DE GRUYTER Adv. Calc. Var. 2024; 17(3): 753–774

Research Article

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A characterization of gauge balls in $\mathbb{H}^n$ by horizontal curvature

https://doi.org/10.1515/acv-2022-0058
Received July 15, 2022; revised November 18, 2022; accepted December 7, 2022

Abstract: In this paper, we aim at identifying the level sets of the gauge norm in the Heisenberg group $\mathbb{H}^n$ via the prescription of their (non-constant) horizontal mean curvature. We establish a uniqueness result in $\mathbb{H}^1$ under an assumption on the location of the singular set, and in $\mathbb{H}^n$ for $n \geq 2$ in the proper class of horizontally umbilical hypersurfaces.

Keywords: Heisenberg group, horizontal curvature, gauge balls

MSC 2010: Primary 53A10; secondary 35R03, 53C21, 32V20

Communicated by: Zoltan Balogh

1 Introduction

If we identify the Heisenberg group $\mathbb{H}^n$ with $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with generic point $\xi = (x, y, t)$ and we choose the group law

$$\xi \circ \xi' = (x, y, t) \circ (x', y', t') = \Big(x + x', y + y', t + t' + 2\sum_{k=1}^{n}(x_k y'_k - y_k x'_k)\Big), \tag{1.1}$$

the so-called homogeneous gauge is the function defined by

$$\rho(\xi) = ((|x|^2 + |y|^2)^2 + t^2)^{\frac{1}{4}}.$$

Such a $\rho(\,\cdot\,)$ is in fact homogeneous of degree 1 with respect to the family of dilations

$$\delta_R(\xi) = (Rx, Ry, R^2 t), \quad R > 0, \tag{1.2}$$

and it provides the defining function of the following *gauge balls* (sometimes called Korányi balls)

$$B_R(\xi_0) = \{\xi \in \mathbb{H}^n : \rho(\xi_0^{-1} \circ \xi) < R\} \quad \text{for } \xi_0 \in \mathbb{H}^n,\ R > 0. \tag{1.3}$$

The gauge function appeared in [22] in the study of singular integrals on homogeneous spaces. Over the years, it has played a crucial role in the analysis of PDEs of sub-elliptic type since the discovery in [15, Theorem 2] that $\rho^{-2n}(\,\cdot\,)$ is, up to a constant, the fundamental solution of the Heisenberg sub-Laplacian $\Delta_{\mathbb{H}^n}$. It is in fact known since [17, Théorème 3] (see also the treatment in [4, Section 5]) the validity of an analogue of the classical Gauss–Koebe theorem saying that the pointwise value of every solution u to $\Delta_{\mathbb{H}^n} u = 0$ can be represented as a weighted average of the values of u on gauge balls B_R. The weight is given by the squared norm of the horizontal gradient of ρ (which is homogeneous of degree 0 but not constant). Gauge balls are actually characterized by such a weighted mean value property for $\Delta_{\mathbb{H}^n}$-harmonic functions as proved by Lanconelli in [23].

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The metric balls B_R defined in (1.3) are not the unique choice of "balls" adapting to the sub-Riemannian features of the Heisenberg group. For instance, the Carnot–Carathéodory balls play somehow the role of the geodesic balls in $\mathbb{H}^n$. Furthermore, very much related to our purposes is the case of the domains bounded by the so-called *Pansu spheres*: they are the cmc-spheres with respect to the relevant notion of horizontal mean curvature (see Definition 2.2 below) and they are the conjectured unique minimizers for the isoperimetric inequality [33]. The solution of the isoperimetric problem in the Heisenberg group, also known as Pansu's conjecture, has generated a great amount of attention and several proofs appeared in the literature under extra-assumptions on the class of competitors; see [24], [5], [13], [29], [30], and [35]. Concerning the related Alexandrov-type problem, it was shown in [36] that Pansu spheres are the only rotationally invariant hypersurfaces with constant horizontal mean curvature. To the best of our knowledge, a result which is reminiscent of the classical Alexandrov theorem [1] is available only in $\mathbb{H}^1$: as a matter of fact, in [37, Theorem 6.10] Ritoré and Rosales proved that Pansu spheres in $\mathbb{H}^1$ are the only C^2-smooth critical points of the horizontal perimeter under volume constraint. For $n \geq 2$, various characterizations of Pansu spheres among horizontally umbilical hypersurfaces were established in [8].

In this paper, we take a new perspective as we address the question of characterizing the gauge balls by prescribing the horizontal mean curvature. In a similar spirit, in a companion paper [28] two of us have dealt with various characterizations of gauge balls through suitable overdetermined problems. To give a better description of the main results, we provide the reader with some initial background on the main notions involved, and we refer to Section 2 for the precise definitions. In $\mathbb{H}^n$, the horizontal distribution is spanned at any point $\xi = (x, y, t)$ by the vector fields

$$X_j = \frac{\partial}{\partial x_j} - 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + 2x_j \frac{\partial}{\partial t}, \qquad j = 1, \dots, n,$$

which are left-invariant with respect to the group law (1.1) and homogeneous of degree one with respect to (1.2). In our notations, we let

$$\mathcal{H}_\xi = \operatorname{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\}.$$

We also set $T = \frac{\partial}{\partial t}$, and we consider in $\mathbb{H}^n$ the Riemannian metric $\langle \cdot , \cdot \rangle$ which makes the basis

$$\mathcal{B} = \{X_1, \dots, X_n, Y_1, \dots, Y_n, T\}$$

orthonormal. If we consider a smooth hypersurface $M \subset \mathbb{H}^n$, a point $\xi \in M$ is said to be characteristic if the tangent space of M at ξ coincides with $\mathcal{H}_\xi$. At any point $\xi \in M$ which is not characteristic, it is thus well-defined the so-called horizontal normal ν^H as the normalized $\langle \cdot , \cdot \rangle$-orthogonal projection on $\mathcal{H}_\xi$ of the metric (outer, whenever possible) unit normal ν. The horizontal mean curvature is the divergence of such ν^H (see Section 2 for the precise definitions), which is therefore well-defined at any non-characteristic point. A simple computation shows that the horizontal mean curvature of $\partial B_R(0) \subset \mathbb{H}^n$ is proportional to the distance to the t-axis, i.e. it is a constant multiple of

$$\sqrt{|x|^2 + |y|^2}$$

at any point $(x, y, t) \in \partial B_R(0)$ (outside of the two poles sitting on the t-axis, which correspond to the only characteristic points for the gauge sphere). In $\mathbb{H}^1$, our main result reads as follows.

Theorem 1.1. *Let M be a smooth surface in $\mathbb{H}^1$ which is connected, orientable, compact, and without boundary. Assume that there are no characteristic points of M outside of the line $\{(0, 0, t) \in \mathbb{H}^1 : t \in \mathbb{R}\}$. If at every non-characteristic point $(x, y, t) \in M$ the horizontal mean curvature of M is proportional to*

$$\sqrt{x^2 + y^2}$$

up to a constant factor $c \neq 0$, then $c > 0$ and there exists $t_0 \in \mathbb{R}$ such that $M = \partial B_R(\xi_0)$ with

$$R = \sqrt{\frac{3}{c}} \quad \textit{and} \quad \xi_0 = (0, 0, t_0).$$

The restriction to the ($n = 1$)-dimensional case in the previous theorem relies on the fact that the 2-dimensional surface $M \subset \mathbb{H}^1$ has only one horizontal tangent vector field at every non-characteristic point, and M is then 'ruled' by its integral curves (as it is clear from the analysis developed in [9, 37]). In higher dimensions, we have the following counterpart, which is a characterization of gauge spheres under the proper prescribed curvature assumption among the class of umbilical hypersurfaces introduced in [8] (see Definition 2.4 below).

Theorem 1.2. *Fix $n \geq 2$. Let M be a smooth hypersurface of $\mathbb{H}^n$ which is connected, orientable, compact, and without boundary. Suppose that M is umbilical and that, at every non-characteristic point $(x, y, t) \in M$, the horizontal mean curvature of M is proportional to*

$$\sqrt{|x|^2 + |y|^2}$$

up to a constant factor $c \neq 0$. Then $c > 0$ and there exists $t_0 \in \mathbb{R}$ such that $M = \partial B_R(\xi_0)$ with

$$R = \sqrt{\frac{1}{c}\frac{2n+1}{2n-1}} \quad \text{and} \quad \xi_0 = (0, 0, t_0).$$

Remark 1.3. We know that the horizontal mean curvature of a generic gauge sphere $\partial B_R(x_0, y_0, t_0)$ in $\mathbb{H}^n$ is a constant multiple of

$$\sqrt{|x - x_0|^2 + |y - y_0|^2}.$$

Since all assumptions involved are invariant by left-translation, we explicitly notice that for any $(x_0, y_0) \in \mathbb{R}^{2n}$ we can reformulate Theorem 1.1 and Theorem 1.2 as characterizations for gauge balls centered at points on the t-axis $\{(x_0, y_0, t) : t \in \mathbb{R}\}$.

The paper is organized as follows. In Section 2, we recall the main definitions involved and we show some basic properties. In Section 3, we give the proofs of Theorem 1.1 and Theorem 1.2 which follow a similar pattern: the main aim is to infer, under the respective assumptions, that the key functions φ_h, φ_{v} introduced below in (3.2) and (3.11) are constant throughout M. Finally, we will show in Corollary 3.6 that Theorems 1.1 and 1.2 imply in particular a rigidity result in the class of cylindrically symmetric hypersurfaces $M \subset \mathbb{H}^n$ for any $n \geq 1$.

2 Definitions and preliminaries

In this section, we collect some preliminary material that will be used in the rest of the paper. We shall recall some known notions for the study of smooth hypersurfaces in $\mathbb{H}^n$, and we refer the reader to [11], [34], [32], [9], [12], [36], [20], [35], [10], [2], [8], and [3] for several insights and different perspectives and approaches to the geometry of submanifolds in various sub-Riemannian settings.

