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Pseudo-moment generating functions: Application to pseudo-Schur constant random vectors

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ABSTRACT

In this note we show that pseudo-analysis tools can be effective in obtaining results in a distorted probability framework. More precisely, we introduce the notion of pseudo-independence and that of pseudo-moment generating function, the latter representing a generalization of the pseudo-Laplace transform, and both aiming at extending the corresponding notions in the usual probabilistic context. We show that these concepts and their properties, and more in general pseudo-analysis, are particularly useful to provide characterization results for a class of bivariate random vectors that we call "pseudo-Schur constant" family which represents an extension of the Schur-constant class.

1. Introduction

Pseudo-analysis is based on the structure of general semirings where a pseudo-sum $\oplus$ and a pseudo-product $\otimes$ substitute and generalize the usual sum and multiplication. In the particular case of a non-idempotent semiring, it has been proved that this is identified by a continuous and invertible function through which both operations can be expressed (see, among the wide literature, [1,28], and [29]). The construction of the pseudo-integral (that is the base of pseudo-analysis) is based on measures with values in a given semiring where the standard σ-additivity is substituted by the σ-$\oplus$-additivity: they are called pseudo-additive measures (see [29]). In this paper, we will focus on the case of a non-idempotent semiring (which corresponds, in our framework, to considering pseudo-additive measures given by distorted probabilities that are called pseudo-probabilities) and for the corresponding pseudo-integral and pseudo-calculus we refer to [28] and [29].

In this framework, we introduce a definition of pseudo-moment generating function that is in line with the pseudo-Laplace transform given in [21] and [25], but considering a more general class of pseudo-exponential functions. Since the measures we are dealing with assume values in a semiring, we introduce the concept of pseudo-independence (in line with the classical concept of independence) and we show that the classical results concerning moment generating functions of a vector of independent random variables and of their sum (that decompose in the product of the moment-generating functions of the single random variables) extend to pseudo-moment generating functions if the random variables involved are pseudo-independent.

Since the semi-copula associated to two pseudo-independent random variables has the same shape as the Archimedean copula expressed in multiplicative form, we investigate the possibility to apply the pseudo-analysis to a classical probabilistic setting with random variables linked by an Archimedean copula. We find results concerning a new class of vectors that can be seen as a generalization of the Schur-constant class.

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A non-negative vector (X, Y) is Schur-constant if its survival function is given by

$$\bar{F}_{X,Y}(x, y) = S(x + y), \quad x, y > 0 \tag{1}$$

where S is a convex survival function (see, for instance, [4,7,17]). A continuous Schur-constant random vector can be specified by

$$(X, Y) \overset{d}{=} (UV, (1 - U), V)$$

where $V \overset{d}{=} X + Y$ and U is an uniformly distributed random variable independent of V (see for instance [6]). Moreover, Kozlova and Salminen [14] and, after, Ta and Van [26], provide a characterization of a bivariate Schur-constant random vector showing that its joint moment-generating function can be written in terms of the integral of the moment generating function of a random variable with the same distribution as $X + Y$.

In this paper, we consider random vectors with survival function of type (1) where the standard sum is replaced by a non-commutative pseudo-sum binary operator: we call them "pseudo-Schur constant" random vectors. We show that this class coincides with that of bivariate random vectors with survival Archimedean dependence and marginal survival distribution functions that are strictly decreasing when positive. Moreover, we provide some characterizations in terms of pseudo-operations, pseudo-integrals and pseudo-moment generating functions, in a suitable pseudo-probability setting. In particular, we extend the above mentioned characterizations of Schur-constant random vectors showing that pseudo-analysis and its applications represent a powerful tool to study some properties on the underlying probability space, based on the fact that there is a one-to-one correspondence between probabilities and pseudo-probabilities. More specifically, we show that they result to be particularly suited and useful in the analysis of dependence structures of Archimedean type.

This paper is organized as follows. In section 2, we recall the main concepts and results concerning semirings, pseudo-calculus, pseudo-derivatives, pseudo-additive measures and pseudo-integrals that will be used in the sequel. In section 3, we introduce the notion of pseudo-independence and we analyze pseudo-independent random variables. In section 4, after giving the definition of pseudo-exponential functions, we introduce the pseudo-moment generating function and we state its main properties, in particular in the case of pseudo-independent random variables. In section 5, we introduce the notion of pseudo-Schur constant random vectors characterizing them using the concepts of pseudo-calculus and pseudo-moment generating function. Section 6 concludes.

2. Preliminaries

In this section we will introduce the main concepts and results concerning semirings, pseudo-calculus, fuzzy measures and pseudo-integrals, that will be used in the sequel.

2.1. Semirings

Semirings have been widely studied and applied to many fields (see for example [11]). In this section, we will introduce a specific type of semiring based on a generator function, in line with the wide literature on this topic (see [1,28,29], and the references therein).

Throughout the paper, we will denote by $[a, b]$ a closed subinterval of $[-\infty, +\infty]$ and by $\bar{\mathbb{R}}_+$ the extended interval $[0, +\infty]$. We will consider the following class of functions:

$$\mathcal{G}_{a,b} = \left\{ g : [a, b] \to \bar{\mathbb{R}}_+ : g \textit{ is strictly monotone, continuous and surjective} \right\}.$$

The elements of $\mathcal{G}_{a,b}$ will be called "generators".

We consider generalizations of the standard sum and product in $[a, b]$. The pseudo-addition, denoted by $\oplus : [a, b] \times [a, b] \to [a, b]$, is defined as

$$x \oplus y = g^{-1}(g(x) + g(y)) \tag{2}$$

and the pseudo-multiplication, denoted by $\otimes : [a, b] \times [a, b] \to [a, b]$ is defined as

$$x \otimes y = g^{-1}(g(x)\, g(y)). \tag{3}$$

$([a, b], \oplus, \otimes)$ is a non-idempotent semiring. In fact, both operations satisfy the associativity and commutativity properties and the pseudo-multiplication is distributive with respect to the pseudo-addition. Moreover, if g is increasing, $g(a) = 0$ and $g(b) = +\infty$ and $\mathbf{0} = a$ is the neutral element of the pseudo-addition; if g is decreasing, $g(b) = 0$ and $g(a) = +\infty$ and $\mathbf{0} = b$ is the neutral element of the pseudo-addition. We have $\mathbf{0} \otimes_g x = \mathbf{0}$ and the unit element for the pseudo-multiplication is $\mathbf{1} = g^{-1}(1)$. The $\oplus$ operation induces a full order on $[a, b]$ through the generator g, that is, $x \preceq y$ if and only if $g(x) \leq g(y)$.

In the sequel, we will use the notation $\oplus_g$ and $\otimes_g$ in order to specify the dependence on g. Clearly, when $h = id$, we recover the standard sum and the standard product on $\bar{\mathbb{R}}_+$.

Starting from the pseudo-sum defined in (2), the pseudo-difference $\ominus_g$ is given by

$$x \ominus_g y = g^{-1}\left(g(x) - g(y)\right), \text{ for } x, y \in [a, b],\ x \succeq y.$$

Similarly, for pseudo-division $\oslash$, we have:

$$x \oslash_g y = g^{-1}(g(x) : g(y)), \text{ for } x, y \in [a,b], \; y \neq \mathbf{0}.$$

By the same way, starting from the pseudo-product in (3), the pseudo-power of a number $x \in [a,b]$ is given by

$$x^{\otimes_g \alpha} = g^{-1}([g(x)]^\alpha),$$

with $\alpha > 0$ and with $\alpha \in \mathbb{R}$ when $x \neq \mathbf{0}$ (see [22], and [12]).

