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Model matching problems for impulsive linear systems with polytopic uncertainties

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ABSTRACT

This work deals with the problem of designing a feedback compensator that forces the output of a linear system with abrupt discontinuities in the state evolution and polytopic uncertainties to match that of a given model with the same features. First, the case in which the system and the model are initialized at zero and output matching is required to be exact is considered. Then, the case in which, for arbitrary initialization, output matching is required to be asymptotic for sufficiently slow sequences of the time instants wherein the state exhibits abrupt discontinuities is studied. In addition, on the assumption that the model is stable for sufficiently slow jump time sequences, also the further requirement that asymptotic output matching be achieved with stability of the compensated system is investigated. Constructive, directly checkable, solvability conditions for the problems addressed are derived by leveraging on appropriate structural notions and geometric tools. Algorithmic procedures for the synthesis of the compensators, when the solvability conditions are met, are devised. Some illustrative examples conclude the work.

1. Introduction

Impulsive systems are dynamical systems whose state evolves with continuity over time except at certain instants, wherein it exhibits abrupt discontinuities. These systems are useful to model phenomena where some variables suddenly change their value due to external events and they have been extensively studied, e.g., in [1–5]. In particular, impulsive linear systems can be described as hybrid systems whose state evolution is ruled by a linear differential equation (flow dynamics) and a linear difference equation (jump dynamics). As to the control of impulsive linear systems, the literature is fairly wide and includes the investigation of problems like stabilization [6–9], output regulation [10–14], state estimation [15,16]; disturbance decoupling [17,18], and model matching [19]. At the same time, when the coefficients of the equations defining the dynamical systems are not exactly known but may vary between a minimum and a maximum, the so-called polytopic representation provides a convenient way to model the system's uncertainties and to deal with robust control design (see, e.g., [20,21]).

Impulsive linear systems with polytopic uncertainties are hybrid systems characterized by a flow dynamics and a jump dynamics whose linear maps, like those describing the output behaviour, are affected by polytopic uncertainties. This means that each associated matrix can be expressed as a linear combination, through the same uncertain parameter vector which may vary within a

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given symplex, of a set of matrices whose entries are real constants. These latter matrices are the vertices of the respective polytopes. Impulsive linear systems with polytopic uncertainties were first considered in the early 2000s in connection with the problem of attenuating a persistent, bounded disturbance [22,23]. In the following years, the literature on impulsive systems with polytopic uncertainties was mainly focused on the development of methods for stability analysis [24–27]. Only in the latest years, the focus has returned on control synthesis [28,29]. In particular, in [28], the authors have tackled the problem of disturbance decoupling: i.e., the problem of annihilating the effect of a disturbance on the system output. Since the signal to be decoupled was assumed to be inaccessible (i.e., neither measurable nor a-priori known), the solution sought for was necessarily based on feedback. Directly checkable solvability conditions were shown and an algorithmic procedure for the synthesis of a robust state feedback was derived by introducing appropriate structural geometric tools. Herein, the authors aim to contribute further to the investigation of control problems for impulsive linear systems with polytopic uncertainties by addressing a different problem, the model matching problem.

Essentially, solving a model matching problem consists in designing a compensator that forces the output of a given system to match that of a given model. This problem has both a methodological relevance and a practical impact, since a number of control design problems can be effectively reformulated and solved in its terms. Namely, given a plant and defined a desired behaviour, a controller that forces the plant to behave accordingly, if any exists, can be found by constructing a model with the required behaviour and by solving the related model matching problem. This approach, widely illustrated, e.g., in [30], was adopted to deal with output tracking [31], model reference adaptive control [32], servomechanism design [33] and many other regulation and control problems. Methodological relevance and practical impact are the main motivations for the huge research effort that the investigation of model matching problems has been attracting for the last fifty years. Indeed, the model matching problem was introduced for linear systems in [34] and received fundamental contributions in this area throughout the seventies and the following decade [35–40]. Starting from the eighties, several authors worked to the characterization of the solvability of model matching problems stated for other classes of dynamical systems, such as nonlinear systems [41,42], time-delay systems [43,44], systems over rings [45], periodic systems [46], 2D systems [47], time-varying systems [48], systems with time-varying delays [49], LPV systems [50,51], switching systems [52,53], and max-plus systems [54].

In the depicted framework, the contribution of this work consists in

i. developing the structural geometric tools needed to study the model matching problem for impulsive linear systems with polytopic uncertainties;
ii. deriving consistent solvability conditions, easy to check by means of properly defined computational algorithms;
iii. devising an algorithmic procedure for the synthesis of a compensator if the solvability conditions are met.

In particular, this work focuses on three different formulations, progressively more demanding, of the model matching problem for impulsive linear systems with polytopic uncertainties. These can be concisely defined as

i. *the exact model matching problem*: the problem of compensating a given impulsive linear system with polytopic uncertainties in such a way that its output exactly matches the output of a given impulsive linear model of the same kind, if both the system and the model are initialized at 0 (the origin of the respective state space), for all the admissible values of the uncertain parameter vector and for all the sequences of the jump time instants;
ii. *the asymptotic model matching problem:* the problem of compensating a given impulsive linear system with polytopic uncertainties in such a way that its output asymptotically matches the output of a given impulsive linear model of the same kind for any initial condition of the system and of the model, for all the admissible values of the uncertain parameter vector and for all the sufficiently slow sequences of the jump time instants;
iii. *the model matching problem with stability:* the problem of compensating a given impulsive linear system with polytopic uncertainties in such a way that its output asymptotically matches the output of a given impulsive linear model of the same kind for any initial condition of the system and of the model and the dynamics of the compensated system is asymptotically stable, provided that the model is such, for all the admissible values of the uncertain parameter vector and for all the sufficiently slow sequences of the jump time instants.

The formal statements of the three problems, the respective solvability conditions, and the mutual relations between them will be the core of this work. These problems will be investigated through a structural geometric approach stemmed from the classic geometric approach [55,56]. Indeed, structural approaches have proven to be particularly effective in solving a number of control problems formulated for impulsive linear systems [11–14,17]. Likewise, they have been successfully extended to the study of dynamical systems with polytopic uncertainties [57–61]. Hence, it is quite natural to search for structural geometric notions and tools that fit with impulsive linear systems with polytopic uncertainties, along the lines first sketched in [28]. In more detail, the hybrid nature of the class of dynamical systems addressed requires that the structural properties of invariance and controlled invariance, which play a crucial role in the solution of the model matching problem, be defined in such a way to take into account both the flow and the jump dynamics, according to the idea first developed for impulsive linear systems with perfectly known mathematical descriptions in [12]. Furthermore, in agreement with the approach first presented for linear time invariant systems with polytopic uncertainties in [60], the parametric uncertainties are handled by requiring that some key structural properties be true for all the systems corresponding to the vertices of the uncertain parameter vector symplex and by deriving properties that are true for the systems corresponding to any value of the uncertain parameter vector within the considered symplex.

The work is organized as follows. In Section 2, the mathematical description of impulsive linear systems with polytopic uncertainties is introduced and the structural geometric background on which the analysis of the considered model matching problems and the derivation of their solvability conditions hinge is set out. In Section 3, the formal statement of the model matching problems addressed is given. In Section 4, the solvability conditions for the structural (i.e., exact) model matching problem and for the problem with stability (whose solvability implies also that of the asymptotic model matching problem) are shown. Moreover, how those solvability conditions can be directly checked by means of the algorithm for the maximal robust hybrid controlled invariant subspace and of a set of linear matrix inequalities (LMIs) is explained. In Section 5, some numerical examples are worked out to illustrate how to perform the computations involved in the check of the solvability conditions and in the synthesis of the compensator. In Section 6, some final considerations are presented.

Notation. The symbols $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ stand for the sets of real numbers, nonnegative real numbers, and natural numbers including 0, respectively. Linear maps between real vector spaces and matrices are denoted by slanted capital letters, like A. Sets, real vector spaces and subspaces are denoted by calligraphic capital letters, like $\mathcal{X}$. Vectors and scalars are denoted by slanted lowercase letters, like v. For a linear map $A : \mathcal{X} \to \mathcal{Y}$, the image and the kernel of A are respectively denoted by $\operatorname{Im} A$ and $\operatorname{Ker} A$. The transpose of A is denoted by $A^\top$. The inverse image with respect to A of a subspace $\mathcal{V} \subseteq \mathcal{Y}$ is the subspace $A^{-1}\mathcal{V} = \{x \in \mathcal{X}$ such that $Ax \in \mathcal{V}\}$. Given a real vector space $\mathcal{X}$ of dimension n, the symbol $\mathcal{X}^{\oplus N}$ denotes the external direct sum of N copies of $\mathcal{X}$, that is the (Nn)-dimensional real vector space which consists of the set of all vectors $[x_1^\top \dots x_N^\top]^\top$, where $x_i \in \mathcal{X}$ with $i = 1, \dots, N$, equipped with the addition and the multiplication by scalars. Given a subspace $\mathcal{V} \subseteq \mathcal{X}$, the symbol $\mathcal{V}^{\oplus N}$ denotes the external direct sum of N copies of $\mathcal{V}$, that is the subspace of $\mathcal{X}^{\oplus N}$ which consists of all vectors $[v_1^\top \dots v_N^\top]^\top$, where $v_i \in \mathcal{V}$ with $i = 1, \dots, N$. Given a set of linear maps $A_i : \mathcal{X} \to \mathcal{Y}$, with $i = 1, \dots, N$, (or, respectively, a set of matrices A_i, with $i = 1, \dots, N$, of consistent dimensions), the symbol $\oplus_{i=1}^N A_i$ denotes a linear map defined as $\oplus_{i=1}^N A_i : \mathcal{X} \to \mathcal{Y}^{\oplus N}$, such that $x \to [(A_1 x)^\top \dots (A_N x)^\top]^\top$ (or, respectively, the matrix defined as $\oplus_{i=1}^N A_i = [A_1^\top \dots A_N^\top]^\top$). Given a vector $x \in \mathbb{R}^n$, $\|x\|_\infty$ denotes the H_∞ norm of x, that is $\|x\|_\infty = \max_{i=1,\dots,n} |x_i|$. Given a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ (or, respectively, a matrix $A \in \mathbb{R}^{m\times n}$), $\|A\|_\infty$ denotes the H_∞ norm of A, that is $\|A\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^m |a_{ij}|$. Given a symmetric matrix $M \in \mathbb{R}^{n\times n}$, the notation $M > 0$ means that M is positive definite, that is $x^\top M x > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$. The identity matrix of dimension n is denoted by I_n.

2. Preliminaries and structural geometric background

In this section, the state space representation of impulsive linear systems with polytopic uncertainties is introduced and some key structural geometric notions are defined and characterized.