With $\langle \cdot, \cdot \rangle$ being the metric defined in Section 1 (with induced norm $|\cdot|$), we denote by ∇ the Levi-Civita connection associated to this metric. A direct computation shows that for any $i, j = 1, \dots, n$ the following holds:

$$\begin{cases} \nabla_{X_i} X_j = 0, & \nabla_{X_i} Y_j = 2\delta_{ij} T, & \nabla_{X_i} T = -2Y_i, \\ \nabla_{Y_i} X_j = -2\delta_{ij} T, & \nabla_{Y_i} Y_j = 0, & \nabla_{Y_i} T = 2X_i, \\ \nabla_T X_i = -2Y_i, & \nabla_T Y_i = 2X_i, & \nabla_T T = 0. \end{cases} \tag{2.1}$$

For any smooth vector field V in the horizontal distribution $\mathcal{H}$, we define

$$J(V) := -\frac{1}{2}\nabla_V T.$$

In this way, we have $J(X_i) = Y_i$ and $J(Y_i) = -X_i$ for all $i \in \{1, \dots, n\}$. Moreover, for any $V, W \in \mathcal{H}$, one can easily see that the following relations hold:

$$\begin{cases} \langle J(V), W \rangle = -\langle V, J(W) \rangle, \\ \langle J(V), J(W) \rangle = \langle V, W \rangle, \\ J(\nabla_V W) = \nabla_V (JW), \\ \langle [V, W], T \rangle = 4\langle J(V), W \rangle. \end{cases} \tag{2.2}$$

For any smooth vector field V in $\mathbb{H}^n$, we will use the notation $\mathcal{P}_H(V)$ to denote its horizontal projection, with $\mathcal{P}_H$ being the orthogonal projection onto $\mathcal{H}$. A special role will be played by the horizontal part of the position vector, i.e.

$$\xi^H := \mathcal{P}_H(\xi) = \sum_{j=1}^{n} x_j X_j + y_j Y_j.$$

We can show that

$$\nabla_Z \xi^H = Z + 2\langle J(Z), \xi^H\rangle T \quad \text{for any } Z \in \mathcal{H}. \tag{2.3}$$

To see (2.3), we just write $Z = \sum_{j=1}^{n}(\alpha_j X_j + \beta_j Y_j)$ and it is straightforward to recognize from (2.1) that

$$\begin{aligned}\nabla_Z \xi^H &= \sum_{j=1}^{n}(Z(x_j)X_j + Z(y_j)Y_j) + \sum_{j,k=1}^{n}(\alpha_k x_j \nabla_{X_k} X_j + \beta_k x_j \nabla_{Y_k} X_j + \alpha_k y_j \nabla_{X_k} Y_j + \beta_k y_j \nabla_{Y_k} Y_j)\\ &= Z + 2\sum_{j=1}^{n}(\alpha_j y_j - \beta_j x_j)T\\ &= Z + 2\langle J(Z), \xi^H\rangle T.\end{aligned}$$

Similarly, we have

$$Z(t) = -2\langle J(Z), \xi^H\rangle \quad \text{for any } Z \in \mathcal{H}, \tag{2.4}$$

since in the same notations we can check that

$$Z(t) = \sum_{j=1}^{n}(-2\alpha_j y_j + 2\beta_j x_j) = -2\langle J(Z), \xi^H\rangle.$$

We now start considering a C^2-smooth codimension 1 submanifold M in $\mathbb{H}^n$. We always assume M to be connected and orientable. We denote by ν a fixed choice for the metric normal with unit length, and by $T_\xi M$ the tangent space at $\xi \in M$. Whenever M is also compact and without boundary, we agree to fix ν as the outward unit normal. The characteristic set is defined by

$$S_M := \{\xi \in M : \mathcal{P}_H(\nu) = 0\} = \{\xi \in M : T_\xi M = \mathcal{H}_\xi\}.$$

Outside of the set S_M, we suppose the hypersurface to be C^∞-smooth. For any point in $M \setminus S_M$ it is well-defined

$$\nu^H = \frac{1}{|\mathcal{P}_H(\nu)|}\mathcal{P}_H(\nu),$$

and we can write

$$\nu = |\mathcal{P}_H(\nu)|\nu^H + \langle \nu, T\rangle T$$

and define the tangent vector field

$$\tau := \langle \nu, T\rangle \nu^H - |\mathcal{P}_H(\nu)|T.$$

We further set

$$\eta = -J\nu^H, \tag{2.5}$$

which clearly belongs to $\mathcal{H} \cap TM$. In case $n > 1$, locally around any point $\xi \in M \setminus S_M$ we can also pick smooth horizontal vector fields V_i, W_i for $i = 1, \dots, n-1$ such that $J(V_i) = W_i$ and

$$\{\eta, \nu^H, V_1, W_1, \dots, V_{n-1}, W_{n-1}\}$$

is an orthonormal basis for $\mathcal{H}_\xi$. With these choices, we have fixed the orthonormal frames for TM and $\mathcal{H} \cap TM$ (outside of characteristic points) as, respectively,

$$\{\tau, \eta, V_1, W_1, \dots, V_{n-1}, W_{n-1}\} \quad \text{and} \quad \{\eta, V_1, W_1, \dots, V_{n-1}, W_{n-1}\}.$$

In our notations, we have the following lemma.

Lemma 2.1. *In $M \smallsetminus S_M$, it holds*

$$\langle \nabla_Z \nu^H, T\rangle = 2\langle \eta, Z\rangle \qquad \text{for every } Z \in \mathcal{H}, \tag{2.6}$$

$$\langle \nabla_\tau \nu^H, \tau\rangle = 0 = \langle \nabla_\nu \nu^H, \nu\rangle, \tag{2.7}$$

$$\langle [Z_1, Z_2], \nu^H\rangle = \frac{-4\langle \nu, T\rangle}{|\mathcal{P}_H(\nu)|}\langle J(Z_1), Z_2\rangle \quad \text{for every } Z_1, Z_2 \in \mathcal{H} \cap TM. \tag{2.8}$$

Proof. Relation (2.6) follows by (2.2) and (2.5) since, for all $Z \in \mathcal{H}$, we have

$$\langle \nabla_Z \nu^H, T\rangle = -\langle \nu^H, \nabla_Z T\rangle = 2\langle \nu^H, J(Z)\rangle = 2\langle -J\nu^H, Z\rangle = 2\langle \eta, Z\rangle.$$

On the other hand, using that $|\nu^H| = 1$ together with (2.1)–(2.2), we obtain

$$\begin{aligned}
\langle \nabla_\nu \nu^H, \nu\rangle &= \langle \nabla_\nu \nu^H, |\mathcal{P}_H(\nu)|\nu^H + \langle \nu, T\rangle T\rangle \\
&= \langle \nu, T\rangle\langle \nabla_\nu \nu^H, T\rangle \\
&= -\langle \nu, T\rangle\langle \nu^H, \nabla_\nu T\rangle \\
&= -\langle \nu, T\rangle\langle \nu^H, \nabla_{|\mathcal{P}_H(\nu)|\nu^H} T\rangle \\
&= 2|\mathcal{P}_H(\nu)|\langle \nu, T\rangle\langle \nu^H, J\nu^H\rangle \\
&= 0,
\end{aligned}$$

and analogously

$$\begin{aligned}
\langle \nabla_\tau \nu^H, \tau\rangle &= \langle \nabla_\tau \nu^H, \langle \nu, T\rangle \nu^H - |\mathcal{P}_H(\nu)| T\rangle \\
&= -|\mathcal{P}_H(\nu)|\langle \nabla_\tau \nu^H, T\rangle \\
&= |\mathcal{P}_H(\nu)|\langle \nu^H, \nabla_\tau T\rangle \\
&= |\mathcal{P}_H(\nu)|\langle \nu^H, \nabla_{\langle \nu, T\rangle \nu^H} T\rangle \\
&= -2|\mathcal{P}_H(\nu)|\langle \nu, T\rangle\langle \nu^H, J\nu^H\rangle \\
&= 0.
\end{aligned}$$

The previous two identities show (2.7). Finally, in order to prove (2.8), we can pick any $Z_1, Z_2 \in \mathcal{H} \cap TM$ and deduce from the property $[Z_1, Z_2] \in TM$ and (2.2) that

$$\begin{aligned}
\langle [Z_1, Z_2], \nu^H\rangle &= \frac{1}{|\mathcal{P}_H(\nu)|}\langle [Z_1, Z_2], |\mathcal{P}_H(\nu)|\nu^H\rangle \\
&= \frac{1}{|\mathcal{P}_H(\nu)|}\langle [Z_1, Z_2], \nu\rangle - \frac{\langle \nu, T\rangle}{|\mathcal{P}_H(\nu)|}\langle [Z_1, Z_2], T\rangle \\
&= -\frac{\langle \nu, T\rangle}{|\mathcal{P}_H(\nu)|}\langle [Z_1, Z_2], T\rangle \\
&= \frac{-4\langle \nu, T\rangle}{|\mathcal{P}_H(\nu)|}\langle J(Z_1), Z_2\rangle.
\end{aligned}$$

This finishes the proof. □

We are then ready to recall the definition of horizontal mean curvature. Such notion arises in the criticality condition for the horizontal perimeter (see [6, 12]).

Definition 2.2 (Horizontal mean curvature). Let $M \subset \mathbb{H}^n$ be as above. For any $\xi \in M \smallsetminus S_M$, we define the horizontal mean curvature of M at ξ by

$$H_M(\xi) = \frac{\operatorname{div}(\nu^H)}{2n-1} = \frac{1}{2n-1}\Big(\langle \nabla_\eta \nu^H, \eta\rangle + \sum_{i=1}^{n-1} \langle \nabla_{V_i} \nu^H, V_i\rangle + \langle \nabla_{W_i} \nu^H, W_i\rangle\Big), \tag{2.9}$$

where div stands for the divergence with respect to the metric $\langle \cdot, \cdot\rangle$. In particular, if $M \subset \mathbb{H}^1$, we simply have

$$H_M(\xi) = \langle \nabla_\eta \nu^H, \eta\rangle \quad \text{in case } n = 1. \tag{2.10}$$

We warn the reader that the second equality in (2.9) is justified by (2.7). The definition of H_M can be (and, in the literature, has been) in fact given in multiple ways. For example, since $\nabla_T T = 0$, it is immediate to recognize that

$$\operatorname{div}(\nu^H) = \sum_{i=1}^{n} \langle \nabla_{X_i} \nu^H, X_i \rangle + \langle \nabla_{Y_i} \nu^H, Y_i \rangle.$$

By noticing that by (2.6) we have

$$\mathcal{P}_H(\nabla_Z \nu^H) = \nabla_Z \nu^H - 2\langle \eta, Z\rangle T \quad \text{for any } Z \in \mathcal{H},$$

we can also recall the notion of horizontal shape operator (see [35]), which will be needed in what follows.

Definition 2.3 (horizontal shape operator). Let $M \subset \mathbb{H}^n$ as above. For any $\xi \in M \smallsetminus S_M$, we can define the symmetric endomorphism $A_M(\cdot)(\xi)$ on $\mathcal{H}_\xi \cap T_\xi M$ by

$$A_M(Z) = \mathcal{P}_H(\nabla_Z \nu^H) - \frac{2\langle \nu, T\rangle}{|\mathcal{P}_H(\nu)|}(J(Z) - \langle \eta, Z\rangle \nu^H)$$

for $Z \in \mathcal{H} \cap TM$.

The fact that $A_M(Z) \in \mathcal{H} \cap TM$ for $Z \in \mathcal{H} \cap TM$ follows by the two identities

$$\langle A_M(Z), T\rangle = 0 = \langle A_M(Z), \nu^H\rangle,$$

which can be easily checked. On the other hand, the symmetry of $A_M(\cdot)$ can be deduced from (2.8) since

$$\begin{aligned}\langle A_M(Z_1), Z_2\rangle - \langle A_M(Z_2), Z_1\rangle &= \langle \nabla_{Z_1} \nu^H, Z_2\rangle - \langle \nabla_{Z_2} \nu^H, Z_1\rangle - \frac{2\langle \nu, T\rangle}{|\mathcal{P}_H(\nu)|}(\langle J(Z_1), Z_2\rangle - \langle J(Z_2), Z_1\rangle)\\ &= -\langle \nu^H, \nabla_{Z_1} Z_2 - \nabla_{Z_2} Z_1\rangle - \frac{4\langle \nu, T\rangle}{|\mathcal{P}_H(\nu)|}\langle J(Z_1), Z_2\rangle\\ &= 0\end{aligned}$$

for any $Z_1, Z_2 \in \mathcal{H} \cap TM$. When $n = 1$, then $\mathcal{H} \cap TM$ is 1-dimensional (and generated by η in our notations) and $A_M(\cdot)$ is nothing but the multiplication by the factor H_M. In higher dimensions, the horizontal mean curvature appears as the normalized trace of A_M since

$$\begin{aligned}\langle A_M(\eta), \eta\rangle + \sum_{i=1}^{n-1} \langle A_M(V_i), V_i\rangle + \langle A_M(W_i), W_i\rangle &= \langle \nabla_\eta \nu^H, \eta\rangle + \sum_{i=1}^{n-1} \langle \nabla_{V_i} \nu^H, V_i\rangle + \langle \nabla_{W_i} \nu^H, W_i\rangle\\ &= (2n-1)H_M.\end{aligned} \tag{2.11}$$

The following notion of horizontally umbilical hypersurface was introduced and studied in [7, 8].

