induces a 2-dimensional FODC on B via the algebra embedding $B \rightarrow A$ and a 1-dimensional bicovariant FODC on H via the quotient map π . This data is a principal covariant calculus on A , see [5] Ex. 5.51.

We next show that for a principal covariant calculus on a principal comodule algebra A the Hopf–Galois extension $B = A^{\text{co}H} \subseteq A$ lifts to a Hopf–Galois extension of graded algebras. Recall that for a bicovariant H -bimodule Γ_H we obtain a graded Hopf algebra $\Omega_H^{\leq 1} := H \oplus \Gamma_H$. The multiplication is given by that in H and by the H -module structure, while the product of degree one elements is trivial. The comultiplication $\Delta_{\Omega_H^{\leq 1}} : \Omega_H^{\leq 1} \rightarrow \Omega_H^{\leq 1} \otimes \Omega_H^{\leq 1}$ has components on degree zero and one $\Delta^0 := \Delta : H \rightarrow H \otimes H$ and

$$\Delta^1 := \Delta_R^{\Gamma_H} + \Delta_L^{\Gamma_H} : \Gamma_H \rightarrow \Gamma_H \otimes H \oplus H \otimes \Gamma_H.$$

The antipode $S_{\Omega_H^{\leq 1}} : \Omega_H^{\leq 1} \rightarrow \Omega_H^{\leq 1}$ has components $S^0 = S : H \rightarrow H$ and

$$S^1 : \Gamma_H \rightarrow \Gamma_H \\ \omega \mapsto -S(\omega_{-1})\omega_0 S(\omega_1).$$

Lemma 3.22. *Consider a principal comodule algebra A with a principal covariant calculus (Γ_A, d_A) . The graded algebra $\Omega_A^{\leq 1} := A \oplus \Gamma_A$ is a graded right $\Omega_H^{\leq 1}$ -comodule algebra with right coaction $\Delta_{\Omega_A^{\leq 1}} : \Omega_A^{\leq 1} \rightarrow \Omega_A^{\leq 1} \otimes \Omega_H^{\leq 1}$ given by its components $\Delta_{\Omega_A^{\leq 1}}^0 := \delta_A : A \rightarrow A \otimes H$ and*

$$\Delta_{\Omega_A^{\leq 1}}^1 := \Delta_R^{\Gamma_A} + \text{ver} : \Gamma_A \rightarrow \Gamma_A \otimes H \oplus A \otimes \Gamma_H. \tag{3.30}$$

Defining $\Omega_B^{\leq 1} := B \oplus \Gamma_B \subseteq \Omega_A^{\leq 1}$, we further have that $\Omega_B^{\leq 1} = (\Omega_A^{\leq 1})^{\text{co}\Omega_H^{\leq 1}}$ is the subalgebra of $\Omega_H^{\leq 1}$ -coinvariants.

Proof. By assumption $\Delta_{\Omega_A^{\leq 1}}^0 = \delta_A : A \rightarrow A \otimes H \subseteq A \otimes \Omega_H^{\leq 1}$ is a right $\Omega_H^{\leq 1}$ -coaction and an algebra map. Similarly also $\Delta_{\Omega_A^{\leq 1}}$ is an $\Omega_H^{\leq 1}$ -coaction and an algebra map. For the coaction property we just have to prove $(\Delta_{\Omega_A^{\leq 1}} \otimes \text{id}) \circ \Delta_{\Omega_A^{\leq 1}} = (\text{id} \otimes \Delta_{\Omega_H^{\leq 1}}) \circ \Delta_{\Omega_A^{\leq 1}}$ on 1-forms. This is a straightforward computation using the right H -covariance of (Γ_A, d_A) , the definition of the vertical map and the bicovariance of (Γ_H, d_H) . The coaction $\Delta_{\Omega_A^{\leq 1}}$ is an algebra map because for all $a, a' \in A$ and $\omega \in \Gamma_A$ we have

$$\Delta_{\Omega_A^{\leq 1}}(a \cdot \omega \cdot a') = \delta_A(a)\Delta_{\Omega_A^{\leq 1}}(\omega)\delta_A(a') = \Delta_{\Omega_A^{\leq 1}}(a)\Delta_{\Omega_A^{\leq 1}}(\omega)\Delta_{\Omega_A^{\leq 1}}(a'),$$

where we used that Γ_A is a right H -covariant A -bimodule and that the vertical map satisfies the compatibility condition (3.28).

The algebra inclusion $B \subseteq A$ implies the graded algebra one $\Omega_B^{\leq 1} = B \oplus \Gamma_B \subseteq \Omega_A^{\leq 1}$. The equality $\Omega_B^{\leq 1} = (\Omega_A^{\leq 1})^{\text{co}\Omega_H^{\leq 1}}$ is obvious in degree zero. In degree one, recalling definition (3.30) and that the codomain of $\Delta_{\Omega_A^{\leq 1}}$ is a direct sum, the coinvariance condition $\Delta_{\Omega_A^{\leq 1}}(\omega) = \omega \otimes 1$ implies $\Delta_R^{\Gamma_A}(\omega) = \omega \otimes 1$ and $\text{ver}(\omega) = 0$. These equalities respectively imply $\omega \in \Gamma_A^{\text{co}H}$ and $\omega \in \Gamma_A^{\text{hor}}$, this latter condition following from exactness of the sequence (3.26). Then, from Theorem 3.17 we obtain $\Gamma_B = \Gamma^{\text{hor}} \cap \Gamma^{\text{co}H}$, which implies $(\Gamma_A)^{\text{co}\Omega_H^{\leq 1}} = \Gamma_B$. In conclusion, $(\Omega_A^{\leq 1})^{\text{co}\Omega_H^{\leq 1}} = \Omega_B^{\leq 1}$. \square

In the following we prove that principal covariant calculi are equivalent to graded Hopf–Galois extensions with compatible differentials.

Theorem 3.23. *Let A be a principal comodule algebra and (Γ_A, d_A) a principal covariant calculus on A , with corresponding bicovariant FODC (Γ_H, d_H) on H . Then*

$$\Omega_B^{\leq 1} = (\Omega_A^{\leq 1})^{\text{co}\Omega_H^{\leq 1}} \subseteq \Omega_A^{\leq 1} \tag{3.31}$$

is a faithfully flat Hopf–Galois extension of graded algebras. Moreover we have the following commutative diagram

$$\begin{array}{ccc} \Gamma_A & \xrightarrow{\Delta_{\Omega_A^{\leq 1}}^1} & \Gamma_{A \otimes H} \\ d_A \uparrow & & \uparrow d_{A \otimes H} \\ A & \xrightarrow{\delta_A} & A \otimes H \end{array} \tag{3.32}$$

where we endow $A \otimes H$ with the tensor product calculus $(\Gamma_{A \otimes H}, d_{A \otimes H})$.

Conversely, if (Γ_A, d_A) is a FODC on a right H -comodule algebra A and (Γ_H, d_H) a bicovariant FODC on H such that (3.31) is a faithfully flat Hopf–Galois extension of graded algebras and the diagram in (3.32) commutes, then A is a principal comodule algebra and (Γ_A, d_A) a principal covariant calculus on A .

Proof. If (Γ_A, d_A) is a principal covariant calculus on A , $B = A^{\text{co}H} \subseteq A$ is a faithfully flat Hopf–Galois extension, the sequence (3.26) is exact and by Lemma 3.22 $\Omega_A^{\leq 1}$ is a graded right $\Omega_H^{\leq 1}$ -comodule algebra. Then from [40] Cor. 5.9, $\Omega_B^{\leq 1} = (\Omega_A^{\leq 1})^{\text{co}\Omega_H^{\leq 1}} \subseteq \Omega_A^{\leq 1}$ is a Hopf–Galois extension which is faithfully flat as a left $\Omega_B^{\leq 1}$ -module. Moreover, the diagram (3.32) commutes by the right H -covariance of (Γ_A, d_A) and the definition (3.27) of the vertical map.

Conversely, if A is a right H -comodule algebra with a FODC (Γ_A, d_A) and a bicovariant FODC (Γ_H, d_H) on H such that (3.31) is a faithfully flat Hopf–Galois extension (so that in particular $B \subseteq A$ is a faithfully flat Hopf–Galois extension) then from [40] Cor. 5.9 the sequence

$$0 \rightarrow A \otimes_B \Gamma_B \rightarrow \Gamma_A \xrightarrow{\Theta} A \square_H \Gamma_H \rightarrow 0 \tag{3.33}$$

is exact, where Θ is the projection of $\Delta_{\Omega_A^{\leq 1}}^1 : \Gamma_A \rightarrow \Omega_A^{\leq 1} \otimes \Omega_H^{\leq 1}$ to $A \otimes \Gamma_H$. If we assume commutativity of the diagram in (3.32) it follows that

$$\Delta_R^{\Gamma_A}(d_A a) + \Theta(d_A a) = d_A(a_0) \otimes a_1 + a_0 \otimes d_H a_1$$

for all $a \in A$, which implies $\Delta_R^{\Gamma_A}(d_A a) = d_A(a_0) \otimes a_1$ and $\Theta(d_A a) = a_0 \otimes d_H a_1$ for all $a \in A$. Since Θ and $\Delta_R^{\Gamma_A}$ are A -bilinear (where $A \otimes H$ is an A -bimodule via δ_A) the first equality implies that (Γ_A, d_A) is a right H -covariant calculus, the second one that Θ is the vertical map. Thus (3.33) is the exact sequence showing the principality of the differential calculus (Γ_A, d_A) . \square

4. A Sheaf-Theoretic Approach to Noncommutative Calculi

In this section we introduce differential calculi on quantum principal bundles in a two-step process. First we define a covariant calculus on a sheaf of comodule algebras \mathcal{F} as a sheaf Υ of \mathcal{F} -bimodules together with a morphism $d : \mathcal{F} \rightarrow \Upsilon$ of sheaves, requiring locally Leibniz rule and a surjectivity condition. In case \mathcal{F} is a quantum principal bundle we demand an additional local compatibility of the calculi on the total sheaf, the base sheaf and the structure Hopf algebra.

4.1. Quantum principal bundles as sheaves. We start by recalling the main notions and results of [4]. Those concern the (function) algebra level. In the remaining sections we generalize the definitions and findings to FODCi.

A *quantum ringed space* (M, \mathcal{O}_M) is a pair consisting of a classical topological space M and a sheaf \mathcal{O}_M on M of noncommutative algebras. We call a sheaf of H -comodule algebras \mathcal{F} an *H -principal bundle* or *quantum principal bundle* (QPB) over (M, \mathcal{O}_M) if there exists an open covering $\{U_i\}$ of M such that:

- (i) $\mathcal{F}(U_i)^{\text{co}H} = \mathcal{O}_M(U_i)$,
- (ii) \mathcal{F} is *locally principal*, that is, $\mathcal{O}_M(U_i) \subseteq \mathcal{F}(U_i)$ is a principal comodule algebra.