In the sequel we will consider also a non-commutative extension of the pseudo-sum (2) with arguments and output in different domains. Given $(k, k_1, k_2) \in \mathcal{G}_{c_0,d_0} \times \mathcal{G}_{c_1,d_1} \times \mathcal{G}_{c_2,d_2}$, we define the non-commutative sum operator (see for instance [23], for a case of non commutative pseudo-operations) as

$$x \;{}_{k_1}\overset{k}{\oplus}_{k_2}\; y = k^{-1}(k_1(x) + k_2(y)), \quad x \in [c_1, d_1], \; y \in [c_2, d_2]. \tag{4}$$

Remark 2.1. Clearly, the triplet (k, k_1, k_2) identifies a sum operator, up to a multiplicative constant: that is, for all $c > 0$, if $\hat{k}(x) = c\,k(x)$, $\hat{k}_1(x) = c\,k_1(x)$, $\hat{k}_2(x) = c\,k_2(x)$, then, (k, k_1, k_2) and $(\hat{k}, \hat{k}_1, \hat{k}_2)$ define the same addition operator.

Similarly as for the commutative case, we can define the generalized pseudo-difference as

$$x \;{}_{k_1}\overset{k}{\ominus}_{k_2}\; y = k^{-1}(k_1(x) - k_2(y)), \; x \in [c_1, d_1], y \in [c_2, d_2], k_1(x) \geq k_2(y). \tag{5}$$

2.2. Pseudo-derivatives

An important tool in pseudo-calculus is the pseudo-derivative of a function (see [19], and [24], among the others).

Definition 2.1. Let $g \in \mathcal{G}_{a,b}$ be a differentiable generator on (a,b) that generates the semiring $([a,b], \oplus_g, \otimes_g)$. Let $f : [c,d] \to [a,b]$ be a function differentiable on (c,d) and with the same monotonicity as g. Then we define the pseudo-derivative of f at $x \in (c,d)$ by:

$$\frac{{}^{\oplus_g}d}{dx} f(x) = g^{-1}\left(\frac{d}{dx}[g \circ f(x)]\right).$$

By iteratively applying the above definition, if f and g are n times differentiable with $\frac{d^n}{dt^n}[g \circ f](t)$ non-negative, we get an expression for the n-th pseudo-derivative of f, that is

$$\frac{{}^{\oplus_g}d^n}{dx^n} f(x) = g^{-1}\left(\frac{d^n}{dx^n}[g \circ f(x)]\right).$$

2.3. Pseudo-additive measures

In this section we recall the notion of fuzzy measure (see, among the wide literature, the seminal paper of Choquet [8], and Wang and Klir [27], for a general presentation).

Definition 2.2. Let $(\Omega, \mathcal{F})$ be a measurable space. A set function $m : \mathcal{F} \to [0, +\infty]$ is called a fuzzy measure if it satisfies the following properties:

1. $m(\emptyset) = 0$,
2. $m(A) \leq m(B)$, whenever $A \subseteq B$, $A, B \in \mathcal{F}$,
3. for every monotone sequence $\{A_i\}_{i=1,2,\dots} \subset \mathcal{F}$, $\lim_{i\to\infty} m(A_i) = m\left(\lim_{i\to\infty} A_i\right)$.

If $m(\Omega) = 1$, m is called "regular".

In what follows, given $g \in \mathcal{G}_{a,b}$, we will focus on measures over the semiring $([a,b], \oplus_g, \otimes_g)$ that satisfy the σ-$\oplus_g$-additive property (see, among the others, [20,29]).

Definition 2.3. Let $(\Omega, \mathcal{F})$ be a measurable space. $m : \mathcal{F} \to [a,b]$ is called a pseudo-additive measure if

1. $m(\emptyset) = \mathbf{0}$,
2. it is σ-$\oplus_g$-additive, that is,

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigoplus_{g\; n=1}^{+\infty} m(A_n)$$

for any sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$.

In order to specify the $\sigma - \oplus_g$-additivity of m, we will use the notation m_g. By Definition 2.3, the set function $g \circ m_g : \mathcal{F} \to [0, +\infty]$ is an additive measure on the space $(\Omega, \mathcal{F})$ and a $\sigma - \oplus_g$-additive fuzzy measure is a distorted measure. Moreover m_g is monotone with respect to the order induced by $\oplus_g$, that is $A \subset B$ implies that $m_g(A) \preceq_g m_g(B)$.

In particular, if $m_g(\Omega) = \mathbf{1}$, $g \circ m_g$ is a probability on $(\Omega, \mathcal{F})$: in this case, we will call m_g "pseudo-probability" (see [16]).

In the sequel, we will often consider, given $b \in (1, +\infty]$, the class of generators

$$\mathcal{M}_{0,b} = \left\{ g \in \mathcal{G}_{0,b} : g \text{ is strictly increasing with } g(1) = 1 \right\}.$$

Remark 2.2. If, for $b \in (1, +\infty]$, $g \in \mathcal{M}_{0,b}$, then, according to Definition 2.2, the pseudo-probability m_g is a regular fuzzy measure satisfying the $\sigma - \oplus_g$-additivity property.

2.4. Pseudo-integrals

The pseudo-integral, with respect to a σ-$\oplus_g$-additive fuzzy measure m_g, is defined assuming the general semiring structure defined above (for more details on the construction see, among the others, [21], and [29], for the general case).

Since we are focusing on non-idempotent semirings, the pseudo-integral of a measurable function X on $(\Omega, \mathcal{F})$ taking values in $[a, b]$ is given by

$$\int_{\Omega}^{\oplus_g} X \otimes_g dm_g = g^{-1}\left(\int_{\Omega} g \circ X \, d(g \circ m_g) \right) \tag{6}$$

where the inner integral is the Lebesgue integral with respect to the measure $g \circ m_g$.

When $\Omega = [x_0, x_1]$, $\mathcal{F} = \mathcal{B}\left([x_0, x_1]\right)$, i.e. the Borel σ-algebra on $[x_0, x_1]$, and $g \circ m_g = \mathcal{L}$, where $\mathcal{L}$ is the Lebesgue measure, then (6) reduces to the so called 'g-integral'.

In the case of a pseudo-probability m_g, the pseudo-expectation of X is defined by

$$\tilde{E}^{m_g}[X] = \int_{\Omega}^{\oplus_g} X \otimes_g dm_g = g^{-1}\left(E^{\mathbb{P}}\left[g(X) \right] \right), \tag{7}$$

where $E^{\mathbb{P}}$ denotes the standard expectation with respect to the probability $\mathbb{P} = g \circ m_g$ (see [2], and [3]).

Many concepts and results of standard analysis have been generalized to the g-semiring framework: among the others, the notion of L^p space has been extended in the following way. If X is a measurable function on the measurable space $\left(\Omega, \mathcal{F}, m_g\right)$, where m_g is a pseudo-additive measure, we say that X belongs to $L^p_{\oplus_g}$ (called "pseudo-L^p space") with $p > 0$ if

$$\left(\int_{\Omega}^{\oplus_g} X^{\otimes_g p} \otimes_g dm_g \right)^{\otimes_g \frac{1}{p}}$$

is finite in the sense of the underlying semiring, that is if $\int_{\Omega}^{\oplus_g} X^{\otimes_g p} \otimes_g dm_g < +\infty$ (see [22]).

3. Random variables and pseudo-independence

In this section we introduce the notion of pseudo-independence as an extension of the classical notion of independence (the latter being recovered by considering the case $g = id$ and $[a, b] = \bar{\mathbb{R}}_+$).

3.1. Pseudo-cumulative distribution functions

Given $(\Omega, \mathcal{F}, m)$, where m is a regular fuzzy measure, we can associate to any random variable the function $F_X(x) = m(X \le x)$, for all $x \in \mathbb{R}$. The function F_X satisfies all the properties of a standard cumulative distribution function but it doesn't identify the regular fuzzy measure induced by X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. If instead, we consider a regular $\sigma - \oplus_g$-additive fuzzy measure m_g, where $g \in \mathcal{M}_{0,b}$, with $b \in (1, +\infty]$ (see Remark 2.2), then the $\sigma - \oplus_g$-additive fuzzy measure $\mu_g^{F_X}$ induced by F_X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is defined through

$$\mu_g^{F_X}((c, d]) = F_X(d) \ominus_g F_X(c)$$

on intervals of type $(c, d]$ and then extended to $\mathcal{B}(\mathbb{R})$ taking into account that

$$g \circ \mu_g^{F_X}((c, d]) = g \circ F_X(d) - g \circ F_X(c)$$

and that $g \circ \mu_g^{F_X}$ extends to a unique probability on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Similarly, considering the more general situation of a space $\left(\Omega, \mathcal{F}, m_g\right)$ where, for $g \in \mathcal{G}_{a,b}$, m_g is a pseudo-probability and X is a random variable, we can consider the associated pseudo-cumulative distribution function defined by $F_X(x) = m_g(X \le x)$ for all $x \in \mathbb{R}$.