Definition 2.4. Let $n \geq 2$. We say that M is umbilical if, in $M \smallsetminus S_M$, it holds

$$A_M(Z) = (l - k)\langle \eta, Z\rangle \eta + kZ \quad \text{for all } Z \in \mathcal{H} \cap TM,$$

for some suitable functions k, l. In particular, at any non-characteristic point ξ, by (2.11) we have

$$H_M(\xi) = \frac{1}{2n-1}(l(\xi) + (2n-2)k(\xi)).$$

The class of umbilical hypersurfaces is wide enough to contain any M which is rotationally symmetric with respect to the vertical t-axes (see in this respect [8, Proposition 3.1], see also the proof of Corollary 3.6 below).

Remark 2.5. It is evident from Definition 2.4 that

$$\text{if } M \text{ is umbilical with } l = 3k, \text{ then } k(\xi) = \frac{2n-1}{2n+1} H_M(\xi) \text{ for all } \xi \in M \smallsetminus S_M. \tag{2.12}$$

The case $l = 3k$ is related to the gauge spheres (see Example 2.6 below). On the other hand, let us mention that the Pansu spheres satisfy the umbilicality property with $l = 2k$: this is the case studied in [8].

Let us compute explicitly the objects previously discussed in the particular case of the gauge spheres.

Example 2.6. Let $n \geq 1$, $\xi_0 = (0, 0, t_0) \in \mathbb{H}^n$ and

$$M = \partial B_R(\xi_0) = \{\xi = (x, y, t) : (|x|^2 + |y|^2)^2 + (t - t_0)^2 = R^4\} \subset \mathbb{H}^n.$$

We use the notation

$$r = r(x, y) = \sqrt{|x|^2 + |y|^2} = |\xi^H|$$

to denote the distance from the t-axis. For any $\xi \in M$, we have

$$|\mathcal{P}_H(\nu)| = \frac{2rR^2}{\sqrt{4r^2R^4 + (t - t_0)^2}} \quad \text{and} \quad \langle \nu, T\rangle = \frac{t - t_0}{\sqrt{4r^2R^4 + (t - t_0)^2}}, \tag{2.13}$$

which is saying in particular that the characteristic set coincides with the intersection of $\partial B_R(\xi_0)$ with the t-axis, i.e. $S_M = \{(0, 0, t_0 \pm R^2)\}$. Outside of these two points, we have

$$\begin{aligned} \nu^H &= \sum_{j=1}^{n} \frac{r^2 x_j - y_j(t - t_0)}{rR^2} X_j + \frac{r^2 y_j + x_j(t - t_0)}{rR^2} Y_j = \frac{r^2\xi^H + (t - t_0)J\xi^H}{rR^2}, \\ \eta = -J\nu^H &= \frac{(t - t_0)\xi^H - r^2 J\xi^H}{rR^2}. \end{aligned} \tag{2.14}$$

A straightforward computation then shows for any $j, k \in \{1, \dots, n\}$,

$$\begin{aligned} \langle \nabla_{X_k}\nu^H, X_j\rangle &= X_k(\langle \nu^H, X_j\rangle) = \frac{1}{r^2R^2}((2x_jx_k + \delta_{jk}r^2 + 2y_jy_k)r - \langle \nu^H, X_j\rangle R^2 x_k), \\ \langle \nabla_{Y_k}\nu^H, X_j\rangle &= Y_k(\langle \nu^H, X_j\rangle) = \frac{1}{r^2R^2}((2x_jy_k - \delta_{jk}(t - t_0) - 2y_jx_k)r - \langle \nu^H, X_j\rangle R^2 y_k), \\ \langle \nabla_{X_k}\nu^H, Y_j\rangle &= X_k(\langle \nu^H, Y_j\rangle) = \frac{1}{r^2R^2}((2y_jx_k + \delta_{jk}(t - t_0) - 2x_jy_k)r - \langle \nu^H, Y_j\rangle R^2 x_k), \\ \langle \nabla_{Y_k}\nu^H, Y_j\rangle &= Y_k(\langle \nu^H, Y_j\rangle) = \frac{1}{r^2R^2}((2y_jy_k + \delta_{jk}r^2 + 2x_jx_k)r - \langle \nu^H, Y_j\rangle R^2 y_k), \end{aligned}$$

which we can rewrite using (2.14) in the following way:

$$\begin{aligned} \mathcal{P}_H(\nabla_Z \nu^H) &= \frac{1}{R^2}\Big(\frac{2}{r}(\langle \xi^H, Z\rangle\xi^H + \langle J\xi^H, Z\rangle J\xi^H) + rZ + \frac{t - t_0}{r} JZ - \frac{R^2}{r^2}\langle \xi^H, Z\rangle \nu^H\Big) \\ &= \frac{1}{R^2}\Big(2r\langle \eta, Z\rangle\eta + rZ + \frac{t - t_0}{r} JZ - \frac{t - t_0}{r}\langle \eta, Z\rangle \nu^H\Big) \end{aligned} \tag{2.15}$$

for any horizontal vector Z. It is then easy to check that

$$H_M(\xi) = \frac{2n + 1}{2n - 1}\frac{r}{R^2}. \tag{2.16}$$

Also, recalling Definition 2.3 and using (2.13) and (2.15), we can recognize

$$A_M(Z) = \frac{2r}{R^2}\langle \eta, Z\rangle\eta + \frac{r}{R^2} Z \quad \text{for all } Z \in \mathcal{H} \cap TM.$$

According to Definition 2.4, when $n \geq 2$, this is saying that M is umbilical with $l(\xi) = 3k(\xi)$ and $k(\xi) = \frac{|\xi^H|}{R^2}$.

It is well known in the literature that, whenever $n \geq 2$, the horizontal and tangent vector fields in $\mathcal{H} \cap TM$ satisfy a Hörmander-type property as they can reproduce any tangent direction via commutation. If M is also umbilical, such information can be made very precise and it is encoded in the following lemma.

Lemma 2.7. *Let $n \geq 2$. For $\xi \in M \setminus S_M$, set*

$$\mathcal{H}^0_\xi = \operatorname{span}\{V_1, W_1, \dots, V_{n-1}, W_{n-1}\}.$$

If M is umbilical, then

$$\operatorname{span}\{Z, [Z_1, Z_2] : Z, Z_1, Z_2 \in \mathcal{H}^0_\xi\} = \operatorname{span}\Big\{V_1, W_1, \dots, V_{n-1}, W_{n-1}, \tau - \frac{k(\xi)|\mathcal{P}_H(\nu)|}{2}\eta\Big\}.$$

Proof. Fix any $Z_1, Z_2 \in \mathcal{H}^0_\xi$. By (2.2) and (2.8), we have

$$\begin{aligned}\langle [Z_1, Z_2], \tau\rangle &= \langle \nu, T\rangle \langle [Z_1, Z_2], \nu^H\rangle - |\mathcal{P}_H \nu| \langle [Z_1, Z_2], \nu^H\rangle \\ &= -4\Big(\frac{\langle \nu, T\rangle^2}{|\mathcal{P}_H \nu|} + |\mathcal{P}_H \nu|\Big)\langle J(Z_1), Z_2\rangle \\ &= \frac{-4}{|\mathcal{P}_H \nu|}\langle J(Z_1), Z_2\rangle,\end{aligned}$$

which says that $\operatorname{span}\{Z, [Z_1, Z_2] : Z, Z_1, Z_2 \in \mathcal{H}^0_\xi\}$ is at least $(2n-1)$-dimensional. On the other hand, using also (2.5) and the commutation property of J and ∇ together with the umbilicality of M, we obtain

$$\begin{aligned}\langle [Z_1, Z_2], \eta\rangle &= \langle \nabla_{Z_1} J Z_2 - \nabla_{Z_2} J Z_1, \nu^H\rangle \\ &= \langle J(Z_1), \nabla_{Z_2} \nu^H\rangle - \langle J(Z_2), \nabla_{Z_1} \nu^H\rangle \\ &= \Big\langle J(Z_1), A_M(Z_2) + \frac{2\langle \nu, T\rangle}{|\mathcal{P}_H \nu|} J(Z_2)\Big\rangle - \Big\langle J(Z_2), A_M(Z_1) + \frac{2\langle \nu, T\rangle}{|\mathcal{P}_H \nu|} J(Z_1)\Big\rangle \\ &= 2k\langle J(Z_1), Z_2\rangle.\end{aligned}$$

Hence we get

$$\Big\langle [Z_1, Z_2], \eta + \frac{k(\xi)|\mathcal{P}_H(\nu)|}{2}\tau\Big\rangle = 0 \quad \text{for every } Z_1, Z_2 \in \mathcal{H}^0_\xi.$$

This implies that $\operatorname{span}\{Z, [Z_1, Z_2] : Z, Z_1, Z_2 \in \mathcal{H}^0_\xi\}$ is exactly $(2n-1)$-dimensional and the vector

$$\tau - \frac{k(\xi)|\mathcal{P}_H(\nu)|}{2}\eta$$

belongs to such vector space as desired. □

3 Darboux-type results

3.1 The case of $\mathbb{H}^1$

In this section, we first treat the $(n = 1)$-dimensional case by providing the proof of Theorem 1.1. As we mentioned in Section 1 and recalled in (2.10), for surfaces M in $\mathbb{H}^1$ the main role is played by the integral curves of the only horizontal and tangent vector field η. A (naive) way to describe our approach to Theorem 1.1 is to draw a parallelism with the classical problem of identifying the pieces of circles as the only smooth connected curves Γ in $\mathbb{R}^2$ with non-zero constant curvature $K = K_\Gamma$. Among the many ways to show this property, a very direct one is to consider (denoting with $p = (p_1, p_2)$ the generic point in $\mathbb{R}^2$ and with N a choice for the unit normal to Γ) the two functions

$$\begin{cases} f_1(p) = Kp_1 - \langle N, \partial_{p_1}\rangle, & f_1 : \Gamma \to \mathbb{R}, \\ f_2(p) = Kp_2 - \langle N, \partial_{p_2}\rangle, & f_2 : \Gamma \to \mathbb{R}. \end{cases} \tag{3.1}$$

By differentiating along a unit tangent vector U and using $K = \langle \nabla_U N, U\rangle$, one recognizes that $Uf_1 = Uf_2 = 0$ on Γ. Thus, there have to exist two constants c_1, c_2 such that $f_i \equiv c_i$, $i = 1, 2$, and we have

$$1 = \langle N, \partial_{p_1}\rangle^2 + \langle N, \partial_{p_2}\rangle^2 = (Kp_1 - c_1)^2 + (Kp_2 - c_2)^2 \quad \text{for } p \in \Gamma,$$

i.e. Γ is contained in the circle of radius $\frac{1}{|K|}$ and center $(\frac{c_1}{K}, \frac{c_2}{K})$. If we bring back the attention to the case of the 2-dimensional surface M in $\mathbb{H}^1$, we emphasize that in Theorem 1.1 we prescribe the curvature $H_M(\xi)$ to be proportional to $|\xi^H|$ (see also (2.16) in Example 2.6). The term $|\xi^H|$ corresponds to the distance (either Euclidean distance or gauge-related distance, as they coincide in this case) to the vertical line L_v defined by

$$L_v = \{(0, 0, t) \in \mathbb{H}^1 : t \in \mathbb{R}\}.$$

Having this in mind, as well as the notations introduced in Section 2, we define the two functions

$$\begin{cases} \varphi_h(\xi) = \frac{1}{3} H_M(\xi)|\xi^H|^2 - \langle \nu^H, \xi^H \rangle, & \varphi_h : M \setminus S_M \to \mathbb{R}, \\ \varphi_{\mathrm{v}}(\xi) = \frac{1}{3} H_M(\xi) \frac{t}{|\xi^H|} - \langle \eta, \frac{\xi^H}{|\xi^H|} \rangle, & \varphi_{\mathrm{v}} : M \setminus (S_M \cup L_{\mathrm{v}}) \to \mathbb{R}. \end{cases} \tag{3.2}$$

The functions φ_h and φ_{v} have different roles. In the following lemma, we show that φ_h is in fact constant along the integral curves of η, whereas the behavior of φ_{v} is subordinate to the one of φ_h.