If the base ring is a field local principality is equivalent to faithfully flatness of the local Hopf–Galois extensions. If these Hopf–Galois extensions on the open cover $\{U_i\}$ are cleft or trivial we say that the QPB is *locally cleft* or *locally trivial*, respectively.

Remark 4.1. (Principal comodule algebras restrict locally) In [11] Lem. 4.1 it is shown that the pushforward of a strong connection $\ell : H \rightarrow A \otimes A$ by a comodule algebra map $\phi : A \rightarrow A'$ is a strong connection $(\phi \otimes \phi) \circ \ell$ on A' . Recalling that if the base ring \mathbb{k} is a field there is a bijective correspondence of strong connections on comodule algebras with principal comodule algebras (cf. the paragraph after Definition 2.8) it follows that the existence of a comodule algebra map $\phi : A \rightarrow A'$ implies that if A is a principal comodule algebra, so is A' .

Since \mathcal{F} is a sheaf of H -comodule algebras, for any inclusion $U \subset U_i$ of an open U of the topology of M in an open U_i of the covering $\{U_i\}$, the restriction $r_{UU_i} : \mathcal{F}(U_i) \rightarrow \mathcal{F}(U)$, which by definition is a comodule algebra map, then implies that $\mathcal{F}(U)$ is a principal comodule algebra.

Let now \mathcal{F}_p be the stalk of \mathcal{F} at $p \in M$. Choosing U_i such that $p \in U_i$ and considering the canonical map $\mathcal{F}(U_i) \rightarrow \mathcal{F}_p$ of right H -comodule algebras, defined by $s \mapsto [(U_i, s)]_p$, where the equivalence class $[(U_i, s)]_p \in \mathcal{F}_p$ is the germ at p of the section $s \in \mathcal{F}(U_i)$, we obtain that the stalk \mathcal{F}_p is a principal comodule algebra.

Our main examples of QPBs have a quantum group as total space. Let G be a complex semisimple algebraic group and P a parabolic subgroup. The quotient G/P is a projective variety and the projection $G \rightarrow G/P$ is a principal bundle. Let $\mathcal{O}_q(G)$ and $\mathcal{O}_q(P) = \mathcal{O}_q(G)/I_P$ be Hopf algebras over $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$, quantizations of $\mathcal{O}(G)$ and $\mathcal{O}(P)$, the coordinate algebras of G and P , respectively (see [4] Sect. 3, [10] Sect. 3). We shall later specialize q to a complex number in order to consider Hopf–Galois extensions with \mathbb{C} the base ring. Once we fix a projective embedding for G/P , obtained through the global sections of a very ample line bundle \mathcal{L} , we have the graded algebra $\mathcal{O}(G/P)$. We say that $s \in \mathcal{O}_q(G)$ is a *quantum section* if

$$(\text{id} \otimes \pi)\Delta(s) = s \otimes \pi(s), \quad s \equiv t \pmod{q - 1},$$

where $\pi : \mathcal{O}_q(G) \rightarrow \mathcal{O}_q(P)$, Δ is the coproduct of $\mathcal{O}_q(G)$ and t is a *classical section*, that is, it lifts to $\mathcal{O}(G)$ the character of P defining \mathcal{L} (see [4] Def. 3.6).

Denote by $\{s_i\}_{i \in \mathcal{I}}$ a choice of linearly independent elements in $\Delta(s) = \sum_{i \in \mathcal{I}} s^i \otimes s_i$, that we assume to generate the homogeneous coordinate ring $\mathcal{O}_q(G/P)$ quantization of the commutative homogeneous coordinate ring $\mathcal{O}(G/P)$ (\mathcal{L} is very ample). Assume furthermore that $S_i = \{s_i^r, r \in \mathbb{Z}_{\geq 0}\}$ is Ore, denote by $\mathcal{O}_q(G)S_i^{-1}$ the Ore localization, and assume that subsequent Ore localizations do not depend on the order. In the specialization for $q = 1$, the s_i 's define an open cover of G , whose projection on $M = G/P$ gives an open cover $\{U_i\}$ of M (see [4], Sect. 4 for more details). Let us define the open sets

$$U_I := U_{i_1} \cap \dots \cap U_{i_r}, \quad I = (i_1, \dots, i_r)$$

with $r = 0, 1, 2, \dots, n$ (here n is the cardinality of \mathcal{I} and the empty set \emptyset corresponds to $r = 0$). We consider on M the topology generated by the open sets U_I , which hence form a basis $\mathcal{B} = \{U_I\}_{I \in \mathcal{II}}$, where \mathcal{II} is the set of ordered multi-indices $I = (i_1, \dots, i_r)$, $1 \leq i_1 < \dots < i_r \leq n$ with $r = 1, 2, \dots, n$ (and we also consider the case $r = 0$ corresponding to the empty set).

The main result in [4] is Theorem 4.8 that states the following.

Theorem 4.2. *Let the notation be as above. Then*

(i) *the assignment*

$$\begin{aligned}
 U_I \mapsto \mathcal{O}_M(U_I) &:= \mathbb{C}_q[s_{k_1}s_{i_1}^{-1}, \dots, s_{k_r}s_{i_r}^{-1}; 1 \leq k_1 \leq n, \dots, 1 \leq k_r \leq n] \\
 &\subset \mathcal{O}_q(G)S_{i_1}^{-1} \dots S_{i_r}^{-1}
 \end{aligned}
 \tag{4.1}$$

defines a sheaf \mathcal{O}_M on $M = G/P$,

(ii) *the assignment*

$$U_I \mapsto \mathcal{F}_G(U_I) := \mathcal{O}_q(G)S_{i_1}^{-1} \dots S_{i_r}^{-1}, \quad I = (i_1, \dots, i_r)
 \tag{4.2}$$

defines a sheaf \mathcal{F}_G of right $\mathcal{O}_q(P)$ -comodule algebras on the topological space M ,

(iii) $\mathcal{F}_G^{\text{co}\mathcal{O}_q(P)} = \mathcal{O}_M$, i.e., the subsheaf $\mathcal{F}_G^{\text{co}\mathcal{O}_q(P)} : U \rightarrow \mathcal{F}_G(U)^{\text{co}\mathcal{O}_q(P)} \subset \mathcal{F}_G(U)$ is canonically isomorphic to the sheaf \mathcal{O}_M .

Remark 4.3. We clarify the sheafification implicitly understood in [4] Thm. 4.8. The restriction morphisms

$$\begin{aligned}
 r_{IJ}^M : \mathcal{O}_M(U_J) &\rightarrow \mathcal{O}_M(U_I) \\
 r_{IJ} : \mathcal{F}_G(U_J) &\rightarrow \mathcal{F}_G(U_I),
 \end{aligned}
 \tag{4.3}$$

$U_I \subseteq U_J$ (i.e. $J \subseteq I$), for \mathcal{O}_M and \mathcal{F}_G are given by the natural morphisms. Thus, (4.1) and (4.2) determine presheaves \mathcal{O}_M and \mathcal{F}_G on the basis \mathcal{B} .

Define $\mathcal{F}_G^{\text{co}\mathcal{O}_q(P)}(U_I) := \mathcal{F}_G(U_I)^{\text{co}\mathcal{O}_q(P)}$ for all I and define the restriction morphisms

$$r_{IJ}^{\text{co}H} := r_{IJ}|_{\mathcal{F}_G^{\text{co}H}(U_J)} : \mathcal{F}_G^{\text{co}H}(U_J) \rightarrow \mathcal{F}_G^{\text{co}H}(U_I),
 \tag{4.4}$$

which are well-defined because H -colinearity of the r_{IJ} in (4.3) implies $r_{IJ}(\mathcal{F}_G^{\text{co}H}(U_J)) \subseteq \mathcal{F}_G^{\text{co}H}(U_I)$. The restriction morphisms $r_{IJ}^{\text{co}H}$ inherit the \mathcal{B} -presheaf properties from the r_{IJ} , so that $\mathcal{F}_G^{\text{co}\mathcal{O}_q(P)}$ is a \mathcal{B} -presheaf. Using the technique of [4] Prop. 4.7, one can prove the \mathcal{B} -presheaf equality $\mathcal{F}_G^{\text{co}\mathcal{O}_q(P)} = \mathcal{O}_M$. Using the following observation these \mathcal{B} -presheaves are extended to sheaves on the full topological space, these are the sheaves of Theorem 4.2.

Observation 4.4. If we have a presheaf $\mathcal{G}_{\mathcal{B}}$ defined on a basis \mathcal{B} for the topology, we can always extend it to the presheaf \mathcal{G} , where $\mathcal{G}(U) := \varprojlim \mathcal{G}_{\mathcal{B}}(V)$ for $V \subset U, V \in \mathcal{B}$, provided the target category admit inverse limits (see [17] Chpt. 0, Sect. 3.2.1). If \mathcal{B} is finite the existence of $\varprojlim \mathcal{G}_{\mathcal{B}}(V)$ is granted. Then, once we have a presheaf, we can always proceed to the sheafification and obtain a sheaf on the topology with the same target category (groups, H -comodule algebras, etc). By universality, a morphism of presheaves on a basis will extend uniquely first to a morphisms of presheaves and then to a morphisms of their sheafification (cf. [20] Chpt. 2 and also [14] for more details on this standard construction). To ease the notation, we shall use the same letter to denote a presheaf on \mathcal{B} , its extension to a presheaf and the sheafification.

The topology of M used in Theorem 4.2 is finite since it is induced by a finite cover $\{U_i\}_{i \in \mathcal{I}}$ of M via the finite basis $\mathcal{B} = \{U_I\}_{I \in \mathcal{I}\mathcal{I}}$. We can then define the *minimal opens*

$$U_p := \bigcap_{U_I \ni p} U_I \in \mathcal{B}.
 \tag{4.5}$$

Given a presheaf \mathcal{F} on M , its stalk at $p \in M$ coincides with its sections on U_p , i.e. $\mathcal{F}_p = \mathcal{F}(U_p)$. This is immediate from the definition of stalk since any open neighbourhood of p includes U_p . Now the stalks of the sheafification of \mathcal{F} coincide with the stalks of the initial presheaf, and so \mathcal{F} coincides with its sheafification on the opens U_p .

4.2. Covariant calculi on sheaves of comodule algebras. We now give the definition of a covariant FODC on a sheaf of comodule algebras. The bimodule property is entirely captured in a sheaf-theoretic language, while we demand the Leibniz rule and surjectivity property of the differential on stalks. To account for coinvariance we have to consider the differential as a morphism of sheaves of comodules. We give some fundamental examples of covariant FODCi on sheaves and discuss the sheafs of base forms, horizontal forms and coinvariant forms.

Given a morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves on M we denote the induced morphism on the stalks at $p \in M$ by $\varphi_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$. We also recall that, for \mathcal{F} a sheaf of algebras on a topological space M , a sheaf of \mathcal{F} -modules associates to each open U in M an $\mathcal{F}(U)$ -module with compatible restriction maps.