An impulsive linear system with polytopic uncertainties Σ_σ has a state space representation of the form

$$\Sigma_\sigma \equiv \begin{cases} \dot{x}(t) &= A(\mu)x(t) + B(\mu)u(t), \qquad \text{with } t \neq \sigma(k),\ k \in \mathbb{N}, \\ x(t) &= J(\mu)x^-(t), \qquad \text{with } t = \sigma(k),\ k \in \mathbb{N}, \\ y(t) &= C(\mu)x(t), \end{cases} \tag{1}$$

where $t \in \mathbb{R}^+$ is the time variable; $x \in \mathcal{X} = \mathbb{R}^n$, $u \in \mathcal{U} = \mathbb{R}^m$, and $y \in \mathcal{Y} = \mathbb{R}^p$ are the state, the control input and the output, respectively. The time axis consists of continuous time intervals interlaced by discrete time instants, whose sequence is determined by a map $\sigma : \mathbb{N} \to \mathbb{R}^+$, which satisfies the condition

$$\tau_\sigma = \inf\{\sigma(0),\ \sigma(k+1) - \sigma(k);\, k \in \mathbb{N},\ \sigma(k+1) \neq \sigma(k)\} > 0. \tag{2}$$

Indeed, condition (2) implies that the image of σ, namely $\operatorname{Im}\sigma = \{t \in \mathbb{R}^+,\ t = \sigma(k)$ for some $k \in \mathbb{N}\}$, is a discrete, finite or countably infinite, ordered subset of $\mathbb{R}^+$, whose subsets (including $\operatorname{Im}\sigma$ itself) have no accumulation points. The set of all σ satisfying (2) is denoted by $\mathscr{S}$. The set of all σ satisfying $\tau_\sigma > \bar{\tau}$, where $\bar{\tau}$ is a positive real constant, is denoted by $\mathscr{S}_{\bar{\tau}}$. The system's matrices, $A(\mu)$, $B(\mu)$, $J(\mu)$, and $C(\mu)$, which are affected by polytopic uncertainties, are expressed as linear combinations, through the elements of the same uncertain parameter vector $\mu = [\mu_1 \dots \mu_N]^\top$, which belongs to the standard $(N-1)$-dimensional symplex Δ^{N-1} in $\mathbb{R}^N$, of respective families of constant real matrices with consistent dimensions

$$\{A_i\}_{i\in\mathcal{I}}, \quad \{B_i\}_{i\in\mathcal{I}}, \quad \{J_i\}_{i\in\mathcal{I}}, \quad \{C_i\}_{i\in\mathcal{I}}, \tag{3}$$

where $\mathcal{I} = \{1, \dots, N\}$. Namely, the system's matrices are expressed as

$$A(\mu) = \sum_{i=1}^N \mu_i A_i, \quad B(\mu) = \sum_{i=1}^N \mu_i B_i, \quad J(\mu) = \sum_{i=1}^N \mu_i J_i, \quad C(\mu) = \sum_{i=1}^N \mu_i C_i, \tag{4}$$

and the constant real matrices of the families in (3) represent the vertices of the respective uncertainty polytopes, since $\mu \in \Delta^{N-1}$ means that $\mu_i \geq 0$ for all $i \in \mathcal{I}$ and $\sum_{i=1}^N \mu_i = 1$. Further, the family $\{\Sigma_i\}_{i\in\mathcal{I}}$ of the impulsive linear systems

$$\Sigma_i \equiv \begin{cases} \dot{x}(t) &= A_i x(t) + B_i u(t), \qquad \text{with } t \neq \sigma(k),\ k \in \mathbb{N}, \\ x(t) &= J_i x^-(t), \qquad \text{with } t = \sigma(k),\ k \in \mathbb{N}, \\ y(t) &= C_i x(t), \end{cases} \tag{5}$$

with $i \in \mathcal{I}$, is the family of the vertex systems (or, simply, the vertices) of Σ_σ.

The state evolution of the linear impulsive system with polytopic uncertainties Σ_σ on the continuous time intervals $\sigma(k) < t < \sigma(k+1)$ (i.e., the flow dynamics of the system) is ruled by the differential state equation in (1). The state behaviour over each discrete time instant (i.e., the jump dynamics) is ruled by the difference equation in (1), where $x^-(t)$ stands for the limit of $x(t-\varepsilon)$ for ε

which goes to 0 from the right, that is $\lim_{\varepsilon\to 0+} x(t-\varepsilon)$. It is worth noting that, on the basis of the given mathematical description, only one jump occurs at each point $\sigma(k)\in\operatorname{Im}\sigma$. Hence, according to (1) and (2), where, in particular, the latter implies that the time when the first jump occurs, $\sigma(0)$, is greater than the initial time, $t=0$, and on the assumption that $x(0)=x_0$, it ensues that, for any value of the uncertain parameter vector $\mu\in\Delta^{N-1}$ and for any sequence $\sigma\in\mathscr{S}$, the state evolution is described by

$$\begin{aligned} x(\sigma(0)) &= J(\mu)e^{A(\mu)\sigma(0)}x(0), \\ x(\sigma(k+1)) &= J(\mu)e^{A(\mu)(\sigma(k+1)-\sigma(k))}x(\sigma(k)), \quad \text{with } k\in\mathbb{N}, \end{aligned}$$

over the sequence of the discrete-time instants and by

$$\begin{aligned} x(t) &= e^{A(\mu)t}x(0), && \text{with } 0\le t<\sigma(0), \\ x(t) &= e^{A(\mu)(t-\sigma(k))}x(\sigma(k)), && \text{with } \sigma(k)\le t<\sigma(k+1),\ k\in\mathbb{N}, \end{aligned}$$

over the continuous-time intervals. It is also worth pointing out that the set of the lengths of the time intervals between consecutive jump time instants is not assumed to have a finite upper bound. In particular, if $\operatorname{Im}\sigma$ is a finite set, there are no jumps on the time interval $[\max_{k\in\mathbb{N}}\sigma(k),+\infty)$ and the dynamics of Σ_σ reduces to the sole flow dynamics on that time interval.

With reference to the linear impulsive system with polytopic uncertainties Σ_σ and in view of the investigation of the model matching problems that will be the object of Section 3, it is of interest to review the definition of some structural geometric concepts and their characterization [28]. These concepts are introduced in such a way to take into account

- the hybrid nature of the system, by stating that the defining conditions must be true (in a compatible fashion) for both the flow dynamics and the jump dynamics;
- the polytopic uncertainties of the system, by stating that the conditions must be true for any value of the uncertain parameter vector $\mu\in\Delta^{N-1}$.

Moreover, these concepts are characterized in terms of the vertices $\{\Sigma_i\}_{i\in\mathcal{I}}$ of the system Σ_σ. Necessary and sufficient conditions involving the linear maps of the vertex systems (thus, directly checkable) are presented for each defined object. The first notion considered is that of robust hybrid invariant subspace and refers to the free dynamics of Σ_σ

Definition 1 (*Robust Hybrid Invariant Subspace*)**.** Given the impulsive linear system with polytopic uncertainties Σ_σ, defined by (1), a subspace $\mathcal{V}\subseteq\mathcal{X}$ is said to be a robust hybrid invariant subspace for Σ_σ if

$$A(\mu)\mathcal{V} \subseteq \mathcal{V}, \tag{6a}$$

$$J(\mu)\mathcal{V} \subseteq \mathcal{V}, \tag{6b}$$

for all $\mu\in\Delta^{N-1}$.

Proposition 1. *Given the impulsive linear system with polytopic uncertainties Σ_σ, defined by (1), whose vertices are the impulsive linear systems of the family $\{\Sigma_i\}_{i\in\mathcal{I}}$, defined by (5), a subspace $\mathcal{V}\subseteq\mathcal{X}$ is a robust hybrid invariant subspace for Σ_σ if and only if*

$$\left(\bigoplus_{i=1}^{N} A_i\right)\mathcal{V} \subseteq \mathcal{V}^{\oplus N}, \tag{7a}$$

$$\left(\bigoplus_{i=1}^{N} J_i\right)\mathcal{V} \subseteq \mathcal{V}^{\oplus N}. \tag{7b}$$

It is worth mentioning that the sufficiency of (7) is immediate consequence of the definitions of $A(\mu)$ and $J(\mu)$ in (4), while the necessity follows by picking $\mu=e_i$, where e_i denotes the ith vector of the natural basis of $\mathbb{R}^N$, with $i=1,\dots,N$, in (6).

To introduce the notion of hybrid robust controlled invariant subspace, it is necessary to consider the impulsive linear system with polytopic uncertainties $\Sigma_\sigma^{\mathscr{F}}$, which results from the application of a state feedback $u(t)=Fx(t)$ to the system Σ_σ: i.e.,

$$\Sigma_\sigma^{\mathscr{F}} \equiv \begin{cases} \dot{x}(t) = (A(\mu)+B(\mu)F)x(t), & \text{with } t\neq\sigma(k),\ k\in\mathbb{N}, \\ x(t) = J(\mu)x^-(t), & \text{with } t=\sigma(k),\ k\in\mathbb{N}, \\ y(t) = C(\mu)x(t). \end{cases} \tag{8}$$

The family of the vertices of $\Sigma_\sigma^{\mathscr{F}}$ is denoted by $\{\Sigma_i^{\mathscr{F}}\}_{i\in\mathcal{I}}$ and the vertices are the impulsive linear systems

$$\Sigma_i^{\mathscr{F}} \equiv \begin{cases} \dot{x}(t) = (A_i+B_iF)x(t), & \text{with } t\neq\sigma(k),\ k\in\mathbb{N}, \\ x(t) = J_ix^-(t), & \text{with } t=\sigma(k),\ k\in\mathbb{N}, \\ y(t) = C_ix(t), \end{cases} \tag{9}$$

with $i\in\mathcal{I}$. It is worthwhile noting that the state feedback applied is independent of the uncertain parameter vector and, consequently, it is independent of the vertex systems. This is consistent with the fact that the value of the uncertain parameter vector is not available.

Definition 2 (*Robust Hybrid Controlled Invariant Subspace*). Given the impulsive linear system with polytopic uncertainties Σ_σ, defined by (1), a subspace $\mathcal{V} \subseteq \mathcal{X}$ is said to be a robust hybrid controlled invariant subspace for Σ_σ if there exists a state feedback $u(t) = Fx(t)$ such that $\mathcal{V}$ is a hybrid robust invariant subspace for the closed-loop system $\Sigma_\sigma^{\mathcal{F}}$ or, which is the same,

$$(A(\mu) + B(\mu)F)\mathcal{V} \subseteq \mathcal{V}, \tag{10a}$$

$$J(\mu)\mathcal{V} \subseteq \mathcal{V}, \tag{10b}$$

for all $\mu \in \Delta^{N-1}$.

Proposition 2. *Given the impulsive linear system with polytopic uncertainties Σ_σ, defined by (1), whose vertices are the impulsive linear systems of the family $\{\Sigma_i\}_{i\in\mathcal{I}}$, defined by (5), a subspace $\mathcal{V} \subseteq \mathcal{X}$ is a robust hybrid controlled invariant subspace for Σ_σ if and only if there exists a state feedback $u(t) = Fx(t)$ such that*

$$\left(\bigoplus_{i=1}^{N}(A_i + B_i F)\right)\mathcal{V} \subseteq \mathcal{V}^{\oplus N}, \tag{11a}$$

$$\left(\bigoplus_{i=1}^{N} J_i\right)\mathcal{V} \subseteq \mathcal{V}^{\oplus N}. \tag{11b}$$

Proposition 2 is a direct consequence of Proposition 1, where the system $\Sigma_\sigma^{\mathcal{F}}$ is considered in place of Σ_σ. Namely, conditions (10) express the fact that $\mathcal{V}$ is a robust hybrid invariant subspace for the closed-loop system $\Sigma_\sigma^{\mathcal{F}}$ and conditions (11) are the counterpart of (7), where the free flow dynamics is replaced by the closed-loop flow dynamics.