Notice that the cumulative distribution function of X with respect to the probability $\mathbb{P} = g \circ m_g$ is $F_X^{\mathbb{P}}(x) = g \circ F_X(x)$. Hence we can again construct the pseudo-probability $\mu_g^{F_X}$ induced by F_X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ following the same procedure described above.

Exactly as for the univariate case, given the random vector (X, Y), if $F_{X,Y}(x, y) = m_g(X \le x, Y \le y)$ for $x, y \in \mathbb{R}$ is the joint pseudo-additive cumulative distribution function with respect to the pseudo-probability m_g, then the joint cumulative distribution function of (X, Y) with respect to the probability $\mathbb{P} = g \circ m_g$ is $F_{X,Y}^{\mathbb{P}}(x, y) = g \circ F_{X,Y}(x, y)$. If $g \in \mathcal{M}_{0,b}$ with $b \in (1, +\infty]$, m_g is a regular $\sigma - \oplus_g$-additive measure according to Remark 2.2, and, as for the additive case, the joint fuzzy pseudo-additive cumulative distribution function can be written in terms of the fuzzy marginal ones and of a semi-copula that links them (see [9], and [10], for the notion of semi-copula and related results). If $S(u, v)$ is the semi-copula associated to $F_{X,Y}$ and $C(u, v)$ the copula associated to $F_{X,Y}^{\mathbb{P}}$, then they are linked by the relation

$$S(u, v) = g^{-1}\left(C(g(u), g(v)\right) \tag{8}$$

and, if g is concave, then S is indeed a copula function (see Proposition 2.8 in [10], and related comments).

3.2. Pseudo-independence

The definition of pseudo-independence is obtained substituting the standard product by the pseudo-one in the usual definition of independence.

Definition 3.1. Let $(\Omega, \mathcal{F}, m)$ be a measurable space provided with a regular fuzzy measure m. Given $b \in (1, +\infty]$ and $g \in \mathcal{M}_{0,b}$, we say that $A, B \in \mathcal{F}$ are g-independent if and only if

$$m(A \cap B) = m(A) \otimes_g m(B). \tag{9}$$

Clearly, dealing with regular fuzzy measures, two different generators that coincide on the interval $[0, 1]$ define the same relation in (9).

According to Definition 3.1, it is natural to introduce the notion of pseudo-independence between random variables. Let X and Y be two real valued random variables defined on $(\Omega, \mathcal{F}, m)$, where m is a regular fuzzy measure: given $b \in (1, +\infty]$ and $g \in \mathcal{M}_{0,b}$, they are said to be g-independent if and only if

$$m\left(X \in B_1, Y \in B_2\right) = m\left(X \in B_1\right) \otimes_g m\left(Y \in B_2\right)$$

for all $B_1, B_2 \in \mathcal{B}(\mathbb{R})$.

The notion can be extended to more than two events and to more than two random variables exactly as for classical independence.

If we consider a pseudo-probability m_g with $g \in \mathcal{G}_{a,b}$, we can similarly say that X and Y are g-independent if and only if

$$m_g\left(X \in B_1, Y \in B_2\right) = m_g\left(X \in B_1\right) \otimes_g m_g\left(Y \in B_2\right) \tag{10}$$

for all $B_1, B_2 \in \mathcal{B}(\mathbb{R})$. But equation (10) can be rewritten as

$$g \circ m_g\left(X \in B_1, Y \in B_2\right) = g \circ m_g\left(X \in B_1\right) \cdot g \circ m_g\left(Y \in B_2\right)$$

for all $B_1, B_2 \in \mathcal{B}(\mathbb{R})$, so (10) holds true if and only if X and Y are independent with respect to the probability $\mathbb{P} = g \circ m_g$. As a consequence, (10) is equivalent to

$$m_g(X \le x, Y \le y) = m_g(X \le x) \otimes_g m_g(Y \le y) \tag{11}$$

for all $x, y \in \mathbb{R}$. In terms of pseudo-cumulative distribution functions, (11) can be rewritten as

$$F_{X,Y}(x, y) = F_X(x) \otimes_g F_Y(y). \tag{12}$$

If, in particular, given $b \in (1, +\infty]$, $g \in \mathcal{M}_{0,b}$, then $S(u, v) = u \otimes_g v$ is the associated semi-copula.

In this case, clearly, $F_{X,Y}$ in (12) is a joint cumulative distribution function if and only if $u \otimes_g v$ for $u, v \in [0, 1]$ is a copula. It is well-known that an Archimedean copula can be written in multiplicative and in additive form as

$$u \otimes_g v = \phi\left(\phi^{-1}(u) + \phi^{-1}(v)\right) \tag{13}$$

where $\phi(t) = g^{-1}\left(e^{-t}\right)$ and that (13) is indeed a copula if and only if $\phi(0) = 1$, $\lim_{t \to +\infty} \phi(t) = 0$ and ϕ is continuous, non-increasing and convex on $[0, +\infty)$ (see for instance Theorem 2 in [15], or [18]). Since $g \in \mathcal{M}_{0,b}$, we have that ϕ satisfies the required properties if and only if additionally $g^{-1}\left(e^{-t}\right)$ is convex.

4. Pseudo-moment generating functions

In this section we will generalize the notion of moment generating function (both at the univariate as well as at the bivariate level) considering the more general setup introduced in Section 2. Given the extreme generality and flexibility of the framework that assumes a general semiring and a pseudo-probability, we start introducing a general notion of exponential function based on which the moment generating function will be defined.

4.1. Pseudo-exponential functions

The exponential function is characterized as the solution of some specific functional or differential equations. When considering the analogous functional and differential equations expressed in terms of pseudo-operations and pseudo-derivatives, we obtain different functions that, clearly, degenerate to the standard exponential function when the involved generators coincide with the identity function.

Proposition 4.1. *Let $g \in \mathcal{G}_{a,b}$, $k \in \mathcal{G}_{c,d}$ and $f : [c,d] \to [a,b]$. Then*

$$f(x \oplus_k y) = f(x) \otimes_g f(y) \tag{14}$$

for all $x, y \in [c,d]$ if and only if $f(x) = g^{-1}(t^{k(x)})$, for some $t > 0$.

Proof. (14) is equivalent to

$$f \circ k^{-1}(k(x) + k(y)) = g^{-1}\left(g \circ f(x) \cdot g \circ f(y)\right),$$

from which, setting $u = k(x)$ and $v = k(y)$, we get

$$g \circ f \circ k^{-1}(u + v) = g \circ f \circ k^{-1}(u) \cdot g \circ f \circ k^{-1}(v),$$

which is true if and only if $g \circ f \circ k^{-1}(x) = t^x$ for some $t > 0$, and the conclusion trivially follows. □

In the setting of the semiring $([a,b], \oplus_g, \otimes_g)$, that is when $k = g$, the resulting pseudo-exponential function consistent with it is the function $f(x) = g^{-1}\left(t^{g(x)}\right)$.

When $k(x) = x$ and $[c,d] = \bar{\mathbb{R}}_+$, we obtain the function $f(x) = g^{-1}(t^x)$, that, when $t = e$, is the solution of the following pseudo-differential equation.