Lemma 3.1. *Let M be a smooth surface in $\mathbb{H}^1$ which is connected and orientable. Let also ω be a relatively open set contained in $M \setminus S_M$. Suppose there exists $c \in \mathbb{R}$ such that $H_M(\xi) = c|\xi^H|$ for $\xi \in \omega$. Then we have*

$$\begin{cases} \eta(\varphi_h) = 0 & \text{in } \omega, \\ \eta(\varphi_{\mathrm{v}}) = \frac{\langle \nu^H, \xi^H \rangle}{|\xi^H|^3} \varphi_h & \text{in } \omega \setminus L_{\mathrm{v}}. \end{cases}$$

Proof. By (2.3) and (2.10), we have

$$\eta(|\xi^H|) = \frac{\langle \eta, \xi^H \rangle}{|\xi^H|} \quad \text{and} \quad \eta(\langle \nu^H, \xi^H \rangle) = H_M \langle \eta, \xi^H \rangle, \tag{3.3}$$

where in the second equality we also exploited the fact that $\mathcal{H}_\xi$ is generated by the two orthogonal unit vectors η and ν^H. Hence, for $\xi \in \omega$, we obtain

$$\eta(\varphi_h) = \eta\Big(\frac{c}{3}|\xi^H|^3 - \langle \nu^H, \xi^H \rangle\Big) = c|\xi^H|^2 \frac{\langle \eta, \xi^H \rangle}{|\xi^H|} - H_M(\xi)\langle \eta, \xi^H \rangle = 0.$$

On the other hand, by (2.4)–(2.5) we have

$$\eta(t) = -2\langle \nu^H, \xi^H \rangle \tag{3.4}$$

and, using also (2.3)–(2.2), we have

$$\eta(\langle \eta, \xi^H \rangle) = 1 + \langle \nabla_\eta \eta, \xi^H \rangle = 1 + \langle \nabla_\eta \nu^H, J\xi^H \rangle = 1 - H_M \langle \nu^H, \xi^H \rangle. \tag{3.5}$$

Recalling that

$$|\xi^H|^2 = \langle \eta, \xi^H \rangle^2 + \langle \nu^H, \xi^H \rangle^2,$$

we then infer

$$\begin{aligned} \eta(\varphi_{\mathrm{v}}) &= \eta\Big(\frac{c}{3} t - \frac{\langle \eta, \xi^H \rangle}{|\xi^H|}\Big) \\ &= \frac{-2c}{3}\langle \nu^H, \xi^H \rangle - \frac{1}{|\xi^H|} + H_M(\xi)\Big\langle \nu^H, \frac{\xi^H}{|\xi^H|}\Big\rangle + \frac{1}{|\xi^H|^3}\langle \eta, \xi^H \rangle^2 \\ &= \Big\langle \nu^H, \frac{\xi^H}{|\xi^H|}\Big\rangle\Big(\frac{-2c}{3}|\xi^H| + H_M(\xi)\Big) - \frac{1}{|\xi^H|^3}(|\xi^H|^2 - \langle \eta, \xi^H \rangle^2) \\ &= \frac{\langle \nu^H, \xi^H \rangle}{|\xi^H|^3}\Big(\frac{1}{3} H_M(\xi)|\xi^H|^2 - \langle \nu^H, \xi^H \rangle\Big) \\ &= \frac{\langle \nu^H, \xi^H \rangle}{|\xi^H|^3}\varphi_h(\xi) \end{aligned}$$

whenever $\xi^H \neq 0$. This completes the proof of the lemma. □

Keeping in mind the comparison between the derivatives of (3.2) along η in Lemma 3.1 and the derivatives along the curve Γ of (3.1), it is no surprise that we want φ_h to vanish identically throughout M. This is exactly what we show in the next lemma. We will deduce this fact from the global properties of the integral curves of η and from the assumption $S_M \subset L_{\mathrm{v}}$.

Lemma 3.2. *Let M be a smooth surface in $\mathbb{H}^1$ which is connected, orientable, compact, and without boundary. Assume that $S_M \subseteq M \cap L_{\mathrm{v}}$, and that there exists $c \neq 0$ such that $H_M(\xi) = c|\xi^H|$ for every point $\xi \in M \setminus S_M$. Then $c > 0$, $\varphi_h \equiv 0$, and every integral curve of η reaches S_M.*

Proof. Let us divide the proof into three steps.

Step I. In the first step, we shall show that $c > 0$. By the compactness of M, the function $\frac{1}{2}|\xi^H|^2$ attains its maximum at a point $\xi_1 \in M \smallsetminus L_{\mathrm{v}}$. Since we have $S_M \subseteq M \cap L_{\mathrm{v}}$, at $\xi = \xi_1$ we have

$$0 = \eta\Big(\frac{1}{2}|\xi^H|^2\Big) = \langle \eta, \xi_1^H \rangle \quad \text{and} \quad 0 = \tau\Big(\frac{1}{2}|\xi^H|^2\Big) = \langle \nu, T \rangle \langle \nu^H, \xi_1^H \rangle.$$

Since

$$\langle \nu^H, \xi_1^H \rangle^2 = \langle \eta, \xi_1^H \rangle^2 + \langle \nu^H, \xi_1^H \rangle^2 = |\xi_1^H|^2 > 0,$$

we have that $\langle \nu, T \rangle = 0$, and therefore

$$\langle \nu^H, \xi_1^H \rangle = \frac{1}{|\mathcal{P}_H \nu|} \langle \nu, \xi_1 \rangle > 0,$$

where the positive sign is a consequence of the maximality condition and the fact that ν is the outward normal. Moreover, we also know that $\eta^2(\frac{1}{2}|\xi^H|^2) \le 0$ at the maximum point $\xi = \xi_1$. This fact, together with the identity

$$1 - c|\xi^H|\langle \nu^H, \xi^H \rangle = 1 - H_M(\xi)\langle \nu^H, \xi^H \rangle = \eta(\langle \eta, \xi^H \rangle) = \eta^2\Big(\frac{1}{2}|\xi^H|^2\Big)$$

provided by (3.5), yields that

$$c \ge \frac{1}{|\xi_1^H|\langle \nu^H, \xi_1^H \rangle} > 0.$$

Step II. We now prove that

$$\varphi_h \equiv 0.$$

By contradiction, we shall assume the existence of $\xi_0 \in M \smallsetminus S_M$ such that $\varphi_h(\xi_0) \neq 0$. Since by definition we have

$$\varphi_h(\xi) = \frac{c}{3}|\xi^H|^3 - \langle \nu^H, \xi^H \rangle, \tag{3.6}$$

it is clear that φ_h vanishes on the vertical line L_{v}. Therefore, we know that $\xi_0 \in M \smallsetminus L_{\mathrm{v}}$. Let us consider the integral curve γ of η starting from ξ_0. Lemma 3.1 implies that φ_h is constant along γ, i.e.

$$\varphi_h(\gamma(s)) = \varphi_h(\xi_0) =: \varphi_0.$$

Since $S_M \subset L_{\mathrm{v}}$ and $\varphi_0 \neq 0$, γ remains in $M \smallsetminus L_{\mathrm{v}}$ and there is no problem in extending the curve indefinitely. We claim that this fact will contradict the boundedness of $M \supset \gamma$. Denote by $t(s)$, $r(s)$, and $\theta(s)$ the three smooth functions defined for $\xi \in \gamma$ respectively by

$$\begin{aligned} &t(s) = \gamma_3(s), \\ &r(s) = (\gamma_1^2(s) + \gamma_2^2(s))^{\frac{1}{2}} = |\xi^H|, \\ &\begin{cases} \cos(\theta(s)) = \Big\langle \nu^H, \dfrac{\xi^H}{|\xi^H|} \Big\rangle, \\ \sin(\theta(s)) = \Big\langle \eta, \dfrac{\xi^H}{|\xi^H|} \Big\rangle. \end{cases} \end{aligned}$$

From (3.6) we readily recognize

$$\frac{c}{3} r^3(s) - r(s)\cos(\theta(s)) = \varphi_0,$$

which implies that along the curve γ the positive function $r(s)$ is in fact a function of $\cos(\theta(s))$ (in the sense that it is uniquely determined by the value $\cos(\theta(s))$). As we will make use of this fact, we set the notation $R(\cos(\theta(s))) = r(s)$. From (3.4) we have that

$$t'(s) = -2r(s)\cos(\theta(s)) = 2\varphi_0 - \frac{2c}{3} r^3(s). \tag{3.7}$$

Thus, if $\varphi_0 < 0$, then $t'(s) \le 2\varphi_0 < 0$ and $t(s)$ would be forced to be unbounded providing an immediate contradiction. We can then assume $\varphi_0 > 0$. Since from (3.3) and (3.5) we have

$$\eta\Big(\arctan\Big(\frac{\langle \eta, \xi^H\rangle}{\langle \nu^H, \xi^H\rangle}\Big)\Big) = \frac{(1 - H_M\langle \nu^H, \xi^H\rangle)\langle \nu^H, \xi^H\rangle - H_M\langle \eta, \xi^H\rangle^2}{\langle \eta, \xi^H\rangle^2 + \langle \nu^H, \xi^H\rangle^2} = \frac{\langle \nu^H, \xi^H\rangle - H_M|\xi^H|^2}{|\xi^H|^2} = \frac{-(2c/3)|\xi^H|^3 - \varphi_h(\xi)}{|\xi^H|^2},$$

we obtain

$$\theta'(s) = \frac{-(2c/3)r^3(s) - \varphi_0}{r^2(s)}. \tag{3.8}$$

The assumption $\varphi_0 > 0$ (together with the boundedness of M) implies that $\theta'(s)$ stays below a strictly negative constant. Thus $\theta(s)$ is strictly decreasing and the angle formed by (the horizontal projections of) ν^H and ξ^H attains every value in $[0, 2\pi]$ infinitely many times along γ. We can then consider a strictly increasing sequence of values $\{s_k\}_{k\in\mathbb{N}}$ such that $\theta(s_k) - \theta(s_{k+1}) = 2\pi$ for all $k \in \mathbb{N}$. By exploiting (3.7) and (3.8), we notice that

