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Carleman estimates for third order operators of KdV and non KdV-type and applications

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Abstract

In this paper we study a class of variable coefficient third order partial differential operators on $\mathbb{R}^{n+1}$, containing, as a subclass, some variable coefficient operators of KdV-type in any space dimension. For such a class, as well as for the adjoint class, we obtain a Carleman estimate and the local solvability at any point of $\mathbb{R}^{n+1}$. A discussion of possible applications in the context of dispersive equations is provided.

Keywords Third order equations with variable coefficients · Variable coefficient KdV-type equations · Carleman estimates · Local solvability

Mathematics Subject Classification 35A01 · 35B45 · 35G05 · 35Q53

Contents

1 Introduction

In this paper, we will continue the investigation of some variable coefficient PDOs (partial differential operators) built from a system of real smooth vector fields, initiated in the works [13, 15, 18, 19] (see also [12, 14, 17]). Since when the celebrated works by Kolmogorov [31] first, and by Hörmander [24] afterwards, about the hypoellipticity of operators written as sums of squares of vector fields were published, a lot of connected problems have been investigated. In particular, Hörmander's hypoelliptic theorem in [24] opened up the study of sub-Laplacians on Lie groups (see [40]), of parametrices for such operators, of sharp estimates in appropriate functional spaces (see the pioneering works [23, 40]), but also, among other

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questions, that of the local solvability of closely related models (see, for instance, [13, 15, 18, 19, 35, 39] and references therein).

In the series of works [13, 15, 18, 19], the authors focus on the local solvability of some classes of degenerate second order PDOs built from a system of real smooth vector fields and a somewhere vanishing function. The use of the vanishing function is twofold: on one side it adds degeneracy to a model which is already degenerate by itself, on the other side it permits to include in the treatment some operators generalizing the Kannai operator, that is operators having a changing sign principal symbol. The introduction of the vanishing function requires a price to pay, price which is given in terms of conditions on the lower order terms of the operators (see [12, 14, 17] for an overview of such results). Nevertheless, the classes studied in these works encompass different lower order terms, which makes it possible to include parabolic-type operators, Schrödinger-type operators, a blend of the two, and some prototypes with non-smooth coefficients.

What we do in the present paper, is to take inspiration by these works and by the form of some operators relevant in physics, and build a class of operators of order three (instead of two) starting from a system of smooth real vector fields. Let us say right away that we will not involve any additional somewhere vanishing function, since the model under consideration is already highly degenerate and complicated without this adjustment.

Besides the connection with the aforementioned previous works, the interest for such kind of operators has several motivations (see [30] and references therein). One of them is represented by the applications to variable coefficient dispersive equations of KdV-type, where we refer to KdV-type operators as those of the form $i\partial_t + \mathcal{L}(t, x, D_x)$, with $\mathcal{L}$ being a third order PDO with smooth coefficients.
In the last decades variable coefficient Schrödinger equations have attracted lots of attention. Smoothing and Strichartz estimates have been proved under different hypotheses (see [2, 11, 20–22, 29, 34, 37, 41] and references therein), nonlinear problems have been solved (see [11, 20, 21, 25, 29, 34]), and uniqueness results have been proved (see [7, 8, 10, 27] in the constant coefficient case, and [3, 16] and references therein for variable coefficient cases). For KdV-type equations the investigation has not been pushed that far, possibly because of the unknown real analogue of this equation in dimensions higher than two, and also because variable coefficient third order equations can be much more challenging to study. Here we wish to start a local analysis of some variable coefficient cases, and prove Carleman type estimates to be employed to reach uniqueness results. Let us underline that such estimates have a pivotal role in establishing uniqueness properties, not only at a local (in space) level, but also for global results. In the context of dispersive equations, results for KdV and ZK (Zakharov-Kuznetsov) equations can be found, for instance, in [4, 9, 26, 28, 36, 44], while for nonlocal operators we refer the interested reader to [27, 38]. Note that all these cases deal with constant coefficients operators, while here we are concerned with the situation where the coefficients are variable. As far as the author knows, there are no results in the space variable coefficients setting, at least when the coefficients appear in the highest order part of the operator of KdV-type. Recently, some results on p-evolution equations in one space dimension have appeared in [1]. In that work the operators under investigation have time-dependent coefficients in the leading part, and the class includes KdV operators (since the space domain is one dimensional) with time variable coefficients in the leading part, but not ZK operators. Here we are concerned with space-dependent coefficients in any space dimension.
An other motivation driving this work is that of the local solvability problem for multiple characteristics PDOs. This problem is very hard to attack, and general results often require precise geometrical conditions on the characteristic set. We refer the interested reader to [43]

for a nice overview of the classical local solvability problem, and to [5] for the resolution of the Nirenberg-Treves local solvability conjecture.

Let us give a description of the family of operators we will be considering in this paper.

Given a system of vector fields $\{iX_j(x, D)\}_{j=0}^N$ on $\mathbb{R}^{n+1}$ - where N is a positive integer not necessarily equal to n or $n+1$, and $D = (D_1, \dots, D_{n+1}) = (-i\partial_{x_1}, \dots, -i\partial_{x_{n+1}})$ - such that

- the vector fields $iX_j(x, D)$, $j = 0, \dots, N$, are *smooth* and *real*;
- the vector fields $iX_j(x, D)$; $j = 1, \dots, N$, form a global involutive distribution, that is, $[X_l, X_m](x, D) = -[iX_l, iX_m]$ is uniquely determined by a linear combination with smooth coefficients of the vector fields $\{iX_j(x, D)\}_{j=1,\dots,N}$, for all $l, m = 1, \dots, N$, and for all $x \in \mathbb{R}^{n+1}$;
- X_1 is nondegenerate, namely it is nowhere vanishing on $\mathbb{R}^{n+1}$;

we define P_1 as the third order PDO of the form

$$P_1(x, D) := X_1 \sum_{j=1}^{N} X_j^* X_j + X_0. \tag{1.1}$$

The latter is the class of operators we will be dealing with, which, depending on N and on the choice of the system of vector fields, includes (linear) KdV-type operators of different kind.

We have already mentioned that our goal is to prove Carleman estimates for such class of operators. To be precise, we will not only obtain Carleman estimates for the class of operators represented by P_1, but also for the adjoint class, that is the family of operators P_1^* obtained by taking the adjoint of P_1, namely

$$P_1^* = \sum_{j=1}^{n} X_j^* X_j X_1^* + X_0^*, \tag{1.2}$$

which coincides with P_1 up to a differential operator of order two.

The main motivation for the Carleman estimate for P_1^*, is our interest in the local solvability of P_1. We recall that the role of Carleman estimates—which is well-known to be determinant in the analysis of unique continuation problems (see, for instance, [32] and [42])—is also crucial to prove local solvability properties of partial differential equations with variable coefficients. An estimate of this sort for an operator P, yields a local solvability result for the adjoint operator P^*. Hence, we can take advantage of the Carleman estimate for P_1^* to get the local solvability of P_1.

The connection between our operator P_1 and some important operators coming from physics, can be seen by analyzing the following two examples: the (linear) KdV operator and its higher dimensional generalization, the ZK operator. The (linear) KdV operator is described by iP_1 with $N = n = 1$, $\mathbb{R}^{n+1} = \mathbb{R}^2_{t,x}$, $iX_0 = \partial_t$. and $iX_1 = -\partial_x$. However, by taking $iX_1(x) = a(x)\partial_x$, with $a(x) \neq 0$ for all $x \in \mathbb{R}$, then we get an operator, again of the form P_1, that describes a variable coefficient KdV operator. If we consider $N = n = 2$, $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^2_x$, $iX_0 = \partial_t$ and $iX_j = -\partial_{x_j}$, $j = 1, 2$, then iP_1 describes the ZK operator. Again, by taking nonvanishing space-variable coefficient real vector fields satisfying condition (H_1) in (2.1) below, we get a variable coefficient ZK model.

Note that both the classical KdV and the ZK operator (with constant coefficients) are built by taking an elliptic operator in $\mathbb{R}^n_x$, i.e. the Laplacian in $\mathbb{R}$ and $\mathbb{R}^2$ respectively, and then using the two vector fields $iX_0 = \partial_t$ and $iX_1 = \partial_{x_1}$ to define a third order operator. Observe that P_1 is built exactly the same way, but employing variable coefficient real vector fields instead of constant coefficient ones. In fact, to define P_1 we take operators being sums of squares, that is $\sum_{j=1}^{N} X_j^* X_j$ (not necessarily elliptic in a subspace of dimension n of the $n+1$-dimensional domain), and cook up a third order operator by using X_1 and X_0. An example of variable coefficient KdV-type operator in a three-dimensional Euclidean space domain, can be constructed, for instance, by using the canonical basis of the Lie algebra of the Heisenberg group. Let

$$iX_0 = \partial_t, \quad iX_1(x,t) = \partial_{x_1} - \frac{x_2}{2}\partial_{x_3}, \quad iX_2(x,t) = \partial_{x_2} + \frac{x_1}{2}\partial_{x_3}, \quad iX_3(x) = \partial_{x_3},$$

be four real smooth vector fields, where iX_1, iX_2 are the generators of the stratified Lie algebra $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ of $\mathbb{H}^1$. Then $\{iX_j\}_{j=1}^{3}$ is a global involutive distribution in $\mathbb{R}^4_{t,x}$,

$$\mathcal{L} = \sum_{j=1}^{3} X_j^* X_j$$

is an elliptic operator on $\mathbb{R}^3_x$, and

$$X_{j_0} \sum_{j=1}^{3} X_j^* X_j + X_0, \quad j_0 = 1, 2, 3,$$

is a variable coefficient KdV-type operator of the form P_1 for every choice of $j_0 = 1, 2, 3$, since X_{j_0} is nondegenerate for all $j_0 = 1, 2, 3$.