Proposition 4.2. *Let $g \in \mathcal{G}_{a,b}$ be differentiable on (a,b) and $f : \mathbb{R} \to [a,b]$ be a differentiable function with the same monotonicity as g. Then*

$$f(x) = \frac{^{\oplus_g} d}{dx} f(x) \tag{15}$$

if and only if $f(x) = g^{-1}(e^x)$.

Proof. By definition of pseudo-derivative, equation (15) is equivalent to

$$g(f(x)) = \frac{d}{dx}[g \circ f(x)]$$

that holds true if and only if $g \circ f(x) = e^x$. □

Since the above pseudo-exponential functions will be used in the sequel in the specific case $t = e$, given $g \in \mathcal{G}_{a,b}$ and $k \in \mathcal{G}_{c,d}$, we define

$$\exp_{g,k}(x) = g^{-1}(e^{k(x)}). \tag{16}$$

If $g = k \in \mathcal{G}_{a,b}$, we get the pseudo-exponential function considered in [12].

4.2. Pseudo-moment generating functions

Based on the pseudo-exponential function of type (16), which depends on the specific choice of $g \in \mathcal{G}_{a,b}$ and $k \in \mathcal{G}_{c,d}$, we can define a family of pseudo-moment generating functions, parametrized by the pair $(g,k) \in \mathcal{G}_{a,b} \times \mathcal{G}_{c,d}$.

Definition 4.1. Let $(g,k) \in \mathcal{G}_{a,b} \times \mathcal{G}_{c,d}$ and $\left(\Omega, \mathcal{F}, m_g\right)$ be a measurable space provided with a pseudo-probability m_g. Let X be a random variable with values in $[c,d]$. We define the (g,k)-pseudo-moment generating function as

$$M_X^{(g,k)}(t)=\tilde{E}^{m_g}\left[\exp_{g,k}^{\otimes_g t}(X)\right] \tag{17}$$

where $\exp_{g,k}^{\otimes_g t}(x)=\left(\exp_{g,k}(x)\right)^{\otimes_g t}$.

When $k(x)=x$ and $[c,d]=\bar{\mathbb{R}}_+$, Definition 4.1 corresponds to the definition of the pseudo-Laplace transform in [21].

By (7), the (g,k)-pseudo-moment generating function defined in (17) can be easily rewritten as

$$\begin{aligned} M_X^{(g,k)}(t)&=\tilde{E}^{m_g}[g^{-1}(e^{tk(X)})]=g^{-1}(E^{\mathbb{P}}[e^{tk(X)}])=\\ &=g^{-1}(M_{k(X)}^{\mathbb{P}}(t))=M_{k(X)}^{(g,id)}(t)\end{aligned} \tag{18}$$

where $M_{k(X)}^{\mathbb{P}}$ is the standard moment-generating function of $k(X)$ with respect to the probability $\mathbb{P}=g\circ m_g$. So, $M_X^{(g,k)}$ and $M_{k(X)}^{\mathbb{P}}$ share the same domain and $M_X^{(g,k)}$ identifies the σ-$\oplus_g$-additive pseudo-probability induced by X on $[c,d]$.

An explicit expression for $M_X^{(g,k)}$ can be trivially recovered in the cases in which we are dealing with random variables whose transformation through k has a known standard moment generating function.

Example 4.1. If, under $\mathbb{P}=g\circ m_g$, $k(X)\overset{d}{=}Poi(\lambda)$, then, clearly, $M_X^{(g,k)}(t)=g^{-1}(e^{\lambda(e^t-1)})$, while, if, under $\mathbb{P}=g\circ m_g$, $k(X)\overset{d}{=}\Gamma(\alpha,\mu)$, then $M_X^{(g,k)}(t)=g^{-1}(\mu^\alpha(\mu-t)^{-\alpha}),t<\mu$.

Given $g\in\mathcal{G}_{a,b}$ and a random variable $X:\Omega\to[a,b]$, if $X\in L^n_{\oplus_g}$, with respect to the pseudo-probability m_g, we can define its n-th pseudo-moment as $\tilde{E}^{m_g}[X^{\otimes_g n}]$.

Pseudo-moments can be determined using the pseudo-moment generating function via pseudo-differentiation.

Proposition 4.3. *Let X be a random variable on $\left(\Omega,\mathcal{F},m_g\right)$ with values in $[a,b]$ such that $M_X^{(g,g)}$ is defined in a neighborhood of 0. Then, for $n\in\mathbb{N}$,*

$$\left.\frac{^{\oplus_g}d^n}{dt^n}M_X^{(g,g)}(t)\right|_{t=0}=\tilde{E}^{m_g}[X^{\otimes_g n}]. \tag{19}$$

Proof. By applying the n-th pseudo-differentiation to (18) with $k=g$, we have that

$$\frac{^{\oplus_g}d^n}{dt^n}M_X^{(g,g)}(t)=g^{-1}\left(\frac{d^n}{dt^n}M_{g(X)}^{\mathbb{P}}(t)\right).$$

But

$$\left.\frac{d^n}{dt^n}M_{g(X)}^{\mathbb{P}}(t)\right|_{t=0}=E^{\mathbb{P}}[(g(X))^n]=g\left(\tilde{E}^{m_g}\left[g^{-1}\left((g(X))^n\right)\right]\right),$$

from which the conclusion follows. □

4.3. The joint pseudo-moment generating function

In this subsection we will extend the definition of pseudo-moment generating function to random vectors, in line with the classical case.

Let $(g,k_1,k_2)\in\mathcal{G}_{a,b}\times\mathcal{G}_{c_1,d_1}\times\mathcal{G}_{c_2,d_2}$ and (X_1,X_2) be a random vector with values in $[c_1,d_1]\times[c_2,d_2]$ defined on $\left(\Omega,\mathcal{F},m_g\right)$ where m_g is a pseudo-probability. We define

$$M_{X_1,X_2}^{(g,k_1,k_2)}(s,t)=\tilde{E}^{m_g}\left[\exp_{g,k_1}^{\otimes_g s}(X_1)\otimes_g\exp_{g,k_2}^{\otimes_g t}(X_2)\right] \tag{20}$$

the "(g,k_1,k_2) pseudo-joint moment generating function".

Exactly as for the one-dimensional version, we have that

$$\begin{aligned} M_{X_1,X_2}^{(g,k_1,k_2)}(s,t)&=g^{-1}\left(M_{k_1(X_1),k_2(X_2)}^{\mathbb{P}}(s,t)\right)=\\ &=M_{k_1(X_1),k_2(X_2)}^{(g,id,id)}(s,t)\end{aligned} \tag{21}$$

where $M_{U,W}^{\mathbb{P}}(s,t)$ is the standard moment generating function of the random vector (U,W) with respect to $\mathbb{P}=g\circ m_g$. Similarly as in the one-dimensional case, $M_{X_1,X_2}^{(g,k_1,k_2)}$ and $M_{k_1(X_1),k_2(X_2)}^{\mathbb{P}}$ share the same domain and $M_{X_1,X_2}^{(g,k_1,k_2)}$ identifies the σ-$\oplus_g$-additive pseudo-probability induced by (X_1,X_2) on $[c_1,d_1]\times[c_2,d_2]$ since the dependence structure between X_1 and X_2 can be directly recovered from that of $k_1(X_1)$ and $k_2(X_2)$.

As the classical moment generating function of two independent random variables decomposes into the product of the marginal moment generating functions, the (g,k_1,k_2)-pseudo-moment generating function in (20) of two g-independent random variables can be written as the pseudo-product $\otimes_g$ of the one-dimensional corresponding (g,k_i)-pseudo-moment generating functions, $i=1,2$, as we show in the following proposition.