Proposition 3. *Given the impulsive linear system with polytopic uncertainties Σ_σ, defined by (1), whose vertices are the impulsive linear systems of the family $\{\Sigma_i\}_{i\in\mathcal{I}}$, defined by (5), a subspace $\mathcal{V} \subseteq \mathcal{X}$ is a robust hybrid controlled invariant subspace for Σ_σ if and only if*

$$\left(\bigoplus_{i=1}^{N} A_i\right)\mathcal{V} \subseteq \mathcal{V}^{\oplus N} + \operatorname{Im}\left(\bigoplus_{i=1}^{N} B_i\right), \tag{12a}$$

$$\left(\bigoplus_{i=1}^{N} J_i\right)\mathcal{V} \subseteq \mathcal{V}^{\oplus N}. \tag{12b}$$

Proposition 3 is a direct consequence of Proposition 2, since condition (12a) is equivalent to condition (11a), as can be shown by simple algebraic manipulations. To this aim, let $V \in \mathbb{R}^{n\times r}$ be a matrix whose column vectors form a basis of the subspace $\mathcal{V}$. Moreover, let $W \in \mathbb{R}^{Nn\times Nr}$ be defined as

$$W = \begin{bmatrix} V & 0 & \dots & 0 \\ 0 & V & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V \end{bmatrix}, \tag{13}$$

so that W is a matrix whose column vectors form a basis of $\mathcal{V}^{\oplus N}$. Hence, (11a) is equivalent to the existence of matrices $L_i \in \mathbb{R}^{r\times r}$, with $i = 1, \dots, N$, such that the following equation holds:

$$\left(\bigoplus_{i=1}^{N} A_i\right)V + \left(\bigoplus_{i=1}^{N} B_i F\right)V = W\left(\bigoplus_{i=1}^{N} L_i\right). \tag{14}$$

Eq. (14) can also be written as

$$\left(\bigoplus_{i=1}^{N} A_i\right)V = W\left(\bigoplus_{i=1}^{N} L_i\right) + \left(\bigoplus_{i=1}^{N} B_i\right)U, \tag{15}$$

where $U = -FV$, which, in turn, is equivalent to (12a).

It is worth noting that the approach adopted herein to introduce the notion of hybrid robust controlled invariant subspace sticks to the principle of introducing the structural geometric objects with reference to the impulsive linear system with polytopic uncertainties and, then, to characterize them in terms of the vertex systems. This approach is opposite, yet formally equivalent, to that adopted in [28], where the definition of hybrid robust controlled invariant subspace was given in terms of the vertex systems and its characterization was given in terms of the existence of a state feedback. In this regard, it is worth stressing that the condition of Definition 2 involves the existence of a state feedback, but it does not provide any information on how to compute it. Instead, Proposition 3 expresses a characterization in terms of the vertex systems and, provided that a candidate subspace be known, it is directly checkable. Moreover, the equivalence between Propositions 2 and 3 leads to the computational procedure for the synthesis of a state feedback, through (14) and (15) in particular.

The last result reviewed in this section introduces a special robust hybrid controlled invariant subspace and provides an algorithm to compute it, thus setting the basis for the formulation of the constructive conditions for the solvability of the model matching problems that will be stated in Section 3.

Proposition 4. *Given the impulsive linear system with polytopic uncertainties Σ_σ, defined by (1), and a subspace $\mathcal{W} \subseteq \mathcal{X}$, the robust hybrid controlled invariant subspaces for Σ_σ contained in $\mathcal{W}$ form a family, denoted by $\mathbf{V_R}(\mathcal{W})$, which has a maximum element, denoted by $\mathcal{V}_R^*(\mathcal{W})$, computable as the last term of the sequence of subspaces recursively generated by*

$$\mathcal{V}_0 = \mathcal{W}, \tag{16a}$$

$$\mathcal{V}_k = \mathcal{V}_{k-1} \cap (\bigoplus_{i=1}^{N} A_i)^{-1}(\mathcal{V}_{k-1}^{\oplus N} + \mathrm{Im}\,(\bigoplus_{i=1}^{N} B_i)) \cap (\bigoplus_{i=1}^{N} J_i)^{-1} \mathcal{V}_{k-1}^{\oplus N}, \quad k = 1, 2, \dots, \bar{k}, \tag{16b}$$

where $\bar{k} \leq \dim(\mathcal{W})$ is the least integer such that $\mathcal{V}_{\bar{k}+1} = \mathcal{V}_{\bar{k}}$.

It is worth mentioning that the existence of a maximal element derives from the fact that the family $\mathbf{V_R}(\mathcal{W})$ is closed with respect to the sum of subspaces. Then, simple arguments of linear algebra show that the last term of the sequence generated by (16) is the maximal robust hybrid controlled invariant subspace for Σ_σ contained in $\mathcal{W}$.

3. Problem statements

The model matching problems studied in this work are stated with reference to an impulsive linear system with polytopic uncertainties $\Sigma_{P\sigma}$, the plant, and an impulsive linear system with polytopic uncertainties $\Sigma_{M\sigma}$, the model. The plant $\Sigma_{P\sigma}$ has the state-space representation

$$\Sigma_{P\sigma} \equiv \begin{cases} \dot{x}_P(t) = A_P(\mu)x_P(t) + B_P(\mu)u_P(t), & \text{with } t \neq \sigma(k),\ k \in \mathbb{N}, \\ x_P(t) = J_P(\mu)x_P^-(t), & \text{with } t = \sigma(k),\ k \in \mathbb{N}, \\ y_P(t) = C_P(\mu)x_P(t), \end{cases} \tag{17}$$

where $x_P \in \mathcal{X}_P = \mathbb{R}^{n_P}$, $u_P \in \mathcal{U}_P = \mathbb{R}^{m_P}$, and $y_P \in \mathcal{Y} = \mathbb{R}^p$ are state, the control input, and the output, respectively. Moreover, the system's matrices are given by

$$A_P(\mu) = \sum_{i=1}^{N} \mu_i A_{Pi}, \quad B_P(\mu) = \sum_{i=1}^{N} \mu_i B_{Pi}, \quad J_P(\mu) = \sum_{i=1}^{N} \mu_i J_{Pi}, \quad C_P(\mu) = \sum_{i=1}^{N} \mu_i C_{Pi}, \tag{18}$$

where $\mu = [\mu_1 \dots \mu_N]^\top \in \Delta^{N-1}$ is the uncertain parameter vector and

$$\{A_{Pi}\}_{i\in\mathcal{I}}, \quad \{B_{Pi}\}_{i\in\mathcal{I}}, \quad \{J_{Pi}\}_{i\in\mathcal{I}}, \quad \{C_{Pi}\}_{i\in\mathcal{I}}, \tag{19}$$

where $\mathcal{I} = \{1, \dots, N\}$, are families of constant real matrices with consistent dimensions. The model $\Sigma_{M\sigma}$ has the state space representation

$$\Sigma_{M\sigma} \equiv \begin{cases} \dot{x}_M(t) = A_M(\mu)x_M(t) + B_M(\mu)u_M(t), & \text{with } t \neq \sigma(k),\ k \in \mathbb{N}, \\ x_M(t) = J_M(\mu)x_M^-(t), & \text{with } t = \sigma(k),\ k \in \mathbb{N}, \\ y_M(t) = C_M(\mu)x_M(t), \end{cases} \tag{20}$$

where $x_M \in \mathcal{X}_M = \mathbb{R}^{n_M}$, $u_M \in \mathcal{U}_M = \mathbb{R}^{m_M}$, and $y_M \in \mathcal{Y} = \mathbb{R}^p$ are state, the input, and the output, respectively. Moreover, the matrices of the model are given by

$$A_M(\mu) = \sum_{i=1}^{N} \mu_i A_{Mi}, \quad B_M(\mu) = \sum_{i=1}^{N} \mu_i B_{Mi}, \quad J_M(\mu) = \sum_{i=1}^{N} \mu_i J_{Mi}, \quad C_M(\mu) = \sum_{i=1}^{N} \mu_i C_{Mi}, \tag{21}$$

where $\mu = [\mu_1 \dots \mu_N]^\top \in \Delta^{N-1}$ is the uncertain parameter vector and

$$\{A_{Mi}\}_{i\in\mathcal{I}}, \quad \{B_{Mi}\}_{i\in\mathcal{I}}, \quad \{J_{Mi}\}_{i\in\mathcal{I}}, \quad \{C_{Mi}\}_{i\in\mathcal{I}}, \tag{22}$$

where $\mathcal{I} = \{1, \dots, N\}$, are families of constant real matrices with consistent dimensions.

In model matching problems, the to-be-controlled variable is the output matching error, denoted by $e(t)$ and defined as the difference between the output of the plant $y_P(t)$ and the output of model $y_M(t)$: namely,

$$e(t) = y_P(t) - y_M(t). \tag{23}$$

The output matching error can be seen as the output of the impulsive linear system with polytopic uncertainties, denoted by $\Sigma_{D\sigma}$, derived from $\Sigma_{P\sigma}$ and $\Sigma_{M\sigma}$ according to the following equations,

$$\Sigma_{D\sigma} \equiv \begin{cases} \dot{x}(t) = A(\mu)x(t) + B(\mu)u_P(t) + H(\mu)u_M(t), & \text{with } t \neq \sigma(k),\ k \in \mathbb{N}, \\ x(t) = J(\mu)x^-(t), & \text{with } t = \sigma(k),\ k \in \mathbb{N}, \\ e(t) = E(\mu)x(t), \end{cases} \tag{24}$$

where $x(t) = [x_P(t)^\top \ x_M(t)^\top]^\top$ and

$$A(\mu) = \begin{bmatrix} A_P(\mu) & 0 \\ 0 & A_M(\mu) \end{bmatrix}, \quad B(\mu) = \begin{bmatrix} B_P(\mu) \\ 0 \end{bmatrix}, \quad H(\mu) = \begin{bmatrix} 0 \\ B_M(\mu) \end{bmatrix}, \tag{25a}$$

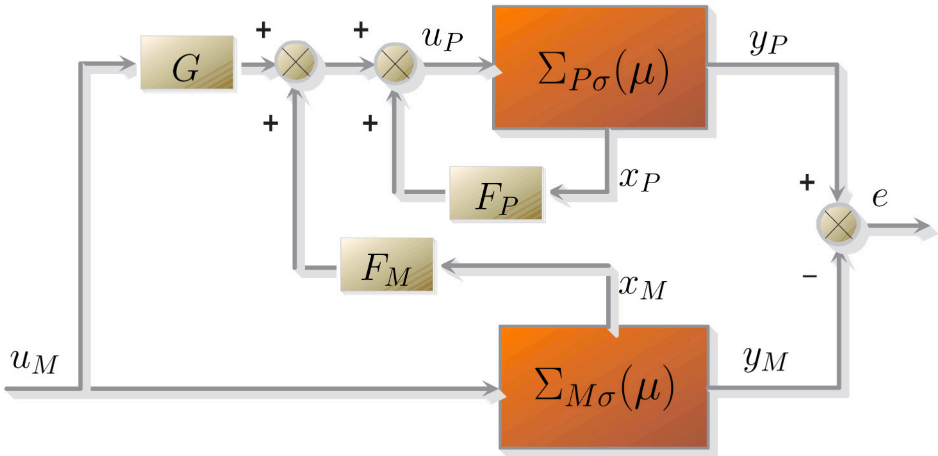

Fig. 1. Block diagram of the overall compensated system $\Sigma_{D\sigma}^{\mathscr{C}}$.