The examples above show that P_1-type operators include KdV and ZK variable coefficient PDOs. However, our class is not limited to be just a generalization of KdV-type operators in any space dimension, since we are free to choose $n \geq 3$, $N \leq n+1$, but also $iX_0 \neq \partial_t$, giving rise to a very wide family of degenerate operators.

Let us finally conclude this introduction with the plan of the paper. Section 2 is devoted to the the main results of this work, specifically Theorem 2.8 about a Carleman estimate for P_1 and for P_1^*, and Theorem 2.10 giving a local solvability result for both the aforementioned operators. Section 3 is dedicated to explicit examples of operators of the form under study and of KdV-type. The last section, Sect. 4, contains final remarks and the discussion of possible applications in the context of dispersive equations.

2 Carleman estimate and local solvability for P_1 and P_1^*

We recall once more that in this section we are concerned with the proofs of the main results of the article: a Carleman estimate for P_1 and P_1^*, and a local solvability result for these operators. The Carleman estimate is the most important result of the paper, in that it is the key tool to get the solvability property. Moreover, it can find applications in other problems of our interest, like in uniqueness/unique continuation problems for dispersive equations. On the other hand, the investigation of the local solvability of highly degenerate operators is quite complicated, thus our results represent a step ahead in the understanding of certain kind of operators.

Before stating and proving our theorems, let us make precise the objects we are working with. Recall that, on $\mathbb{R}^{n+1}_x$, $x = (x_1, \ldots, x_n, x_{n+1})$, $n \geq 1$, we define the operator P_1 as

$$P_1 = X_1 \sum_{j=1}^{N} X_j^* X_j + X_0, \quad 1 \leq N \leq n+1,$$

where

$$X_j(x,D_x)=\sum_{k=1}^{n+1}a_{jk}(x_1,\dots,x_{n+1})D_k,\quad \forall j=1,\dots,N$$

$$X_j^*(x,D_x)=\sum_{k=1}^{n+1}a_{jk}(x_1,\dots,x_{n+1})D_k+\mathrm{d}_j(x_1,\dots,x_{n+1})$$

$$=X_j(x,D)+\mathrm{d_j}(x_1,\dots,x_{n+1}),\quad \mathrm{d_j}(x):=\sum_{k=1}^{n+1}D_ka_{jk}(x_1,\dots,x_{n+1}),$$

$$\forall j=1,\dots,N$$

with $D_k=-i\partial_k:=-i\partial_{x_k}$, and $a_{jk}\in C^\infty(\mathbb{R}^{n+1})$, for all $j,k=1,\dots,n+1$.

The condition we shall assume on the system of vector fields on $\mathbb{R}^{n+1}$ which make up P_1 is the following:

(H_1) We say that a system of real smooth vector fields $\{iX_l\}_{l=1,\dots,N}$, $1\le N\le n+1$, satisfies condition (H_1) if the distribution $\Gamma(\{iX_l\}_{l=1,\dots,N})$ is a global involutive distribution, that is, for all $j,k=1,\dots,N$, there exist (uniquely determined) $c_l^{jk}\in C^\infty(\mathbb{R}^{n+1},i\mathbb{R})$, $l=1,\dots,N$, such that

$$[X_j,X_k](x,D)=\sum_{l=1}^{N}c_l^{jk}(x)X_l(x,D),\quad \forall x\in\mathbb{R}^{n+1},\tag{2.1}$$

where $[X,Y]:=XY-YX$ denotes the commutator of the operators X and Y.

In other words (H_1) amounts to the global involutivity of the system of real smooth vector fields $\{iX_j\}_{j=1}^N$ on $\mathbb{R}^{n+1}$.

Remark 2.1 Due to the global involutivity of the system, and the fact that $N\ge 1$, there exists at least one vector field $iX_{j_0}\in\{iX_j\}_{j=1}^N$ being nondegenerate on $\mathbb{R}^{n+1}$, or, equivalently, a nowhere vanishing vector field on $\mathbb{R}^{n+1}$ in $\{iX_j\}_{j=1}^N$. By possibly renaming the vector fields, we can always assume $j_0=1$, so we will use this convention in the rest of the paper when assuming condition (H_1) on a system of vector fields. This assumption will be crucial to derive the Carleman estimate below, and thus the solvability result as well. Note also that, if $N\le n+1$ and $\Gamma(\{iX_j\})$ is a global involutive distribution of rank $=N$, then X_j is nondegenerate in $\mathbb{R}^{n+1}$ for all $j=1,\dots,N$, and we are free to take any vector field in the system as the one playing the role of X_1. Finally, observe that the requirement $N\le n+1$ is not restrictive. Indeed, when the system is globally involutive and the rank of the distribution is $M\le n+1$, then if $N>n+1$ we can rewrite P_1 in terms of $M\le n+1<N$ vector fields in the second order part.

Remark 2.2 Notice that formula (2.1) holds true independently of the rank of the distribution. If the system of vector fields satisfies (H_1), and the rank of $\Gamma(\{iX_l\}_{l=1,\dots,N})=M\le N\le n+1$, then

$$[X_j,X_k](x,D)=\sum_{l=1}^{M}c_l^{jk}(x)X_l(x,D)=\sum_{l=1}^{N}c_l^{jk}(x)X_l(x,D),$$

where some coefficients will be identically zero. Since we will not care about the precise rank of the distribution, we will simply use (2.1).

Remark 2.3 An additional remark is that, by imposing only (H_1) on the system of vector fields, P_1 generates a class of operators larger than the one of KdV-type operators, in that the latter is modeled by using a system of vector fields satisfying the additional conditions

- $X_0(x, D) \neq 0, \quad \forall x \in \mathbb{R}^{n+1}$,
- $[X_j, X_0] \equiv 0, \quad \forall j = 1, \dots, N$,
- $X_j(x, D) = X_j(x_1, \dots, x_n, D_1, \dots, D_n), \quad \forall j = 1, \dots, N$,
- $N = n$ and $\sum_{j=1}^{n} X_j^* X_j$ is elliptic on $\mathbb{R}^n_{(x_1,\dots,x_n)}$.

Since we will be giving a local solvability result for P_1 and P_1^*, for completeness we recall here the definition of $H^s - H^{s'}$ locally solvable partial differential operator at a point $x_0 \in \Omega$, where, adopting the usual notation, $H^s(\Omega)$ will denote the standard Sobolev space of order s.

Definition 2.4 Let P be a partial differential operator with smooth coefficients defined on an open subset Ω of $\mathbb{R}^n$, and let $x_0 \in \Omega$. We say that P is $H^s - H^{s'}$ locally solvable at x_0 if there exists a compact K containing x_0 in its interior U_K, such that, for every $f \in H^{-s}_{loc}(\Omega)$ there exists $u \in H^{-s'}_{loc}(\Omega)$ for which $Pu = f$ in U_K. We say that P is $H^s - H^{s'}$ locally solvable in Ω, if it is locally solvable at each point of Ω.

When P is $H^s - H^{s'}$ locally solvable at x_0 with $s = s' = 0$, we will just say that the operator is $L^2 - L^2$ locally solvable at x_0.

Notations. Below N will always be a positive integer, λ will be a real number, whereas $A = \{a_{ji}(x)\}_{\substack{j=0,\dots,N \\ i=1,\dots,n+1}}$ will denote the variable coefficient matrix containing the coefficients of the vector fields iX_j, for $j = 0, \dots, N$. For a matrix A as above and $\alpha \in \mathbb{N}^{n+1} \bigcup \{0\}^{n+1}$, we define

$$\|\nabla^\alpha A\|_{L^\infty(\Omega)} := \sup_{\substack{j=1,\dots,N \\ i=1,\dots,n+1}} \|\partial^\alpha a_{ji}\|_{L^\infty(\Omega)}, \quad \forall \Omega \subset \mathbb{R}^{n+1},$$

where $\partial^\alpha f = \partial_1^{\alpha_1} \dots \partial_{n+1}^{\alpha_{n+1}}$. Throughout the paper we will write $\|\cdot\|_{L^\infty}$, $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^s}$ without specifying the set Ω where the $L^\infty(\Omega)$, $L^2(\Omega)$, and the $H^s(\Omega)$ norms are taken. Given that we will be working with compactly supported functions, the set Ω will be the fixed support of the function (or its interior), so we omit it for simplicity. However, note that we can take $\|\nabla^\alpha A\|_{L^\infty} = \|\nabla^\alpha A\|_{L^\infty(\mathbb{R}^{n+1})}$ in the coming estimates when A has $C_b^\infty(\mathbb{R}^{n+1})$ coefficients.

Given a partial differential operator P, we will write $(Pf)g$ to indicate the multiplication of Pf and g, meaning that the operator P is applied to the function f only, while Pfg means that P acts on the product fg, that is on everything appearing on the right of the operator. For example, for two vector fields iX and iY, we will have $Xfg = (Xf)g + f(Xg)$, and $(Xf)Yg = (Xf)(Yg)$.

Finally, we shall use the notation $\Gamma(\{iX_j\}_{j=1}^N)$ for the distribution spanned by the system of vector fields $\{iX_j\}_{j=1}^N$.