Theorem 4.2. *Let $(g,k_1,k_2)\in\mathcal{G}_{a,b}\times\mathcal{G}_{c_1,d_1}\times\mathcal{G}_{c_2,d_2}$ and X_1 and X_2 be two random variables defined on $(\Omega,\mathcal{F},m_g)$, where m_g is a pseudo-probability, with values in $[c_1,d_1]$ and $[c_2,d_2]$, respectively. They are g-independent if and only if*

$$M^{(g,k_1,k_2)}_{X_1,X_2}(s,t)=M^{(g,k_1)}_{X_1}(s)\otimes_g M^{(g,k_2)}_{X_2}(t) \tag{22}$$

and this is true for all choices of $(k_1,k_2)\in\mathcal{G}_{c_1,d_1}\times\mathcal{G}_{c_2,d_2}$.

Proof. By (21) and (18) and the fact that g-independence is equivalent to $\mathbb{P}=g\circ m_g$-independence, we have that

$$\begin{aligned}M^{(g,k_1,k_2)}_{X_1,X_2}(s,t)&=g^{-1}\left(M^{\mathbb{P}}_{k_1(X_1),k_2(X_2)}(s,t)\right)=\\&=g^{-1}\left(M^{\mathbb{P}}_{k_1(X_1)}(s)\cdot M^{\mathbb{P}}_{k_2(X_2)}(t)\right)=\\&=g^{-1}\left(g(M^{(g,k_1)}_{X_1}(s))\cdot g(M^{(g,k_2)}_{X_2}(t))\right).\quad\square\end{aligned}$$

Example 4.2. Let X_1 and X_2 be g-independent so that, under $\mathbb{P}=g\circ m_g$, $k_i(X_i)$ is Bernoulli distributed with parameter π_i, $i=1,2$. Then, by Theorem 4.2,

$$M^{(g,k_1,k_2)}_{X_1,X_2}(s,t)=g^{-1}\left((e^t\pi_1+1-\pi_1)(e^t\pi_2+1-\pi_2)\right).$$

In the standard probability case, the moment generating function of the sum of two independent random variables decomposes into the product of the corresponding one-dimensional moment generating functions. A similar result can be obtained for a general non-commutative pseudo-sum of g-independent random variables.

Proposition 4.4. *Let $(g,k_1,k_2)\in\mathcal{G}_{a,b}\times\mathcal{G}_{c_1,d_1}\times\mathcal{G}_{c_2,d_2}$ and X_1 and X_2 be two g-independent random variables defined on $(\Omega,\mathcal{F},m_g)$ and with values in $[c_1,d_1]$ and $[c_2,d_2]$, respectively. Then, for any $k\in\mathcal{G}_{c_0,d_0}$,*

$$M^{(g,k)}_{X_1\,{}_{k_1}\overset{k}{\oplus}_{k_2}X_2}(t)=M^{(g,k_1)}_{X_1}(t)\otimes_g M^{(g,k_2)}_{X_2}(t). \tag{23}$$

Proof. Since

$$\begin{aligned}M^{(g,k_1,k_2)}_{X_1,X_2}(t,t)&=\tilde{E}^{m_g}\left[g^{-1}\left(e^{t\left(k_1(X_1)+k_2(X_2)\right)}\right)\right]=\\&=\tilde{E}^{m_g}\left[g^{-1}\left(e^{tk\left(X_1\,{}_{k_1}\overset{k}{\oplus}_{k_2}X_2\right)}\right)\right]=\\&=M^{(g,k)}_{X_1\,{}_{k_1}\overset{k}{\oplus}_{k_2}X_2}(t),\end{aligned}$$

the conclusion follows from Theorem 4.2. $\square$

From (23), depending on the choice of the generators, we can obtain several interesting particular cases:

1. If $k=k_1=k_2=id$ with $[c_i,d_i]=\bar{\mathbb{R}}_+$ for $i=0,1,2$, we get
$$M^{(g,id)}_{X_1+X_2}(t)=M^{(g,id)}_{X_1}(t)\otimes_g M^{(g,id)}_{X_2}(t) \tag{24}$$
that is the equivalent of the statement of Theorem 3 in [21] where the authors extend the relationship between Laplace transform and convolution to the pseudo-Laplace transform and the pseudo-convolution.
2. If $g=id$, with $[a,b]=\bar{\mathbb{R}}_+$,
$$M^{(id,k)}_{X_1\,{}_{k_1}\overset{k}{\oplus}_{k_2}X_2}(t)=M^{(id,k_1)}_{X_1}(t)\cdot M^{(id,k_2)}_{X_2}(t)$$
that corresponds to the case in which the underlying pseudo-additive fuzzy measure is indeed a probability.
3. In the semiring framework, that is when $g=k=k_1=k_2\in\mathcal{G}_{a,b}$, we obtain
$$M^{(g,g)}_{X_1\oplus_g X_2}(t)=M^{(g,g)}_{X_1}(t)\otimes_g M^{(g,g)}_{X_2}(t).$$

Example 4.3. Here we will provide some examples of the above cases.

- If, under $\mathbb{P} = g\circ m_g$, for $i = 1,2$, $k_i(X_i)$ is Poisson distributed with parameter λ_i, with X_1 and X_2 g-independent, then

$$M^{(g,k)}_{X_1 {}_{k_1}\overset{k}{\oplus}_{k_2} X_2}(t) = g^{-1}(e^{(\lambda_1+\lambda_2)(e^t-1)}).$$

In particular, if $[a,b] = \bar{\mathbb{R}}_+$, $[c_i,d_i] = \bar{\mathbb{R}}_+$ for $i = 0,1,2$, $g(x) = \frac{\log(x(e^\theta-1)+1)}{\theta}$, $\theta \neq 0$ and $k_i = id$, $i = 1,2$ and $k = id$ we get

$$M^{(g,id)}_{X_1+X_2}(t) = \frac{\exp\left(\theta e^{(\lambda_1+\lambda_2)(e^t-1)}\right) - 1}{e^\theta - 1}.$$

While, if, under $\mathbb{P} = g\circ m_g$, for $i = 1,2$, $k_i(X_i)$ is Gamma distributed, that is $k_i(X_i) \overset{d}{=} \Gamma(\alpha_i,\mu)$, with X_1 and X_2 g-independent, then

$$M^{(g,k)}_{X_1 {}_{k_1}\overset{k}{\oplus}_{k_2} X_2}(t) = g^{-1}(\mu^{\alpha_1+\alpha_2}(\mu-t)^{-(\alpha_1+\alpha_2)}).$$

- Let $[a,b] = \bar{\mathbb{R}}_+$ and $g(x) = \frac{\log(x(e^\theta-1)+1)}{\theta}$, $\theta \neq 0$. Let X_1 and X_2 be g-independent and assume that, under $\mathbb{P} = g\circ m_g$, $g(X_i)$ is Bernoulli distributed with parameter π_i, $i = 1,2$. Then

$$M^{(g,g)}_{X_1\oplus_g X_2}(t) = \frac{e^{\theta(e^t\pi_1+1-\pi_1)(e^t\pi_2+1-\pi_2)} - 1}{e^\theta - 1}.$$

The first pseudo-moment of $X_1 \oplus_g X_2$, using equation (19), is given by

$$\tilde{E}^{m_g}[X_1 \oplus_g X_2] = \frac{e^{\theta(\pi_1+\pi_2)} - 1}{e^\theta - 1}.$$

4.4. Pseudo-moment generating functions and dependence structure

As discussed at the end of subsection 3.1, there is a one-to-one correspondence between the semi-copula associated to a vector (X_1,X_2) with respect to the regular σ-$\oplus_g$-additive fuzzy measure m_g (see Remark 2.2) and the copula associated with respect to the probability $\mathbb{P} = g\circ m_g$ (see formula (8)). Choosing in the appropriate way $k_1,k_2 \in \mathcal{G}_{c_1,d_1} \times \mathcal{G}_{c_2,d_2}$, we will show that we can separate, in the expression of the joint moment generating function, the contribution of the marginal fuzzy distributions from that of the dependence structure.