$$J(\mu) = \begin{bmatrix} J_P(\mu) & 0 \\ 0 & J_M(\mu) \end{bmatrix}, \quad E(\mu) = \begin{bmatrix} C_P(\mu) & -C_M(\mu) \end{bmatrix}. \tag{25b}$$

As to the control scheme, it is assumed that the state of the plant $x_P(t)$ and the state of the model $x_M(t)$ are both available, along with the input of the model $u_M(t)$. Hence, the control law consists of a feedback action, exploiting $x_P(t)$ and $x_M(t)$, and of a feedforward action, exploiting $u_M(t)$, applied to the control input $u_P(t)$ of the plant $\Sigma_{P\sigma}$. Namely, the control law can be written as

$$u_P(t) = F_P x_P(t) + F_M x_M(t) + G u_M(t). \tag{26}$$

Consequently, the overall compensated system is the impulsive linear system with polytopic uncertainties, denoted by $\Sigma_{D\sigma}^{\mathscr{C}}$, which is obtained by taking into account that a control law of the form (26) is applied to $\Sigma_{P\sigma}$. Namely, with the same notation of (24), the overall compensated system $\Sigma_{D\sigma}^{\mathscr{C}}$ can be described by

$$\Sigma_{D\sigma}^{\mathscr{C}} \equiv \begin{cases} \dot{x}(t) &= A_F(\mu)x(t) + B_G(\mu)u_M(t), & \text{with } t \neq \sigma(k),\ k \in \mathbb{N}, \\ x(t) &= J(\mu)x^-(t), & \text{with } t = \sigma(k),\ k \in \mathbb{N}, \\ e(t) &= E(\mu)x(t), \end{cases} \tag{27}$$

where

$$A_F(\mu) = \begin{bmatrix} A_P(\mu) + B_P(\mu)F_P & B_P(\mu)F_M \\ 0 & A_M(\mu) \end{bmatrix}, \quad B_G(\mu) = \begin{bmatrix} B_P(\mu)G \\ B_M(\mu) \end{bmatrix}, \tag{28}$$

while $J(\mu)$ and $E(\mu)$ were defined in (25b). A schematic diagram illustrating the structure of the compensator and of the overall compensated system is shown in Fig. 1. With the notation introduced so far, the model matching problems investigated in this work can be stated as follows.

Problem 1 (*Exact Model Matching Problem, EMMP*)**.** Given the impulsive linear system with polytopic uncertainties $\Sigma_{P\sigma}$, defined by (17), and the impulsive linear model with polytopic uncertainties $\Sigma_{M\sigma}$, defined by (20), the Exact Model Matching Problem consists in finding a control law $u_P(t)$, of the form (26), such that the overall compensated system $\Sigma_{D\sigma}^{\mathscr{C}}$, defined by (27) with (28), satisfies the requirement

$\mathcal{R}$.1. $e(t)=0$ for all $t \geq 0$, with initial state $x(0)=0$, for all $\mu \in \Delta^{N-1}$ and for all $\sigma \in \mathscr{S}$.

The EMMP is a merely structural problem, in the sense that it presupposes that the system and the model are initially synchronized and it only focuses on the behaviour of the output matching error by expressing a condition that must be satisfied for all time instants. Consequently, as it will be clarified in Section 4, its solvability depends only on the structure of the linear maps defining the system and the model, while it does not depend on their qualitative properties related to stability, nor does it depend on the sequence of the time instants when the state evolution shows abrupt discontinuities.

Problem 2 (*Asymptotic Model Matching Problem, AMMP*)**.** Given the impulsive linear system with polytopic uncertainties $\Sigma_{P\sigma}$, defined by (17), and the impulsive linear model with polytopic uncertainties $\Sigma_{M\sigma}$, defined by (20), the Asymptotic Model Matching Problem consists in finding a control law $u_P(t)$, of the form (26), such that the overall compensated system $\Sigma_{D\sigma}^{\mathscr{C}}$, defined by (27) with (28), satisfies the requirement

$\mathcal{R}$.1. $\lim_{t\to+\infty} e(t)=0$, for any initial state $x(0)=x_0$, for all $\mu \in \Delta^{N-1}$ and for all $\sigma \in \mathscr{S}_\tau$.

The AMMP does not presuppose that the system and the model are initially synchronized, but still it solely focuses on the behaviour of the output matching error by expressing a condition that must be satisfied asymptotically, for sufficiently slow sequences of the jump time instants. Therefore, its solvability depends not only on the structure of the linear maps defining the system and the model, but also on their qualitative properties related to stability, which should guarantee that the transient of the output matching error vanishes for sufficiently slow sequences of the jump time instants. Actually, in control system design, the focus is not only on the behaviour of the output matching error, but also on the stability, defined in a proper sense, of the compensated system, given that, in general, it is of interest to match a stable, in the proper sense, model. For this reason, in the next problem formulation, the model is assumed to be asymptotically stable for sufficiently slow sequences of the jump time instants and the compensated system is required to be asymptotically stable in the same sense.

Problem 3 (*Model Matching Problem with Stability, SMMP*)**.** Given the impulsive linear system with polytopic uncertainties $\Sigma_{P\sigma}$, defined by (17), and the impulsive linear model with polytopic uncertainties $\Sigma_{M\sigma}$, defined by (20), on the assumption that $\Sigma_{M\sigma}$ be asymptotically stable for all $\mu \in \Delta^{N-1}$ and for all $\sigma \in \mathscr{S}_{\tau'}$, for some $\tau' > 0$, the Model Matching Problem with Stability consists in finding a control law $u_P(t)$, of the form (26), such the compensated system $\Sigma_{D\sigma}^{\mathscr{C}}$, defined by (27) with (28), satisfies the requirements

$\mathcal{R}.1.$ $e(t)=0$ for all $t \geq 0$, with initial state $x(0)=0$, for all $\mu \in \Delta^{N-1}$ and for all $\sigma \in \mathscr{S}$;

$\mathcal{R}.2.$ $\Sigma_{D\sigma}^{\mathscr{C}}$ is asymptotically stable for all $\mu \in \Delta^{N-1}$ and for all $\sigma \in \mathscr{S}_{\tau}$, for some $\tau \geq \tau'$.

The SMMP is a stronger formulation of the AMMP, since the qualitative properties of the linear maps defining the system and the model (the latter assumed to be asymptotically stable for sufficiently slow sequences of the jump time instants) should guarantee that also the transient of the state evolution of the overall compensated system $\Sigma_{D\sigma}^{\mathscr{C}}$ vanishes for sufficiently slow sequences of the jump time instants.

It is worth noting that, due to the upper block-triangular structure of the dynamic matrices of $\Sigma_{D\sigma}^{\mathscr{C}}$ and to the stability assumption on the model $\Sigma_{M\sigma}$, Requirement $\mathcal{R}.2.$ of the SMMP is equivalent to the requirement that the dynamics of the closed-loop plant $\Sigma_{P\sigma}^{\mathscr{F}_P}$, with the obvious notation, is asymptotically stable for sufficiently slow sequences of the jump time instants.

Finally, it is worth noting that Requirement $\mathcal{R}.1.$ of the SMMP matches Requirement $\mathcal{R}.1.$ of the EMMP. Moreover, Requirement $\mathcal{R}.2.$ of the SMMP includes Requirement $\mathcal{R}.1.$ of the AMMP, which means that any solution of the SMMP also solves the AMMP. For these reasons, only the solvability conditions for the EMMP and the SMMP will be investigated in Section 4.

4. Main results

In this section, the solvability of the problems formulated in Section 3 is expressed by conditions which can be directly checked by properly-devised algorithmic procedure. This section is organized in three parts. In Section 4.1, a structural, sufficient solvability condition is given for the EMMP (Theorem 1). This condition is constructive, in the sense that its proof illustrates how to compute the feedback and the feedforward components of the control law. A comparison between the structural solvability condition for decoupling an accessible signal (condition which is derived herein as a byproduct of the proof of Theorem 1) and the structural solvability condition for decoupling an inaccessible signal or, equivalently, a disturbance in the proper sense (condition which was derived in [28, Theorem 1]), is presented in Section 4.2. Finally, a sufficient solvability condition for the SMMP (hence, for the AMMP) is given in Section 4.3 (Theorem 2). In this case, the computation of the feedback and feedforward components of the control law follows the same lines as those illustrated in the proof of Theorem 1, with additional conditions which guarantee that the feedback from the plant state achieves the plant stabilization (Condition $\mathcal{C}.2$ of Theorem 2).

4.1. Solvability condition for the EMMP

First, a sufficient condition for the solvability of the EMMP, the merely structural problem, is given. To this aim, a specific maximal robust hybrid controlled invariant subspace is introduced. Let $\Sigma_{E\sigma}$ denote the impulsive linear system with polytopic uncertainties derived from the system $\Sigma_{D\sigma}$, defined by (24), by disregarding the input $u_M(t)$: i.e.,

$$\Sigma_{E\sigma} \equiv \begin{cases} \dot{x}(t) &=& A(\mu)x(t) + B(\mu)u_P(t), & \text{with } t \neq \sigma(k),\ k \in \mathbb{N}, \\ x(t) &=& J(\mu)x^-(t), & \text{with } t = \sigma(k),\ k \in \mathbb{N}, \\ e(t) &=& E(\mu)x(t), \end{cases} \tag{29}$$

where $A(\mu)$, $B(\mu)$, $J(\mu)$, and $E(\mu)$ were defined as functions of the respective matrices of the system $\Sigma_{P\sigma}$ and of the model $\Sigma_{M\sigma}$ in (25). With reference to $\Sigma_{E\sigma}$, let

$$\mathcal{K} = \bigcap_{i=1}^{N} \operatorname{Ker} E_i = \bigcap_{i=1}^{N} \operatorname{Ker} [\ C_{Pi} \quad -C_{Mi}\ ] \tag{30}$$

and let $\mathcal{V}_R^*$ denote the maximal robust hybrid controlled invariant subspace for $\Sigma_{E\sigma}$ contained in $\mathcal{K}$: i.e.,

$$\mathcal{V}_R^* = \mathcal{V}_R^*(\mathcal{K}).$$

Then, a sufficient condition for the solvability of the EMMP is given as follows.