Definition 4.5. Let \mathcal{F} be a sheaf of algebras on a topological space M . A FODC on \mathcal{F} is a sheaf Υ of \mathcal{F} -bimodules on M with a morphism

$$d : \mathcal{F} \longrightarrow \Upsilon$$

of sheaves, such that for all $p \in M$ the induced maps on stalks $d_p : \mathcal{F}_p \rightarrow \Upsilon_p$ satisfy

- (i) the Leibniz rule $d_p(fg) = (d_p f)g + f d_p g$ for all $f, g \in \mathcal{F}_p$
- (ii) the surjectivity condition $\Upsilon_p = \mathcal{F}_p d_p \mathcal{F}_p$.

If H is a Hopf algebra and \mathcal{F} a sheaf of right H -comodule algebras over M we call a FODC (Υ, d) on \mathcal{F} *right H -covariant* if Υ is a sheaf of right H -covariant \mathcal{F} -bimodules and d is a morphism of sheaves of right H -comodules.

Similarly one defines left H -covariant and H -bicovariant FODCi on sheaves of left H -comodule algebras and H -bicomodule algebras, respectively.

Example 4.6. (i) Let M be an algebraic variety and G an (affine) algebraic group acting on M . Then, \mathcal{O}_M , the structural sheaf of M , carries an $H = \mathcal{O}(G)$ coaction, where $\mathcal{O}(G)$ denotes the global sections of the structural sheaf of G , which carries a natural Hopf algebra structure. Define Ω as the sheaf of Kähler differentials with $d : \mathcal{O}_M \longrightarrow \Omega$ as in [20], Sec. 8, II. As one can readily check, using the results in [20], (Ω, d) is a FODC on \mathcal{O}_M .

- (ii) Let the algebraic variety $M = G$ be a simple complex algebraic group and let P be a parabolic subgroup of G . We can view the principal bundle $\pi : G \longrightarrow G/P$ in the sheaf-theoretic language. An example of FODC is again given by the sheaf of Kähler differentials on G . Since G is a principal bundle, we can also define base forms as the differential forms over G/P : they are obtained by considering horizontal forms invariant under the natural P action. We are going to give a quantum version of this Kähler differential construction in Sect. 4.4. The explicit example of $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ is discussed in Sect. 4.5.

We now turn to examine base, horizontal and coinvariant forms in the sheaf-theoretic context.

Let \mathcal{F} be a sheaf of right H -comodule algebras. As for the \mathcal{B} -presheaf $\mathcal{F}_G^{\text{co}O_q(P)}$, we define the presheaf $\mathcal{F}^{\text{co}H}$ of algebras by assigning to every open U of M the algebra $\mathcal{F}^{\text{co}H}(U) := \mathcal{F}(U)^{\text{co}H}$ with restriction morphisms, cf. (4.4),

$$r_{VU}^{\text{co}H} := r_{VU}|_{\mathcal{F}^{\text{co}H}(U)} : \mathcal{F}^{\text{co}H}(U) \rightarrow \mathcal{F}^{\text{co}H}(V). \tag{4.6}$$

The presheaf $\mathcal{F}^{\text{co}H}$ is a subsheaf of \mathcal{F} , indeed it is the kernel of the sheaf morphism $\delta_R - \text{id} \otimes 1_H : \mathcal{F} \rightarrow \mathcal{F} \otimes H$.

Definition 4.7. Let (Υ, d) be a FODC on a sheaf \mathcal{F} of right H -comodule algebras over M . For U open in M we define the $\mathcal{F}^{\text{co}H}(U)$ -subbimodules of

$$\begin{aligned} \text{base forms} \quad \quad \quad \Upsilon_M(U) &:= \mathcal{F}^{\text{co}H}(U)d\mathcal{F}^{\text{co}H}(U), \\ \text{horizontal forms} \quad \quad \quad \Upsilon^{\text{hor}}(U) &:= \mathcal{F}(U)\Upsilon_M(U). \end{aligned}$$

If (Υ, d) is a right H -covariant FODC on \mathcal{F} we define the $\mathcal{F}^{\text{co}H}(U)$ -subbimodule of

$$\text{right } H\text{-coinvariant forms} \quad \Upsilon^{\text{co}H}(U) := \{\omega \in \Upsilon(U) \mid \Delta_R^U(\omega) = \omega \otimes 1\},$$

where $\Delta_R^U : \Upsilon(U) \rightarrow \Upsilon(U) \otimes H$ is the right H -coaction on $\Upsilon(U)$.

Proposition 4.8. *The assignments*

$$\Upsilon_M : U \mapsto \Upsilon_M(U), \quad \Upsilon^{\text{hor}} : U \mapsto \Upsilon^{\text{hor}}(U), \quad \Upsilon^{\text{co}H} : U \mapsto \Upsilon^{\text{co}H}(U),$$

with restriction morphisms

$$\begin{aligned} r_{VU}^{\Upsilon_M} &:= r_{VU}^{\Upsilon}|_{\Upsilon_M(U)} : \Upsilon_M(U) \rightarrow \Upsilon_M(V), \\ r_{VU}^{\text{hor}} &:= r_{VU}^{\Upsilon}|_{\Upsilon^{\text{hor}}(U)} : \Upsilon^{\text{hor}}(U) \rightarrow \Upsilon^{\text{hor}}(V), \\ r_{VU}^{\Upsilon^{\text{co}H}} &:= r_{VU}^{\Upsilon}|_{\Upsilon^{\text{co}H}(U)} : \Upsilon^{\text{co}H}(U) \rightarrow \Upsilon^{\text{co}H}(V), \end{aligned} \tag{4.7}$$

where $r_{VU}^{\Upsilon} : \Upsilon(U) \rightarrow \Upsilon(V)$ denotes the restriction morphism of Υ , define sheaves of $\mathcal{F}^{\text{co}H}$ -bimodules.

Proof. We first prove that the maps in (4.7) are well-defined, that is they map in the claimed codomains (which are submodules of $\Upsilon(V)$). We show for example $r_{VU}^{\Upsilon_M}(\Upsilon_M(U)) \subseteq \Upsilon_M(V)$. Let $f^i, g_i \in \mathcal{F}^{\text{co}H}(U) \subseteq \mathcal{F}(U)$ we have

$$r_{VU}^{\Upsilon_M}(f^i d_U g_i) = r_{VU}^{\Upsilon}(f^i d_U g_i) = r_{VU}(f^i) r_{VU}^{\Upsilon}(d_U g_i) = r_{VU}^{\text{co}H}(f^i) d_V r_{VU}^{\text{co}H}(g_i) \in \Upsilon_M(V),$$

where we used that r^{Υ} is left \mathcal{F} -linear, d is a morphism of sheaves and (4.6). Similarly r_{VU}^{hor} is well-defined. The proof for $r_{VU}^{\Upsilon^{\text{co}H}}$ is straightforward from H -colinearity of r_{VU}^{Υ} . The maps in (4.7) are $\mathcal{F}^{\text{co}H}(U)$ -bimodule maps (the $\mathcal{F}^{\text{co}H}(U)$ -bimodule structure on the images being given via $r_{VU}^{\text{co}H}$) because they are restrictions of the $\mathcal{F}(U)$ -bimodule map r_{VU}^{Υ} . Finally, the morphisms in (4.7) for V, U open in M define the presheaves $\Upsilon_M, \Upsilon^{\text{hor}}, \Upsilon^{\text{co}H}$ since they are restrictions of the morphisms r_{VU}^{Υ} defining the presheaf Υ . Since $\Upsilon^{\text{co}H}$ is the kernel of the sheaf morphism $\Delta_R - \text{id} \otimes 1_H : \Upsilon \rightarrow \Upsilon \otimes H$, it is a sheaf. By common abuse of notation we shall denote by Υ_M and Υ^{hor} the sheafifications of the corresponding presheaves. These are sheaves in the category of $\mathcal{F}^{\text{co}H}$ -bimodules. \square

Corollary 4.9. *Let (Υ, d) be a FODC on a sheaf \mathcal{F} of right H -comodule algebras over M . Then (Υ_M, d_M) with $d_M := d|_{\mathcal{F}^{\text{co}H}} : \mathcal{F}^{\text{co}H} \rightarrow \Upsilon_M$ is a FODC on $\mathcal{F}^{\text{co}H}$.*

Proof. We already noted that the presheaf $\mathcal{F}^{\text{co}H}$ is a subsheaf of \mathcal{F} . Since d_M is the restriction of $d : \mathcal{F} \rightarrow \Upsilon$ to $\mathcal{F}^{\text{co}H}$ it is a morphism of sheaves and satisfies the Leibniz rule on stalks. On the level of presheaves we have the surjectivity condition on stalks $\mathcal{F}_p^{\text{co}H} (d_M)_p \mathcal{F}_p^{\text{co}H} = (\Upsilon_M)_p$, for any $p \in M$. Since the stalks of the presheaf coincide with the stalks of its sheafification we conclude that the surjectivity condition of the sheaf morphism d_M is satisfied (cf. the analogue of (ii) in Definition 4.5). \square

Let $\Upsilon^{\text{co}H} \cap \Upsilon^{\text{hor}}$ denote the sheafification of the presheaf obtained intersecting the subsheaves $\Upsilon^{\text{co}H}$ and Υ^{hor} of Υ .

Theorem 4.10. *Let the base ring \mathbb{k} be a field. For any right H -covariant FODC (Υ, d) on a QPB \mathcal{F} we have an isomorphism*

$$\Upsilon_M \cong \Upsilon^{\text{co}H} \cap \Upsilon^{\text{hor}}$$

of sheaves of \mathcal{O}_M -bimodules.

Proof. The natural inclusion $\Upsilon_M \rightarrow \Upsilon^{\text{co}H} \cap \Upsilon^{\text{hor}}$ (by which we mean the sheafification of the corresponding morphism of presheaves) is a morphism of sheaves. Recalling from Remark 4.1 that stalks are principal comodule algebras, the induced inclusion on stalks, $(\Upsilon_M)_p \rightarrow \Upsilon_p^{\text{co}H} \cap \Upsilon_p^{\text{hor}}$, is an isomorphism thanks to Theorem 3.17. Thus $\Upsilon_M \rightarrow \Upsilon^{\text{co}H} \cap \Upsilon^{\text{hor}}$ is an isomorphism of sheaves. \square

4.3. Ore extension of covariant calculi. As a tool for the following sections, we now discuss the Ore extension of covariant calculi. We recall that if $a \in A$ is a (right) Ore element, we denote by AS^{-1} or by $A[a^{-1}]$ the Ore localization of A at the multiplicative right Ore set $S = \{a^m\}_{m \in \mathbb{Z}_{\geq 0}}$ (for more details see [28] Chpt. 4, Sect. 10 and [24] Sect. I.7.).