$$\begin{aligned}
t(s_{k+1}) - t(s_k) &= \int_{s_k}^{s_{k+1}} t'(s)ds \\
&= -2\int_{s_k}^{s_{k+1}} r(s)\cos(\theta(s))ds = 2\int_{s_k}^{s_{k+1}} \frac{r^3(s)\cos(\theta(s))}{(2c/3)r^3(s) + \varphi_0}\theta'(s)ds \\
&= \frac{2}{c}\int_{s_k}^{s_{k+1}} \frac{(cr^3(s) - r(s)\cos(\theta(s)) + r(s)\cos(\theta(s)))\cos(\theta(s))}{(2c/3)r^3(s) + \varphi_0}\theta'(s)ds \\
&= \frac{2}{c}\int_{s_k}^{s_{k+1}} \cos(\theta(s))\theta'(s)ds + \frac{2}{c}\int_{s_k}^{s_{k+1}} \frac{r(s)\cos^2(\theta(s))}{(2c/3)r^3(s) + \varphi_0}\theta'(s)ds \\
&= \frac{2}{c}\int_{s_k}^{s_{k+1}} \frac{r(s)\cos^2(\theta(s))}{(2c/3)r^3(s) + \varphi_0}\theta'(s)ds \\
&= -\frac{2}{c}\int_{\theta(s_{k+1})}^{\theta(s_k)} \frac{R(\cos(\sigma))\cos^2(\sigma)}{(2c/3)R^3(\cos(\sigma)) + \varphi_0}d\sigma \\
&= -\frac{2}{c}\int_{\theta(s_1)-2\pi}^{\theta(s_1)} \frac{R(\cos(\sigma))\cos^2(\sigma)}{(2c/3)R^3(\cos(\sigma)) + \varphi_0}d\sigma
\end{aligned}$$

for every $k \in \mathbb{N}$. This implies, also in the case $\varphi_0 > 0$, the unboundedness of $t(s)$ since

$$t(s_{k+1}) = t(s_1) - k\frac{2}{c}\int_{\theta(s_1)-2\pi}^{\theta(s_1)} \frac{R(\cos(\sigma))\cos^2(\sigma)}{(2c/3)R^3(\cos(\sigma)) + \varphi_0}d\sigma \to -\infty \quad \text{as } k \to \infty.$$

Therefore, under both the assumptions $\varphi_0 < 0$ and $\varphi_0 > 0$, we have reached a contradiction. This completes the proof of the identity $\varphi_h \equiv 0$.

Step III. We finally show that

$$S_M \neq \emptyset \text{ and every integral curve } \gamma \text{ of } \eta \text{ starting from any } \xi_0 \in M \setminus S_M \text{ reaches } S_M.$$

Let us exploit the same notations as in step II. Arguing again by contradiction, we can assume that the curve γ can be extended indefinitely. We stress that $r(s)$ can vanish (at points in $L_{\mathrm{v}} \setminus S_M$), but only at isolated points on the curve since $\eta_{\bar{\xi}}$ belongs to $\operatorname{span}\{\partial_x, \partial_y\}$ at points $\bar{\xi} \in L_{\mathrm{v}} \setminus S_M$. Also, the functions $r(s)$ and $\theta(s)$ are smooth outside L_v. Since $S_M \subseteq M \cap L_v$, two situations might occur: either there exists s_0 such that $\inf_{s\in(s_0,\infty)} r(s) > 0$, or there exists a strictly increasing sequence of values $\{s_k\}_{k\in\mathbb{N}}$ such that, for every $k \in \mathbb{N}$, we have $r(s_k) = 0$ and $r(s) > 0$ for $s \in (s_k, s_{k+1})$. Since by step II and (3.7) we have

$$t'(s) = -\frac{2c}{3} r^3(s), \tag{3.9}$$

we deduce that the occurrence of the first case leads to an immediate contradiction because

$$t'(s) \le -\frac{2c}{3} (\inf_s r(s))^3 < 0 \quad \text{for } s > s_0,$$

and $t(s)$ would be unbounded. Hence, we can assume the existence of the sequence $\{s_k\}_{k\in\mathbb{N}}$ satisfying the above assumptions. Fix any $k \in \mathbb{N}$ and consider $s \in (s_k, s_{k+1})$. Using step II, (3.8), and step I, we have

$$\cos(\theta(s)) = \frac{c}{3} r^2(s) > 0 \quad \text{and} \quad \theta'(s) = -\frac{2c}{3} r(s) < 0. \tag{3.10}$$

This yields

$$\begin{cases} (\cos(\theta(s)), \sin(\theta(s))) \to (0, +1) & \text{as } s \to s_k^+ \\ (\cos(\theta(s)), \sin(\theta(s))) \to (0, -1) & \text{as } s \to s_{k+1}^-. \end{cases}$$

Hence we infer

$$t(s_{k+1}) - t(s_k) = \int_{s_k}^{s_{k+1}} t'(s)ds = -2 \int_{s_k}^{s_{k+1}} r(s)\cos(\theta(s))ds = \frac{3}{c} \int_{s_k}^{s_{k+1}} \theta'(s)\cos(\theta(s))ds = \frac{-6}{c}.$$

In other words, each time the curve γ re-joins the vertical line L_v, the t-component of the curve drops by a fixed amount. Since we are assuming that γ is reaching L_v an infinite number of times, this fact is in contradiction with the compactness of M. The proof is then complete. □

We are now ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We start by noticing that, under our assumptions, the set S_M (which is non-empty by Lemma 3.2) consists of isolated points. As a matter of fact, since $S_M \subset L_v$, if we had a sequence of points in S_M converging to $\bar{\xi} \in S_M$, then such sequence would be in L_v and at the point $\bar{\xi}$ the vector field T would be tangent. On the other hand, the tangent space at the characteristic points coincides with the horizontal distribution which is $\text{span}\{\partial_x, \partial_y\}$ on L_v. This argument ensures the fact that the characteristic points are isolated. Therefore, there exist $t_1 < t_2 < \cdots < t_p$ for some finite $p \in \mathbb{N}$ such that

$$S_M = \{(0, 0, t_1), \ldots, (0, 0, t_p)\}.$$

Consider now any point $\xi_0 \in M \cap \{t < t_2\}$ such that $\xi_0 \neq (0, 0, t_1)$, and consider the integral curve γ of η starting from ξ_0. Using the same notations as in Lemma 3.2, we know that $t(s)$ is decreasing (see (3.9)), so that $\gamma \subset M \cap \{t < t_2\}$. Exploiting Lemma 3.1 together with the identity $\varphi_h \equiv 0$ showed in Lemma 3.2, we have that the function

$$\varphi_v(\xi) = \frac{c}{3} t - \left\langle \eta, \frac{\xi^H}{|\xi^H|} \right\rangle$$

is constant along γ, i.e.

$$\frac{c}{3} t(s) - \sin(\theta(s)) = \varphi_v(\xi_0).$$

We stress that the previous identity holds true in $\gamma \setminus L_v$, and it can then be extended by continuity on the whole γ. By Lemma 3.2, we have that γ reaches S_M, and in particular as $\gamma(s) \to (0, 0, t_1)$, we have (see also (3.10) in this respect)

$$t(s) \to t_1 \quad \text{and} \quad \sin(\theta(s)) \to -1.$$

Hence,

$$\varphi_v(\xi_0) = \frac{c}{3} t_1 + 1.$$

By the arbitrariness of $\xi_0 \in (M \cap \{t < t_2\}) \smallsetminus \{(0,0,t_1)\}$, we have

$$\varphi_v(\xi) = \frac{c}{3}t_1 + 1 \quad \text{for all } \xi \in (M \cap \{t < t_2\}) \smallsetminus \{(0,0,t_1)\}.$$

The two identities $\varphi_h \equiv 0$ and $\varphi_v \equiv \frac{c}{3}t_1 + 1$ can be rewritten as

$$\Big\langle \nu^H, \frac{\xi^H}{|\xi^H|} \Big\rangle = \frac{c}{3}|\xi^H|^2 \quad \text{and} \quad \Big\langle \eta, \frac{\xi^H}{|\xi^H|} \Big\rangle = \frac{c}{3}\Big(t - t_1 - \frac{3}{c}\Big),$$

which implies

$$1 = \Big\langle \nu^H, \frac{\xi^H}{|\xi^H|} \Big\rangle^2 + \Big\langle \eta, \frac{\xi^H}{|\xi^H|} \Big\rangle^2 = \Big(\frac{c}{3}|\xi^H|^2\Big)^2 + \Big(\frac{c}{3}\Big(t - t_1 - \frac{3}{c}\Big)\Big)^2$$

for any $\xi \in (M \cap \{t < t_2\}) \smallsetminus \{(0,0,t_1)\}$. By the very definition of gauge sphere, this shows that

$$(M \cap \{t < t_2\}) \smallsetminus \{(0,0,t_1)\} \subset \partial B_R(0,0,t_0),$$

where

$$R^2 = \frac{3}{c} \quad \text{and} \quad t_0 = t_1 + \frac{3}{c}.$$

Since M is a smooth connected surface with no boundary and since the gauge sphere has only two characteristic points at $(0,0,t_0 - R^2) = (0,0,t_1)$ and $(0,0,t_0 + R^2) = (0,0,t_1 + \frac{6}{c})$, we can conclude that $M = \partial B_R(0,0,t_0)$, as desired. □

3.2 The case of $\mathbb{H}^n$, $n \geq 2$

Let us now turn the attention to the case $n \geq 2$ and to the proof of Theorem 1.2. By pushing further the parallelism with the classical Euclidean framework, we can say that the higher-dimensional analogue of the planar argument sketched in (3.1) is effective if one requires the hypersurface to be (locally) umbilical: it provides in fact a proof of the classical characterization of umbilical surfaces also known as Darboux theorem [14] (see [31] for an expository text, see also [18, 26] for different but related settings). In our Theorem 1.2, the main assumption is the umbilicality of M with respect to Definition 2.4. We warn the reader that in Definition 2.4 there is no information about the relationship between the two functions l and k, and therefore a characterization is possible only under a prescription of the curvature (see in this respect [8] for the case of constant σ_k-curvatures). Having this is mind, together with the fact that we are prescribing $H_M(\xi) = c|\xi^H|$, our aim is to provide a Darboux-type approach to Theorem 1.2. We define

$$\begin{cases} \varphi_h(\xi) = \dfrac{2n-1}{2n+1}H_M(\xi)|\xi^H|^2 - \langle \nu^H, \xi^H \rangle, & \varphi_h : M \smallsetminus S_M \to \mathbb{R}, \\ \varphi_v(\xi) = \dfrac{2n-1}{2n+1}H_M(\xi)\dfrac{t}{|\xi^H|} - \Big\langle \eta, \dfrac{\xi^H}{|\xi^H|} \Big\rangle, & \varphi_v : M \smallsetminus (S_M \cup L_v) \to \mathbb{R}, \end{cases} \tag{3.11}$$

where we have kept the notation

$$L_v = \{(0,0,t) \in \mathbb{H}^n : t \in \mathbb{R}\}$$

to denote the t-axis. With the following lemma, we realize that the constancy of the key function φ_h along η is tied to the vanishing of $l - 3k$ (in Example 2.6 we saw that for the gauge spheres $l = 3k$ by a direct computation).