Theorem 1. *Given the impulsive linear system with polytopic uncertainties $\Sigma_{P\sigma}$, defined by (17), and the impulsive linear model with polytopic uncertainties $\Sigma_{M\sigma}$, defined by (20), Problem 1 is solvable if*

C.1. the following inclusion holds:

$$\mathrm{Im}\left(\bigoplus_{i=1}^{N}\begin{pmatrix}0\\ B_{Mi}\end{pmatrix}\right)\subseteq(\mathcal{V}_R^*)^{\oplus N}+\mathrm{Im}\left(\bigoplus_{i=1}^{N}\begin{pmatrix}B_{Pi}\\ 0\end{pmatrix}\right). \tag{31}$$

Proof. Let $\{B_i\}_{i\in\mathcal{I}}$ and $\{H_i\}_{i\in\mathcal{I}}$ denote the families of constant real matrices associated with $B(\mu)$ and $H(\mu)$ defined in (25a). Let $V\in\mathbb{R}^{n\times r}$ be a matrix whose column vectors form a basis of $\mathcal{V}_R^*$ and let W have the same form as in (13), so that W is a matrix whose column vectors form a basis of $(\mathcal{V}_R^*)^{\oplus N}$. Hence, there exist a family of matrices $\{M_i\}_{i\in\mathcal{I}}$, where $M_i\in\mathbb{R}^{r\times m_M}$ with $i=1,\dots,N$, and a matrix $G\in\mathbb{R}^{m_P\times m_M}$ such that (31) can be written (in a more compact form, with the notation just introduced) as

$$\left(\bigoplus_{i=1}^{N}H_i\right)=W\left(\bigoplus_{i=1}^{N}M_i\right)-\left(\bigoplus_{i=1}^{N}B_i\right)G. \tag{32}$$

Moreover, (32) can be written as

$$\left(\bigoplus_{i=1}^{N}(H_i+B_iG)\right)=W\left(\bigoplus_{i=1}^{N}M_i\right). \tag{33}$$

In turn, (33) is equivalent to

$$\mathrm{Im}\left(\bigoplus_{i=1}^{N}(H_i+B_iG)\right)\subseteq(\mathcal{V}_R^*)^{\oplus N}. \tag{34}$$

Then, consider the system $\Sigma_{D\sigma}$, defined by (24), and apply the control law

$$u_P(t)=Gu_M(t)+v(t), \tag{35}$$

where G is the matrix introduced in (32) and $v\in\mathcal{U}_P$ is a new, independent, control input. The resulting system, denoted by $\Sigma_{D\sigma}^{\mathcal{G}}$, is described by

$$\Sigma_{D\sigma}^{\mathcal{G}}\equiv\begin{cases}\dot{x}(t) = A(\mu)x(t)+B(\mu)v(t)+(B(\mu)G+H(\mu))u_M(t), & \text{with } t\neq\sigma(k),\ k\in\mathbb{N},\\ x(t) = J(\mu)x^-(t), & \text{with } t=\sigma(k),\ k\in\mathbb{N},\\ e(t) = E(\mu)x(t).\end{cases} \tag{36}$$

Note that $\mathcal{V}_R^*$ in (31), hence in (34), is the maximal robust hybrid controlled invariant subspace for $\Sigma_{E\sigma}$, defined by (29), or, which is the same, for $\Sigma_{D\sigma}^{\mathcal{G}}$, defined by (36), where the input $u_M(t)$ is disregarded. Since (34) holds, there exists a state feedback

$$v(t)=Fx(t), \tag{37}$$

where F is a friend of $\mathcal{V}_R^*$, which, applied to $\Sigma_{D\sigma}^{\mathcal{G}}$, annihilates the effect of $u_M(t)$ on $e(t)$ for all $t\geq 0$ [28, Theorem 1]. In fact, by applying (37) to $\Sigma_{D\sigma}^{\mathcal{G}}$, defined by (36), one obtains the compensated system $\Sigma_{D\sigma}^{\mathcal{C}}$ described by

$$\Sigma_{D\sigma}^{\mathcal{C}}\equiv\begin{cases}\dot{x}(t) = (A(\mu)+B(\mu)F)x(t)+(B(\mu)G+H(\mu))u_M(t), & \text{with } t\neq\sigma(k),\ k\in\mathbb{N},\\ x(t) = J(\mu)x^-(t), & \text{with } t=\sigma(k),\ k\in\mathbb{N},\\ e(t) = E(\mu)x(t),\end{cases} \tag{38}$$

which is the same as (27) with (28), where $F=[F_P\ F_M]$ and G have been properly determined above. Now, consider a change of basis $x(t)=Tz(t)$, where $T=[V\ T_1]$ and V, as defined earlier, is a matrix whose column vectors form a basis of $\mathcal{V}_R^*$. With respect to the new coordinates, where the state $z(t)$ is partitioned as $z(t)=[z_1(t)^\top\ z_2(t)^\top]^\top$ according to T, the system $\Sigma_{D\sigma}^{\mathcal{C}}$ in (38) can be written as

$$\Sigma_{D\sigma}^{\mathcal{C}}\equiv\begin{cases}\begin{bmatrix}\dot{z}_1(t)\\ \dot{z}_2(t)\end{bmatrix} = \begin{bmatrix}A_{11}(\mu) & A_{12}(\mu)\\ 0 & A_{22}(\mu)\end{bmatrix}\begin{bmatrix}z_1(t)\\ z_2(t)\end{bmatrix}+\begin{bmatrix}B_1(\mu)\\ 0\end{bmatrix}u_M(t), & \text{with } t\neq\sigma(k),\ k\in\mathbb{N},\\ \begin{bmatrix}z_1(t)\\ z_2(t)\end{bmatrix} = \begin{bmatrix}J_{11}(\mu) & J_{12}(\mu)\\ 0 & J_{22}(\mu)\end{bmatrix}\begin{bmatrix}z_1^-(t)\\ z_2^-(t)\end{bmatrix}, & \text{with } t=\sigma(k),\ k\in\mathbb{N},\\ e(t) = \begin{bmatrix}0 & C_2(\mu)\end{bmatrix}\begin{bmatrix}z_1(t)\\ z_2(t)\end{bmatrix}.\end{cases} \tag{39}$$

The matrices of $\Sigma_{D\sigma}^{\mathcal{C}}$ in (39) exhibit structural zero blocks which show that the output $e(t)$ is decoupled from the input $u_M(t)$. Namely, the output $e(t)$ only depends on the component $z_2(t)$ of the state and the latter is not affected by $u_M(t)$ as a consequence of the structure of the input matrix and of the upper block-triangular structure of the dynamic matrices. In terms of the original system $\Sigma_{P\sigma}$ and model $\Sigma_{M\sigma}$, this means that the desired output matching is achieved. □

Remark 1. The condition of Theorem 1 is only sufficient, in general. In fact, it hinges on the structural condition for achieving disturbance decoupling (i.e., for achieving decoupling of a signal which is not accessible) in the same class of dynamical systems and that condition was shown to be only sufficient, in general (see [28, Remark 2]).

Remark 2. The condition of Theorem 1 can be checked directly. Moreover, it is constructive. In fact, in (31), the subspace $\mathcal{V}_R^*$ can be computed by the recursive algorithm (16) and the inclusion can be checked by standard methods of linear algebra. Then, if the condition is met, a friend of $\mathcal{V}_R^*$ can be derived by exploiting the equivalence between Propositions 2 and 3 (through (14) and (15), in particular), while the feedforward matrix G can be obtained from (32).

Remark 3. Theorem 1 encompasses the special case of systems and models which do not exhibit any impulsive behaviour (i.e., $J_{Pi} = I_{n_P}$ and/or $J_{Mi} = I_{n_M}$ for all $i \in \mathcal{I}$). Moreover, Theorem 1 encompasses the special case of systems and models which are not affected by polytopic uncertainties (i.e., the plant $\Sigma_{P\sigma}$ and/or the model $\Sigma_{M\sigma}$ do not depend on the uncertain parameter vector μ). In particular, it is of interest to consider the case in which the plant is an impulsive linear system with polytopic uncertainties (according to the general assumptions made in this work), while the model is not affected by any polytopic uncertainty and it only exhibits an impulsive behaviour or it does not even exhibit any impulsive behaviour. In this case, the model matching problem can be seen as the problem of forcing the output of the impulsive linear system with polytopic uncertainties to match that of a given linear (impulsive or even time-invariant) model in a robust way with respect to the system's uncertainties. These special cases will be illustrated by the numerical examples of Section 5.

4.2. Comparison between the solvability conditions for structural decoupling

As a by-product of the proof of Theorem 1, it ensues that the more general problem of decoupling the output $y(t)$ of an impulsive linear system with polytopic uncertainties $\Sigma_{C\sigma}$, described by

$$\Sigma_{C\sigma} \equiv \begin{cases} \dot{x}(t) &=& A(\mu)x(t) + B(\mu)u(t) + D(\mu)d(t), & \text{with } t \neq \sigma(k),\ k \in \mathbb{N}, \\ x(t) &=& J(\mu)x^-(t), & \text{with } t = \sigma(k),\ k \in \mathbb{N}, \\ y(t) &=& C(\mu)x(t), \end{cases} \tag{40}$$

from the accessible disturbance $d(t)$, with $d \in \mathcal{W}_D = \mathbb{R}^s$ and $D(\mu) = \sum_{i\in\mathcal{I}} \mu_i D_i$, by a control law of the form

$$u(t) = Fx(t) + Gd(t) \tag{41}$$

is solvable if

$$\operatorname{Im}\left(\bigoplus_{i=1}^{N} D_i\right) \subseteq (\mathcal{V}_R^*)^{\oplus N} + \operatorname{Im}\left(\bigoplus_{i=1}^{N} B_i\right). \tag{42}$$

Instead, if the disturbance $d(t)$ is not accessible, the feedforward term which appears in the control law (41) does not apply. The control law reduces to the sole state feedback and takes the form $u(t) = Fx(t)$. Consistently, the term $\operatorname{Im}\left(\bigoplus_{i=1}^{N} B_i\right)$ which appears in (42) is dropped and the solvability condition reduces to the more restrictive

$$\operatorname{Im}\left(\bigoplus_{i=1}^{N} D_i\right) \subseteq (\mathcal{V}_R^*)^{\oplus N}. \tag{43}$$

It is worth noting that (43) is equivalent to the existence of a family of matrices $\{M_i\}_{i\in\mathcal{I}}$, with $M_i \in \mathbb{R}^{r\times s}$, such that

$$\left(\bigoplus_{i=1}^{N} D_i\right) = W\left(\bigoplus_{i=1}^{N} M_i\right), \tag{44}$$

with the usual meaning of the matrix W as a basis matrix of $(\mathcal{V}_R^*)^{\oplus N}$. Eq. (44) clearly shows that (43) can be equivalently expressed as the set of inclusions

$$\operatorname{Im} D_i \subseteq \mathcal{V}_R^*, \quad \text{for all } i \in \mathcal{I}, \tag{45}$$

which is also the formulation of the solvability condition for the disturbance decoupling problem given in [28, Theorem 1]. On the other hand, (42) is equivalent to the existence of a family of matrices $\{M_i\}_{i\in\mathcal{I}}$, with $M_i \in \mathbb{R}^{r\times s}$, and *only one* matrix L such that

$$\left(\bigoplus_{i=1}^{N} D_i\right) = W\left(\bigoplus_{i=1}^{N} M_i\right) + \left(\bigoplus_{i=1}^{N} B_i\right)L. \tag{46}$$

Eq. (46) plainly shows that a reformulation of (42) as a set of inclusions of the kind of

$$\operatorname{Im} D_i \subseteq \mathcal{V}_R^* + \operatorname{Im} B_i, \quad \text{for all } i \in \mathcal{I}, \tag{47}$$

is *not* possible, since (42) (or, equivalently, (46)) is actually more demanding than the set of inclusions (47).

4.3. Solvability condition for the SMMP

In this section, a sufficient condition for the solvability of the SMMP is presented. In line with the considerations which conclude Section 3, the solvability condition for the SMMP, which is twofold since it consists of a structural condition and of a stabilizability condition, has the following features:

i. the structural condition of the SMMP coincides with the solvability condition of the EMMP (i.e., Condition $\mathcal{C}$.1. of Theorem 1), because Requirement $\mathcal{R}$.1. of Problem 3 (i.e., the structural requirement) matches Requirement $\mathcal{R}$.1. (i.e., the only requirement) of Problem 1;
ii. the stabilizability condition of the SMMP is focused on the stabilizability of the plant, because the model is assumed to be asymptotically stable for sufficiently slow sequences of the jump time instants and because the dynamic matrices of the overall compensated system have an upper block-triangular form where the lower right block pertains to the model dynamics — see $\Sigma_{D\sigma}^{\mathscr{C}}$ in (27) with $A_F(\mu)$ and $J(\mu)$ given by (28) and (25b), respectively;
iii. the structural and the stabilizability conditions for the solvability of the SMMP are also sufficient for the solvability of the AMMP, being the AMMP a weaker formulation of the SMMP: namely, if Requirement $\mathcal{R}$.2. of Problem 3 is satisfied, then Requirement $\mathcal{R}$.1. (i.e., the only requirement) of Problem 2 is also satisfied.