Let (Γ, d) be a FODC on an algebra A . Let a be an Ore element in A . We define the $A[a^{-1}]$ -bimodule

$$\Gamma_a := A[a^{-1}] \Gamma A[a^{-1}] := A[a^{-1}] \otimes_A \Gamma \otimes_A A[a^{-1}] \tag{4.8}$$

and the \mathbb{k} -linear map

$$d_a : A[a^{-1}] \longrightarrow \Gamma_a, \quad d_a(a') = \begin{cases} da' & a' \in A \\ -a^{-1} da a^{-1} & a' = a^{-1} \end{cases}, \tag{4.9}$$

where we extend d_a to $A[a^{-1}]$ by the Leibniz rule.

Lemma 4.11. (Γ_a, d_a) is a FODC on $A[a^{-1}]$.

Proof. We need to verify the properties of Definition 3.1. The first two properties are satisfied by definition. Moreover by the equality

$$a' da'' a^{-n} = a' d_a(a'' a^{-n}) - a' a'' d_a(a^{-n}),$$

where $a', a'' \in A$ and $n \in \mathbb{N}$, we see that we can express any element of Γ_a as an element in $A[a^{-1}] d_a A[a^{-1}]$. \square

With a slight abuse of terminology, we call (Γ_a, d_a) the *localization* of (Γ, d) via the Ore element $a \in A$.

We now show that the covariant properties of the FODC Γ are preserved under localization.

Lemma 4.12. *Let (Γ, d) be a right H -covariant FODC on a right H -comodule algebra (A, δ_R) and $a \in A$ an Ore element. Assume that $\delta_R(a) \in A \otimes H$ is invertible. Then*

- (i) δ_R extends to a right H -coaction $\delta_R^a : A[a^{-1}] \rightarrow A[a^{-1}] \otimes H$, structuring $(A[a^{-1}], \delta_R^a)$ as a right H -comodule algebra;
- (ii) the FODC (Γ_a, d_a) on $(A[a^{-1}], \delta_R^a)$ is right H -covariant;

Proof. (i) We set $\delta_R^a(a') := \delta_R(a')$ for all $a' \in A$, furthermore $\delta_R^a(a^{-1}) := \delta_R(a)^{-1}$ and extend δ_R^a as an algebra homomorphism to $A[a^{-1}]$. From the coaction properties of δ_R it follows that δ_R^a is a right H -coaction on $A[a^{-1}]$ and by construction it is compatible with the algebra structure of $A[a^{-1}]$.

- (ii) There is a canonical right H -coaction Δ_R^a on $\Gamma_a := A[a^{-1}] \otimes_A \Gamma \otimes_A A[a^{-1}]$ given by the diagonal coaction. We show that Δ_R^a structures Γ_a as a right H -covariant $A[a^{-1}]$ -bimodule such that d_a is right H -colinear. The claim that (Γ_a, d_a) is a right H -covariant FODC on $A[a^{-1}]$ then follows from Proposition 3.6. Consider an arbitrary element $b^i \otimes_A \omega^i \otimes_A c^i \in \Gamma_a$ with $b^i, c^i \in A[a^{-1}]$ and $\omega^i \in \Gamma$, sum over i understood. Since, according to (i), δ_R^a is an algebra homomorphism, we obtain

$$\begin{aligned} \Delta_R^a(b \cdot (b^i \otimes_A \omega^i \otimes_A c^i) \cdot c) &= (bb^i)_0 \otimes_A \omega_0^i \otimes_A (c^i c)_0 \otimes (bb^i)_1 \omega_1^i (c^i c)_1 \\ &= b_0 (b_0^i \otimes_A \omega_0^i \otimes_A c_0^i) c_0 \otimes b_1 (b_1^i \omega_1^i c_1^i) c_1 \\ &= \delta_R^a(b) \Delta_R^a(b^i \otimes_A \omega^i \otimes_A c^i) \delta_R^a(c) \end{aligned}$$

for all $b, c \in A[a^{-1}]$. That is Γ_a is a right H -covariant $A[a^{-1}]$ -bimodule. Next, applying $d_a \otimes \text{id}_H$ to the equation $1_A \otimes 1_H = \delta_R(a) \delta_R^a(a^{-1})$ we obtain

$$\begin{aligned} 0 &= (da_0 \otimes a_1) \cdot \delta_R^a(a^{-1}) + \delta_R^a(a) \cdot (d_a a_0^{-1} \otimes a_1^{-1}) \\ &= \Delta_R(da) \cdot \delta_R^a(a^{-1}) + \delta_R^a(a) \cdot (d_a a_0^{-1} \otimes a_1^{-1}), \end{aligned}$$

where in the last expression we used right H -colinearity of d . This implies

$$\begin{aligned} d_a a_0^{-1} \otimes a_1^{-1} &= -\delta_R^a(a^{-1}) \cdot \Delta_R(da) \cdot \delta_R^a(a^{-1}) = -\Delta_R^a(a^{-1} d(a) a^{-1}) \\ &= \Delta_R^a(d_a(a^{-1})). \end{aligned} \tag{4.10}$$

The Leibniz rule of d_a , the algebra homomorphism property of δ_R^a , the A -bilinearity of Δ_R^a and (4.10) then imply that d_a is right H -colinear on any power of a^{-1} and furthermore on any element of $A[a^{-1}]$. □

Let A be an algebra, $a_i \in A, i = 1, \dots, n$ be Ore elements and $S_i = \{a_i^r, r \in \mathbb{Z}_{\geq 0}\}$. For a multi-index $I = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$ we use the notation $A_I = AS_{i_1}^{-1} \dots S_{i_k}^{-1}$, or equivalently the notation

$$A_I = A_{(i_1, \dots, i_k)} := A[a_{i_1}^{-1}, \dots, a_{i_k}^{-1}] := (\dots (A[a_{i_1}^{-1}]) \dots)[a_{i_k}^{-1}]$$

for the k -fold Ore localization. Given a FODC (Γ, d) on A we similarly define the k -fold Ore localization of the A -bimodule Γ by

$$\begin{aligned} \Gamma_I &= \Gamma_{(i_1, \dots, i_k)} := (\cdots (\Gamma_{a_{i_1}})_{a_{i_2}} \cdots)_{a_k} \\ &:= A_I \cdots \otimes_{A_{(i_1, i_2)}} (A_{(i_1, i_2)} \otimes_{A_{i_1}} (A_{i_1} \otimes_A \Gamma \otimes_A A_{i_1})) \otimes_{A_{i_1}} A_{(i_1, i_2)} \otimes_{A_{(i_1, i_2)}} \cdots A_I \end{aligned}$$

and of the exterior derivative by

$$d_I = d_{(i_1, \dots, i_k)} := (\cdots (d_{a_{i_1}})_{a_{i_2}} \cdots)_{a_k} : A_I \rightarrow \Gamma_I. \tag{4.11}$$

Lemma 4.13. *Let A be an algebra and $a_i \in A, i = 1, \dots, n$ be Ore elements such that subsequent localizations do not depend on the order. Then, subsequent localizations of a FODC on A are independent of the order, as well.*

Proof. Consider a FODC (Γ, d) on A . For two Ore elements $a_i, a_j \in A$, using that $A_{(i, j)} = A_{(j, i)}$ we have

$$\begin{aligned} (\Gamma_{a_i})_{a_j} &= A_{(i, j)} \otimes_{A_i} (A_i \otimes_A \Gamma \otimes_A A_i) \otimes_{A_i} A_{(i, j)} \\ &= A_{(i, j)} \otimes_A \Gamma \otimes_A A_{(i, j)} \\ &= A_{(j, i)} \otimes_{A_j} (A_j \otimes_A \Gamma \otimes_A A_j) \otimes_{A_j} A_{(j, i)} \\ &= (\Gamma_{a_j})_{a_i} \end{aligned}$$

as an equation of $A_{(i, j)}$ -bimodules. In particular,

$$d_{(i, j)}, d_{(j, i)} : A_{(i, j)} \rightarrow \Gamma_{(i, j)}$$

have the same domain and codomain. They coincide on A and by definition (4.9) they also agree on a_i^{-1} and a_j^{-1} . Both differentials are extended by the Leibniz rule, therefore $d_{(i, j)} = d_{(j, i)}$. An easy inductive argument on the number of Ore localization then proves the lemma. \square

This lemma shows in particular that the k -fold Ore localization of the A -bimodule Γ is $\Gamma_I = A_I \otimes_A \Gamma \otimes_A A_I$, that for short we frequently write as $\Gamma_I = A_I \Gamma A_I$. Recalling also Lemma 4.12 the following proposition is straightforward.

Proposition 4.14. *Let (Γ, d) be a FODC on a right H -comodule algebra (A, δ_R) and $a_i \in A, i = 1, \dots, n$ be Ore elements such that $\delta_R(a_i)$ are invertible elements of $A_i \otimes H$. Then*

$$\Gamma_I = A_I \otimes_A \Gamma \otimes_A A_I, \quad d_I : A_I \rightarrow \Gamma_I$$

is a right H -covariant FODC on A_I . If subsequent Ore localizations of A are independent on the order also the subsequent localizations of the FODC (Γ, d) are order independent.

Remark 4.15. We have considered FODC on Ore extended algebras. In Sect. 3.5, associated with a FODC (Γ_A, d_A) there is the graded algebra of 0- and 1-forms $\Omega_A^{\leq 1} = A \oplus \Gamma_A$. It is natural to consider also Ore extensions of this graded algebra (which are necessarily by degree zero elements). In this case an Ore element a is in particular Ore in the zero degree subalgebra A and we also require to be able to order on the right (and on the left) of 1-forms the negative powers a . Thus, if $a \in A$ is Ore in $\Omega_A^{\leq 1}$ the graded algebra $A_a \oplus \Gamma_a$ with Γ_a defined in (4.8) is the Ore extension of $A \oplus \Gamma$ corresponding to a . This Ore condition on 1-forms is met in the examples we shall consider.

4.4. *Covariant differential calculi on sheaves over quantum projective varieties.* In this section we consider all modules over the complex numbers \mathbb{C} . When considering the quantizations $\mathcal{O}_q(G), \mathcal{O}_q(P)$ we therefore specialize q to be in \mathbb{C} . Fix a complex semisimple algebraic group G with parabolic subgroup P , quantizations $\mathcal{O}_q(G), \mathcal{O}_q(P)$ and the sheaves \mathcal{F}_G and \mathcal{O}_M as in Theorem 4.2, where the QPB property of \mathcal{F}_G is with respect to the finite open cover $\{U_i\}_{i \in \mathcal{I}}, \mathcal{I} = \{1, \dots, n\}$ determined by a quantum section. Consider the (finite) topology generated by the open cover; it will have topological basis $\mathcal{B} = \{U_I\}_{I \in \mathcal{II}}, \mathcal{II}$ is the set of ordered multi-indices $I = (i_1, \dots, i_r), 1 \leq i_1 < \dots < i_r \leq n$ with $r = 1, 2, \dots, n$ (and we also consider the case $r = 0$ corresponding to the empty set) and $U_I := \cap_{i \in I} U_i$.