Lemma 3.3. *Fix $n \geq 2$. Let M be a smooth hypersurface in $\mathbb{H}^n$ which is connected and orientable. Also, let ω be a relatively open set contained in $M \smallsetminus S_M$. Suppose there exists $c \in \mathbb{R}$ such that $H_M(\xi) = c|\xi^H|$ for $\xi \in \omega$. If M is umbilical, then we have*

$$\eta(\varphi_h) = \frac{2n-2}{2n+1}\langle \eta, \xi^H \rangle (3k - l) \quad \text{in } \omega.$$

Proof. The umbilicality condition in Definition 2.4 implies that

$$\langle \nabla_\eta \nu^H, \xi^H \rangle = l(\xi)\langle \eta, \xi^H \rangle. \tag{3.12}$$

For $\xi \in \omega$, we can then exploit the assumption $H_M(\xi) = c|\xi^H|$, together with (3.12) and (2.3), to deduce that

$$\begin{aligned}\eta(\varphi_h) &= \eta\Big(c\frac{2n-1}{2n+1}|\xi^H|^3 - \langle \nu^H, \xi^H\rangle\Big)\\ &= 3c\frac{2n-1}{2n+1}|\xi^H|^2\frac{\langle \eta, \xi^H\rangle}{|\xi^H|} - l(\xi)\langle \eta, \xi^H\rangle\\ &= \langle \eta, \xi^H\rangle\Big(\frac{3(2n-1)}{2n+1}H_M(\xi) - l(\xi)\Big).\end{aligned}$$

Keeping in mind Definition 2.4, we obtain

$$\eta(\varphi_h) = \langle \eta, \xi^H\rangle\Big(\frac{3(2n-2)}{2n+1}k(\xi) + \frac{2-2n}{2n+1}l(\xi)\Big) = \frac{2n-2}{2n+1}\langle \eta, \xi^H\rangle(3k(\xi) - l(\xi)),$$

as desired. □

We now show that in fact $\varphi_h \equiv 0$ and $l \equiv 3k$. There are two main tools in the proof: the use of the Codazzi equations found in [8], and the analysis of the global behavior of the auxiliary function $|\xi^H|^{2n-2}\varphi_h(\xi)$ (a weighted version of φ_h).

Lemma 3.4. *Fix $n \geq 2$. Let M be a smooth hypersurface in $\mathbb{H}^n$ which is connected, orientable, compact, and without boundary. Assume that M is umbilical, and suppose that there exists $c \neq 0$ such that $H_M(\xi) = c|\xi^H|$ for every point $\xi \in M \smallsetminus S_M$. Then, for all $\xi \in M \smallsetminus S_M$, we have*

$$\begin{cases}\langle \nu^H, \xi^H\rangle = |\xi^H|^2 k(\xi),\\ \langle \eta, \xi^H\rangle = 2|\xi^H|^2\dfrac{\langle \nu_\xi, T\rangle}{|\mathcal{P}_H(\nu_\xi)|},\\ \langle V_j, \xi^H\rangle = \langle W_j, \xi^H\rangle = 0 \quad \text{for } j \in \{1, \dots, n-1\}.\end{cases} \tag{3.13}$$

We also have that $c > 0$, $S_M = M \cap L_{\mathrm{v}} \neq \emptyset$, and

$$\varphi_h \equiv 0 \equiv l - 3k. \tag{3.14}$$

Proof. Let us divide the proof into multiple steps.

Step I. We first show the validity of (3.13). Let

$$\alpha = \frac{2\langle \nu, T\rangle}{|\mathcal{P}_H \nu|}.$$

By [8, Proposition 4.2], we know that

$$V_j(k) = W_j(k) = 0 = V_j(l) = W_j(l) \quad \text{for } j \in \{1, \dots, n-1\}, \tag{3.15}$$

and

$$\eta(k) = (l - 2k)\alpha, \quad \eta(\alpha) = k^2 - \alpha^2 - kl. \tag{3.16}$$

For $\xi \in M \smallsetminus (S_M \cup L_{\mathrm{v}})$, from (3.15) and the identity $(2n-1)H_M = (2n-2)k + l$, we obtain

$$\frac{\langle V_j, \xi^H\rangle}{|\xi^H|} = V_j(|\xi^H|) = \frac{1}{c}V_j(H_M(\xi)) = \frac{2n-2}{c(2n-1)}V_j(k) + \frac{1}{c(2n-1)}V_j(l) = 0$$

for $j \in \{1, \dots, n-1\}$. The same holds for $\langle W_j, \xi^H\rangle$. This shows that

$$\langle V_j, \xi^H\rangle = \langle W_j, \xi^H\rangle = 0 \quad \text{for } j \in \{1, \dots, n-1\} \text{ and } \xi \in M \smallsetminus S_M. \tag{3.17}$$

We then deduce that the function $|\xi^H|$ is constant even along the commutators of the vector fields in

$$\operatorname{span}\{V_1, W_1, \dots, V_{n-1}, W_{n-1}\}.$$

Exploiting Lemma 2.7, this implies

$$\langle \tau, \xi^H \rangle = \frac{k(\xi)|\mathcal{P}_H(\nu)|}{2} \langle \eta, \xi^H \rangle.$$

Recalling that $\tau = \langle \nu, T\rangle \nu^H - |\mathcal{P}_H(\nu)| T$ and using $\langle T, \xi^H \rangle = 0$, we infer

$$\alpha(\xi)\langle \nu^H, \xi^H \rangle = k(\xi)\langle \eta, \xi^H \rangle \quad \text{for } \xi \in M \setminus S_M. \tag{3.18}$$

From (2.3), (2.2), and the umbilicality condition in Definition 2.4, we can compute

$$\begin{aligned} \eta(\langle \eta, \xi^H \rangle) &= 1 + \langle \nabla_\eta \eta, \xi^H \rangle \\ &= 1 + \langle \nabla_\eta J\eta, J\xi^H \rangle \\ &= 1 + \langle \nabla_\eta \nu^H, J\xi^H \rangle \\ &= 1 + l(\xi)\langle \eta, J\xi^H \rangle \\ &= 1 - l(\xi)\langle \nu^H, \xi^H \rangle. \end{aligned} \tag{3.19}$$

If we now differentiate the identity (3.18) along η, relations (3.12), (3.16), and (3.19) yield

$$\begin{aligned} 0 &= \eta(\alpha)\langle \nu^H, \xi^H \rangle + \alpha\eta(\langle \nu^H, \xi^H \rangle) - \eta(k)\langle \eta, \xi^H \rangle - k\eta(\langle \eta, \xi^H \rangle) \\ &= (k^2 - \alpha^2 - kl)\langle \nu^H, \xi^H \rangle + \alpha l \langle \eta, \xi^H \rangle + (2k - l)\alpha\langle \eta, \xi^H \rangle + kl\langle \nu^H, \xi^H \rangle - k \\ &= (k^2 - \alpha^2)\langle \nu^H, \xi^H \rangle + 2k\alpha\langle \eta, \xi^H \rangle - k \\ &= (k^2 + \alpha^2)\langle \nu^H, \xi^H \rangle - k + 2\alpha(k\langle \eta, \xi^H \rangle - \alpha\langle \nu^H, \xi^H \rangle). \end{aligned}$$

Keeping in mind (3.18), this says that

$$\langle \nu^H, \xi^H \rangle = \frac{k(\xi)}{k^2(\xi) + \alpha^2(\xi)} \quad \text{and} \quad \langle \eta, \xi^H \rangle = \frac{\alpha(\xi)}{k^2(\xi) + \alpha^2(\xi)} \tag{3.20}$$

at least for any $\xi \in M \setminus S_M$ where $k(\xi) \neq 0$. Notice that in our assumptions we have $k^2 + \alpha^2 > 0$ in $M \setminus S_M$ (see [7, Theorem B (a)], and keep in mind that $\alpha \not\equiv 0$ due to the boundedness of M). Also notice that, if k vanishes at a point $\bar{\xi} \in M \setminus (S_M \cup L_\mathrm{v})$, then

$$\eta(k)(\bar{\xi}) = l(\bar{\xi})\alpha(\bar{\xi}) = (2n-1)c|\bar{\xi}^H|\alpha(\bar{\xi}) \neq 0.$$

Hence the relations (3.20) hold true by continuity throughout $M \setminus S_M$. This implies that

$$\begin{aligned} \frac{1}{k^2(\xi) + \alpha^2(\xi)} &= \langle \nu^H, \xi^H \rangle^2 + \langle \eta, \xi^H \rangle^2 \\ &= \langle \nu^H, \xi^H \rangle^2 + \langle \eta, \xi^H \rangle^2 + \sum_{j=1}^{n-1} \langle V_j, \xi^H \rangle^2 + \langle W_j, \xi^H \rangle^2 \\ &= |\xi^H|^2, \end{aligned}$$

where in the second equality we used (3.17). Inserting the last identity in (3.20), we get

$$\langle \nu^H, \xi^H \rangle = |\xi^H|^2 k(\xi) \quad \text{and} \quad \langle \eta, \xi^H \rangle = |\xi^H|^2 \alpha(\xi) \qquad \text{for } \xi \in M \setminus S_M. \tag{3.21}$$

The combination of (3.17) and (3.21) completes the proof of (3.13). In particular, since the function α^2 tends to ∞ only at characteristic points, from (3.21) we can also deduce that

$$S_M = M \cap L_\mathrm{v}. \tag{3.22}$$

As a matter of fact, the inclusion $M \cap L_\mathrm{v} \subseteq S_M$ follows from the fact that, at non-characteristic points,

$$|\xi^H|^{-2} = k^2(\xi) + \alpha^2(\xi)$$

is finite, whereas the inclusion $S_M \subseteq M \cap L_\mathrm{v}$ is a consequence of the boundedness of

$$\alpha^2(\xi) \leq k^2(\xi) + \alpha^2(\xi) = |\xi^H|^{-2}$$

outside of L_v.

Step II. We now show that

$$\xi \mapsto \phi(\xi) := |\xi^H|^{2n-2}\varphi_h(\xi) \quad \text{is constant throughout } M \smallsetminus S_M.$$

To this end, using (3.21), we can rewrite the function φ_h in the following way:

$$\begin{aligned}\varphi_h(\xi) &= \frac{2n-1}{2n+1}H_M(\xi)|\xi^H|^2 - k(\xi)|\xi^H|^2\\ &= |\xi^H|^2\Big(\frac{(2n-2)k(\xi)+l(\xi)}{2n+1} - k(\xi)\Big)\\ &= \frac{l(\xi)-3k(\xi)}{2n+1}|\xi^H|^2,\end{aligned} \tag{3.23}$$

so that

$$\phi(\xi) = \frac{l(\xi)-3k(\xi)}{2n+1}|\xi^H|^{2n}.$$

It is clear from (3.15) and (3.17) that

$$V_j(\phi) = W_j(\phi) = 0 \quad \text{for all } j \in \{1,\dots,n-1\},$$

which also implies by Lemma 2.7 that

$$\tau(\phi) - \frac{k|\mathcal{P}_H(\nu)|}{2}\eta(\phi) = 0.$$

On the other hand, by using (3.23) and Lemma 3.3, we obtain

$$\begin{aligned}\eta(\phi) &= \eta(|\xi^H|^{2n-2}\varphi_h)\\ &= (2n-2)|\xi^H|^{2n-4}\langle\eta,\xi^H\rangle\varphi_h + |\xi^H|^{2n-2}\eta(\varphi_h)\\ &= \frac{2n-2}{2n+1}|\xi^H|^{2n-2}\langle\eta,\xi^H\rangle(l-3k) + \frac{2n-2}{2n+1}|\xi^H|^{2n-2}\langle\eta,\xi^H\rangle(3k-l)\\ &= 0.\end{aligned}$$

This says that the function ϕ is constant along every tangent vector field in $M \smallsetminus S_M$ and concludes the proof of the current step.