Hence, a sufficient condition for the solvability of the SMMP is given as follows.

Theorem 2. *Given the impulsive linear system with polytopic uncertainties $\Sigma_{P\sigma}$, defined by (17), and the impulsive linear model with polytopic uncertainties $\Sigma_{M\sigma}$, defined by (20), on the assumption that $\Sigma_{M\sigma}$ be asymptotically stable for all $\mu \in \Delta^{N-1}$ and for all $\sigma \in \mathscr{S}_{\tau'}$, for some $\tau' > 0$, Problem 3 is solvable if*

$\mathcal{C}$.1. the inclusion (31) holds;
$\mathcal{C}$.2. there exists a friend $F = [F_P \;\, F_M]$ of $\mathcal{V}_R^*$ such that the condition

$$\left(A_{Pi} + B_{Pi}F_P\right)^\top P + P\left(A_{Pi} + B_{Pi}F_P\right) < 0 \tag{48}$$

holds for all $i \in \mathcal{I}$ and for some positive definite matrix $P \in \mathbb{R}^{n_P \times n_P}$.

Proof. Condition $\mathcal{C}$.1. implies that Requirement $\mathcal{R}$.1. of Problem 3 is met, as was shown in the proof of Theorem 1 with reference to Problem 1. Hence, to show that also Requirement $\mathcal{R}$.2. of Problem 3 is satisfied, assume that Condition $\mathcal{C}$.2. holds and note that this means that the flow dynamics of the vertex systems $\Sigma_{Pi}^{\mathscr{F}_P}$, with $i \in \mathcal{I}$, of the compensated plant $\Sigma_{P\sigma}^{\mathscr{F}_P}$ are asymptotically stable and that, in addition, they have a common quadratic Lyapunov function, which is $V(x(t)) = x_P(t)^\top P x_P(t)$. By virtue of [28, Proposition 5], this implies that $V(x(t))$ is a common quadratic Lyapunov function for the flow dynamics given by $A_P(\mu) + B_P(\mu)F_P$ for all $\mu \in \Delta^{N-1}$ and, in turn, this implies that, for any given $\mu \in \Delta^{N-1}$, the corresponding flow dynamics is asymptotically stable. With a reasoning similar to that developed in the proof of [28, Theorem 2], where $A_P(\mu) + B_P(\mu)F_P$ and $J_P(\mu)$ should replace $A(\mu) + B(\mu)F$ and $J(\mu)$, respectively, one can prove that the uncertain impulsive linear dynamics of $\Sigma_{P\sigma}^{\mathscr{F}_P}$ is asymptotically stable for all $\mu \in \Delta^{N-1}$ and for all $\sigma \in \mathscr{S}_{\tau''}$ for some $\tau'' > 0$. Then, in light of the assumption on the asymptotic stability of $\Sigma_{M\sigma}$ for all $\mu \in \Delta^{N-1}$ and for all $\sigma \in \mathscr{S}_{\tau'}$ for some $\tau' > 0$ and in light of the upper block-diagonal structure of the dynamic matrices of the overall compensated system $\Sigma_{D\sigma}^{\mathscr{C}}$, one can conclude that Requirement $\mathcal{R}$.2. of Problem 3 is satisfied. □

Remark 4. Both condition (31) and condition (48) can be directly checked: the first one, by using the algorithm for constructing $\mathcal{V}_R^*$ given in Proposition 4 and proceeding as described in Remark 2; the second one, by reducing it to a set of LMIs with a reasoning similar to that described in [28, Section 4].

Remark 5. As to the computation of a suitable value for the positive real constant τ such that the overall compensated system $\Sigma_{D\sigma}^{\mathscr{C}}$ is asymptotically stable for all $\mu \in \Delta^{N-1}$ and for all $\sigma \in \mathscr{S}_{\tau}$, this issue is not encompassed in Theorem 2. Indeed, the proof of Theorem 2 is a proof of the existence of some τ'' such that the compensated plant $\Sigma_{P\sigma}^{\mathscr{F}_P}$ is asymptotically stable for all $\mu \in \Delta^{N-1}$ and for all $\sigma \in \mathscr{S}_{\tau''}$ and, on that basis, one can infer the existence of the mentioned τ. However, that proof, like the proof of [28, Theorem 2] to which it refers, is not constructive in regard to τ. Hence, only heuristic and ad hoc methods can be suggested to compute τ. A demonstration of how to perform such computations will be given in the worked-out examples of Section 5.

Remark 6. An obvious necessary condition for the solvability of the EMMP and of the SMMP is that the specific problem is solvable for each one of the vertex systems. This can be algorithmically checked by using the solvability characterization given in [19, Theorem 2] for impulsive linear systems which are not subject to uncertainties.

On a final note, it may be worth observing that the problem formulations to which Theorems 1 and 2 provide sufficient solvability conditions are neat and sharp, in the sense that zero output matching error and, in addition, stability are required to be achieved, with constant feedback and feedforward matrices, for all the values of the uncertain parameters within the given polytope and for all the admissible sequences of jump time instants. This explains the fact that the solvability conditions are likewise strong.

5. Illustrative examples

In this section, two numerical examples are worked out with the aim of illustrating, step by step, how to check the solvability conditions, how to design the control law and how to compute a lower bound for the length of the smallest time interval between two consecutive jump time instants, so as to guarantee the asymptotic stability of the overall compensated system. The first example is focused on the problem of matching the output of a given impulsive linear plant with polytopic uncertainties with the output of a given impulsive linear model perfectly known (i.e., not subject to uncertainties). The second example is a variant of the former, where the model is assumed to be a given linear time-invariant system (i.e., neither subject to polytopic uncertainties nor to impulsive behaviour). Simulations show the performance of the designed control systems, with nonzero initial conditions, in the presence of a finite perturbation and in the presence of a persistent input, respectively.

5.1. Example 1

Let $\Sigma_{P\sigma}$ be a given impulsive linear plant with polytopic uncertainties, of the form (17), defined by $\sigma \in \mathscr{S}$, $\mathcal{I}=\{1,2\}$, $\mu=[\mu_1\ \mu_2]^\top \in \Delta^1$, with

$$A_{P1}=\begin{bmatrix}-2 & 2\\ -3 & -2\end{bmatrix},\ A_{P2}=\begin{bmatrix}-2 & 3\\ -3 & -2\end{bmatrix},\ B_{P1}=B_{P2}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},\ J_{P1}=J_{P2}=\begin{bmatrix}-3 & 1\\ 0 & -1\end{bmatrix},\ C_{P1}=C_{P2}=\begin{bmatrix}2 & 0\end{bmatrix}. \tag{49}$$

Let $\Sigma_{M\sigma}$ be a given impulsive linear model, of the form (20) – specialized to the case in which there are no polytopic uncertainties, therefore, simply defined by

$$A_M=\begin{bmatrix}-2 & 1\\ 0 & -3\end{bmatrix},\quad B_M=\begin{bmatrix}-1\\ 1\end{bmatrix},\quad J_M=\begin{bmatrix}-1 & 2\\ 3 & 0\end{bmatrix},\quad C_M=\begin{bmatrix}-1 & 1\end{bmatrix}. \tag{50}$$

In this case, the uncertainty concerns only the plant and, in particular, the coefficient that describes how the time derivative of the first state variable, $\dot{x}_{P1}(t)$, depends on the second state variable, $x_{P2}(t)$, in the flow equation. This coefficient may range between 2 and 3. Since the model is not subject to uncertainties, the related EMMP can be seen as the problem of forcing the output of the plant $\Sigma_{P\sigma}$ to match that of the model $\Sigma_{M\sigma}$ in a robust way with respect to the plant's uncertainty, for all $\sigma \in \mathscr{S}$, if the initial conditions of both the plant and the model are equal to 0.

Computations performed according to Proposition 4 show that the maximal robust hybrid controlled invariant subspace for the system $\Sigma_{E\sigma}$, constructed as in (29) with (25), contained in $\mathcal{K}$, defined as in (30), is given by

$$\mathcal{V}_R^* = \operatorname{Im} V = \operatorname{Im}\begin{bmatrix}1 & 0 & -1 & 1\end{bmatrix}^\top \subseteq \mathcal{X}_P \oplus \mathcal{X}_M = \mathbb{R}^4. \tag{51}$$

The sufficient condition for the solvability of the EMMP is given by Theorem 1. Herein, condition (31) reduces to

$$\operatorname{Im}\begin{bmatrix}0\\ B_M\end{bmatrix} \subseteq \mathcal{V}_R^* + \operatorname{Im}\begin{bmatrix}B_P\\ 0\end{bmatrix}, \tag{52}$$

where $B_P = B_{P1} = B_{P2}$, and, taking (51) into account, it is easy to see that it is satisfied. In particular,

$$\begin{bmatrix}0\\ B_M\end{bmatrix} = V - \begin{bmatrix}B_P\\ 0\end{bmatrix} G, \quad \text{with } G=\begin{bmatrix}-1\\ 0\end{bmatrix}. \tag{53}$$

A friend of $\mathcal{V}_R^*$ can be derived from (14) and a possible choice is given by

$$F=\begin{bmatrix}F_P & F_M\end{bmatrix}=\left[\begin{array}{cc|cc}1 & 1 & 0 & 0\\ 3 & -2 & 0 & 0\end{array}\right]. \tag{54}$$

Therefore, a solution of the EMMP is given by the control law (26), where G, F_P and F_M are given by (53) and (54): i.e.,

$$u_P(t)=\begin{bmatrix}1 & 1\\ 3 & -2\end{bmatrix} x_P(t) + \begin{bmatrix}-1\\ 0\end{bmatrix} u_M(t). \tag{55}$$

Indeed, the control law (55), applied to the output difference system $\Sigma_{D\sigma}$, constructed as in (24) with (25), leads to the overall compensated system $\Sigma_{D\sigma}^{\mathscr{C}}$, of the form (27) with (28), which, taking into account that $\mu_1 = 1-\mu_2$, results to have the matrices

$$A_F(\mu_2)=\left[\begin{array}{cc|cc}-3 & 1+\mu_2 & 0 & 0\\ 0 & -4 & 0 & 0\\ \hline 0 & 0 & -2 & 1\\ 0 & 0 & 0 & -3\end{array}\right],\quad B_G=\left[\begin{array}{c}1\\ 0\\ \hline -1\\ 1\end{array}\right],\quad J=\left[\begin{array}{cc|cc}-3 & 1 & 0 & 0\\ 0 & -1 & 0 & 0\\ \hline 0 & 0 & -1 & 2\\ 0 & 0 & 3 & 0\end{array}\right],\quad E=\left[\begin{array}{cc|cc}2 & 0 & 1 & -1\end{array}\right], \tag{56}$$

where $\mu_2 \in [0,\,1]$. In light of (51), the matrices in (56) show that

$$\operatorname{Im} B_G \subseteq \mathcal{V}_R^* \subseteq \operatorname{Ker} E. \tag{57}$$

In other words, (51) and (56) show that the forced state evolution of $\Sigma_{D\sigma}^{\mathscr{C}}$ is contained in the null space of the output matrix for all $\sigma \in \mathscr{S}$ and for all $\mu \in \Delta^1$. Hence, provided that $x_P(0)=0$ and $x_M(0)=0$ in the EMMP, it ensues that

$$e(t) = E\,x(t) = y_P(t) - y_M(t) = 0. \tag{58}$$

Namely, the output $y_P(t)$ of the plant is equal to the output $y_M(t)$ of the model for all $\sigma \in \mathscr{S}$, all $\mu \in \Delta^1$, all $u_M(t)$ and all $t \in \mathbb{R}^+$.