Given a right $\mathcal{O}_q(P)$ -covariant FODC (Γ, d) on $\mathcal{O}_q(G)$ we induce a $\mathcal{O}_q(P)$ -covariant FODC on \mathcal{F} according to Sect. 4.3 as follows. We set on the basis \mathcal{B} of open sets U_I

$$\Upsilon_G(U_I) := \mathcal{F}_G(U_I) \otimes_{\mathcal{O}_q(G)} \Gamma \otimes_{\mathcal{O}_q(G)} \mathcal{F}_G(U_I), \tag{4.12}$$

and define restriction morphisms $r_{JI}^\Upsilon : \Upsilon_G(U_I) \rightarrow \Upsilon_G(U_J)$ for all $U_J \subseteq U_I$ by

$$r_{JI}^\Upsilon(f \otimes_{\mathcal{O}_q(G)} \omega \otimes_{\mathcal{O}_q(G)} g) := r_{JI}(f) \otimes_{\mathcal{O}_q(G)} \omega \otimes_{\mathcal{O}_q(G)} r_{JI}(g), \tag{4.13}$$

where $f, g \in \mathcal{F}_G(U_I)$ and $\omega \in \Gamma$. Since r_{JI} are algebra homomorphisms which are the identity on $\mathcal{O}_q(G)$ the expression (4.13) is well-defined on the algebraic tensor product. The right $\mathcal{O}_q(P)$ -comodule algebra map r_{JI} defines an $\mathcal{F}_G(U_I)$ -bimodule structure on $\Upsilon_G(U_J)$ and $r_{JI}^\Upsilon : \Upsilon_G(U_I) \rightarrow \Upsilon_G(U_J)$ is a map of right $\mathcal{O}_q(P)$ -covariant $\Upsilon_G(U_J)$ -bimodules, i.e., it is $\mathcal{O}_q(P)$ -colinear and $r_{JI}^\Upsilon(f \cdot \theta \cdot g) = r_{JI}(f)r_{JI}^\Upsilon(\theta)r_{JI}(g)$ for all $f, g \in \mathcal{F}_G(U_I)$ and $\theta \in \Upsilon_G(U_I)$. The equality $r_{KJ}^\Upsilon \circ r_{JI}^\Upsilon = r_{KI}^\Upsilon$ for any three opens $U_K \subseteq U_J \subseteq U_I$ then follows from that for the restriction morphisms of \mathcal{F}_G . We have defined the \mathcal{B} -presheaf Υ_G of right $\mathcal{O}_q(P)$ -covariant \mathcal{F}_G -bimodules. We denote its sheafification by the same symbol Υ_G .

Theorem 4.16. (Induced calculus on the sheaf \mathcal{F}_G) *Let (Γ, d) be a right covariant FODC on the Hopf algebra $\mathcal{O}_q(G)$ and $\mathcal{F}_G, \Upsilon_G$ the sheaves of right $\mathcal{O}_q(P)$ -comodule algebras and of right $\mathcal{O}_q(P)$ -covariant \mathcal{F}_G -bimodules defined above.*

(i) *The linear maps*

$$d_I : \mathcal{F}_G(U_I) \longrightarrow \Upsilon_G(U_I),$$

defined in (4.11) for $A_I = \mathcal{F}_G(U_I)$ and $\Gamma_I = \Upsilon_G(U_I)$, define a morphism of sheaves of right $\mathcal{O}_q(P)$ -comodules $d : \mathcal{F}_G \rightarrow \Upsilon_G$. This gives a right $\mathcal{O}_q(P)$ -covariant FODC (Υ_G, d) on the sheaf \mathcal{F}_G .

- (ii) *The FODC (Υ_G, d) on the sheaf \mathcal{F}_G induces a FODC (Υ_M, d_M) on the sheaf $\mathcal{F}_G^{\text{co}\mathcal{O}_q(P)} = \mathcal{O}_M$.*
- (iii) *If \mathcal{F}_G is a QPB the sheaf of base forms is isomorphic, as a sheaf of \mathcal{O}_M -bimodules, to the intersection of that of horizontal and H -coinvariant forms: $\Upsilon_M \cong \Upsilon_G^{\text{hor}} \cap \mathcal{F}_G^{\text{co}\mathcal{O}_q(P)}$.*

Proof. Let \mathcal{F}_G be the \mathcal{B} -presheaf defined in (4.2) and Υ_G the \mathcal{B} -presheaf defined in (4.12), (4.13). We show that the assignment $U_I \mapsto d_I$ defines a morphism of \mathcal{B} -presheaves $d : \mathcal{F}_G \rightarrow \Upsilon_G$. That is, $d_J(r_{JI}(f)) = r_{JI}^\Upsilon(d_I f)$ for all $f \in \mathcal{F}_G(U_I)$

and $U_J \subseteq U_I$. This is clear if $f \in \mathcal{O}_q(G) \subseteq \mathcal{F}_G(U_I)$. Also for $f = s_{i_\ell}^{-1}$, where $1 \leq \ell \leq n$, $i_\ell \in I = (i_1, \dots, i_r)$, we obtain

$$\begin{aligned} r_{JI}^\Upsilon(d_I s_{i_\ell}^{-1}) &= -r_{JI}^\Upsilon(s_{i_\ell}^{-1} \otimes_{\mathcal{O}_q(G)} ds_{i_\ell} \otimes_{\mathcal{O}_q(G)} s_{i_\ell}^{-1}) \\ &= -r_{JI}(s_{i_\ell}^{-1}) \otimes_{\mathcal{O}_q(G)} ds_{i_\ell} \otimes_{\mathcal{O}_q(G)} r_{JI}(s_{i_\ell}^{-1}) \\ &= -s_{i_\ell}^{-1} \otimes_{\mathcal{O}_q(G)} ds_{i_\ell} \otimes_{\mathcal{O}_q(G)} s_{i_\ell}^{-1} \\ &= d_J(s_{i_\ell}^{-1}) \\ &= d_J(r_{JI}(s_{i_\ell}^{-1})). \end{aligned}$$

By the Leibniz rule, cf. Proposition 4.14, this \mathcal{B} -presheaf property extends to any element $f \in \mathcal{F}_G(U_I)$, $d_J(r_{JI}(f)) = r_{JI}^\Upsilon(d_I f)$.

Recalling from Proposition 4.14 that $(\Upsilon_G(U_I), d_I)$ is a right $\mathcal{O}_q(P)$ -covariant FODC and that \mathcal{F}_G and Υ_G are \mathcal{B} -presheaves of right $\mathcal{O}_q(P)$ -comodules, we see that the \mathcal{B} -presheaf morphism $d : \mathcal{F}_G \rightarrow \Upsilon_G$ is compatible with the right $\mathcal{O}_q(P)$ -comodule structure. Furthermore, the Leibniz rule and surjectivity condition are satisfied for all opens U_I in \mathcal{B} . We sheafify according to Observation 4.4 and obtain the sheaf Υ_G of right $\mathcal{O}_q(P)$ -covariant \mathcal{F}_G -bimodules with the sheaf morphism $d : \mathcal{F}_G \rightarrow \Upsilon_G$. Recalling that stalks are preserved under sheafification and that the stalks at p are respectively $(\mathcal{F}_G)_p = \mathcal{F}_G(U_p)$, $(\Upsilon_G)_p = \Upsilon_G(U_p)$ we infer that the Leibniz rule and the surjectivity condition for the sheafified morphism $d : \mathcal{F}_G \rightarrow \Upsilon_G$ are satisfied on the stalks. This proves (i). Recalling that the sheaf equality $\mathcal{F}_G^{\text{co}\mathcal{O}_q(P)} = \mathcal{O}_M$ was proven in Theorem 4.2 we have that (ii) follows directly from Corollary 4.9. The last point (iii) follows from Theorem 4.10. \square

4.5. Principal covariant calculi on quantum principal bundles. A principal covariant calculus on a QPB \mathcal{F} is a right H -covariant FODC (Υ, d) on the sheaf \mathcal{F} (see Definition 4.5) with an exact sequence on the stalks.

Definition 4.17. Consider a QPB \mathcal{F} on a quantum ringed space (M, \mathcal{O}_M) and a FODC (Υ, d) on \mathcal{F} . Given a left covariant FODC (Γ_H, d_H) on H we call (Υ, d) a *principal calculus* on \mathcal{F} if for all $p \in M$ we have the exact sequence of stalks

$$0 \rightarrow \mathcal{F}_p \otimes_{(\mathcal{O}_M)_p} (\Upsilon_M)_p \rightarrow \Upsilon_p \xrightarrow{\text{ver}_p} \mathcal{F}_p \square_H \Gamma_H \rightarrow 0, \tag{4.14}$$

where ver_p is the vertical map defined in Sect. 3.5 and Υ_M denotes the sheaf of base forms defined in Proposition 4.8. If, in addition, (Υ, d) is right H -covariant and (Γ_H, d_H) is bicovariant, we say that (Υ, d) is a *principal covariant calculus* on the sheaf \mathcal{F} .

We next apply Theorem 4.16 and consider three FODCi on QPBs over $\mathbf{P}^1(\mathbb{C})$, the second one being a principal calculus and the third one a principal covariant calculus. The first two have total space algebra $\mathcal{O}_q(\text{SL}_2(\mathbb{C}))$ with total space calculus the 4D and 3D ones of Examples 3.10 and 3.11. The third example has total space algebra $\mathcal{O}_q(\text{GL}_2(\mathbb{C}))$ with total space calculus the 4D one of Example 3.9.

4.5.1. *First order differential calculus via ore localization of 4D calculus on $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$*
 Consider the principal bundle $\wp : \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_2(\mathbb{C})/P \simeq \mathbf{P}^1(\mathbb{C})$, where P is the upper Borel in $\mathrm{SL}_2(\mathbb{C})$. Take V_i to be the subset of matrices in $\mathrm{SL}_2(\mathbb{C})$ with entry $(i, 1)$ not equal to zero. We observe that $\{V_1, V_2\}$ is an open cover of $\mathrm{SL}_2(\mathbb{C})$. Define $U_i = \wp(V_i)$ and observe that $\{U_1, U_2\}$ is an open cover of $\mathbf{P}^1(\mathbb{C})$ since \wp is an open map. This is the cover obtained via the quantum section $\alpha \in \mathcal{O}(\mathrm{SL}_2(\mathbb{C}))$ with coproduct $\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma$. In fact $V_1 = \{g \in \mathrm{SL}_2(\mathbb{C}) \mid \alpha(g) \neq 0\}$ and $V_2 = \{g \in \mathrm{SL}_2(\mathbb{C}) \mid \gamma(g) \neq 0\}$. Explicitly, U_1, U_2 are the opens obtained by removing the north or south pole. The coaction

$$\delta_R : \mathcal{O}(\mathrm{SL}_2(\mathbb{C})) \rightarrow \mathcal{O}(\mathrm{SL}_2(\mathbb{C})) \otimes \mathcal{O}(P), \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix}$$

uniquely extends to coactions on the localizations $\mathcal{O}(\mathrm{SL}_2)[\alpha^{-1}]$ and $\mathcal{O}(\mathrm{SL}_2)[\gamma^{-1}]$ by defining $\delta_R \alpha^{-1} = \alpha^{-1} \otimes t^{-1}$ and $\delta_R \gamma^{-1} = \gamma^{-1} \otimes t^{-1}$, respectively. The coinvariant subalgebras for $\mathcal{O}(\mathrm{SL}_2)[\alpha^{-1}]$ and $\mathcal{O}(\mathrm{SL}_2)[\gamma^{-1}]$ respectively are isomorphic to $\mathbb{C}[u]$ and $\mathbb{C}[v]$ with $u := \alpha^{-1}\gamma$ and $v := \alpha\gamma^{-1}$ and they are the coordinate rings of the affine algebraic varieties corresponding to the opens U_1, U_2 in $\mathrm{SL}_2(\mathbb{C})/P \simeq \mathbf{P}^1(\mathbb{C})$.