Step III. In this step, we show that

$$c > 0.$$

We argue similarly to step I in the proof of Lemma 3.2. By the compactness of M, the function $\frac{1}{2}|\xi^H|^2$ attains its maximum at a point $\xi_1 \in M \smallsetminus L_\nu$, and we know from (3.22) that $\xi_1 \notin S_M$. Then, at $\xi = \xi_1$, we have

$$0 = \eta\Big(\frac{1}{2}|\xi^H|^2\Big) = \langle\eta,\xi_1^H\rangle \quad\text{and}\quad 0 = \tau\Big(\frac{1}{2}|\xi^H|^2\Big) = \langle\nu,T\rangle\langle\nu^H,\xi_1^H\rangle.$$

Since by (3.17) we have

$$\langle\nu^H,\xi_1^H\rangle^2 = \langle\eta,\xi_1^H\rangle^2 + \langle\nu^H,\xi_1^H\rangle^2 + \sum_{j=1}^{n}\langle V_j,\xi_1^H\rangle^2 + \langle W_j,\xi_1^H\rangle^2 = |\xi_1^H|^2 > 0,$$

we deduce that $\langle\nu,T\rangle = 0$, and therefore

$$k(\xi_1) = \frac{1}{|\xi_1^H|^2}\langle\nu^H,\xi_1^H\rangle = \frac{1}{|\xi_1^H|^2|\mathcal{P}_H\nu|}\langle\nu,\xi_1\rangle > 0,$$

where the positive sign is a consequence of the maximality condition and the fact that ν is the outward normal. Moreover, at the maximum point $\xi = \xi_1$, from (3.19) we obtain

$$1 - l(\xi_1)\langle\nu^H,\xi_1^H\rangle = \eta^2\Big(\frac{1}{2}|\xi^H|^2\Big) \le 0,$$

which says that

$$l(\xi_1) \ge \frac{1}{\langle\nu^H,\xi_1^H\rangle} > 0.$$

Therefore, keeping in mind Definition 2.4, we have

$$c = \frac{l(\xi_1)}{(2n-1)|\xi_1^H|} + \frac{(2n-2)k(\xi_1)}{(2n-1)|\xi_1^H|} > 0.$$

Step IV. We now show that

$$\phi(\xi) = |\xi^H|^{2n-2}\varphi_h(\xi) \equiv 0 \quad \text{for } \xi \in M \setminus S_M.$$

We already know from step II that ϕ is identically equal to a constant value ϕ_0. By contradiction, we shall assume that $\phi_0 \neq 0$. Since from the definition of φ_h in (3.11) it is clear that φ_h and ϕ tend to 0 as ξ approaches $M \cap L_{\mathrm{v}}$, and we know from (3.22) that $M \cap L_{\mathrm{v}} = S_M$, we have that $S_M = M \cap L_{\mathrm{v}} = \emptyset$ (so that there exist $0 < r_m \leq r_M < \infty$ satisfying $r_m \leq |\xi^H| \leq r_M$). If we consider the integral curve γ of η starting from any point $\xi_0 \in M$, we can then extend γ indefinitely. Arguing similarly to step II in the proof of Lemma 3.2 (from which we also borrow the analogous notations for the smooth functions $t(s)$, $r(s)$, and $\theta(s)$), we want to infer that the assumption $\phi_0 \neq 0$ leads to the unboundedness of $\gamma(s)$ (which contradicts the compactness of M). We can rewrite

$$\frac{c(2n-1)}{(2n+1)} r^{2n+1}(s) - r^{2n-1}(s)\cos(\theta(s)) = \phi_0,$$

which implies in particular that along the curve γ we have $r(s) = R(\cos(\theta(s)))$ (i.e. the positive function $r(s)$ is uniquely determined by the value $\cos(\theta(s))$). From (2.4) we infer that

$$t'(s) = -2r(s)\cos(\theta(s)) = 2\phi_0 r^{2-2n}(s) - \frac{2c(2n-1)}{(2n+1)} r^3(s). \tag{3.24}$$

Since $c > 0$ by step III and $r(s)$ is bounded, if $\phi_0 < 0$, then $t'(s) \leq 2\phi_0 r_M^{2-2n} < 0$ and $t(s)$ would be forced to be unbounded. We can then assume $\phi_0 > 0$. Exploiting (3.12), (3.17), and (3.19), we recognize that

$$\begin{aligned}
\eta\Big(\arctan\Big(\frac{\langle\eta, \xi^H\rangle}{\langle\nu^H, \xi^H\rangle}\Big)\Big) &= \frac{(1 - l(\xi)\langle\nu^H, \xi^H\rangle)\langle\nu^H, \xi^H\rangle - l(\xi)\langle\eta, \xi^H\rangle^2}{\langle\eta, \xi^H\rangle^2 + \langle\nu^H, \xi^H\rangle^2} \\
&= \frac{\langle\nu^H, \xi^H\rangle - l(\xi)|\xi^H|^2}{|\xi^H|^2} \\
&= \frac{\langle\nu^H, \xi^H\rangle - (2n+1)\varphi_h(\xi) - 3k(\xi)|\xi^H|^2}{|\xi^H|^2} \\
&= \frac{-2\langle\nu^H, \xi^H\rangle - (2n+1)\varphi_h(\xi)}{|\xi^H|^2},
\end{aligned}$$

where in the last two equalities we have made use of (3.23) and (3.21). The previous identity yields

$$\theta'(s) = \frac{2n-1}{r^2(s)}\big(r(s)\cos(\theta(s)) - cr^3(s)\big) = \frac{2n-1}{r^2(s)}\Big(-\phi_0 r^{2-2n}(s) - \frac{2c}{2n+1} r^3(s)\Big). \tag{3.25}$$

Since we know that $c > 0$ and M is bounded, the assumption $\phi_0 > 0$ implies that $\theta'(s)$ stays below a strictly negative constant. Thus $\theta(s)$ is strictly decreasing and $\theta(s) \to -\infty$ as $s \to \infty$. We can then pick a strictly increasing sequence of values $\{s_k\}_{k\in\mathbb{N}}$ such that $\theta(s_k) - \theta(s_{k+1}) = 2\pi$ for all $k \in \mathbb{N}$. From (3.24) and (3.25), we get

$$\begin{aligned}
t(s_{k+1}) - t(s_k) &= \int_{s_k}^{s_{k+1}} t'(s)ds \\
&= -2\int_{s_k}^{s_{k+1}} r(s)\cos(\theta(s))ds \\
&= \frac{2}{2n-1}\int_{s_k}^{s_{k+1}} \frac{r^3(s)\cos(\theta(s))}{\phi_0 r^{2-2n}(s) + \frac{2c}{2n+1} r^3(s)}\theta'(s)ds \\
&= \frac{2}{(2n-1)c}\int_{s_k}^{s_{k+1}} \frac{(cr^3(s) - r(s)\cos(\theta(s)) + r(s)\cos(\theta(s)))\cos(\theta(s))}{\phi_0 r^{2-2n}(s) + \frac{2c}{2n+1} r^3(s)}\theta'(s)ds \\
&= \frac{2}{(2n-1)c}\int_{s_k}^{s_{k+1}} \cos(\theta(s))\theta'(s)ds + \frac{2}{(2n-1)c}\int_{s_k}^{s_{k+1}} \frac{r(s)\cos^2(\theta(s))}{\phi_0 r^{2-2n}(s) + \frac{2c}{2n+1} r^3(s)}\theta'(s)ds
\end{aligned}$$

$$= -\frac{2}{(2n-1)c} \int_{\theta(s_{k+1})}^{\theta(s_k)} \frac{R(\cos(\sigma))\cos^2(\sigma)}{\phi_0 R^{2-2n}(\cos(\sigma)) + \frac{2c}{2n+1}R^3(\cos(\sigma))} d\sigma$$
$$= -\frac{2}{(2n-1)c} \int_{\theta(s_1)-2\pi}^{\theta(s_1)} \frac{R(\cos(\sigma))\cos^2(\sigma)}{\phi_0 R^{2-2n}(\cos(\sigma)) + \frac{2c}{2n+1}R^3(\cos(\sigma))} d\sigma$$

for every $k \in \mathbb{N}$. Since the term $t(s_{k+1}) - t(s_k)$ is strictly negative and independent of k, we conclude as in Lemma 3.2 that $t(s_k) \to -\infty$ as $k \to \infty$. Hence we have reached a contradiction in both scenarios $\phi_0 < 0$ and $\phi_0 > 0$. This ensures the validity of $\phi \equiv 0$.

Step V. In this final step, we finish the proof of the desired statements. Keeping the same notations as before, if we insert the information $\phi \equiv 0$ proved in step IV in (3.24), we infer

$$t'(s) = -\frac{2c(2n-1)}{(2n+1)} r^3(s) \leq 0. \tag{3.26}$$

If the sets S_M and $M \cap L_{\mathrm{v}}$ were empty, we could extend the integral curves γ of η indefinitely and we would have the contradicting property $t(s) \to -\infty$ as $s \to \infty$. Therefore, by (3.22), it has to be

$$S_M = M \cap L_{\mathrm{v}} \neq \emptyset. \tag{3.27}$$

Finally, exploiting again (3.22) and the identity $\phi \equiv 0$ in $M \setminus S_M$, we have

$$\varphi_h \equiv 0 \quad \text{in } M \setminus S_M,$$

and, by (3.23), also

$$l - 3k \equiv 0 \quad \text{in } M \setminus S_M.$$

This concludes the proof of (3.14), and of the lemma. □

We are finally ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. With Lemma 3.3 and Lemma 3.4 in hand, in order to complete the proof we can follow closely the arguments of Theorem 1.1. In fact, we deduce from (3.27) that there exists $p \in \mathbb{N}$ such that

$$S_M = \{(0,0,t_1), \dots, (0,0,t_p)\}$$

for some $t_1 < t_2 < \cdots < t_p$ (we stress that points in $S_M = M \cap L_{\mathrm{v}}$ cannot accumulate since T is aligned with the normal direction at characteristic points). If we consider any point $\xi_0 \in M \cap \{t < t_2\}$ such that $\xi_0 \neq (0,0,t_1)$, we can look at the integral curve γ of η starting from ξ_0. With the same notations as in Lemma 3.4, we know from (3.26) that $\gamma \subset M \cap \{t < t_2\}$ and

$$\gamma(s) \text{ reaches } (0,0,t_1).$$

As $r(s) > 0$ and it is reaching 0, we obtain from the identity $\varphi_h \equiv 0$ proved in Lemma (3.2) that

$$\cos(\theta(s)) = \frac{(2n-1)c}{2n+1} r^2(s) > 0 \quad \text{and} \quad \cos(\theta(s)) \to 0.$$

Recalling (3.25), this implies that $\sin(\theta(s))$ is decreasing and

$$\sin(\theta(s)) \to -1.$$

This yields

$$\frac{(2n-1)c}{2n+1} t(s) - \sin(\theta(s)) \to \frac{(2n-1)c}{2n+1} t_1 + 1 \quad \text{as } \gamma(s) \text{ approaches } (0,0,t_1). \tag{3.28}$$