In addition, as is easy to check, with the matrix F_P computed above, condition (48) of Theorem 2, with $i=1,2$, is satisfied, e.g., with the symmetric positive definite matrix

$$P=\begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{bmatrix}. \tag{59}$$

Indeed, with the matrix F_P computed above, one can obtain P by solving the system of two linear matrix inequalities of the form (48), with $i=1,2$, in the unknown P, with P symmetric and positive definite. This can be done, for instance, with the LMI solver available with MATLAB® Robust Control Toolbox. Therefore, the control law (55) also solves the SMMP (hence, the AMMP) for the considered plant and model. Let $\tau>0$ denote a lower bound for the length of the smallest time interval between two consecutive jump time instants of an admissible σ, in such a way that $\Sigma^{\mathscr{C}}_{D\sigma}$ be asymptotically stable for all $\mu\in\Delta^{N-1}$ and for all $\sigma\in\mathscr{S}_\tau$. In this example, a value for $\tau>0$ can be computed explicitly, as will be shown below, since the dynamic matrix $A_F(\mu_2)$ in (56) is diagonalizable. To this aim, let $\bar\tau>0$ be the length of the time interval between two consecutive jump times $\sigma(k)$ and $\sigma(k+1)$, with $\sigma(k+1)\neq\sigma(k)$, then

$$x^-(\sigma(k+1)) = e^{A_F(\mu_2)\,\bar\tau}\,x(\sigma(k)) = W(\mu_2)\,e^{D\,\bar\tau}\,W^{-1}(\mu_2)\,x(\sigma(k))$$

with $D=\mathrm{diag}\{-4,-3,-3,-2\}$ and

$$W(\mu_2)=\begin{bmatrix} -\mu_2-1 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 1\\ 0 & 0 & 1 & 0\end{bmatrix},\quad W(\mu_2)^{-1}=\begin{bmatrix} 0 & 1 & 0 & 0\\ 1 & \mu_2+1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}.$$

Consequently, the following inequalities holds

$$\begin{aligned}\|x^-(\sigma(k+1))\|_\infty &= \left\|W(\mu_2)\,e^{D\,\bar\tau}W^{-1}(\mu_2)\,x(\sigma(k))\right\|_\infty \le \left\|W(\mu_2)\,e^{D\,\bar\tau}W^{-1}(\mu_2)\right\|_\infty \|x(\sigma(k))\|_\infty\\ &\le \left\|W(\mu_2)\right\|_\infty \left\|e^{D\,\bar\tau}\right\|_\infty \left\|W^{-1}(\mu_2)\right\|_\infty \|x(\sigma(k))\|_\infty \le 9\,e^{-2\bar\tau}\,\|x(\sigma(k))\|_\infty\,,\end{aligned} \tag{60}$$

where, at the last step, $\|W(\mu_2)\|_\infty$ and $\|W^{-1}(\mu_2)\|_\infty$ were replaced by their respective upper bounds for $\mu_2\in[0,1]$. Since the state at the jump time $\sigma(k+1)$ is given by $x(\sigma(k+1))=J\,x^-(\sigma(k+1))$, where J was given in (56) and is independent of μ_2, it ensues that

$$\|x(\sigma(k+1))\|_\infty = \|J\,x^-(\sigma(k+1))\|_\infty \le \|J\|_\infty\|\,x^-(\sigma(k+1))\|_\infty = 4\,\|\,x^-(\sigma(k+1))\|_\infty. \tag{61}$$

Then, from (60) and (61), it follows that

$$\|x(\sigma(k+1))\|_\infty \le 36\,e^{-2\bar\tau}\,\|x(\sigma(k))\|_\infty\,. \tag{62}$$

By imposing that the coefficient in (62) be smaller than 1, one gets $\bar\tau>1.7918$. Hence, a positive real constant τ greater than or equal to, e.g., 1.8 ensures that the free state evolution of $\Sigma^{\mathscr{C}}_{D\sigma}$ converges to 0 asymptotically for all $\mu\in\Delta^1$. Thus, one can say that the SMMP (and the AMMP as well) is solved by the control law (55) with $\tau\ge1.8$. However, due to the inequalities used in evaluating $\bar\tau$, the previous statement is conservative. For instance, the simulation (run with MATLAB™) detailed below shows that the control law (55) achieves model matching with asymptotic stability for a periodic $\sigma\in\mathscr{S}_\tau$, whose period is equal to $\tau=1$. Fig. 2 shows the behaviour of $e(t)$ for different values of the parameter, $\mu_2=0,\ 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1$, when the initial conditions are $x_P(0)=[1\ 2]^\top$ and $x_M(0)=[0\ 3]^\top$, the input is $u_M(t)=H(t)-H(t-1)$, where $H(t)$ denotes the Heaviside step function, and, as said, $\tau=1$. The corresponding behaviour of the states $x_P(t)$ and $x_M(t)$ is shown in Fig. 3, where both trajectories converge to 0 asymptotically.

5.2. Example 2

Let $\Sigma_{P\sigma}$ be a given impulsive linear plant with polytopic uncertainties, of the form (17), defined by $\sigma\in\mathscr{S}$, $\mathcal{I}=\{1,2\}$, $\mu=[\mu_1\ \ \mu_2]^\top\in\Delta^1$, and let Σ_M be a given linear time-invariant model, of the form (20) – specialized to the case where neither polytopic uncertainties nor impulsive behaviours are present. Let the matrices of $\Sigma_{P\sigma}$ and of Σ_M be respectively defined as in (49) and (50), except for the jump matrices, which are

$$J_{P1}=J_{P2}=\begin{bmatrix}1 & 10\\ 0 & -1\end{bmatrix},\quad J_M=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \tag{63}$$

The related EMMP can be seen as the problem of forcing the output of the plant Σ_{P_σ} to match that of the model Σ_M in a robust way with respect to the plant's uncertainty, for all $\sigma\in\mathscr{S}$, if the initial conditions of both the plant and the model are equal to 0. The computations performed according to Proposition 4 show that the maximal robust hybrid controlled invariant subspace for the system $\Sigma_{E\sigma}$, constructed as in (29) with (25), contained in $\mathcal{K}$, defined as in (30), is the same as that of (51). The solvability condition (31) in Theorem 1, which reduces to (52), is satisfied and, in particular, (53) holds with the same G. A friend of $\mathcal{V}^*_R$ is, e.g., the same F of (54). Hence, a solution of the EMMP is given by the same control law as in (55). In fact, the control law (55), applied to the output difference system $\Sigma_{D\sigma}$, constructed as in (24) with (25), yields the overall compensated system $\Sigma^{\mathscr{C}}_{D\sigma}$, of the form (27) with (28), which, taking into account that $\mu_1=1-\mu_2$, has the same matrices $A_F(\mu_2)$, B_G and E as in (56), while

$$J=\left[\begin{array}{cc|cc}1 & 10 & 0 & 0\\ 0 & -1 & 0 & 0\\ \hline 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]. \tag{64}$$

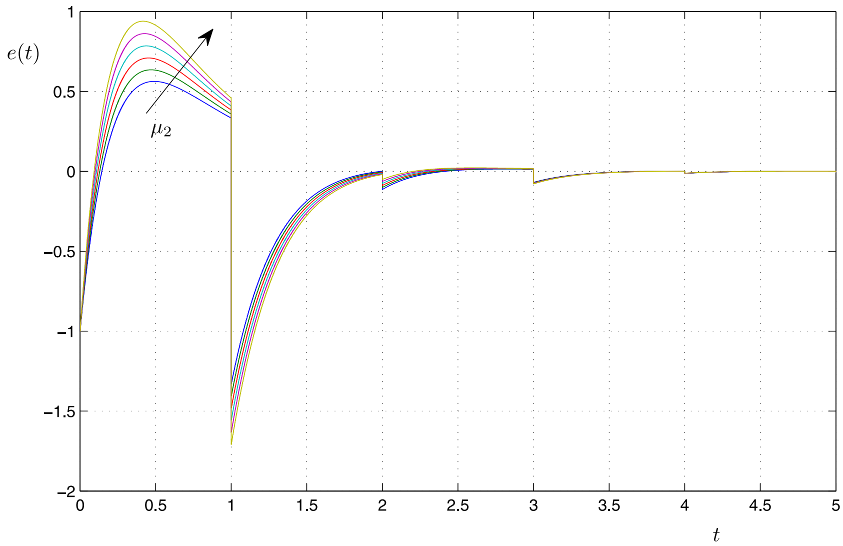

Fig. 2. Example 1. Plot of $e(t)$ with $\sigma(k) = k$, $k \in \mathbb{N}$, and different values of the parameter, $\mu_2 = 0, 0.2, 0.4, 0.6, 0.8, 1$; initial conditions $x_P(0) = [1\ \ 2]^\top$ and $x_M(0) = [0\ \ 3]^\top$; input $u_M(t) = H(t) - H(t-1)$.

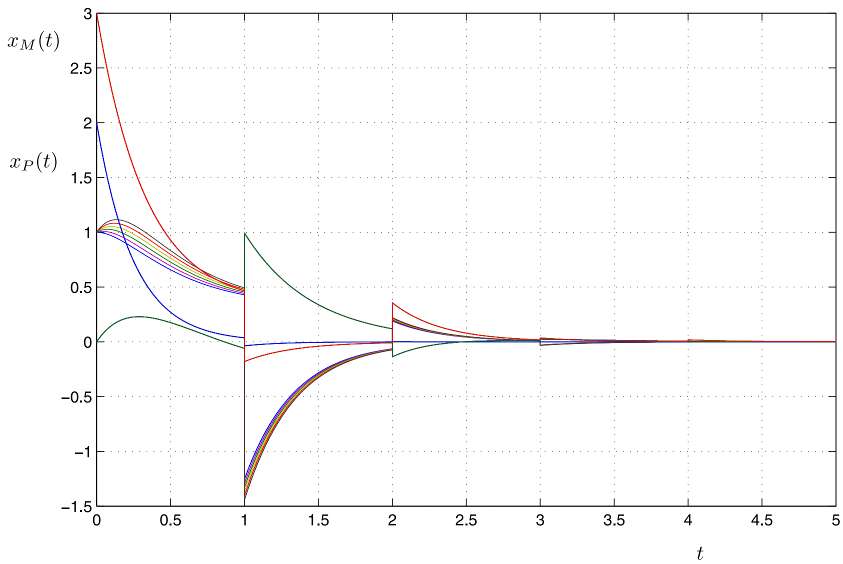

Fig. 3. Example 1. Plots of $x_P(t)$ and $x_M(t)$ with $\sigma(k) = k$, $k \in \mathbb{N}$, and different values of the parameter, $\mu_2 = 0, 0.2, 0.4, 0.6, 0.8, 1$; initial conditions $x_P(0) = [1\ \ 2]^\top$ and $x_M(0) = [0\ \ 3]^\top$; input $u_M(t) = H(t) - H(t-1)$.