The quantum deformation of this construction has been studied in [4] as an example of QPB over projective base. In summary, starting with the Hopf algebra $A := \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ and its parabolic quotient Hopf algebra $H := \mathcal{O}_q(P)$ as in (3.16) we define the noncommutative localizations $A_1 := A[\alpha^{-1}]$ and $A_2 := A[\gamma^{-1}]$ and structure them as right H -comodule algebras in complete analogy to the classical setting. The subalgebras of right H -coinvariants are given by

$$B_1 = \mathbb{C}_q[\gamma\alpha^{-1}] \simeq \mathbb{C}_q[u], \quad B_2 = \mathbb{C}_q[\alpha\gamma^{-1}] \simeq \mathbb{C}_q[v]$$

and the ringed space $(\mathbf{P}^1(\mathbb{C}), \mathcal{O}_{\mathbf{P}^1(\mathbb{C})})$ can then be constructed as

$$\mathcal{O}_{\mathbf{P}^1(\mathbb{C})}(U_i) := B_i, \quad \mathcal{O}_{\mathbf{P}^1(\mathbb{C})}(U_{12}) := B_{12} := B_1[u^{-1}], \quad \mathcal{O}_{\mathbf{P}^1(\mathbb{C})}(\mathbf{P}^1(\mathbb{C})) := \mathbb{C}, \quad (4.15)$$

$i \in \{1, 2\}$, with the only non-trivial restriction map given by $r_{12,2} : B_2 \rightarrow B_{12}, v \mapsto u^{-1}$. Moreover we observe that

$$\begin{aligned} \mathcal{F}_{\mathrm{SL}_q}(U_i) &:= A_i, & \mathcal{F}_{\mathrm{SL}_q}(U_{12}) &:= A_{12} := A_1[\gamma^{-1}] = A_2[\alpha^{-1}], \\ \mathcal{F}_{\mathrm{SL}_q}(\mathbf{P}^1(\mathbb{C})) &:= A, & \mathcal{F}_{\mathrm{SL}_q}(\emptyset) &:= \{0\}, \end{aligned}$$

$i \in \{1, 2\}$, defines a sheaf of right $\mathcal{O}_q(P)$ -comodule algebras on $\mathbf{P}^1(\mathbb{C})$, which we denote by $\mathcal{F}_{\mathrm{SL}_q}$. Endowed with the cleaving maps,

$$\begin{aligned} j_1 : H &\rightarrow A_1, & \begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix} &\mapsto \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, \\ j_2 : H &\rightarrow A_2, & \begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix} &\mapsto \begin{pmatrix} \gamma & \delta \\ 0 & \gamma^{-1} \end{pmatrix}, \end{aligned} \quad (4.16)$$

$\mathcal{F}_{\mathrm{SL}_q}$ becomes a QPB on $\mathbf{P}^1(\mathbb{C})$ (on A_{12} we can choose $j_{12} = j_1 : H \rightarrow A_1 \subset A_{12}$ as a cleaving map or alternatively j_2). Notice that the stalk $(\mathcal{F}_{\mathrm{SL}_q})_p$ of $\mathcal{F}_{\mathrm{SL}_q}$ at $p \in \mathbf{P}^1(\mathbb{C})$ equals $\mathcal{F}_{\mathrm{SL}_q}(U_{12})$ if $p \in U_{12}$ and $\mathcal{F}_{\mathrm{SL}_q}(U_i)$ if $p \in U_i \setminus U_{12}$. Further notice that $\mathcal{B} = \{\emptyset, U_1, U_2, U_{12}\}$. Recalling the induced calculus on a sheaf of Theorem 4.16 we have

Proposition 4.18. *Let $(\Gamma, d) := (\Gamma_{SL}^+, d_{SL}^+)$ be the 4-dimensional bicovariant FODC on $\mathcal{O}_q(SL_2(\mathbb{C}))$ of Example 3.10. The induced right $\mathcal{O}_q(P)$ -covariant FODC (Υ_{SL_q}, d) on \mathcal{F}_{SL_q} is a free left \mathcal{F}_{SL_q} -module of dimension 4.*

Proof. The A_I -bimodules $\Upsilon_{SL_q}(U_I) = \Gamma_I = A_I \Gamma A_I$ of the induced calculus of Theorem 4.16 are constructed in Proposition 4.14. We show that $\Gamma_I = A_I \Gamma$. The proof then follows using that Γ is a free left A -module and ordering the invertible elements of A_I on the left. The only nontrivial cases are $\Gamma_i, i \in \{1, 2\}$, and Γ_{12} . From the commutation relations (3.11) (recall that the differential and the bimodule structure of the 4-dimensional bicovariant calculi on $SL_q(2)$ and $GL_q(2)$ coincide up to the identification $\det_q = 1$) it immediately follows that

$$\begin{aligned} \omega^1 \alpha^{-1} &= q^{-1} \alpha^{-1} \omega^1, & \omega^2 \alpha^{-1} &= \alpha^{-1} \omega^2, \\ \omega^3 \alpha^{-1} &= \alpha^{-1} \omega^3 + q^{-3} \lambda \alpha^{-2} \beta \omega^1, & \omega^4 \alpha^{-1} &= q \alpha^{-1} \omega^4 + \lambda \alpha^{-2} \beta \omega^2 \end{aligned} \tag{4.17}$$

in Γ_1 and

$$\begin{aligned} \omega^1 \gamma^{-1} &= q^{-1} \gamma^{-1} \omega^1, & \omega^2 \gamma^{-1} &= \gamma^{-1} \omega^2, \\ \omega^3 \gamma^{-1} &= \gamma^{-1} \omega^3 + q^{-3} \lambda \gamma^{-2} \delta \omega^1, & \omega^4 \gamma^{-1} &= q \gamma^{-1} \omega^4 + \lambda \gamma^{-2} \delta \omega^2 \end{aligned} \tag{4.18}$$

in Γ_2 . Recalling that a general element of A_1 is of the form $\sum_j a^j \alpha^{-n_j} \in A_1$ for $n_j \in \mathbb{N}, a^j \in A$ and that Γ is an A -bimodule which is freely generated as a left A -module by $\omega^1, \omega^2, \omega^3, \omega^4$, the commutation relations (4.17) imply $\Gamma A_1 \subseteq A_1 \Gamma$. Thus, $\Gamma_1 = A_1 \Gamma A_1 \subseteq A_1 \Gamma$ which gives $\Gamma_1 = A_1 \Gamma$ (the inclusion $\Gamma_1 \supseteq A_1 \Gamma$ being trivial). Similarly we prove $\Gamma_2 = A_2 \Gamma$, and $\Gamma_{12} = A_{12} \Gamma$ using both (4.17) and (4.18). \square

In the proof of Proposition 4.18 it was shown that for all $U_I \in \mathcal{B}$ the right H -covariant A_I -bimodule $\Upsilon_{SL_q}(U_I) = \text{span}_{A_I} \{\omega^1, \omega^2, \omega^3, \omega^4\}$ is freely generated from the left. This implies that $\Upsilon_{SL_q}(U_1 \cup U_2) = \Gamma$, i.e. that the presheaf constructed by the Ore extensions is already a sheaf and there is no sheafification required.

Proposition 4.19. *The induced FODC $(\Upsilon_{\mathbf{P}^1(\mathbb{C})}, d_{\mathbf{P}^1(\mathbb{C})})$ of base forms on $\mathcal{O}_q(\mathbf{P}^1(\mathbb{C}))$ (described in Corollary 4.9) is determined by*

$$\begin{aligned} \Upsilon_{\mathbf{P}^1(\mathbb{C})}(U_1) &:= \Gamma_{B_1} = B_1 d_1 B_1 = \text{span}_{\mathbb{C}} \{u^k d_1 u \mid k \in \mathbb{N}\}, \\ \Upsilon_{\mathbf{P}^1(\mathbb{C})}(U_2) &:= \Gamma_{B_2} = B_2 d_2 B_2 = \text{span}_{\mathbb{C}} \{v^k d_2 v \mid k \in \mathbb{N}\}, \\ \Upsilon_{\mathbf{P}^1(\mathbb{C})}(U_{12}) &:= \Gamma_{B_{12}} = B_{12} d_{12} B_{12} = \text{span}_{\mathbb{C}} \{u^k d_{12} u \mid k \in \mathbb{Z}\}. \end{aligned} \tag{4.19}$$

Moreover, $\Gamma_{B_1} = \text{span}_{B_1} \{\alpha^{-2} \omega^2\}$, $\Gamma_{B_2} = \text{span}_{B_2} \{\gamma^{-2} \omega^2\}$ and $\Gamma_{B_{12}} = \text{span}_{B_{12}} \{\alpha^{-2} \omega^2\}$ are free left modules and we have the sheaf isomorphism

$$\Upsilon_{\mathbf{P}^1(\mathbb{C})} \cong \Upsilon_{SL_q}^{\text{coH}} \cap \Upsilon_{SL_q}^{\text{hor}}. \tag{4.20}$$

Proof. Using (3.12), (4.17) and $\delta = \alpha^{-1} + q^{-1}\alpha^{-1}\beta\gamma$ we obtain

$$\begin{aligned} d_1u &= d_1(\gamma\alpha^{-1}) \\ &= d(\gamma)\alpha^{-1} - \gamma\alpha^{-1}d(\alpha)\alpha^{-1} \\ &= \left(\frac{q-1}{\lambda}\gamma\omega^1 + \frac{q^{-1}-1}{\lambda}\gamma\omega^4 - \delta\omega^2\right)\alpha^{-1} \\ &\quad - \gamma\alpha^{-1}\left(\frac{q-1}{\lambda}\alpha\omega^1 + \frac{q^{-1}-1}{\lambda}\alpha\omega^4 - \beta\omega^2\right)\alpha^{-1} \\ &= -\delta\omega^2\alpha^{-1} + \gamma\alpha^{-1}\beta\omega^2\alpha^{-1} \\ &= -\alpha^{-2}\omega^2 - q^{-1}\alpha^{-1}\beta\gamma\alpha^{-1}\omega^2 + \gamma\alpha^{-1}\beta\alpha^{-1}\omega^2 \\ &= -\alpha^{-2}\omega^2. \end{aligned}$$

Similarly $d_2v = d_2(\alpha\gamma^{-1}) = -\gamma^{-2}\omega^2$. The generators $u = \gamma\alpha^{-1}$ and $v = \alpha\gamma^{-1}$ of B_1 and B_2 have therefore the commutation relations

$$(d_1u)u = q^2ud_1u, \quad (d_2v)v = q^{-2}vd_2v. \tag{4.21}$$

Then the equalities in (4.19) follow, the last one also recalling that $v = u^{-1}$ in B_{12} .