On the other hand, if we differentiate along η the function

$$\varphi_{\mathrm{v}}(\xi) = \frac{2n-1}{2n+1} H_M(\xi) \frac{t}{|\xi^H|} - \frac{\langle \eta, \xi^H \rangle}{|\xi^H|} = \frac{(2n-1)c}{2n+1} t - \frac{\langle \eta, \xi^H \rangle}{|\xi^H|}$$

for $\xi \in M \setminus S_M$, by using (2.4) and (3.19), we obtain

$$\begin{aligned}\eta(\varphi_{\mathrm{v}})(\xi) &= -\frac{2(2n-1)c}{2n+1}\langle \nu^H, \xi^H\rangle - \frac{1 - l\langle \nu^H, \xi^H\rangle}{|\xi^H|} + \frac{\langle \eta, \xi^H\rangle^2}{|\xi^H|^3}\\ &= \frac{1}{|\xi^H|^3}\Big(-\frac{2(2n-1)H_M}{2n+1}|\xi^H|^2\langle \nu^H, \xi^H\rangle - |\xi^H|^2 + l|\xi^H|^2\langle \nu^H, \xi^H\rangle + \langle \eta, \xi^H\rangle^2\Big)\\ &= \frac{\langle \nu^H, \xi^H\rangle}{|\xi^H|^3}\Big(-\frac{2(2n-1)H_M}{2n+1}|\xi^H|^2 - \langle \nu^H, \xi^H\rangle + l|\xi^H|^2\Big) - \frac{\sum_{j=1}^{n-1}\langle V_j, \xi^H\rangle^2 + \langle W_j, \xi^H\rangle^2}{|\xi^H|^3}.\end{aligned}$$

We can now use the properties (3.13)–(3.14) established in Lemma 3.4 together with (2.12), and we deduce

$$\eta(\varphi_{\mathrm{v}})(\xi) = \frac{\langle \nu^H, \xi^H\rangle}{|\xi^H|^3}(-2k(\xi)|\xi^H|^2 - \langle \nu^H, \xi^H\rangle + 3k(\xi)|\xi^H|^2) = 0.$$

Hence φ_{v} is constant along γ. From (3.28) we know that such constant has to be equal to $\frac{(2n-1)c}{2n+1}t_1 + 1$. The arbitrariness of $\xi_0 \in (M \cap \{t < t_2\}) \setminus \{(0, 0, t_1)\}$ (which is the starting point of γ) yields

$$\varphi_{\mathrm{v}} \equiv \frac{(2n-1)c}{2n+1}t_1 + 1 \quad \text{in } (M \cap \{t < t_2\}) \setminus \{(0, 0, t_1)\}.$$

The previous identity and the identity $\varphi_h \equiv 0$ can be rewritten, keeping in mind the definitions in (3.11), as

$$\Big\langle \nu^H, \frac{\xi^H}{|\xi^H|}\Big\rangle = \frac{(2n-1)c}{2n+1}|\xi^H|^2 \quad \text{and} \quad \Big\langle \eta, \frac{\xi^H}{|\xi^H|}\Big\rangle = \frac{(2n-1)c}{2n+1}\Big(t - t_1 - \frac{2n+1}{(2n-1)c}\Big).$$

This implies

$$1 = \Big\langle \nu^H, \frac{\xi^H}{|\xi^H|}\Big\rangle^2 + \Big\langle \eta, \frac{\xi^H}{|\xi^H|}\Big\rangle^2 = \Big(\frac{(2n-1)c}{2n+1}|\xi^H|^2\Big)^2 + \Big(\frac{(2n-1)c}{2n+1}\Big(t - t_1 - \frac{2n+1}{(2n-1)c}\Big)\Big)^2$$

for any $\xi \in (M \cap \{t < t_2\}) \setminus \{(0, 0, t_1)\}$, which shows that

$$(M \cap \{t < t_2\}) \setminus \{(0, 0, t_1)\} \subset \partial B_R(0, 0, t_0)$$

with

$$R^2 = \frac{2n+1}{(2n-1)c} \quad \text{and} \quad t_0 = t_1 + \frac{2n+1}{(2n-1)c}.$$

This allows, as in Theorem 1.1, to conclude the proof of the desired statement. □

3.3 The axially symmetric case

As a concrete application of our main theorems, we want to single out a relevant class of hypersurfaces in which we have a uniqueness result for the gauge spheres. We already mentioned in Section 1 (see also [21], [25], [38], [16], [27], and [19] for related settings) that it is quite typical to require some a priori symmetry in terms of rotational invariances. More precisely, we can recall the following well-known class of symmetric domains (which is consistent with the type of prescription of the horizontal curvature H_M we are dealing with).

Definition 3.5. For $n \geq 1$, we say that a smooth hypersurface $M \subset \mathbb{H}^n$ is cylindrically symmetric if, locally around any point of M, there exists a defining function f for M which can be written as

$$f(x, y, t) = w(|x|^2 + |y|^2, t) \tag{3.29}$$

for some smooth function w.

We have the following corollary.

Corollary 3.6. *Fix $n \geq 1$. Let M be a smooth hypersurface of $\mathbb{H}^n$ which is connected, orientable, compact, and without boundary. Suppose that M is cylindrically symmetric with respect to Definition 3.5 and that, at every* *non-characteristic point $(x, y, t) \in M$, the horizontal mean curvature of M is proportional to*

$$\sqrt{|x|^2 + |y|^2}$$

up to a constant factor $c \neq 0$. Then $c > 0$ and there exists $t_0 \in \mathbb{R}$ such that $M = \partial B_R(\xi_0)$ with

$$R = \sqrt{\frac{1}{c}\frac{2n+1}{2n-1}} \quad \text{and} \quad \xi_0 = (0, 0, t_0).$$

Proof. Fix an open neighborhood $U \subset \mathbb{H}^n$ where $U \cap M$ is described, as in (3.29), by the zero-level set of a smooth function f with non-null gradient. Pick the sign of f such that the outward normal ν at $\xi \in U \cap M$ is equal to

$$\nu = \frac{TfT + \sum_{j=1}^n X_jfX_j + Y_jfY_j}{((Tf)^2 + \sum_{j=1}^n (X_jf)^2 + (Y_jf)^2)^{1/2}}.$$

By exploiting (3.29), we have

$$Tf = w_2, \quad X_jf = 2x_jw_1 - 2y_jw_2, \quad Y_jf = 2y_jw_1 + 2x_iw_2,$$

which implies that

$$|\mathcal{P}_H\nu|^2 = \frac{4(|x|^2 + |y|^2)(w_1^2 + w_2^2)}{4(|x|^2 + |y|^2)(w_1^2 + w_2^2) + w_2^2}.$$

This is saying that for cylindrically symmetric hypersurfaces we can have characteristic points only when $|x|^2 + |y|^2 = 0$, i.e.

$$S_M \subseteq M \cap L_v.$$

Therefore, in case $n = 1$ we can apply Theorem 1.1 to infer the desired statement.

Fix then $n \geq 2$. We want to check that the cylindrically symmetric assumption implies that M is in fact umbilical. Since we have

$$\nu^H = \sum_{j=1}^n \frac{x_jw_1 - y_jw_2}{\sqrt{|x|^2 + |y|^2}\sqrt{w_1^2 + w_2^2}}X_j + \frac{y_jw_1 + x_jw_2}{\sqrt{|x|^2 + |y|^2}\sqrt{w_1^2 + w_2^2}}Y_j$$

and

$$\eta = \sum_{j=1}^n \frac{y_jw_1 + x_jw_2}{\sqrt{|x|^2 + |y|^2}\sqrt{w_1^2 + w_2^2}}X_j + \frac{y_jw_2 - x_jw_1}{\sqrt{|x|^2 + |y|^2}\sqrt{w_1^2 + w_2^2}}Y_j,$$

for any $j, k \in \{1, \dots, n\}$ we can directly compute

$$\begin{aligned}
\langle \nabla_{X_k}\nu^H, X_j\rangle = X_k(\langle \nu^H, X_j\rangle) &= \frac{\delta_{jk}w_1 + 2x_j(x_kw_{11} - y_kv_{12}) - 2y_j(x_kw_{12} - y_kw_{22})}{\sqrt{|x|^2 + |y|^2}\sqrt{w_1^2 + w_2^2}} \\
&\quad - \langle \nu^H, X_j\rangle\Big(\frac{x_k}{|x|^2 + |y|^2} + \frac{2w_1(x_kw_{11} - y_kw_{12}) + 2w_2(x_kw_{12} - y_kw_{22})}{w_1^2 + w_2^2}\Big), \\
\langle \nabla_{X_k}\nu^H, Y_j\rangle = X_k(\langle \nu^H, Y_j\rangle) &= \frac{\delta_{jk}w_2 + 2x_j(x_kw_{12} - y_kv_{22}) + 2y_j(x_kw_{11} - y_kw_{12})}{\sqrt{|x|^2 + |y|^2}\sqrt{w_1^2 + w_2^2}} \\
&\quad - \langle \nu^H, Y_j\rangle\Big(\frac{x_k}{|x|^2 + |y|^2} + \frac{2w_1(x_kw_{11} - y_kw_{12}) + 2w_2(x_kw_{12} - y_kw_{22})}{w_1^2 + w_2^2}\Big), \\
\langle \nabla_{Y_k}\nu^H, X_j\rangle = Y_k(\langle \nu^H, X_j\rangle) &= \frac{-\delta_{jk}w_2 + 2x_j(y_kw_{11} + x_kv_{12}) - 2y_j(y_kw_{12} + x_kw_{22})}{\sqrt{|x|^2 + |y|^2}\sqrt{w_1^2 + w_2^2}} \\
&\quad - \langle \nu^H, X_j\rangle\Big(\frac{y_k}{|x|^2 + |y|^2} + \frac{2w_1(y_kw_{11} + x_kw_{12}) + 2w_2(y_kw_{12} + x_kw_{22})}{w_1^2 + w_2^2}\Big),
\end{aligned}$$

$$\langle \nabla_{Y_k} \nu^H, Y_j \rangle = Y_k(\langle \nu^H, Y_j \rangle) = \frac{\delta_{jk} w_1 + 2x_j(y_k w_{12} + x_k v_{22}) + 2y_j(y_k w_{11} + x_k w_{12})}{\sqrt{|x|^2 + |y|^2}\sqrt{w_1^2 + w_2^2}} - \langle \nu^H, Y_j \rangle \Big(\frac{y_k}{|x|^2 + |y|^2} + \frac{2w_1(y_k w_{11} + x_k w_{12}) + 2w_2(y_k w_{12} + x_k w_{22})}{w_1^2 + w_2^2} \Big).$$

From the previous relations, we can recognize by a straightforward computation that

$$\mathcal{P}_H(\nabla_Z \nu^H) - \frac{2\langle \nu, T \rangle}{|\mathcal{P}_H(\nu)|} J(Z) = kZ \quad \text{for any } Z \in \mathcal{H} \text{ such that } \langle Z, \nu^H \rangle = \langle Z, \eta \rangle = 0,$$

and

$$\mathcal{P}_H(\nabla_\eta \nu^H) = l\eta,$$

where

$$k = \frac{w_1}{\sqrt{|x|^2 + |y|^2}\sqrt{w_1^2 + w_2^2}}$$

and

$$l = \frac{w_1}{\sqrt{|x|^2 + |y|^2}\sqrt{w_1^2 + w_2^2}} + \frac{2\sqrt{|x|^2 + |y|^2}}{(w_1^2 + w_2^2)^{3/2}} (w_{11} w_2^2 + w_{22} w_1^2 - 2w_{12} w_1 w_2).$$

A direct comparison with Definition 2.3 and Definition 2.4 tells us that M is umbilical. We can then apply Theorem 1.2 and complete the proof of the corollary. □

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