Also in this case, (57) holds, which means that the forced state evolution of $\Sigma^{\mathscr{C}}_{D\sigma}$ is contained in the null space of the output matrix for all $\sigma \in \mathscr{S}$ and for all $\mu \in \Delta^1$. Hence, provided that $x_P(0) = 0$ and $x_M(0) = 0$, it ensues that (58) is met or, equivalently, the output $y_P(t)$ of the plant is equal to the output $y_M(t)$ of the model for all $\sigma \in \mathscr{S}$, all $\mu \in \Delta^1$, all $u_M(t)$ and all $t \in \mathbb{R}^+$. Namely, the EMMP is solved. Moreover, since the vertices of the flow dynamics of the compensated plant, $A_{Pi} + B_{Pi} F_P$ with $i = 1, 2$, are the same as those of Example 1, condition (48) of Theorem 2, with $i = 1, 2$, is satisfied with the same symmetric positive definite matrix P given in (59). Thus, the control law (55) also solves the SMMP (hence, the AMMP) for the given plant and model. As to the computation of a positive real constant $\tau > 0$ such that the overall compensated system $\Sigma^{\mathscr{C}}_{D\sigma}$ is asymptotically stable for all $\mu \in \Delta^1$ and for all $\sigma \in \mathscr{S}_\tau$, first, note that, since the flow dynamics of $\Sigma^{\mathscr{C}}_{D\sigma}$ is given by the same $A_F(\mu_2)$ as in (56), the inequalities derived in (60) also hold herein. Furthermore, as to the behaviour of the state of $\Sigma^{\mathscr{C}}_{D\sigma}$ at the jump time $\sigma(k+1)$, from $x(\sigma(k+1)) = J\, x^-(\sigma(k+1))$, it ensues that

$$\|x(\sigma(k+1))\|_\infty = \|J\, x^-(\sigma(k+1))\|_\infty \leq \|J\|_\infty \|\, x^-(\sigma(k+1))\|_\infty = 11\, \|\, x^-(\sigma(k+1))\|_\infty. \tag{65}$$

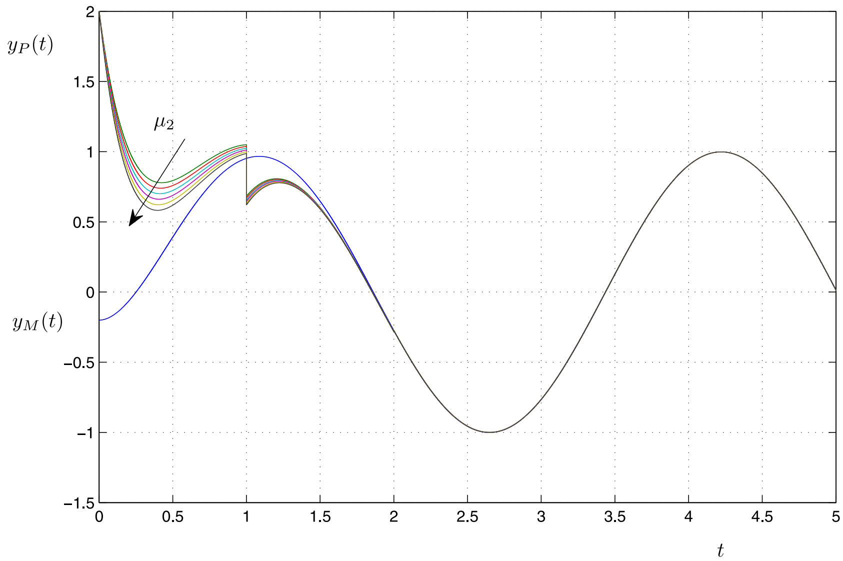

Fig. 4. Example 2. Plots of $y_P(t)$ and $y_M(t)$ with $\sigma(k) = k$, $k \in \mathbb{N}$, and different values of the parameter, $\mu_2 = 0, 0.2, 0.4, 0.6, 0.8, 1$; initial conditions $x_P(0) = [1\ -1]^\top$ and $x_M(0) = [0.4\ \ 0.2]^\top$; input $u_M(t) = 1.8\sin(2t)$.

Then, from (60) and (65), it follows that

$$\|x(\sigma(k+1))\|_\infty \le 99\, e^{-2\bar{\tau}}\ \|x(\sigma(k))\|_\infty . \tag{66}$$

By imposing that the coefficient in (66) be smaller than 1, one gets $\bar{\tau} > 2.2976$. Hence, a positive real constant $\tau \ge 2.3$ ensures that the free state evolution of $\Sigma^{\mathscr{C}}_{D\sigma}$ converges asymptotically to 0 for all $\mu \in \Delta^1$. Thus, one can state that the SMMP (as well as the AMMP) is solved by the control law (55) with $\tau \ge 2.3$. Like in Example 1, the previous statement is conservative. In fact, for instance, the simulation described below shows that the control law (55) achieves model matching with asymptotic stability for a periodic $\sigma \in \mathscr{S}_\tau$, whose period is equal to $\tau = 1$. Fig. 4 shows a fast asymptotic convergence of the output $y_P(t)$ of the plant to the output $y_M(t)$ of the model for different values of the parameter, $\mu_2 = 0, 0.2, 0.4, 0.6, 0.8, 1$, when the initial conditions are $x_P(0) = [1\ -1]^\top$ and $x_M(0) = [0.4\ 0.2]^\top$, the input is $u_M(t) = 1.8\ \sin(2t)$ and, as said, $\tau = 1$.

The asymptotic convergence of the matching error $e(t)$ to 0 is illustrated in Fig. 5. It is also worth considering the state plot in Fig. 6, which shows a steady-state oscillatory behaviour of the states of the model, along with the first component of the state of the plant, while the second component of the state of the plant goes asymptotically to zero (in these figures, the magnitude of the jump is too small, in comparison with the scale, to make it easily visible except at $t = 1$).

Remark 7. The examples presented in this section have a methodological focus and were not motivated by any specific application. Nevertheless, any mechanical system which consists of masses, springs and dampers, where, in particular, the coefficients of the springs and of the dampers are affected by bounded errors with respect to the nominal values (which is the rule in any commercial component) and where the masses involved are subject to collisions, can be modelled as described in this work. Further, as mentioned in Section 1, model matching is an extensively used approach to the formalization of the problem of searching for a compensator that forces the plant to behave according to desired specifications — see, e.g., the mentioned Ref. [30].

6. Conclusion

The matching problem for impulsive linear systems with polytopic uncertainties has been studied and solvability conditions have been derived by means of a structural geometric approach. The advantage of this approach is that of providing sufficient solvability conditions that can be directly checked. Ad hoc procedures to derive a lower bound for the length of the smallest time interval between two consequent jump time instants which ensures asymptotic stability of the compensated system for all the values of the uncertain parameter vector within the considered polytope and for all the jump time sequences satisfying the condition on the lower bound have been shown for specific numerical examples. Nevertheless, the development of a procedure for determining, at least in specific situations, such lower bound will be addressed in future work.

CRediT authorship contribution statement

Elena Zattoni: Conceptualization, Methodology, Writing – original draft, Writing – review & editing. **Naohisa Otsuka:** Conceptualization, Methodology, Writing – original draft, Writing – review & editing. **Anna Maria Perdon:** Conceptualization, Methodology, Writing – original draft, Writing – review & editing. **Giuseppe Conte:** Conceptualization, Methodology, Writing – original draft, Writing – review & editing.

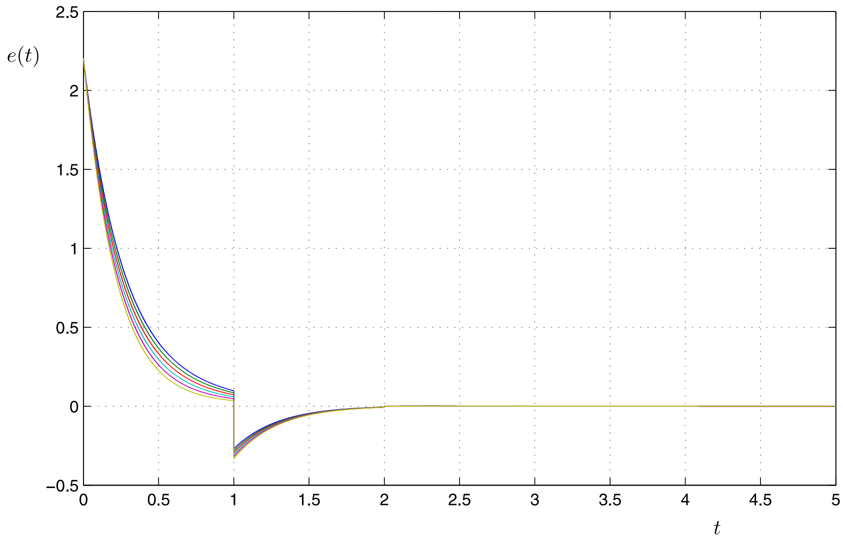

Fig. 5. Example 2. Plot of $e(t)$ with $\sigma(k) = k$, $k \in \mathbb{N}$, and different values of the parameter, $\mu_2 = 0, 0.2, 0.4, 0.6, 0.8, 1$; initial conditions $x_P(0) = [1\ \ -1]^\top$ and $x_M(0) = [0.4\ \ 0.2]^\top$; input $u_M(t) = 1.8\sin(2t)$.

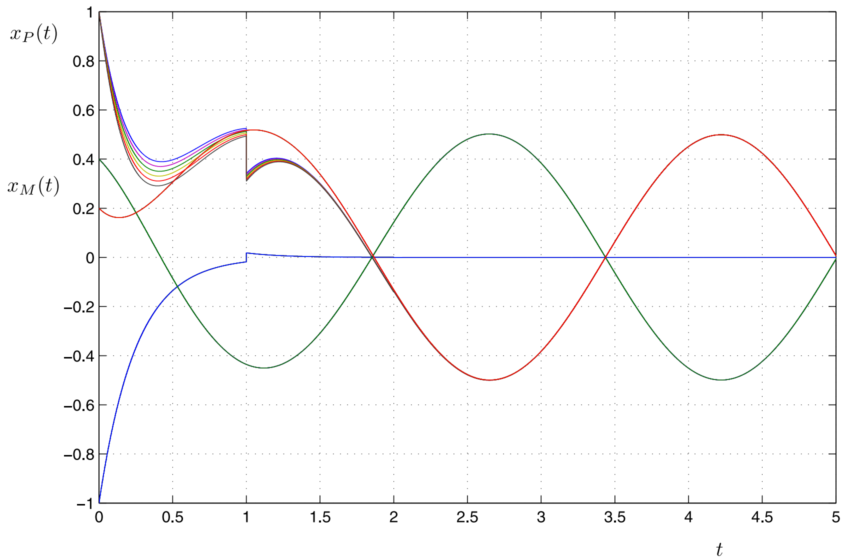

Fig. 6. Example 2. Plots of $x_P(t)$ and $x_M(t)$ with $\sigma(k) = k$, $k \in \mathbb{N}$, and different values of the parameter, $\mu_2 = 0, 0.2, 0.4, 0.6, 0.8, 1$; initial conditions $x_P(0) = [1\ -1]^\top$ and $x_M(0) = [0.4\ \ 0.2]^\top$; input $u_M(t) = 1.8\sin(2t)$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.nahs.2024.101465.

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