Let $(\Omega,\mathcal{F},m_g)$ be a measurable space with m_g a pseudo-probability and (X_1,X_2) a random vector with values in $[c_1,d_1]\times[c_2,d_2]$. We assume that with respect to $\mathbb{P} = g\circ m_g$, its joint cumulative distribution function is, for $x \in [c_1,d_1]$, $y \in [c_2,d_2]$, $F(x,y) = C\left(F_{X_1}(x),F_{X_2}(y)\right)$, where C is a copula and F_{X_i}, $i = 1,2$ are the corresponding marginal cumulative distributions that we assume to be continuous, strictly increasing on $[c_i,d_i]$, with $F_{X_i}(c_i) = 0$, $i = 1,2$.

Let G be a benchmark continuous cumulative distribution function with $G(0) = 0$ and strictly increasing on $\mathbb{R}_+$. If $k_i(x) = G^{-1}\circ F_{X_i}(x) \in \mathcal{G}_{c_i,d_i}$ for $i = 1,2$, then, for every $z > 0$, $F_{k_i(X_i)}(z) = G(z)$. It follows that, if under $\mathbb{P}$ the cumulative distribution function of the vector (V_1,V_2) is given by $F_{V_1,V_2}(x,y) = C\left(G(x),G(y)\right)$, then

$$\begin{aligned} M^{(g,k_1,k_2)}_{X_1,X_2}(s,t) &= g\left(M^{\mathbb{P}}_{k_1(X_1),k_2(X_2)}(s,t)\right) = \\ &= g\left(M^{\mathbb{P}}_{V_1,V_2}(s,t)\right) = \\ &= M^{(g,id,id)}_{V_1,V_2}(s,t). \end{aligned}$$

This way, in $M^{g,G^{-1}\circ F_{X_1},G^{-1}\circ F_{X_2}}_{X_1,X_2}$, we have separated the marginal distributions from the dependence structure, since $M^{(g,id,id)}_{V_1,V_2}$ depends only on the latter.

Example 4.4. In the particular case in which the random variables are g- independent and the reference distribution is the exponential one with parameter equal to one, for $k_i(x) = -\log(1-F_{X_i}(x))$, $i = 1,2$, we get

$$M^{(g,k_1,k_2)}_{X_1,X_2}(s,t) = M^{(g,k_1)}_{X_1}(s) \otimes_g M^{(g,k_2)}_{X_2}(t) = g^{-1}\left(\frac{1}{(1-s)(1-t)}\right), s,t < 1.$$

5. Pseudo-Schur-constant random vectors

In this section we show that some characterization of Schur-constant random vectors can be extended to a more general family of random vectors with Archimedean dependence using pseudo-operations and pseudo-integrals. More precisely, we show that the tools related to pseudo-additive measures can be used to provide characterization results for the underlying probability measure.

We start recalling the notion of Schur-constant bivariate random vector (see, among the others, [4,7,17] and [6]).

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let X and Y be two continuous positive random variables. Then the random vector (X, Y) is Schur-constant if and only if there exists a convex survival function S such that

$$\bar{F}_{X,Y}(x,y) = P(X > x, Y > y) = S(x+y),\ \forall (x,y) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{25}$$

The definition implies that the random vector is exchangeable with survival marginal distribution S and survival copula of Archimedean type with generator S.

In this section we consider the subclass of $\mathcal{G}_{a,b}$ of strictly increasing generators, that is

$$\mathcal{S}_{a,b} = \{g \in \mathcal{G}_{a,b} : g \textit{ is strictly increasing}\}.$$

Starting from the general non-commutative pseudo-sum defined in (4), we generalize (25) considering joint survival distributions of type

$$\bar{F}_{X,Y}(x,y) = S\left(x_{k_1} \overset{k}{\oplus}_{k_2} y\right), \tag{26}$$

with $x \in [c_1, d_1]$, $y \in [c_2, d_2]$ and where $\left(k, k_1, k_2\right) \in \mathcal{G}_{c_0,d_0} \times \mathcal{S}_{c_1,d_1} \times \mathcal{S}_{c_2,d_2}$ and $S \circ k^{-1}$ is a convex survival function on $[0, +\infty)$: it follows that $S : [c_0, d_0] \to [0, 1]$ is monotone and, if k is strictly increasing, S is decreasing with $S(c_0) = 1$ and $S(d_0) = 0$, while, if k is strictly decreasing, S is increasing with $S(c_0) = 0$ and $S(d_0) = 1$. The requirement that $\left(k_1, k_2\right) \in \mathcal{S}_{c_1,d_1} \times \mathcal{S}_{c_2,d_2}$ is necessary in order to guarantee that $\bar{F}_X(x) = \bar{F}_{X,Y}(x, c_2)$ and $\bar{F}_Y(y) = \bar{F}_{X,Y}(c_1, y)$, that implies $k_i(c_i) = 0$, for $i = 1, 2$.

A random vector (X, Y) with survival distribution (26) will be called "pseudo-Schur-constant".

We notice that representation (26) is not unique. In addition to Remark 2.1, we have that, if $\left(k, \hat{k}, k_1, k_2\right) \in \mathcal{G}_{c_0,d_0} \times \mathcal{G}_{\hat{c}_0,\hat{d}_0} \times \mathcal{S}_{c_1,d_1} \times \mathcal{S}_{c_2,d_2}$ and if S and $\hat{S}$ are two monotone functions from $[c_0, d_0]$ and $[\hat{c}_0, \hat{d}_0]$, respectively, to $[0, 1]$, so that $S \circ k^{-1}$ and $\hat{S} \circ \hat{k}^{-1}$ are convex survival functions, then

$$S\left(x_{k_1} \overset{k}{\oplus}_{k_2} y\right) = \hat{S}\left(x_{k_1} \overset{\hat{k}}{\oplus}_{k_2} y\right)$$

if and only if $\hat{S}(z) = S \circ k^{-1} \circ \hat{k}(z)$.

It can be easily checked that the family in (26) coincides with that of survival distribution functions of continuous random pairs (X, Y) with a survival dependence structure given by an Archimedean copula with survival marginal distribution functions that are strictly decreasing when positive. In fact, from (26) we get that marginal survival distribution functions are $\bar{F}_X(x) = S \circ k^{-1} \circ k_1(x)$ for $x \in [c_1, d_1]$ and $\bar{F}_Y(y) = S \circ k^{-1} \circ k_2(y)$ for $y \in [c_2, d_2]$, respectively, and that the survival copula function is of Archimedean type with generator $S \circ k^{-1}$. Conversely, let ϕ be an Archimedean generator (that is $\phi(0) = 1$, $\phi(+\infty) = 0$ and ϕ decreasing and convex), with pseudo inverse $\phi^{(-1)}$ and $\bar{F}_{X,Y}(x,y) = \phi\left(\phi^{(-1)}\left(\bar{F}_X(x)\right) + \phi^{(-1)}\left(\bar{F}_Y(y)\right)\right)$ for $x \in [c_1, d_1]$ and $y \in [c_2, d_2]$ be the survival distribution function of (X, Y). If $x_0 = \inf\{x \in [c_1, d_1] : \bar{F}_X(x) = 0\}$ and $y_0 = \inf\{y \in [c_2, d_2] : \bar{F}_Y(y) = 0\}$, we can choose $k_1 \in \mathcal{S}_{c_1,d_1}$ and $k_2 \in \mathcal{S}_{c_2,d_2}$ so that $k_1(x) = \phi^{(-1)}\left(\bar{F}_X(x)\right)$ for $x \in [c_1, x_0]$ and $k_2(x) = \phi^{(-1)}\left(\bar{F}_Y(y)\right)$ for $y \in [c_2, y_0]$. Then, for $k \in \mathcal{G}_{c_0,d_0}$, we get $\bar{F}_{X,Y}(x,y) = S \circ k^{-1}\left(k_1(x) + k_2(y)\right)$, where $S = \phi \circ k$.

