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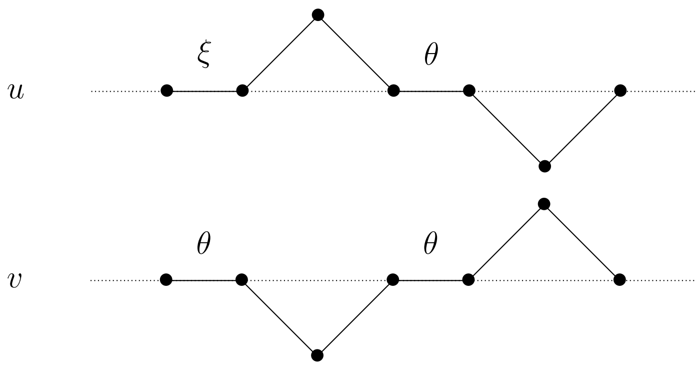
ERRATUM TO "LEFSCHETZ THEORY FOR EXTERIOR ALGEBRAS AND FERMIONIC DIAGONAL COINVARIANTS"

JONGWON KIM, ROBERTO PAGARIA, AND BRENDON RHOADES

This erratum corrects the proof of the main result [1, Thm. 5.2] of [1, Sec. 5]. While this result is correct as stated, its proof is flawed. We adopt the notation of [1, Sec. 5].

The total order $\prec$ is not a term order, so that [1, Lem. 5.3] loses meaning. In particular, [1, Lem. 5.1] is false because the depth $d(\sigma)$ is not multiplicative.

Example 1. For $n = 6$, the elements $u, v \in \wedge\{\Theta_6, \Xi_6\}$ given by $u = \xi_1\theta_3\xi_3\theta_4\theta_5\xi_5$ and $v = \theta_1\theta_2\xi_2\theta_4\theta_6\xi_6$ have the following lattice path representations as in [1, Sec. 5]:



Both u and v have degree 6 and depth -1. So $u \succ v$ because $\xi_1 \succ \theta_1$. However $\xi_4 u \prec \xi_4 v$ because $d(\xi_1\theta_3\xi_3\theta_4\xi_4\theta_5\xi_5) = -2$ and $d(\theta_1\theta_2\xi_2\theta_4\xi_4\theta_6\xi_6) = -1$. Therefore the depth is not multiplicative and $\prec$ is not a term order.

We correct the proof of [1, Thm. 5.2] as follows. We shall calculate a Gröbner basis for the ideal $I_n = \langle \delta_n \rangle \subset \wedge\{\Theta_n, \Xi_n\}$ where $\delta_n = \sum_{i=1}^n \theta_i \xi_i$ with respect to the lexicographical term order $<_{\text{lex}}$.

For each Motzkin path σ as in [1, Sec. 5], we define $j(\sigma)$ to be the x-coordinate where the depth $d(\sigma)$ is achieved the first time. We have $d(\sigma) = 0$ if and only if $j(\sigma) = 0$. If u, v are the Motzkin paths (or monomials) in Example 1 then $j(u) = 5$, and $j(v) = 2$.

Given a Motzkin path $\sigma = (s_1, s_2, \ldots, s_n)$ and $i \leq n$ we define k_i to be the difference between the y-coordinate of the starting point of s_i and $d(\sigma)$. For example $k_1 = -d(\sigma)$ and $k_{j(\sigma)+1} = 0$. For $i \leq j(\sigma)$ we introduce the exterior algebra elements

$$p_i(\sigma) := \begin{cases} (\sum_{l>i} \theta_l \xi_l) - k_i \theta_i \xi_i & s_i = (1,1) \text{ is an up-step} \\ \theta_i & s_i = (1,0) \text{ is decorated by } \theta \\ \xi_i & s_i = (1,0) \text{ is decorated by } \xi \\ 1 & s_i = (1,-1) \text{ is a down-step} \end{cases}$$

and let $p(\sigma) := p_1(\sigma) p_2(\sigma) \cdots p_{j(\sigma)}(\sigma)$ be their product. In Example 1 we have $p(u) = \xi_1 (\sum_{l=3}^{6} \theta_l \xi_l - \theta_2 \xi_2) \theta_4$ and $p(v) = \theta_1$. The definition of $p(\sigma)$ is motivated by the following identities

$$\begin{aligned} \delta_n^k &= k \theta_1 \xi_1 \delta_{n-1}^{k-1} + \delta_{n-1}^k \\ \theta_1 \delta_n^k &= \theta_1 \delta_{n-1}^k \\ \xi_1 \delta_n^k &= \xi_1 \delta_{n-1}^k \\ (\delta_{n-1} - k \theta_1 \xi_1) \delta_n^k &= \delta_{n-1}^{k+1} \end{aligned}$$

where $\delta_{n-1} = \sum_{l=2}^{n} \theta_l \xi_l$. Those identities are fundamental in the proof of the following theorem.

Theorem 2. *The initial ideal* $\mathrm{in}_{lex}(\delta_n^k)$ *with respect the lexicographical term order contains all monomials* σ *with depth* $d(\sigma) \leq -k$*.*

Proof. We claim that the leading monomial of $p(\sigma) \delta_n^{-d(\sigma)}$ divides the monomial $\mathrm{wt}(\sigma)$ and we prove this statement for all n by induction on $j(\sigma)$. The base case $j(\sigma) = 0$ is trivial because all monomials belong to the ideal generated by $\delta_n^0 = 1$.