In order to prove our Carleman inequality for P_1 and for P_1^*, we will make use of the subsequent fundamental lemma.

Lemma 2.5 *Let $f \in C^\infty(\mathbb{R}^{n+1})$, $N \geq 1$, and $\lambda \geq 1$. Let also P_1 be as in (1.1) and $\{iX_j\}_{j=1}^N$ satisfying* (H_1). *Then, for every compact set K of $\mathbb{R}^{n+1}$, there exists a positive constant $C_K = C_K\left(\{\|\partial^\alpha f\|_{L^\infty(K)}\}_{|\alpha|=0,1,2}, \ \{\|\nabla^\alpha A\|_{L^\infty(K)}\}_{|\alpha|=0,1,2}\right)$ such that, for all $u \in C_0^\infty(K)$,*

$\mathrm{Im}(e^{-\lambda f} P_1 u, e^{-\lambda f} u)$

$$\geq \sum_{j=1}^{N}(Q_j e^{-\lambda f}u, e^{-\lambda f}u) + \lambda^3 \sum_{j=1}^{N}\big((-iX_1 f)|X_j f|^2 e^{-\lambda f}u, e^{-\lambda f}u\big) - \lambda^2 C_K \|e^{-\lambda f}u\|^2_{L^2},$$

where

$$Q_j = Q_j(x, D) := \big(\lambda(-iX_1 f) + id_1/2\big)X_j^* X_j + \sum_{k=1}^{N} i c_k^{j1} X_k^* X_j$$
$$- i\lambda(X_j X_1 f)X_j + i/2\Big(c_j' - (X_1 d_j) + 2(X_j d_1) + d_1 d_j\Big)X_j, \tag{2.2}$$

with c_k^{j1}, *as in* (2.1)*, and* $c_j' := \sum_{k=1}^{N}(\sum_{l=1}^{N} c_j^{lk} + (X_k c_j^{k1}) + d_k c_j^{k1} - 2d_k c_k^{j1}) \in C^\infty(\mathbb{R}^{n+1}, i\mathbb{R})$ *for all* $j, k = 1, \ldots, N$.

Proof We start the proof by computing the quantity

$$\mathrm{Im}(e^{-\lambda f}P_1 u, e^{-\lambda f}u).$$

Writing

$$e^{-\lambda f}P_1(x, D)u = e^{-\lambda f}P_1(x, D)e^{\lambda f}e^{-\lambda f}u = P_1^f(x, D)e^{-\lambda f}u,$$

then

$$P_1^f(x, D) = P_1(x, D + \lambda Df) = (X_1 + \lambda(X_1 f))\sum_{j=1}^{N}(X_j + \lambda(X_j f))$$
$${}^*(X_j + \lambda(X_j f)) + X_0 + \lambda(X_0 f)$$
$$= (X_1 + \lambda(X_1 f))\sum_{j=1}^{N}(X_j^* - \lambda(X_j f))(X_j + \lambda(X_j f)) + X_0 + \lambda(X_0 f)$$
$$= (X_1 + \lambda(X_1 f))\sum_{j=1}^{N}\Big[X_j^* X_j + \lambda X_j^*(X_j f) - \lambda(X_j f)X_j - \lambda^2 (X_j f)^2\Big] + X_0 + \lambda(X_0 f)$$
$$= (X_1 + \lambda(X_1 f))\sum_{j=1}^{N}\Big[X_j^* X_j + \lambda d_j(X_j f) + \lambda(X_j^2 f) - \lambda^2(X_j f)^2\Big] + X_0 + \lambda(X_0 f)$$
$$= P_1 + X_1\sum_{j=1}^{N}\Big[\lambda d_j(X_j f) + \lambda(X_j^2 f) - \lambda^2(X_j f)^2\Big] +$$
$$+ \lambda(X_1 f)\sum_{j=1}^{N}\Big[X_j^* X_j + \lambda d_j(X_j f) + \lambda(X_j^2 f) - \lambda^2(X_j f)^2\Big] + \lambda(X_0 f)$$
$$= P_1 + \sum_{j=1}^{N}\Big[\lambda d_j(X_j f) + \lambda(X_j^2 f) - \lambda^2(X_j f)^2\Big]X_1 +$$
$$+ \sum_{j=1}^{N}\Big[\lambda(X_1 d_j)(X_j f) + \lambda d_j(X_1 X_j f) + \lambda(X_1 X_j^2 f) - 2\lambda^2(X_j f)(X_1 X_j f)\Big]$$

$$+\lambda(X_1 f)\sum_{j=1}^{N} X_j^* X_j + (X_1 f)\sum_{j=1}^{N}\Big[\lambda^2 d_j(X_j f) + \lambda^2(X_j^2 f) + \lambda^3(iX_j f)^2\Big] + \lambda(X_0 f)$$
$$= P_1 + L_2 + L_1 + L_0,$$

where L_2, L_1, L_0 are the partial differential differential operators of order 2, 1, and 0, given, respectively, by

$$L_2 = L_2(x, D) := \lambda(X_1 f)\sum_{j=1}^{N} X_j^* X_j$$

$$L_1 = L_1(x, D) := \sum_{j=1}^{N}\Big[\lambda d_j(X_j f) + \lambda(X_j^2 f) - \lambda^2(X_j f)^2\Big] X_1$$

$$L_0 = L_0(x) := \sum_{j=1}^{N}\Big[\lambda(X_1 d_j)(X_j f) + \lambda d_j(X_1 X_j f) + \lambda(X_1 X_j^2 f) - 2\lambda^2(X_j f)(X_1 X_j f) +$$
$$+ (X_1 f)(\lambda^2 d_j(X_j f) + \lambda^2(X_j^2 f) + \lambda^3(iX_j f)^2)\Big] + \lambda(X_0 f)$$

By the calculations above

$$\mathrm{Im}(e^{-\lambda f} P_1 u, e^{-\lambda f} u) = \mathrm{Im}(P_1 e^{-\lambda f} u, e^{-\lambda f} u) + \mathrm{Im}(L_2 e^{-\lambda f} u, e^{-\lambda f} u)$$
$$+ \mathrm{Im}(L_1 e^{-\lambda f} u, e^{-\lambda f} u) + \mathrm{Im}(L_0 e^{-\lambda f} u, e^{-\lambda f} u),$$

so, for simplicity, we handle each term on the right-hand side separately.

Recall that, since the system of vector fields $\{iX_j\}_{j=1,\dots,N}$ is globally involutive, that is satisfies (H_1), for all $j = 1, \dots, N$, there exist $c_k^{j1} \in C^\infty(\mathbb{R}^{n+1}, i\mathbb{R})$, $k = 1, \dots, N$, such that

$$[X_j, X_1] = \sum_{k=1}^{N} c_k^{j1} X_k.$$

By the same token, there exist some $c_l^{jk} \in C^\infty(\mathbb{R}^{n+1}, i\mathbb{R})$, $l = 1, \dots, N$, such that

$$[X_j, [X_j, X_1]] = \sum_{k,l=1}^{N} c_k^{j1} c_l^{jk} X_l.$$

Now, on putting $v := e^{-\lambda f} u$, and using the previous observation, we have

$$2i\mathrm{Im}(P_1 e^{-\lambda f} u, e^{-\lambda f} u) = 2i\sum_{j=1}^{N}\mathrm{Im}(X_1 X_j^* X_j v, v) + 2i\,\mathrm{Im}(X_0 v, v)$$
$$= \sum_{j=1}^{N}\Big((X_1 X_j^* X_j - X_j^* X_j X_1 - X_j^* X_j d_1)v, v\Big) + \big((X_0 - X_0^*)v, v\big)$$
$$= -\sum_{j=1}^{N}\big((2[X_j, X_1]X_j + [X_j, [X_j, X_1]] + d_j[X_j, X_1] - (X_1 d_j)X_j)v, v\big)$$
$$-\sum_{j=1}^{N}\Big(\Big(d_1 X_j^* X_j + 2(X_j d_1)X_j + d_1 d_j X_j + (X_j^* X_j d_1) + (X_j d_1)d_j\Big)v, v\Big) - (d_0 v, v)$$

$$= -\sum_{j,k=1}^{N}(2c_k^{j1}X_kX_jv,v) - d_1\sum_{j=1}^{N}X_j^*X_j - \sum_{j,k,l=1}^{N}(c_l^{jk}X_lv,v) - \sum_{j,k=1}^{N}\Big((X_jc_k^{j1})X_kv,v\Big)$$
$$-\sum_{j,k=1}^{N}(d_jc_k^{j1}X_kv,v) - \sum_{j=1}^{n}\Big(\Big(-(X_1d_j)+2(X_jd_1)+d_1d_j\Big)X_jv,v\Big)$$
$$-\sum_{j=1}^{N}\Big(\Big((X_j^*X_jd_1)+(X_jd_1)d_j\Big)\,v,v\Big) - (d_0v,v).$$