Remark 5.1. Notice that, if S is strictly decreasing,

$$\bar{F}_{X,Y}(x,y) = S\left(x_{k_1} \overset{k}{\oplus}_{k_2} y\right) = \bar{F}_X(x) \otimes_\psi \bar{F}_Y(y)$$

with $\psi \in \mathcal{S}_{0,b}$ with $b \in (1, +\infty]$ such that $\psi(t) = e^{-k \circ S^{-1}(t)}$ for $t \in [0, 1]$. In this case, $\bar{F}_{X,Y}$ can be interpreted as the pseudo-survival distribution function of (X, Y) with respect to a regular $\sigma - \oplus_\psi$-additive fuzzy measure m_ψ (see Remark 2.2) under which X and Y are ψ-independent.

As a consequence of Remark 5.1 and Theorem 4.2 we obtain the following characterization and decomposition:

Proposition 5.1. *(X, Y) is a pseudo-Schur constant random vector under $\mathbb{P}$ with survival function (26), where S is strictly decreasing, if and only if there exist $\psi \in \mathcal{S}_{0,b}$ with $b \in (1, +\infty)$ and a regular σ-$\oplus_\psi$ additive fuzzy measure μ_ψ that generates the same joint pseudo-survival distribution function and under which*

$$M_{X,Y}^{(\psi,k_1,k_2)}(s,t) = M_X^{(\psi,k_1)}(s) \otimes_\psi M_Y^{(\psi,k_2)}(t).$$

Proposition 5.7 in [14] (see also Theorem 2.5 in [26], for an alternative proof) provides a characterization of the Schur-constancy of a random vector (X,Y) in terms of the moment generating function of the sum $X+Y$. More precisely, it states that (X,Y) is Schur-constant if and only if there exists a random variable Z with the same distribution as $X+Y$ such that, for all $s,t<0$, $s\neq t$

$$\mathbb{E}\left[e^{sX+tY}\right]=\frac{\int_t^s \mathbb{E}\left[e^{rZ}\right]dr}{s-t}.$$

This result can be generalized to pseudo-Schur-constant random vectors using the pseudo-moment generating functions defined in (17) and (20) and the pseudo-integral given in (6).

Theorem 5.1. *(X,Y) is a pseudo-Schur-constant random vector with joint survival distribution function (26) if and only if there exists $g\in\mathcal{G}_{a,b}$ such that, under $m_g=g^{-1}\circ\mathbb{P}$, for all $s,t<0$,*

$$M_{X,Y}^{(g,k_1,k_2)}(s,t)=\begin{cases}\left(\int_{(s,t)}^{\oplus_g} M_Z^{(g,k)}\otimes_g d\mathcal{L}_g\right)\oslash_g g^{-1}(t-s), & s<t\\ \left(\int_{(t,s)}^{\oplus_g} M_Z^{(g,k)}\otimes_g d\mathcal{L}_g\right)\oslash_g g^{-1}(s-t), & s>t,\end{cases}\tag{27}$$

where $Z\overset{d}{=}X_{k_1}\overset{k}{\oplus}_{k_2}Y$ and where $\mathcal{L}_g=g^{-1}\circ\mathcal{L}$ with $\mathcal{L}$ the Lebesgue measure on the real line. If there exists $g\in\mathcal{G}_{a,b}$ for which (27) is true, then (27) holds true for all $g\in\mathcal{G}_{\hat{a},\hat{b}}$ for all $-\infty\le\hat{a}<\hat{b}\le+\infty$.

Proof. Clearly, (26) is equivalent to the fact that $(k_1(X),k_2(Y))$ is a Schur-constant random vector with survival function $S\circ k^{-1}$. Thanks to the mentioned result in [14] and [26], we have that (26) is equivalent to

$$M_{k_1(X),k_2(Y)}^{\mathbb{P}}(s,t)=\frac{1}{t-s}\int_s^t M_V^{\mathbb{P}}(r)dr,\quad s,t<0,\ s\neq t,$$

where $V\overset{d}{=}k_1(X)+k_2(Y)$. Then, given any $g\in\mathcal{G}_{a,b}$ and considering the pseudo-probability $m_g=g^{-1}\circ\mathbb{P}$, by (21) and (18),

$$M_{X,Y}^{(g,k_1,k_2)}(s,t)=g^{-1}\left(\frac{1}{t-s}\int_s^t M_V^{\mathbb{P}}(r)\,dr\right)=$$
$$=g^{-1}\left(\frac{1}{t-s}\int_s^t g\left(M_Z^{(g,k)}(r)\right)\,dr\right)$$

and the thesis trivially follows from (6). □

An immediate consequence of this result is that, for $v<0$,

$$M_X^{(g,k_1)}(v)=M_Y^{(g,k_2)}(v)=\left[\int_{(v,0)}^{\oplus_g} M_Z^{(g,k)}(r)\otimes_g d\mathcal{L}_g\right]\oslash_g\ g^{-1}(-v).\tag{28}$$

Since the distribution of Z depends on both k_1 and k_2 while each marginal pseudo-moment generating function depends only on one of them, the marginal distributions of X and Y only depend on Z through the dependence structure between X and Y.

From (27) and (28), we have that

$$M_{X,Y}^{(g,k_1,k_2)}(s,t)=$$
$$=\begin{cases}g^{-1}\left(\frac{-t}{t-s}\right)\otimes_g M_Y^{(g,k_2)}(t)\ominus_g\ g^{-1}\left(\frac{-s}{t-s}\right)\otimes_g M_X^{(g,k_1)}(s), & t<s\\ g^{-1}\left(\frac{-s}{s-t}\right)\otimes_g M_X^{(g,k_1)}(s)\ominus_g\ g^{-1}\left(\frac{-t}{s-t}\right)\otimes_g M_Y^{(g,k_2)}(t), & t>s\end{cases}$$

and the joint pseudo-moment generating function can be written in terms of the marginal pseudo-moment generating functions.

It is well known that (X,Y) is a continuous Schur-constant random vector if and only if there exist a random variable U uniformly distributed in the interval $(0,1)$ and a positive random variable V such that

$$(X,Y)\overset{d}{=}(U\,V,(1-U)\,V),\tag{29}$$

with U independent of V and $V\overset{d}{=}X+Y$ (see, for example, Theorem 2.1 in [6]). Equivalence (29) can be generalized to the case in which (X,Y) is a pseudo-Schur constant random vector, by considering the non-commutative pseudo-sum in place of the classical sum and the pseudo-multiplication in place of the standard one, as we show in the following Proposition.

Proposition 5.2. *(X,Y) is a pseudo-Schur constant random vector with survival function (26) if and only if there exists a random variable W with values in $\left(c_1, k_1^{-1}(1)\right)$ and cumulative distribution function k_1 such that*

$$(X,Y) \stackrel{d}{=} \left(W \otimes_{k_1} Z_1, \left(1_{k_1} \overset{k_2}{\ominus}_{k_1} W\right) \otimes_{k_2} Z_2\right), \tag{30}$$

where $Z_1 \stackrel{d}{=} X_{k_1} \overset{k_1}{\oplus}_{k_2} Y$ and $Z_2 \stackrel{d}{=} X_{k_1} \overset{k_2}{\oplus}_{k_2} Y$ are both independent of W.