For the inductive step, we remove the first step s_1 from σ to get a new path $\tau = (s_2, \dots, s_n)$ involving only the variables $\theta_2, \dots, \theta_n, \xi_2, \dots, \xi_n$. Notice that $p(\sigma) = p_1(\sigma) p(\tau)$. We divide proof in three cases according to the first step s_1.

Case 1: $s_1 = (1,1)$ *is an up step.*

We have $d(\tau) = d(\sigma) - 1$, $\mathrm{wt}(\sigma) = \mathrm{wt}(\tau)$, and

$$\begin{aligned} p(\sigma) \delta_n^{-d(\sigma)} &= p(\tau) \left((\sum_{l>1} \theta_l \xi_l) - (-d(\sigma)) \theta_1 \xi_1 \right) \delta_n^{-d(\sigma)} \\ &= p(\tau) \delta_{n-1}^{-d(\sigma)+1} = p(\tau) \delta_{n-1}^{-d(\tau)}. \end{aligned}$$

By induction, the leading term of $p(\sigma) \delta_n^{-d(\sigma)} = p(\tau) \delta_{n-1}^{-d(\tau)}$ divides $\mathrm{wt}(\sigma) = \mathrm{wt}(\tau)$.

Case 2: $s_1 = (1,0)$ *is a horizontal step.*

We assume that the horizontal step s_1 is labelled with θ; the other case is identical. We have $d(\tau) = d(\sigma)$, $\mathrm{wt}(\sigma) = \theta_1 \mathrm{wt}(\tau)$, and

$$p(\sigma) \delta_n^{-d(\sigma)} = \theta_1 p(\tau) \delta_n^{-d(\sigma)} = \theta_1 p(\tau) \delta_{n-1}^{-d(\tau)}.$$

Notice that $\theta_1 \cdot \mathrm{LM}(p(\tau) \delta_{n-1}^{-d(\tau)}) \neq 0$ and so the leading monomial

$$\mathrm{LM}(p(\sigma) \delta_n^{-d(\sigma)}) = \theta_1 \cdot \mathrm{LM}(p(\tau) \delta_{n-1}^{-d(\tau)})$$

divides $\theta_1 \cdot \mathrm{wt}(\tau) = \mathrm{wt}(\sigma)$ by the inductive hypothesis.

Case 3: $s_1 = (1, -1)$ *is a down step.*

We have $d(\tau) = d(\sigma) + 1$, $\mathrm{wt}(\sigma) = \theta_1\xi_1\mathrm{wt}(\tau)$, $p(\sigma) = p(\tau)$, and

$$p(\sigma)\delta_n^{-d(\sigma)} = -d(\sigma)p(\tau)\theta_1\xi_1\delta_{n-1}^{-d(\sigma)-1} + p(\tau)\delta_{n-1}^{-d(\sigma)}.$$

The leading monomial of the element $p(\tau)\theta_1\xi_1\delta_{n-1}^{-d(\tau)}$ is equal to $\theta_1\xi_1 \cdot \mathrm{LM}(p(\tau)\delta_{n-1}^{-d(\tau)}) \neq 0$. Moreover, the monomial $\theta_1\xi_1 \cdot \mathrm{LM}(p(\tau)\delta_{n-1}^{-d(\tau)})$ is bigger than every monomial appearing in $p(\tau)\delta_{n-1}^{-d(\sigma)}$ because we are using the lexicographical term order. Hence the leading monomial $\mathrm{LM}(p(\sigma)\delta_n^{-d(\sigma)}) = \theta_1\xi_1\mathrm{LM}(p(\tau)\delta_{n-1}^{-d(\tau)})$ divides the monomial $\theta_1\xi_1 wt(\tau) = wt(\sigma)$ by inductive hypothesis.

We conclude that $\mathrm{wt}(\sigma) \in \mathrm{in}_{\mathrm{lex}}(\delta_n^{-d(\sigma)}) \subseteq \mathrm{in}_{\mathrm{lex}}(\delta_n^k)$ for all $k \leq -d(\sigma)$ and the proof is complete. □

The above theorem substitutes [1, Lem. 5.3]. The second part of the proof of [1, Thm. 5.2] is correct and can be left unchanged.

Corollary 3. *The set* $\{p(\sigma)\delta_n \mid \sigma \text{ s.t. } d(\sigma) = -1\}$ *is a Gröbner basis for the ideal* $I_n = \langle \delta_n \rangle$ *with respect to the lexicographical term order.*

References

[1] Kim, Jongwon; Rhoades, Brendon *Lefschetz theory for exterior algebras and fermionic diagonal coinvariants.* Int. Math. Res. Not. IMRN 2022, no. 4, 2906–2933.

Jongwon Kim
Google, Inc., 1600 Ampitheatre Parkway, Mountain View, CA, USA
Email address: jk1093@sas.upenn.edu

Roberto Pagaria
Università di Bologna, Dipartimento di Matematica, Piazza di Porta San Donato 5 - 40126 Bologna, Italy.
Email address: roberto.pagaria@unibo.it

Brendon Rhoades
University of California, San Diego, Department of Mathematics, 9500 Gilman Dr., La Jolla, CA, USA.
Email address: bprhoades@ucsd.edu