Adopting the notation $\|\cdot\|_{L^\infty} := \|\cdot\|_{L^\infty(K)}$, we get

$$\mathrm{Im}(P_1v,v) = \sum_{j,k=1}^{N}(ic_k^{j1}X_kX_jv,v) + \frac{i}{2}d_1\sum_{j=1}^{N}(X_j^*X_jv,v)$$
$$+\frac{i}{2}\sum_{j=1}^{N}\Big(\sum_{k=1}^{N}\Big(\sum_{l=1}^{N}c_j^{lk}+(X_kc_j^{k1})+d_kc_j^{k1}\Big)X_jv,v\Big)$$
$$+\frac{i}{2}\sum_{j=1}^{n}\Big(\Big(-(X_1d_j)+2(X_jd_1)+d_1d_j\Big)X_jv,v\Big)$$
$$+\frac{i}{2}\sum_{j=1}^{N}\Big(\Big((X_j^*X_jd_1)+(X_jd_1)d_j\Big)\,v,v\Big) + \frac{i}{2}(d_0v,v)$$
$$\geq \sum_{j=1}^{N}\Big(\Big(\sum_{k=1}^{N}ic_k^{j1}X_k^*X_j + \frac{i}{2}d_1X_j^*X_j + i/2\Big(c_j' - (X_1d_j)+2(X_jd_1)+d_1d_j\Big)X_j\Big)v,v\Big)$$
$$-C_K\|v\|_{L^2}^2, \tag{2.3}$$

where $c_j'(x) := \sum_{k=1}^{N}(\sum_{l=1}^{N}c_j^{lk} + (X_kc_j^{k1}) + d_kc_j^{k1} - 2d_kc_k^{j1})(x)$, while C_K denotes a constant depending on the $L^\infty(K)$-norm of some smooth functions, specifically functions depending on the coefficients of the vector fields iX_j and on their derivatives. Abusing of the notation, below we shall write C_K for any constant of this form, that is any constant depending on the $L^\infty(K)$-norms of f, of the coefficients of the vector fields iX_j, $j=0,\dots,N$, and of the derivatives of these functions. Sometimes C_K will also depend on the $L^\infty(K)$-norm of c_j^{kl}, $k,l=1,\dots,N$, $j=\dots,n+1$.

As for the term containing L_2, recalling that f is a real function and that X_1f takes purely imaginary values, we obtain

$$\mathrm{Im}(L_2v,v) = \mathrm{Im}\Big(\lambda(X_1f)\sum_{j=1}^{N}X_j^*X_jv,v\Big)$$
$$= \frac{-i}{2}\lambda\sum_{j=1}^{N}\Big(\Big((X_1f)X_j^*X_j - X_j^*X_j(-X_1f)\Big)v,v\Big)$$
$$= \lambda\sum_{j=1}^{N}\Big[\Big((-iX_1f)X_j^*X_jv,v\Big) - i\,\Big((X_jX_1f)X_jv,v\Big)$$
$$-\frac{i}{2}\Big(\Big((X_j^2X_1f)+(X_jX_1f)+d_j(X_jX_1f)\Big)v,v\Big)\Big]$$

$$\geq \lambda \sum_{j=1}^{N} \Big((-iX_1 f) X_j^* X_j v, v) - i((X_j X_1 f) X_j v, v \Big) - C_K \|v\|_{L^2}^2. \tag{2.4}$$

By similar considerations

$$\begin{aligned}\operatorname{Im}(L_1 v, v) &= \frac{1}{2i} \sum_{j=1}^{n} \bigg[\lambda \Big(\big(d_1 d_j (X_j f) - d_1 (X_j^2 f) + (X_1 d_j (X_j f)) - (X_1 X_j^2 f) \big) v, v \Big) \\ &\quad + \lambda^2 \Big(\big(d_1 (X_j f)^2 + 2(X_j f)(X_1 X_j f) \big) v, v \Big) \bigg] \\ &\geq -\lambda^2 C_K \|v\|_{L^2}^2, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\operatorname{Im}(L_0 v, v) &\geq \lambda^3 \sum_{j=1}^{N} \big((-iX_1 f) |X_j f|^2 v, v \big) + \lambda \big(-i(X_0 f) v, v \big) - \lambda^2 C_K \|v\|_{L^2}^2 \\ &\geq \lambda^3 \sum_{j=1}^{N} \big((-iX_1 f) |X_j f|^2 v, v \big) - \lambda^2 C_K \|v\|_{L^2}^2. \end{aligned} \tag{2.6}$$

Finally, by (2.3), (2.4), (2.5), (2.6), and the fact that $\lambda \geq 1$, we can find a new suitable positive constant C_K such that

$$\operatorname{Im}(e^{-\lambda f} P_1 u, e^{-\lambda f} u) \geq \sum_{j=1}^{N} (Q_j v, v) + \lambda^3 \sum_{j=1}^{N} \big((-iX_1 f) |X_j f|^2 v, v \big) - \lambda^2 C_K \|v\|_{L^2}^2,$$
$$\forall u \in C_0^\infty(K),$$

with

$$\begin{aligned} Q_j = Q_j(x, D) &:= \big(\lambda(-iX_1 f) + i d_1/2 \big) X_j^* X_j + \sum_{k=1}^{N} i c_k^{j1} X_k^* X_j \\ &\quad - i\lambda (X_j X_1 f) X_j + i/2 \Big(c_j' - (X_1 d_j) + 2(X_j d_1) + d_1 d_j \Big) X_j. \end{aligned}$$

This completes the proof. □

The same result as Lemma 2.5, with suitable small adjustments, is still valid for the operator P_1^*, the adjoint of P_1. We state the precise result in Lemma 2.6 below.

Lemma 2.6 *Let $f \in C^\infty(\mathbb{R}^{n+1})$, $N \geq 1$, and $\lambda \geq 1$. Let also P_1^* be as in (1.2), and $\{iX_j\}_{j=1}^N$ satisfying* (H$_1$). *Then, for every compact set K of $\mathbb{R}^{n+1}$, there exists a positive constant $C_K = C_K \big(\{\|\partial^\alpha f\|_{L^\infty(K)}\}_{|\alpha|=0,1,2}, \ \{\|\nabla^\alpha A\|_{L^\infty(K)}\}_{|\alpha|=0,1,2} \big)$ such that, for all $u \in C_0^\infty(K)$,*

$$\begin{aligned} &\operatorname{Im}(e^{-\lambda f} P_1^* u, e^{-\lambda f} u) \\ &\quad \geq \sum_{j=1}^{N} (Q_j' e^{-\lambda f} u, e^{-\lambda f} u) + \lambda^3 \sum_{j=1}^{N} \big((-iX_1 f) |X_j f|^2 e^{-\lambda f} u, e^{-\lambda f} u \big) - \lambda^2 C_K \|e^{-\lambda f} u\|_{L^2}^2 \end{aligned}$$

where

$$
\begin{aligned}
Q'_j(x,D) &= Q_j(x,D) - i\sum_{k=1}^{N} c_k^{j1}(X_j^*X_k + X_k^*X_j) - ic''_j X_j \\
&= \big(\lambda(-iX_1 f) + id_1/2\big)X_j^*X_j - \sum_{k=1}^{N} ic_k^{j1}X_j^*X_k \\
&\quad - i\lambda(X_jX_1 f)X_j + i/2\Big(c'_j - c''_j - (X_1 d_j) + 2(X_j d_1) + d_1 d_j\Big)X_j
\end{aligned}
$$

with c_k^{j1}, *as in* (2.1), $c''_j := \sum_{k=1}^{N}\Big((X_k c_k^{j1}) + (X_k^* c_k^{j1}) + (X_k c_j^{k1}) + (X_k^* c_j^{k1})\Big) \in C^\infty(\mathbb{R}^{n+1}, i\mathbb{R})$ *for all* $j,k = 1,\dots,N$, *and* Q_j, c'_j, *as in* (2.2) *for all* $j = 1,\dots,N$.

Proof To prove this lemma we make use of Lemma 2.5. By standard arguments we have

$$
\begin{aligned}
P_1^*(x,D) &= P_1(X,D) + d_1\sum_{j=1}^{n} X_j^*X_j + \sum_{j,k=1}^{N}\Big[c_k^{j1}(X_j^*X_k + X_k^*X_j) - c_k^{j1}d_k X_j + (X_k^* c_j^{k1})X_j\Big] \\
&\quad - \sum_{j=1}^{N}(X_1 d_j)X_j + \sum_{j=1}^{N}\big((X_j d_1)d_j + (X_j^*X_1 d_1)\big) + d_0,
\end{aligned}
$$

where $c_k^{j1} \in C^\infty(\mathbb{R}^{n+1}, i\mathbb{R})$ are such that

$$
[X_j, X_1] = \sum_{k=1}^{N} c_k^{j1} X_k .
$$

Next, as in the proof of Lemma 2.5, we compute

$$
\operatorname{Im}(e^{-\lambda f}P_1^* u, e^{-\lambda f}u) = \operatorname{Im}(P_1^{*f} e^{-\lambda f}u, e^{-\lambda f}u),
$$

where

$$
\begin{aligned}
P_1^{*f} &= e^{-\lambda f}P_1^* e^{\lambda f} = P_1^*(x, D + \lambda Df) \\
&= P_1^f + d_1\sum_{j=1}^{n} X_j^*X_j + \sum_{j,k=1}^{N}\Big[c_k^{j1}(X_j^*X_k + X_k^*X_j) + \big(-c_k^{j1}d_k + (X_k^* c_j^{k1})\big)X_j\Big] \\
&- \sum_{j=1}^{N}(X_1 d_j)X_j \\
&\quad + \sum_{j=1}^{N}\big((X_j d_1)d_j + (X_j^*X_1 d_1)\big) + d_0 + d_1\sum_{j=1}^{N}\Big[\lambda d_j(X_j f) + \lambda(X_j^2 f) - \lambda^2(X_j f)^2\Big] \\
&\quad + \sum_{j,k=1}^{N} c_k^{j1}\Big\{\lambda(X_k f)(X_j^* - X_j) \\
&\quad + \lambda(X_j f)(X_k^* - X_k) + \lambda(X_j^*X_k f) + \lambda(X_k^*X_j f) - 2\lambda^2(X_j f)(X_k f)\Big\} \\
&\quad - \lambda\sum_{j=1}^{N}\left\{\sum_{k=1}^{N}\big(c_k^{j1}d_k - (X_k^* c_j^{k1})\big)(X_j f) + (X_1 d_j)(X_j f)\right\}.
\end{aligned}
$$