Proof. Thanks to (29), the fact that $(k_1(X), k_2(Y))$ is a Schur-constant random vector is equivalent to

$$\left(k_1(X), k_2(Y)\right) \stackrel{d}{=} (U\,V, (1-U)\,V),$$

with U uniformly distributed in $(0,1)$ and independent of $V \stackrel{d}{=} k_1(X) + k_2(Y)$. Then, we have

$$(X,Y) \stackrel{d}{=} \left(k_1^{-1}\left(U\;V\right), k_2^{-1}\left((1-U)\;V\right)\right)$$

that is equivalent to

$$(X,Y) \stackrel{d}{=} \left(k_1^{-1}(U) \otimes_{k_1} Z_1, k_2^{-1}(1-U) \otimes_{k_2} Z_2\right),$$

where $Z_1 \stackrel{d}{=} X_{k_1} \overset{k_1}{\oplus}_{k_2} Y$, $Z_2 \stackrel{d}{=} X_{k_1} \overset{k_2}{\oplus}_{k_2} Y$ and the conclusion follows. □

Remark 5.2. In the semiring framework, that is when only one generator $g \in \mathcal{S}_{a,b}$ is involved, we obtain exactly the same results as for the Schur-constant case, just substituting the standard addition and multiplication with the pseudo-ones. More precisely, if $g \in \mathcal{S}_{a,b}$, the following conditions are equivalent:

- under $\mathbb{P}$, (X,Y) is distributed according to the survival distribution function

  $$\bar{F}(x,y) = S\left(x \oplus_g y\right), x, y \in [a,b] \tag{31}$$

  or, equivalently, under $m_g = g^{-1} \circ \mathbb{P}$, according to the pseudo-survival distribution

  $$\bar{F}(x,y) = \hat{S}\left(x \oplus_g y\right), x, y \in [a,b]$$

  where $\hat{S} = g^{-1} \circ S$;
- the joint pseudo-moment generating function with respect to $m_g = g^{-1} \circ \mathbb{P}$ is of the form, for $s, t < 0$,

  $$M_{X,Y}^{(g,g,g)}(s,t) =$$
  $$= \begin{cases} \left[\int_{(s,t)}^{\oplus} M_Z^{(g,g)}(r) \otimes_g d\mathcal{L}_g\right] \oslash_g g^{-1}(t-s), & \text{when } s < t \\ \left[\int_{(t,s)}^{\oplus} M_Z^{(g,g)}(r) \otimes_h d\mathcal{L}_g\right] \oslash_g g^{-1}(s-t), & \text{when } s > t \end{cases}$$

  with $Z \stackrel{d}{=} X \oplus_g Y$ and $\mathcal{L}_g = g^{-1} \circ \mathcal{L}$, where $\mathcal{L}$ is the Lebesgue measure on the real line;
- under $m_g = g^{-1} \circ \mathbb{P}$, there exists a continuous random variable W with pseudo-cumulative distribution function $F_W(\omega) = \omega$, for $0 \le \omega \le 1$, and a random variable $Z \stackrel{d}{=} X \oplus_g Y$ such that W and Z are g-independent and

  $$(X,Y) \stackrel{d}{=} (W \otimes_g Z, \left(1 \ominus_g W\right) \otimes_g Z).$$

Notice that (31) coincides with the time-transformed exponential model introduced in [5].

Remark 5.3. Analogous characterizations to those provided in Theorem 5.1 and Proposition 5.2 can be obtained for pseudo-Schur constant discrete positive random variables by extending the corresponding results for Schur-constant discrete positive random vectors proved in [13].

6. Conclusions

In this paper, using pseudo-operations and pseudo-calculus, we have introduced a generalization of the notion of the moment generating function (both for random variables as well as for random vectors) with respect to a pseudo-probability and we have analyzed their properties generalizing the corresponding ones in the standard probability case. As an extension of the notion of Schur-constant bivariate random vectors in the usual probability space, we have introduced a more general class of vectors using a non-commutative version of the pseudo-sum: this class coincides with that of bivariate distributions with Archimedean dependence with marginal survival distribution functions that are strictly decreasing when positive. Finally, we have proved that the pseudo-calculus framework as well as the introduced concepts of pseudo-moment generating functions allow to extend some well known characterizations of Schur-constant random vectors to the new more general class.

CRediT authorship contribution statement

Sabrina Mulinacci: Writing – review & editing, Supervision, Methodology, Conceptualization. **Massimo Ricci:** Writing – original draft, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

[1] H. Agahi, R. Mesiar, Y. Ouyang, E. Pap, M. Štrboja, Hölder and Minkowski type inequalities for pseudo-integral, Appl. Math. Comput. 217 (2011) 8630–8639.
[2] H. Agahi, R. Mesiar, A. Babakhani, Generalized expectation with general kernels on g-semirings and its applications, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 111 (2017) 863–875.
[3] H. Agahi, H.M. Dehnavi, Pseudo-cumulative distribution function with applications, Soft Comput. 25 (2021) 9693–9702.
[4] R.E. Barlow, M.B. Mendel, De Finetti-type representation for life distributions, J. Am. Stat. Assoc. 87 (1992) 1116–1122.
[5] B. Bassan, F. Spizzichino, Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes, J. Multivar. Anal. 93 (2005) 313–339.
[6] Y. Chi, Y. Qi, J. Yang, Decomposition of Schur-constant model and its applications, Insur. Math. Econ. 4 (2009) 398–408.
[7] L. Caramellino, F. Spizzichino, Dependence and aging properties of lifetimes with Schur-constant survival functions, Probab. Eng. Inf. Sci. 8 (1994) 103–111.
[8] G. Choquet, Theory of capacities, Ann. Inst. Fourier 5 (1953) 131–295.
[9] F. Durante, C. Sempi, Copula and semicopula transforms, Int. J. Math. Math. Sci. 4 (2005) 645–655.
[10] F. Durante, F. Spizzichino, Semi-copulas, capacities and families of level sets, Fuzzy Sets Syst. 161 (2010) 269–276.
[11] J.S. Golan, Theory of Semirings with Applications in Mathematics and Computer Sciences, Addison-Wesley Longman Ltd, Edinburgh Gate Harlow, Essex CM20 2JE, United Kingdom, 1992.
[12] P. Jain, Classical inequalities for pseudo-integral, Georgian Math. J. 29 (3) (2022) 373–385.
[13] N. Kolev, S. Mulinacci, New characterizations of bivariate discrete Schur-constant models, Stat. Probab. Lett. 180 (2022) 109233.
[14] M. Kozlova, P. Salminen, Diffusion local time storage, Stoch. Process. Appl. 114 (2004) 211–229.
[15] A.J. McNeil, J. Nešlehová, Multivariate Archimedean copulas, d-monotone functions and l1-norm symmetric distributions, Ann. Stat. 37 (2009) 3059–3097.
[16] N. Nedovic, B. Mihailovic, N.M. Ralevic, Some properties of pseudo-measures and pseudo-probability, in: Proceedings of SISY 5th International Symposium on Intelligent Systems and Informatics, 2007, pp. 155–159.
[17] R.B. Nelsen, Some properties of Schur-constant survival models and their copulas, Braz. J. Probab. Stat. 19 (2005) 179–190.
[18] R.B. Nelsen, An Introduction to Copulas, second edition, Springer, 2006.
[19] E. Pap, g-Calculus, Univ. Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 23 (1) (1993) 145–156.
[20] E. Pap, Pseudo-additive measures and their applications, Handb. Meas. Theory 2 (2002) 1403–1468.
[21] E. Pap, N. Ralevic, Pseudo-Laplace transform, Nonlinear Anal. 33 (1998) 533–550.
[22] E. Pap, M. Štrboja, I. Rudas, Pseudo-L^p space and convergence, Fuzzy Sets Syst. 238 (2014) 113–128.
[23] E. Pap, D. Vivona, Non-commutative and non-associative pseudo-analysis and its applications on nonlinear partial differential equations, J. Math. Anal. Appl. 246 (2000) 390–408.
[24] N. Ralevic, Some new properties of the g-Calculus, Univ. Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 24 (1) (1994) 139–157.
[25] N. Ralevic, A generalization of the pseudo-Laplace transform, Novi Sad J. Math. 37 (2) (2007) 13–23.
[26] B.Q. Ta, C.P. Van, Some properties of bivariate Schur-constant distributions, Stat. Probab. Lett. 124 (2017) 69–76.
[27] Z. Wang, G.J. Klir, Fuzzy Measure Theory, first edition, Springer, New York, 1993.
[28] D. Zhang, E. Pap, Fubini theorem and generalized Minkowski inequality for the pseudo-integral, Int. J. Approx. Reason. 122 (2020) 9–23.
[29] D. Zhang, R. Mesiar, E. Pap, Pseudo-integral and generalized Choquet integral, Fuzzy Sets Syst. 446 (2022) 193–221.