From $d_1u = -\alpha^{-2}\omega^2$, $d_2v = -\gamma^{-2}\omega^2$ and the freeness of the left modules Γ_1 , Γ_2 and Γ_{12} it further follows that $\Gamma_{B_1} = \text{span}_{B_1}\{\alpha^{-2}\omega^2\}$, $\Gamma_{B_2} = \text{span}_{B_2}\{\gamma^{-2}\omega^2\}$ and $\Gamma_{B_{12}} = \text{span}_{B_{12}}\{\alpha^{-2}\omega^2\}$ are free left modules. The sheaf equality (4.20) follows directly from Theorem 4.10. \square

Note that the commutation relations (4.21) agree with those obtained by Chu, Ho and Zumino in [9].

Remark 4.20. Consider the right H -covariant FODC $(\Upsilon_{\text{SL}_q}, d)$ on $\mathcal{F}_{\text{SL}_q}$ together with the bicovariant quotient calculus (Γ_H, d_H) on H induced from the 4-dimensional bicovariant FODC (Γ, d) on A . The sequences

$$0 \rightarrow A_I \otimes_{B_I} \Gamma_{B_I} \rightarrow \Gamma_I \xrightarrow{\text{ver}_I} A_I \square_H \Gamma_H \rightarrow 0 \tag{4.22}$$

are well-defined for all $U_I \in \mathcal{B}$ with the vertical map given by $\text{ver}_I : \Gamma_I \rightarrow A_I \square_H \Gamma_H$, $a^i \omega^j \mapsto a_0^i \otimes a_1^j [\omega^j]$ for all $a^i \in A_I$ (the proof is as in Lemma 3.19). However $(\Upsilon_{\text{SL}_q}, d)$ is *not* a principal calculus on the QPB $\mathcal{F}_{\text{SL}_q}$. For example the sequence (4.22) for $I = 1$ is *not* exact. Indeed first observe that, since Γ_1 and Γ_{B_1} are free modules, $\omega^1 \notin A_1 \Gamma_{B_1} = \text{span}_{A_1}\{\omega^2\}$. Then, since ω^1 is in the kernel of the projection $\Gamma \rightarrow \Gamma_H$ it follows that $\text{ver}_1(\omega^1) = 1 \otimes [\omega^1] = 0$ and therefore $A_1 \Gamma_{B_1} \subsetneq \ker \text{ver}_1$.

4.5.2. Principal calculus via ore localization of 3D calculus on $\mathcal{O}_q(\text{SL}_2(\mathbb{C}))$ In this section we show that the failure of (4.22) to be an exact sequence can be cured by considering a 3-dimensional left covariant FODC instead of a 4-dimensional bicovariant FODC on $A = \mathcal{O}_q(\text{SL}_2(\mathbb{C}))$ as the global calculus of $\mathcal{F}_{\text{SL}_q}$. We choose a left covariant calculus on A so to canonically obtain a left covariant calculus on the quotient $H = \mathcal{O}_q(P)$. As in the previous section we consider the QPB $\mathcal{F}_{\text{SL}_q}$ with the cleaving maps (4.16).

Lemma 4.21. *Let $(\Gamma, d) = (\Gamma_{SL}, d_{SL})$ be the 3-dimensional left covariant FODC on A of Example 3.11. The induced FODC (Υ_{SL_q}, d) on \mathcal{F}_{SL_q} is a free left \mathcal{F}_{SL_q} -module of dimension 3.*

Proof. We first provide the commutation relations of α^{-1} and γ^{-1} with the basis $\{\omega^0, \omega^1, \omega^2\}$ of the free A -module Γ . From those in (3.19) we deduce

$$\begin{aligned} \omega^0 \alpha^{-1} &= q^{-3} \alpha^{-1} \omega^0, & \omega^1 \alpha^{-1} &= q^{-2} \alpha^{-1} \omega^1 + (q^{-6} - q^{-4}) \alpha^{-2} \beta \omega^2, \\ \omega^2 \alpha^{-1} &= q^{-3} \alpha^{-1} \omega^2, & \omega^0 \gamma^{-1} &= q^{-3} \gamma^{-1} \omega^0, \\ \omega^1 \gamma^{-1} &= q^{-2} \gamma^{-1} \omega^1 + (q^{-6} - q^{-4}) \gamma^{-2} \delta \omega^2, \\ \omega^2 \gamma^{-1} &= q^{-3} \gamma^{-1} \omega^2. \end{aligned} \tag{4.23}$$

Then, proceeding as in the proof of Proposition 4.18, we conclude that $\Gamma_I = A_I \Gamma$ is a free left module (with $I = 1, 2, 12$). \square

As in Proposition 4.19 the sheaf of base forms (Υ_M, d_M) is determined by the free left modules

$$\begin{aligned} \Gamma_{B_1} &= B_1 d_1 B_1 = \text{span}_{B_1} \{\alpha^{-2} \omega^2\}, & \Gamma_{B_2} &= B_2 d_2 B_2 = \text{span}_{B_2} \{\gamma^{-2} \omega^2\}, \\ \Gamma_{B_{12}} &= B_{12} d_{12} B_{12} = \text{span}_{B_{12}} \{\alpha^{-2} \omega^2\}. \end{aligned}$$

With $u = \gamma \alpha^{-1} \in B_1$ and $v = \alpha \gamma^{-1} \in B_2$ we have $d_1 u = q^{-3} \alpha^{-2} \omega^2$ and $d_2 v = -q^{-2} \gamma^{-2} \omega^2$. The Γ_{B_I} bimodule relations are then determined by

$$(d_1 u)u = q^2 u d_1 u, \quad (d_2 v)v = q^{-2} v d_2 v.$$

We prove for example the properties of Γ_{B_1} . We have

$$\begin{aligned} d_1 u &= d_1(\gamma \alpha^{-1}) \\ &= (d\gamma) \alpha^{-1} - \gamma \alpha^{-1} d(\alpha) \alpha^{-1} \\ &= \gamma \omega^1 \alpha^{-1} + \delta \omega^2 \alpha^{-1} - \gamma \alpha^{-1} (\alpha \omega^1 + \beta \omega^2) \alpha^{-1} \\ &= q^{-3} (\delta \alpha^{-1} - \gamma \alpha^{-1} \beta \alpha^{-1}) \omega^2 \\ &= q^{-3} \alpha^{-2} \omega^2 \end{aligned}$$

where we used (3.18), (4.23) and $\delta \alpha^{-1} - q^{-1} \alpha^{-1} \beta \gamma \alpha^{-1} = \alpha^{-2}$. Again by (4.23) the commutation relation

$$(d_1 u)u = q^{-3} \alpha^{-2} \omega^2 \gamma \alpha^{-1} = q^{-3} \alpha^{-2} \gamma \alpha^{-1} \omega^2 = q^{-1} \gamma \alpha^{-3} \omega^2 = q^2 u d_1 u$$

follows. Thus, $\Gamma_{B_1} = B_1 d_1 B_1 = \text{span}_{B_1} \{\alpha^{-2} \omega^2\}$. Similarly for Γ_{B_2} and $\Gamma_{B_{12}}$.

Proposition 4.22. *Let (Υ_{SL_q}, d) be the FODC on the locally trivial QPB \mathcal{F}_{SL_q} of Lemma 4.21. Let (Γ_H, d_H) be the left covariant FODC on H induced from the 3-dimensional left covariant FODC (Γ, d) on A of Example 3.11. The sequences*

$$0 \rightarrow A_I \otimes_{B_I} \Gamma_{B_I} \rightarrow \Gamma_I \xrightarrow{\text{ver}_I} A_I \square_H \Gamma_H \rightarrow 0, \quad U_I \in \mathcal{B} = \{\emptyset, U_1, U_2, U_{12}\} \tag{4.24}$$

are exact and (Υ_{SL_q}, d) is a principal calculus on \mathcal{F}_{SL_q} .

Proof. Recalling Lemma 3.19, that Γ_I are free left A_I -modules and the left linearity of the vertical map (3.28), we see that the vertical maps in (4.24) are given by $\text{ver}_I : \Gamma_I \rightarrow A_I \square_H \Gamma_H, a^\ell \omega^\ell \mapsto a_0^\ell \otimes a_1^\ell[\omega^\ell]$, where $a^\ell \in A_I$. In particular, they are well-defined. We first show that the second arrow in (4.24) is injective in the case of U_1 . The results for the other opens in \mathcal{B} follow analogously. Recall that Γ_{B_1} is generated as a left B_1 -module by $\alpha^{-2}\omega^2$ (cf. the discussion following Lemma 4.21). An arbitrary element of $A_1 \otimes_{B_1} \Gamma_{B_1}$ is therefore of the form $a^i \otimes_{B_1} b^i \alpha^{-2}\omega^2$ with $a^i \in A_1$ and $b^i \in B_1$ (sum understood). Since, by Lemma 4.21, Γ_1 is a free left A_1 -module with basis $\{\omega^0, \omega^1, \omega^2\}$, it follows that $a^i b^i \alpha^{-2}\omega^2 = 0$ if and only if $a^i b^i \alpha^{-2} = 0$, i.e., if and only if, $a^i b^i = 0$, which proves injectivity. We next prove exactness in Γ_I . The condition $0 = \text{ver}_I(a^\ell \omega^\ell) = a_0^\ell \otimes a_1^\ell[\omega^\ell]$ (sum understood) holds if and only if $a^0 = a^1 = 0$, indeed $[\omega^2] = 0$, while $[\omega^0]$ and $[\omega^1]$ form a basis of the free left H -module Γ_H , cf. Example 3.11. This proves $\ker \text{ver}_I = \text{span}_{A_I} \{\omega_2\} = A_I \Gamma_{B_1}$. In order to prove surjectivity of $\text{ver}_I : \Gamma_I \rightarrow A_I \square_H \Gamma_H$ let $a^{i\ell} \otimes h^i[\omega^\ell]$ (sums understood) be an arbitrary element in $A_I \square_H \Gamma_H$. Since the forms $[\omega^\ell]$ are left H -coinvariant, by definition of cotensor product we have $a_0^{i\ell} \otimes a_1^{i\ell} \otimes h^i = a^{i\ell} \otimes h_1^i \otimes h_2^i$. Then

$$\text{ver}_I(a^{i\ell} \epsilon(h^i) \omega^\ell) = a_0^{i\ell} \otimes a_1^{i\ell} \epsilon(h^i) [\omega^\ell] = a^{i\ell} \otimes h^i [\omega^\ell]$$

and the sequences in (4.24) are exact. □

As in the previous section the freeness of the presheaf Υ_{SL_q} implies that there is no sheafification required and in particular $\Upsilon_{\text{SL}_q}(\mathbf{P}^1(\mathbb{C})) = \Gamma$.