Using the same notations as in the proof of Lemma 2.5, we have

$$\begin{aligned}\operatorname{Im}(P_1^{*f}v,v)\geq&\operatorname{Im}(P_1^f v,v)+\sum_{j,k=1}^{N}\operatorname{Im}\Big(\big(c_k^{j1}(X_j^*X_k+X_k^*X_j)-(c_k^{j1}d_k-(X_k^*c_j^{k1}))X_j\big)v,v\Big)\\&-\sum_{j=1}^{N}\operatorname{Im}\big((X_1d_j)X_jv,v\big)-\lambda^2C_K\|v\|_{L^2}^2\\\geq&\sum_{j=1}^{N}(Q_jv,v)+\lambda^3\|\,|X_1f|^{\frac{3}{2}}v\|_{L^2}^2+\sum_{j,k=1}^{N}\operatorname{Im}\left(c_k^{j1}(X_j^*X_k+X_k^*X_j)v,v\right)\\&-\sum_{j=1}^{N}\operatorname{Im}\Big(\sum_{k=1}^{N}\big(c_k^{j1}d_k-(X_k^*c_j^{k1})\big)X_jv,v\Big)\\&-\sum_{j=1}^{N}\operatorname{Im}\big((X_1d_j)X_jv,v\big)-\lambda^2C_K\|v\|_{L^2}^2,\end{aligned}\tag{2.7}$$

where Q_j is as in (2.2).

Next, we define $g_j(x):=\sum_{k=1}^{N}(c_k^{j1}d_k-(X_k^*c_j^{k1}))$, and have that, since g_j and (X_1d_j) are real valued functions, and since the c_k^{j1} take purely imaginary values, then

$$\operatorname{Im}\left(g_jX_jv,v\right)=-\frac{1}{2i}\Big(\big((X_jg_j)+(d_jg_j)\big)v,v\Big)\geq-C_K\|v\|_{L^2}^2,\tag{2.8}$$

$$\operatorname{Im}((X_1d_j)X_jv,v)=-\frac{1}{2i}\Big(\big((X_jX_1d_j)+(X_1d_j)d_j\big)v,v\Big)\geq-C_K\|v\|_{L^2}^2,\tag{2.9}$$

and

$$\begin{aligned}\operatorname{Im}\left(c_k^{j1}(X_j^*X_k+X_k^*X_j)v,v\right)=&\big(-ic_k^{j1}(X_j^*X_k+X_k^*X_j)v,v\big)\\&-\frac{i}{2}\bigg(\Big(\Big((X_jc_k^{j1})+(X_j^*c_k^{j1})\Big)X_k+\Big((X_kc_k^{j1})+(X_k^*c_k^{j1})\Big)X_j\Big)v,v\bigg)\\&-\frac{i}{2}\Big(\big((X_jc_k^{j1})d_k+(X_kc_k^{j1})d_j+(X_k^*X_jc_k^{j1}+X_j^*X_kc_k^{j1})\big)v,v\Big),\end{aligned}$$

which gives

$$\begin{aligned}\sum_{j,k=1}^{N}\operatorname{Im}\big(c_k^{j1}(X_j^*X_k+X_k^*X_j)v,v\big)\geq&\sum_{j,k=1}^{N}\big(-ic_k^{j1}(X_j^*X_k+X_k^*X_j)v,v\big)\\&-\frac{i}{2}\sum_{j=1}^{N}(c_j''X_jv,v)-C_K\|v\|_{L^2}^2\end{aligned}\tag{2.10}$$

where

$$c_j'':=\sum_{k=1}^{N}\Big((X_kc_k^{j1})+(X_k^*c_k^{j1})+(X_kc_j^{k1})+(X_k^*c_j^{k1})\Big).$$

Inserting the last three inequalities, that is (2.10), (2.9) and (2.8), into (2.7), and using that $\lambda \geq 1$, by rearranging the indices we conclude

$$\operatorname{Im}(P_1^{*f} v, v) \geq \sum_{j=1}^{N} (Q'_j v, v) + \lambda^3 \| \, |X_1 f|^{\frac{3}{2}} v\|_{L^2}^2 - \lambda^2 C_K \|v\|_{L^2}^2,$$

with

$$\begin{aligned} Q'_j(x, D) &= Q_j(x, D) - i \sum_{k=1}^{N} c_k^{j1} (X_j^* X_k + X_k^* X_j) - i c''_j X_j \\ &= \big(\lambda(-i X_1 f) + i d_1/2\big) X_j^* X_j - \sum_{k=1}^{N} i c_k^{j1} X_j^* X_k \\ &\quad - i\lambda (X_j X_1 f) X_j + i/2 \Big(c'_j - c''_j - (X_1 d_j) + 2(X_j d_1) + d_1 d_j\Big) X_j, \quad (2.11) \end{aligned}$$

that is Q'_j differs from Q_j in the coefficients of the second term in (2.11), and in the appearance of the function $c'_j - c''_j$ instead of c'_j in the last term in (2.11). This completes the proof. □

Remark 2.7 The estimates proved so far do not necessitate of the nondegeneracy of X_1. The global nonvanishing requirement on X_1 will come into play in the subsequent results.

In the next theorem we will consider both the case when the system of real smooth vector fields $\{iX_i\}_{i=1}^N$ satisfies (H_1), and the case when the system has the additional property to be locally elliptic, that is in an open set. Recall that the system is elliptic at a point x_0 whenever $iX_1(x_0, D), \ldots, iX_N(x_0, D)$ generate the whole tangent space at that point, namely, in our case, $\mathbb{R}^{n+1}$. When this happens, the operator given as the sum of the squares of the vector fields, i.e. $\sum_{j=1}^N X_j^* X_j$, is elliptic in a sufficiently small neighborhood of x_0. Of course, the system is said to be elliptic in an open set U if it is elliptic at each point of U. The global ellipticity property - the ellipticity at each point of $\mathbb{R}^{n+1}$ - is stronger than the global involutivity in (H_1). We can have globally involutive systems which are not globally elliptic. An easy example is given by the vector fields $iX_1 = \partial_{x_1}$ and $iX_2 = x_2 \partial_{x_2}$ in $\mathbb{R}^2$, which commute and generate a globally involutive system, but form an elliptic system at any point of $\mathbb{R}^2 \backslash \{x \in \mathbb{R}^2; x_2 = 0\}$ and not on the whole $\mathbb{R}^2$.

Theorem 2.8 (Carleman estimate for P_1 and P_1^*) *Let $f \in C^\infty(\mathbb{R}^{n+1})$, $\{iX_i\}_{i=1}^N$ be a system of real smooth vector fields satisfying* (H_1)*, and K a compact subset of $\mathbb{R}^{n+1}$. Then,*

(i) *If $f \in C^\infty(\mathbb{R}^n)$ is such that there exists $C_0 > 0$ for which*

$$-iX_1 f(x) \geq C_0 > 0, \quad \forall x \in K,$$

then there exist $\lambda_0 = \lambda_0(K) \geq 1$ and $C = C(K) > 0$ such that, for every $\lambda \geq \lambda_0$,

$$\|e^{-\lambda f} P_1^* u\|_{L^2}^2, \|e^{-\lambda f} P_1 u\|_{L^2}^2 \geq C\lambda \|e^{-\lambda f} u\|_{L^2}^2, \quad \forall u \in C_0^\infty(K).$$

(ii) *If the system of vector fields $\{iX_i\}_{i=1}^n$ is elliptic in a neighborhood of K, and if there exists $C_0 > 0$ for which*

$$-iX_1 f(x) \geq C_0, \quad \forall x \in K,$$

then there exist $\lambda_0 = \lambda_0(K) \geq 1$ and $C = C(K) > 0$ such that, for every $\lambda \geq \lambda_0$,

$$\|e^{-\lambda f} P_1^* u\|_{L^2}^2, \|e^{-\lambda f} P_1 u\|_{L^2}^2 \geq C\lambda \|e^{-\lambda f} u\|_{H^1}^2, \quad \forall u \in C_0^\infty(K).$$

Proof The proof relies on the use of Lemmas 2.5 and 2.6 when dealing with the statement for P_1 and for P_1^* respectively. Below we shall give a detailed proof of the result for P_1, and we omit the one for P_1^*. This is done because the proofs of the estimates for P_1^* and for P_1 are exactly the same, up to the appearance of different constants depending on the $L^\infty(K)$-norm of some smooth functions (because the difference between the estimates in Lemmas 2.5 and 2.6 is just in the appearance of possibly different smooth coefficients in Q'_j and Q_j). Since there is no substantial change in the two cases, we focus on P_1.

By Lemma 2.5, for all $\lambda \geq 1$, given a compact K of $\mathbb{R}^n$ on which $-iX_1 f \geq C_0 > 0$, we have

$$\begin{aligned}&\mathrm{Im}(e^{-\lambda f}P_1 u, e^{-\lambda f}u)\\&\quad\geq \sum_{j=1}^{N}(Q_j e^{-\lambda f}u, e^{-\lambda f}u) + \lambda^3 \| \, |X_1 f|^{\frac{3}{2}} e^{-\lambda f}u\|_{L^2}^2 - \lambda^2 C_K \|e^{-\lambda f}u\|_{L^2}^2, \quad \forall u \in C_0^\infty(K),\end{aligned}$$

with

$$\begin{aligned}Q_j = Q_j(x,D) &:= \big(\lambda(-iX_1 f) + id_1/2\big)X_j^* X_j + \sum_{k=1}^{N} ic_k^{j1} X_k^* X_j\\&\quad + \Big(-i\lambda(X_j X_1 f) + i/2(c'_j - (X_1 d_j) + 2(X_j d_1) + d_1 d_j)\Big)X_j,\end{aligned}$$

where $ic_k^{j1}, ic'_k, k = 1, \ldots, N$, are smooth and real valued functions. Therefore, once λ is big enough, the proof of (i) and of (ii) depends on the estimate we can prove for the second order term $\sum_{j=1}^{n}\mathrm{Re}(Q_j e^{-\lambda f}u, e^{-\lambda f}u)$. Then, let us focus on this term and show how different hypotheses lead to different lower bounds.

In the sequel we will use again the notation $v := e^{-\lambda f}u$. First observe that, by Cauchy–Schwarz inequality,

$$\begin{aligned}\lambda\sum_{j=1}^{N}\mathrm{Re}\big((-iX_1 f)X_j^* X_j v, v\big) &\geq \lambda\left(C_0 - \frac{\delta_0}{2}\right)\sum_{j=1}^{N}\|X_j v\|_{L^2}^2 - \frac{\lambda N}{2\delta_0}\max_{j=1,\ldots,N}\|X_j X_1 f\|_{L^\infty}\|v\|_{L^2}^2\\&\geq \lambda\left(C_0 - \frac{\delta_0}{2}\right)\sum_{j=1}^{N}\|X_j v\|_{L^2}^2 - \lambda c_0\|v\|_{L^2}^2, \forall \delta_0 \in (0,1],\end{aligned}$$

with $c_0 = c_0(\delta_0, \{\|\partial^\alpha f\|_{L^\infty}\}_{|\alpha|=0,1,2}, \{\|\nabla^\alpha A\|_{L^\infty}\}_{|\alpha|=0,1,2})$, and

$$\begin{aligned}&\frac{i}{2}\sum_{j=1}^{N}(d_1 X_j^* X_j v, v) - \sum_{j,k=1}^{N} ic_k^{j1}(X_k^* X_j v, v)\\&\quad= \frac{i}{2}\sum_{j=1}^{N}\Big((d_1 X_j v, X_j v) - (X_j v, (X_j d_1)v)\Big) - \sum_{j,k=1}^{N}(ic_k^{j1} X_j v, X_k v)\\&\quad - \sum_{j}^{N}\big(X_j v, \sum_{k=1}^{N}(X_k ic_k^{j1})v\big)\\&\quad\geq -\left(\|d_1\|_{L^\infty} + C_1 N + \frac{\delta'_0}{2}\right)\sum_{j=1}^{N}\|X_j v\|_{L^2}^2 - \frac{C_K}{\delta'_0}\|v\|_{L^2}^2,\end{aligned}$$

with a new positive constant $C_1 := \max_{j,k=1,\dots,N}\{|c_k^{j1}|\}$, and where C_K is a positive constant depending on all the L^∞-norms of the functions $X_j d_1$ and $X_k c_k^{j1}$. By similar considerations,

$$\Big(\Big(-i\lambda(X_jX_1f)+i/2\big(c_j'-(X_1d_j)+2(X_jd_1)+d_1d_j\big)\Big)X_jv, v\Big)$$
$$\geq \tfrac{\delta_1}{2}\textstyle\sum_{j=1}^N \|X_jv\|_{L^2}^2-\lambda^2\frac{C_K}{\delta_1}\|v\|_{L^2}^2, \quad \forall\delta_1\in(0,1],$$

where C_K is a different suitable positive constant depending on the L^∞-norms of f, a_{jk}, c_j' and of their derivatives. Therefore,

$$\sum_{j=1}^N (Q_jv, v)\geq\lambda\left(C_0-\frac{\delta_0}{2}-\frac{\delta_0'}{2}-\frac{1}{\lambda}\|d_1\|_{L^\infty}-\frac{1}{\lambda}C_1N-\frac{\delta_1}{2}\right)\|X_jv\|_{L^2}^2-\lambda^2c_1\|v\|_{L^2}^2,$$

with $c_1 = c_1(\delta_0, \delta_0', \delta_1, \{\|\partial^\alpha f\|_{L^\infty}\}_{|\alpha|=0,1,2}, \{\|\nabla^\alpha A\|_{L^\infty}\}_{|\alpha|=0,1,2})$. Then, choosing $\lambda > 4(C_1N+\|d_1\|_{L^\infty})C_0^{-1}$ big enough, and $\delta_0, \delta_0', \delta_1$ sufficiently small, so that

$$C_0-\frac{\delta_0+\delta_0'}{2}-\frac{1}{\lambda}(C_1N+\|d_1\|_{L^\infty})-\frac{\delta_1}{2}\geq\frac{C_0}{2},$$

we get

$$\sum_{j=1}^N (Q_jv, v)\geq\lambda\frac{C_0}{2}\sum_{j=1}^N\|X_jv\|_{L^2}^2-\lambda^2C_K\|v\|_{L^2}^2,$$

where C_K is a new positive constant since we have fixed $\delta_0, \delta_0', \delta_1$.

The estimate above yields

$$\mathrm{Im}(e^{-\lambda f}P_1u, e^{-\lambda f}u)\geq\lambda\frac{C_0}{2}\sum_{j=1}^N\|X_jv\|_{L^2}^2+\lambda^3\|\,|X_1f|^{\frac{3}{2}}v\|_{L^2}^2-\lambda^2C_K\|v\|_{L^2}^2,$$

hence, if $\lambda > \max\{4(C_1N+\|d_1\|_{L^\infty})C_0^{-1}, 2C_K/C_0^3\} =: \lambda_0 = \lambda_0(K)$, we have

$$\mathrm{Im}(e^{-\lambda f}P_1u, e^{-\lambda f}u)\geq\lambda\frac{C_0}{2}\sum_{j=1}^N\|X_jv\|_{L^2}^2+\lambda^2\frac{C_K}{2}\|v\|_{L^2}^2,$$

where, recall, C_K is a constant depending on some fixed constants $N, \delta_0, \delta_0', \delta_1$ and on the L^∞-norms on K of f, a_{jk}, c_j', and of their derivatives.

Finally, by Cauchy-Schwarz inequality,

$$\|e^{-\lambda f}P_1u\|_{L^2}^2\geq 2\delta\Big(\lambda\frac{C_0}{2}\sum_{j=1}^N\|X_jv\|_{L^2}^2+(\lambda^2\frac{C_K}{2}-\frac{\delta}{2})\|v\|_{L^2}^2\Big),$$

which gives, taking δ small enough and not necessarily depending on K (since we can always assume $\frac{C_K}{2}\cdot\lambda > 1$ and take, for instance, $\delta = 1/2 < \lambda^2\frac{C_K}{4}$), we reach

$$\|e^{-\lambda f}P_1u\|_{L^2}^2\geq\lambda\frac{C_0}{2}\sum_{j=1}^N\|X_jv\|_{L^2}^2+\lambda^2\frac{C_K}{4}\|v\|_{L^2}^2, \tag{2.12}$$

which proves part (i) of the theorem.

To prove part (ii) of the statement, we apply the Gårding inequality on the first term on the right hand side of (2.12), that is

$$\sum_{j=1}^{N} \|X_j v\|_{L^2}^2 \geq C\|v\|_{H^1}, \quad \forall v \in C_0^\infty(K).$$

This completes the proof. □

Remark 2.9 (The case $N = n = 1$) Note that when $N = n = 1$ (H_1) is trivially satisfied. For the same reason, result (ii) does not apply to this case, since X_1 alone cannot generate $\mathbb{R}^2$.

An immediate application of the Carleman estimates in Theorem 2.8 is the following local solvability result.