We have seen that the differential calculus $(\Upsilon_{\text{SL}_q}, d)$ is principal. Note that it is not principal covariant since the induced left covariant FODC (Γ_H, d_H) on H is not bicovariant. This is the case because the ideal $I = \text{span}_H \{(t-1)(t-q^2), p^2, (t-q^2)p\}$ characterizing (Γ_H, d_H) is not closed under the adjoint right coaction, e.g. $\text{Ad}_R(p^2) \notin I \otimes H$ (it contains a polynomial in t with roots different from 1 and q^2). See [47] for the classification of bicovariant FODCi on Hopf algebras in terms of ideals closed under the right adjoint coaction).

4.5.3. Principal covariant calculus via Ore localization of 4D calculus on $\mathcal{O}_q(\text{GL}_2(\mathbb{C}))$

Consider the principal bundle $\text{GL}_2(\mathbb{C}) \rightarrow \text{GL}_2(\mathbb{C})/P_{\text{GL}} \cong \mathbf{P}^1(\mathbb{C})$ with P_{GL} the upper Borel in $\text{GL}_2(\mathbb{C})$. As before, localizing with respect to α and γ gives the opens U_1 and U_2 with corresponding topology $\{\emptyset, U_1, U_2, U_{12}, \mathbf{P}^1(\mathbb{C})\}$ on $\mathbf{P}^1(\mathbb{C})$. Although $\text{GL}_2(\mathbb{C})$ is not semisimple, a quantum deformation of this bundle, which is a QPB $\mathcal{F}_{\text{GL}_q}$ on $\mathbf{P}^1(\mathbb{C})$, is obtained via Ore extensions of the quantum group $A = \mathcal{O}_q(\text{GL}_2(\mathbb{C}))$. Explicitly, we define the sheaf of algebras by $\mathcal{F}_{\text{GL}_q}(\emptyset) := \{0\}, \mathcal{F}_{\text{GL}_q}(\mathbf{P}^1(\mathbb{C})) := A$,

$$\begin{aligned} \mathcal{F}_{\text{GL}_q}(U_1) &:= A_1 := A[\alpha^{-1}], & \mathcal{F}_{\text{GL}_q}(U_2) &:= A_2 := A[\gamma^{-1}], \\ \mathcal{F}_{\text{GL}_q}(U_1 \cap U_2) &:= A_{12} := A[\alpha^{-1}, \gamma^{-1}]. \end{aligned}$$

The Hopf algebra quotient $A \rightarrow H = \mathcal{O}_q(P_{\text{GL}}) = A/\langle \gamma \rangle$ defined from Example 3.9 induces a natural comodule algebra structure on $\mathcal{F}_{\text{GL}_q}$. This is a locally trivial QPB with cleaving maps $j_1 : H \rightarrow A_1, j_2 : H \rightarrow A_2$ and $j_{12} = j_1 : H \rightarrow A_1 \subset A_{12}$ given by

$$\begin{aligned} j_1 \begin{pmatrix} t & p \\ 0 & s \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \det_q \end{pmatrix}, & j_2 \begin{pmatrix} t & p \\ 0 & s \end{pmatrix} &= \begin{pmatrix} \gamma & \delta \\ 0 & \gamma^{-1} \det_q \end{pmatrix}, \\ j_1(r') &= r, & j_2(r') &= r, \end{aligned} \tag{4.25}$$

where $r \in A$ and $r' \in H$ are the inverses of the quantum determinants. It is straightforward to verify that the sheaf $\mathcal{O}_{\mathbf{P}^1(\mathbb{C})}$ of coinvariants of the QPB \mathcal{F}_{GL_q} coincides with the one defined in (4.15).

Proposition 4.23. *The Ore extension of the bicovariant 4-dimensional FODC $(\Gamma, d) := (\Gamma_{GL}, d_{GL})$ of Example 3.9 gives a right H -covariant FODC (Υ_{GL_q}, d) on the QPB \mathcal{F}_{GL_q} . Explicitly, the sheaf Υ_{GL_q} of right H -covariant \mathcal{F}_{GL_q} -bimodules is defined by $\Upsilon_{GL_q}(\emptyset) = \{0\}$, $\Upsilon_{GL_q}(\mathbf{P}^1(\mathbb{C})) = \Gamma$ and*

$$\begin{aligned} \Upsilon_{GL_q}(U_1) &= A_1 \otimes_A \Gamma \otimes_A A_1, & \Upsilon_{GL_q}(U_2) &= A_2 \otimes_A \Gamma \otimes_A A_2, \\ \Upsilon_{GL_q}(U_1 \cap U_2) &= A_{12} \otimes_A \Gamma \otimes_A A_{12}. \end{aligned} \tag{4.26}$$

Together with the pullback calculus on the sheaf $\mathcal{O}_{\mathbf{P}^1(\mathbb{C})}$ of coinvariants and the bicovariant quotient calculus on H this gives a principal covariant calculus on \mathcal{F}_{GL_q} .

Proof. Clearly (4.26) defines a presheaf Υ_{GL_q} of right H -covariant \mathcal{F}_{GL_q} -bimodules with restriction morphisms as in (4.13). Repeating the computations of Proposition 4.18 and Proposition 4.19 one shows that

- (i) $\Upsilon_{GL_q}(U_I)$ is a free left $\mathcal{F}_{GL_q}(U_I)$ -module generated by $\{\omega^1, \omega^2, \omega^3, \omega^4\}$,
- (ii) the base forms $(\Upsilon_{\mathbf{P}^1(\mathbb{C})}, d_{\mathbf{P}^1(\mathbb{C})})$ are determined by $\Gamma_{B_1} = \text{span}_{B_1}\{-\alpha^{-2}\omega^2\}$, $\Gamma_{B_2} = \text{span}_{B_2}\{-\gamma^{-2}\omega^2\}$ and $\Gamma_{B_{12}} = \text{span}_{B_{12}}\{-\alpha^{-2}\omega^2\}$.

Using the freeness of the presheaf Υ_{GL_q} it is not hard to check that $\Upsilon_{GL_q}(U_1 \cup U_2) = \Upsilon_{GL_q}(\mathbf{P}^1(\mathbb{C}))$, proving that Υ_{GL_q} is a sheaf of right H -covariant \mathcal{F}_{GL_q} -bimodules. By Proposition 4.14 the extended differential $d_I : A_I \rightarrow \Upsilon_{GL_q}(U_I)$ defines a right H -covariant FODC for each open $U_I \in \mathcal{B}$ and it is straightforward to verify that this determines a morphism $d : \mathcal{F}_{GL_q} \rightarrow \Upsilon_{GL_q}$ of sheaves of right H -comodules.

We next prove that the sequences

$$0 \rightarrow A_I \otimes_A \Gamma_{B_I} \rightarrow \Gamma_I \xrightarrow{\text{ver}_I} A_I \square_H \Gamma_H \rightarrow 0 \tag{4.27}$$

are exact for all $U_I \in \mathcal{B}$. From Theorem 3.17 we have injectivity of the second arrow. Recall from Example 3.9 that the bicovariant quotient calculus (Γ_H, d_H) on H is 3-dimensional (instead of 2-dimensional in the framework of Proposition 4.18) with left coinvariant basis $\{[\omega^1], [\omega^3], [\omega^4]\}$ and that by Lemma 3.19 the vertical map $\text{ver}_I : \Gamma_I \rightarrow A_I \square_H \Gamma_H$ is well-defined for all $U_I \in \mathcal{B}$. Now take an arbitrary element $a^\ell \omega^\ell \in \Gamma_I$ (sum understood). Because of freeness $0 = \text{ver}_I(a^\ell \omega^\ell) = a_0^\ell \otimes a_1^\ell [\omega^\ell]$ if and only if $a^1 = a^3 = a^4 = 0$. Thus $\ker \text{ver}_I = \text{span}_{A_I}\{\omega^2\} = A_I \Gamma_{B_I}$. Surjectivity of ver_I is verified as in Proposition 4.22. Hence the sequences (4.27) are exact and (Υ_{GL_q}, d) is a principal covariant calculus on \mathcal{F}_{GL_q} . \square

As a corollary, applying Theorem 4.10 we have that $\Upsilon_{\mathbf{P}^1(\mathbb{C})} \cong \Upsilon_{GL_q}^{\text{co}H} \cap \Upsilon_{GL_q}^{\text{hor}}$ are isomorphic sheaves.

The sheaf \mathcal{F}_{GL_q} is locally trivial since $A_I \cong B_I \# H$ according to (4.25). Considering the FODC on $(\Upsilon_{\mathbf{P}^1(\mathbb{C})}, d_{\mathbf{P}^1(\mathbb{C})})$ and the bicovariant calculus (Γ_H, d_H) on H , we can construct locally the smash product calculus $(\Gamma_{B_I \# H}, d_{B_I \# H})$ of Sect. 3.4, which can be pulled back to A_I via $\varphi_I : A_I \rightarrow B_I \# H$ (the module of one forms being $\Gamma_{B_I \# H}$ since φ_I are isomorphisms). However this local data does not give a FODC on \mathcal{F}_{GL_q} . Indeed for that to be the case the restriction morphisms $r_{I,J}^{\Upsilon\#}$ of the would be sheaf $\Upsilon_{\#}$ of covariant \mathcal{F}_{GL_q} -bimodules would have to be left \mathcal{F}_{GL_q} -linear and compatible with the

differentials: $d_{B_I\#H} \circ \varphi_I \circ r_{IJ} = r_{IJ}^{\#} \circ d_{B_J\#H} \circ \varphi_J$. Recalling that $\Gamma_{B_I\#H}$ are free left $B_I\#H$ -bimodules it follows from simple straightforward computations that there is not such map $r_{12,2}^{\#} : \Gamma_{B_2\#H} \rightarrow \Gamma_{B_{12}\#H}$.

As a consequence we have that in this example the Ore extended differential calculi (Γ_{A_I}, d_I) on $A_I \cong B_I\#H$ are not isomorphic to the smash product calculi $(\Gamma_{B_I\#H}, d_{B_I\#H})$. While the latter, from a bottom up approach, could seem to be the canonical choice, it turns out that it is the first one to provide a FODC on the sheaf \mathcal{F}_{GL_q} . This is yet another instance of the non uniqueness of the differential calculus construction.

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