Theorem 2.10 *Let $\{iX_j\}_{j=1}^N$ be a system of vector fields satisfying* (H_1) *such that $X_1(x, D) \neq 0$ for all $x \in \mathbb{R}^{n+1}$. Let also P_1 and P_1^* be as in* (1.1) *and* (1.2) *respectively. Then the following properties hold.*

(i) *P_1 and P_1^* are $L^2 - L^2$ locally solvable at any point of $\mathbb{R}^{n+1}$.*
(ii) *If $\{iX_j\}_{j=1}^N$ is elliptic at $x_0 \in \mathbb{R}^{n+1}$, P_1 and P_1^* are $H^{-1} - L^2$ locally solvable at x_0.*

Proof The proof is based on the use of Theorem 2.8 and of some standard functional analysis argument. More specifically, the $H^s - L^2$ local solvability of P_1 at a point $x \in \mathbb{R}^{n+1}$, is equivalent to the validity of the following *solvability estimate* for the adjoint P_1^*: there exists $C > 0$ and a compact K containing x in its interior U_K, such that

$$\|P_1^* u\|_{L^2} \geq C\|u\|_{H^{-s}}, \quad \forall u \in C_0^\infty(U_K).$$

The solvability estimate, Hahn-Banach and Riesz representation theorems, alltogether give the result (see, for instance, Lemma 1.2.30 in [33]). Since X_1 is nondegenerate (see also Remark 2.1), given $C_0 > 0$ and $x_0 \in \mathbb{R}^{n+1}$, it is always possible to find a smooth function f and a compact set $K \subset \mathbb{R}^{n+1}$ containing x_0 in its interior, such that $iX_1 f(x) \geq C_0 > 0$ for all $x \in K$. Moreover, if $\{iX_j\}_{j=1}^N$ is elliptic at x_0, one can always choose K sufficiently small so that the system of vector fields is elliptic in a neighborhood of K. Hence, by the Carleman estimate for P_1^*, for $x_0 \in \mathbb{R}^{n+1}$, and for the compact K chosen as above and containing x_0 in its interior, there exists $\lambda_0(K) > 1$ and $C > 0$ such that, for all $\lambda > \lambda_0$,

$$\|e^{-\lambda f} P_1^*\|_{L^2} \geq C\|e^{-\lambda f} u\|_{H^{-s}}, \quad \forall u \in C_0^\infty(K), \tag{2.13}$$

where $s = 0$ in case (i) and $s = -1$ in case (ii). By estimating from above and from below $e^{-\lambda f}$ on K with positive constants, (2.13) easily implies the solvability estimate with $s = 0$ under the hypothesis in part (i), and with $s = -1$ under the hypothesis in part (ii). From the solvability estimate, by the standard considerations mentioned at the beginning of the proof (see [33]) we get the $L^2 - L^2$ and the $H^{-1} - L^2$ local solvability of P_1 at x_0 according to the hypotheses in (i) and (ii) respectively. Finally, since (2.13) with $s = 0$ does not depend on the choice of x_0, we conclude the $L^2 - L^2$ local solvability of P_1 at any point of $\mathbb{R}^{n+1}$. This completes the proof (i) and of the theorem.

For the local solvability of P_1^*, one repeats the same steps reversing the roles of P_1 and P_1^*. □

Remark 2.11 When the vector field iX_1 is nondegenerate only on a compact $K \subset \mathbb{R}^{n+1}$, using the previous strategy we can reach the local solvability of P_1 and of P_1^* at any point of the interior U_K of K.

Remark 2.12 Note that Theorem 2.10 says that we can have different types of local solvability at different points. At any point we have the $L^2 - L^2$ local solvability, but only at elliptic points - the points where the system is elliptic - we have a better local solvability, that is the $H^{-1} - L^2$.

3 Examples and applications: KdV and non KdV-type operators with variable coefficients

In this section we would like to list a few examples of operators of P_1-type. We pointed out more than once that we are particularly interested in dispersive models, therefore we will mainly focus on such cases here, even though the class is more general and we could exhibit different type of operators.

Variable coefficient KdV operators. Let $N = n = 1$, $\mathbb{R}^{n+1} = \mathbb{R}^2 = \mathbb{R}_t \times \mathbb{R}_x$, and

$$X_0(t, x, D_t, D_x) = D_t, \quad \text{and} \quad X_1(t, x, D_t, D_x) = a(x)D_x,$$

where $a \in C^\infty(\mathbb{R}_x; \mathbb{R})$, $a(x) \neq 0$ for all $x \in \mathbb{R}$. Then

$$\begin{aligned} P_1(t, x, D_t, D_x) &= D_t + \big(a(x)D_x\big)\big(a(x)D_x\big)^*\big(a(x)D_x\big) \\ &= -i\bigg(\partial_t + \big(a(x)\partial_x\big)\big(a(x)\partial_x\big)^*\big(a(x)\partial_x\big)\bigg) \end{aligned}$$

which is a KdV operator with space-variable coefficients. Our solvability theorem applies to this model, so we can assert that this operator, as well as iP_1 of course, is $L^2 - L^2$ locally solvable at any point of $\mathbb{R}^2$. More generally, one can consider the coefficient a as a function of both (t, x), hence $a(t, x)$, or even just a time dependent function, provided that it is everywhere nonvanishing. In this last case one can also describe some of the operators treated in [1].

Variable coefficient ZK operators. Let $N = n = 2$, $\mathbb{R}^{n+1} = \mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}^2_x$, and

$$X_0(t, x, D_t, D_x) = D_t, \quad \text{and} \quad X_j(t, x, D_t, D_x) = X_j(x, D_x), \quad j = 1, 2,$$

with X_1, X_2 such that $X_1{}^*X_1(x, D_x) + X_2{}^*X_2(x, D)$ is an elliptic operator on $\mathbb{R}^2_x$. In other words, we are assuming the vector fields $iX_j(x, D)$ to be nondegenerate and linearly independent at any point $x \in \mathbb{R}^2_x$, so they form a global involutive distribution both in $\mathbb{R}^2$ and in $\mathbb{R}^3$. Under these assumptions the operator

$$P_1 = D_t + X_1(x, D_x)\left(X_1^*X_1(x, D_x) + X_2^*X_2(x, D)\right)$$

is a prototype of a variable coefficient ZK-operator to which our solvability theorem applies. Hence, we have the $L^2 - L^2$ local solvability property for P_1 in $\mathbb{R}^{n+1}$ by Theorem 2.10.

Notice that the ”ellipticity in space” of $X_1{}^*X_1(x, D_x) + X_2{}^*X_2(x, D)$ implies the global involutivity of the system $\{iX_1, iX_2\}$, thus any prototype of this sort fits in our class. On the other hand, we assumed the ellipticity in space of $X_1^*X_1(x, D_x) + X_2{}^*X_2(x, D)$ just in order to represent a variable coefficient analogue of the ZK operator, and not because P_1-type operators need to satisfy this requirement. In fact, we can take more general cases, provided that the global involutivity is satisfied. In general, to describe a variable coefficient P_1-type ZK operator, one can take X_1, X_2 of the form

$$\begin{aligned} &X_1(x, D_x) = a_{11}(x)D_1 + a_{12}(x)D_2, \quad a_{11}(x) \neq 0 \quad \text{or} \quad a_{12}(x) \neq 0 \quad \forall x \in \mathbb{R}^2,\\ &X_2(x, D_x) = a_{21}(x)D_1 + a_{22}(x)D_2\\ &[X_1, X_2](x, D_x) = c_1(x)X_1(x, D_x) + c_2(x)X_2(x, D_x), \qquad c_1, c_2 \in C^\infty(\mathbb{R}^2; i\mathbb{R}),\\ &\qquad \forall x \in \mathbb{R}^2_x, \end{aligned}$$

with $a_{ij} \in C^\infty(\mathbb{R}^2; \mathbb{R})$ for all $i, j = 1, 2$, where the global involutivity is given by the third condition. For instance, the example of globally involutive but not globally elliptic system given after Remark 2.7, that is $X_1 = D_1$, $X_2 = x_2 D_2$, where $[X_1, X_2] = 0$, fits in these cases. Here, by Theorem 2.10, we have $L^2 - L^2$ local solvability on $\mathbb{R}^2$, and $H^{-1} - L^2$ local solvability at the elliptic points, i.e. at any point of $\mathbb{R}^2 \setminus \{x \in \mathbb{R}^2; x_2 = 0\}$.

Non KdV-type operators. To give a more complete picture of the objects of this paper, we provide immediate examples of non KdV-type operators of P_1-type.

Let $N \leq n$ and $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}_x$. Let also $\{iX_j\}_{j=1}^N$ be a global involutive system of real smooth vector fields in $\mathbb{R}^n_x$ (depending only on the space variables and on the space derivatives), X_1 nondegenerate on $\mathbb{R}^{n+1}$, and $X_0 \equiv 0$. Then

$$P_1(t, x, D_t, D_x) = P_1(x, D_x) = X_1 \sum_{j=1}^N X_j^* X_j,$$

is of P_1-type both on $\mathbb{R}^n$ and on $\mathbb{R}^{n+1}$ but not of KdV-type on $\mathbb{R}^{n+1}$, since there is no time-evolution here. Note that we could also take $N \leq n+1$ and define an operator whose leading part has variable coefficients depending both on space and time.

More degenerate variable coefficient KdV-type operators. Still in the case $n = 2$, one can consider models where $N = 1$, $iX_1(x, D_t, D_x) = X_1(x, D_x)$ is a nonvanishing real vector field with smooth coefficients, and $X_0 = D_t$. In such a case we have no ellipticity in space for $X_1^* X_1$, since $X_1^* X_1$ cannot be elliptic under our assumption that iX_1 has real coefficients. Nevertheless, all the hypotheses, namely (H1) and the nondegeneracy of X_1, are satisfied, hence our results apply - the Carleman and the solvability theorem - and the operator is locally solvable at any point of $\mathbb{R}^3$.

This example can be generalized in any dimension $n \geq 2$. By taking $\{iX_j(x, D_x)\}_{j=1}^N$, $N \leq n$, being a globally involutive system of real smooth vector fields (indexed in such a way that X_1 is nondegenerate), and taking $iX_0 = \partial_t$, the corresponding operator P_1 is built from an operator $\mathcal{L} := \sum_{j=1}^N X_j^* X_j$ which is not necessarily elliptic in space.

In the rest of this section we will give concrete examples of KdV and non KdV P_1-type operators constructed via some Lie algebras, specifically stratified Lie algebras. These cases are taken into account due to their global involutive structure. One can, of course, design many other different examples.

KdV-type operators built via the Heisenberg Lie algebra. Let us start with the example mentioned in the introduction and related to the Heisenberg group. We restrict ourselves to $N = n = 3$ - hence P_1 is defined on $\mathbb{R}^4_{t,x}$- which can be immediately generalized to the case $N = n = 2k + 1$, for any positive integer k.

We take, as before, $X_0 = D_t$ and iX_j, $j = 1, 2, 3$, as the vector fields giving the canonical basis of the Lie algebra of $\mathbb{H}^1$. Recall that the Lie algebra $\mathfrak{h}^1$ is stratified of step 2, and that X_1, X_2 are the so-called "generators" of the stratified Lie algebra. These vector fields, together with their commutator, generate the whole Lie algebra $\mathfrak{h}^1 = \text{Span}\{X_1, X_2, X_3\}$. In fact, $[X_1, X_2] = X_3$ and $[X_1, X_3] = [X_2, X_3] = 0$ for all $x \in \mathbb{R}^3$. This guarantees that the vector fields X_j, $j = 1, 2, 3$, form a global involutive structure. For completeness, we recall the expression of the vector fields below

$$iX_1(t,x,D_t,D_x) = iX_1(x,D_x) = \partial_{x_1} - \frac{x_2}{2}\partial_{x_3},$$
$$iX_2(t,x,D_t,D_x) = iX_2(x,D_x) = \partial_{x_2} + \frac{x_1}{2}\partial_{x_3},$$
$$iX_3(t,x,D_t,D_x) = iX_3(x,D_x) = \partial_{x_3}.$$

Note that $X_j = X_j^*$, thus $P_1 = P_1^*$ in this case. Moreover, $X_j(x,D)$ is nondegenerate for all $j = 0,\dots,3$, implying that the operators

$$P_{1,j_0}(t,x,D_t,D_x) = X_{j_0}(x,D_x)\sum_{j=1}^{3} X_j^2(x,D_x) + D_t, \quad j_0 = 1,2,3, \quad (t,x)\in\mathbb{R}^4,$$

are locally solvable at any point of $\mathbb{R}^4_{t,x}$ by Theorem 2.10. To define the same kind of operators in higher dimension it suffices to use the Heisenberg Lie algebra $\mathfrak{h}^k$ of dimension $2k+1$, and take $N = 2k+1 = n$.

KdV-type operators built via the Heisenberg Lie algebra in higher dimensional spaces. By taking the same vector fields as above generating $\mathfrak{h}^1$, one can cook up more singular operators in higher dimensional spaces. Take $N = 3 < n$, $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^{n-3}_y$, and P_{1,j_0} as before, but now defined on $\mathbb{R}^{n+1}$, that is

$$P_{1,j_0}(t,x,y,D_t,D_x,D_y) = X_{j_0}(x,D_x)\sum_{j=1}^{3} X_j^2(x,D_x) + D_t, \quad j_0 = 1,2,3,$$
$$(t,x,y)\in\mathbb{R}_t\times\mathbb{R}^3_x\times\mathbb{R}^{n-3}_y.$$

Then we have that the system of vector fields $\{iX_j(x,D_x)\}_{j=1,2,3}$ is still globally involutive in $\mathbb{R}^{n+1}_{t,x,y}$, and the hypotheses of Theorem 2.10 are satisfied. Hence, we get local solvability for these operators at any point of $\mathbb{R}^{n+1}_{t,x,y}$.

Similar operators can be written by using the Heisenberg Lie algebra $\mathfrak{h}^k$, for every $k \geq 1$, and $N = 2k+1 < n$.

KdV-type operators built via stratified Lie algebras: general construction. Following the previous constructions, we can formalize the procedure to build operators as above with any stratified Lie algebra of step r on $\mathbb{R}^m$. Let $\{iX_j\}_{j=1}^m$ be the canonical basis of $\mathfrak{g}$ (via the expnential map), and

$$\mathfrak{g} = \oplus_{j=1}^r \mathfrak{g}_j, \quad [\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{k+j},$$

where $\mathfrak{g}_1$ generates $\mathfrak{g}$ as an algebra, that is, linear combinations of the elements of $\mathfrak{g}_1$ and of their iterated commutators up to length r, generate the whole $\mathfrak{g}$. In particular, assuming $n_0 = 0$, one has that

$$n_j := \dim(\mathfrak{g}_j) = \dim(\mathrm{Span}\{iX_{n_{j-1}+1}(x,D_x),\dots,iX_{n_j}(x,D_x)\}), \quad \forall j\in\{1,\dots,r\},$$

and

$$\sum_{j=1}^{r} n_j = m.$$

Let now $N = m \leq n$, $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^m_x \times \mathbb{R}^{n-m}_y$, $j_0 = 1,\dots,m$, and

$$P_{1,j_0}(t,x,y,D_t,D_x,D_y) = X_{j_0}(x,D_x)\sum_{j=1}^{m} X_j^*(x,D_x)X_j(x,D_x) + D_t,$$
$$(t,x,y) \in \mathbb{R}_t \times \mathbb{R}_x^3 \times \mathbb{R}_y^{n-3}.$$

Then, once again, we have that $\{iX_j\}_{j=1}^m$ is a global involutive system of vector fields on $\mathbb{R}^{n+1}$. Moreover, the vector fields, being elements of the canonical basis of the Lie algebra, are all nondegenerate in $\mathbb{R}^{n+1}$. Hence, for every $j_0 = 1, \ldots, m$, the operator P_{1,j_0} satisfies the hypotheses of Theorem 2.10, leading to the local solvability property for all these models.

Non KdV-type operators built via Lie algebras. One can get immediate examples of non KdV-type operators just by taking the previous examples with $X_0 \equiv 0$.

Remark 3.1 In our examples above we have focused primarily on space-variable coefficient KdV-type operators. We want to stress that several KdV-type operators, with coefficients depending both on (t,x) or just on t, are still included in our class P_1.

4 Final remarks and applications to dispersive equations

In this conclusive section we would like to discuss possible applications in the context of dispersive equations of KdV-type, as well as some open connected questions.

Uniqueness problems. Uniqueness problems, or Hardy uncertainty principles (see [7]), for dispersive equations, are strictly related to Carleman estimates.

In general, when dealing with dispersive equations, one aims at global in space and local or global in time results. Global in time results are more challenging an not always attainable, so it is a natural common rule to investigate local in time properties first, and, afterwards, to extend, if possible, the result at all times.

The same considerations apply to Carleman estimates. This means that one needs global in space and local in time estimates if one wants to apply them in the context of dispersive equations. This is usually done via a combination of strategies. One of this strategies consists in considering the operator with bounded space-variable coefficients with a certain decay property. This condition is actually not just a technical condition, but is also related to the validity of smoothing properties for dispersive operators (see, for instance, [6], [29], [22], [37], [34]). Once the Carleman estimate is made global (in space) through a series of conditions in spirit as the one above, then one can attain uniqueness results via a precise cutting off procedure according to the assumptions considered on the solutions.

That said, it should be possible to reach global (in all but at most one direction) Carleman estimates for P_1 and P_1^* from our local estimates in Theorem 2.8. These global estimates can be applied to study uniqueness problems for dispersive operators belonging to our class. For what we just explained above, by introducing suitable conditions on the coefficients of the operator and on the weight function f, one should be able to produce the desired global inequality. However, even if the road map to get a global inequality is in the proof of Theorem 2.8, its derivation and the application to uniqueness problems is nontrivial.

Let us also stress that, if one is interested in the study of local unique continuation properties, then our estimates are already strong enough to pursue this goal, and one should focus more on the study of the geometric conditions leading to the result, that is, in other words, on the choice of the suitable weight function f.

Other KdV-type models and relative questions. In our discussion, we have motivated our analysis with the connection of the class P_1 with KdV-type operators. However, it must be said that we have a rigorous derivation of the ZK equation as a sort of generalization of the KdV one in (space) dimension 2, but for the higher dimensional case the argument is much harder and not established to the author knowledge. On the other hand, it is known that third order equations play a central role in water waves, nonlinear optic and related fields. A detailed interesting description of third order dispersive equations and of their physical role is contained in [30], where a class of third order operators with constant coefficients is studied. Let us remark that a subclass of the class in [30] of constant coefficient operators is certainly contained in our general variable coefficient class P_1. We also recall that P_1 is defined in any space dimension, while the class in [30] concerns the case $\mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}^2_x$. That said, it would be interesting to investigate Carleman estimates, local solvability, and connected problems, for operators in the very general form

$$P_2(x, D) := \sum_{j=1}^{M} \sum_{k=1}^{N} X_j X_k{}^* X_k + X_0, \quad 1 \leq M \leq N,\ x \in \mathbb{R}^{n+1},$$

or

$$P_3(x, D) := \sum_{j=1}^{M} \sum_{k=1}^{N} X_j X_k{}^* X_k + \sum_{i,l=1}^{L} c_{il}\, X_i{}^* X_l + X_0, \quad 1 \leq M \leq N,\ L \leq N,\ c_{il} \in \mathbb{R},$$
$$x \in \mathbb{R}^{n+1},$$

under a sort of ellipticity requirement in the spirit of condition (1.5) in [30], condition also appearing in the examples provided above to describe KdV-type operators. Note that these classes, which generalize and contain P_1, clearly still contain KdV-type operators, both with constant and with variable coefficients, as well as the whole class studied in [30], at least in the case described by P_3. Due to these considerations, it is natural to wonder whether P_2 or P_3 is the right generalization of (linear) KdV-type operators, or if P_1 already provides the best description, being clear that P_2 and P_3 are classes wider than P_1. In any case, the validity of Carleman estimates in this generality is still an open problem.

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