ON A PARTICULAR SPIRAL SIMILARITY

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Abstract. One of the most well–known geometric transformations, constituting an important problem solving tool, is the so–called *spiral similarity* of a geometric figure. It allows the transformation, by means of a spiral movement around a point, of a figure F into another similar figure F′, that is generated by a rotation and a simultaneous expansion of F.
Together with the rotation, the inverse transformation of spiral similarity produces, instead, a contraction of the original figure. Aim of this work is precisely the study of properties of this inverse transformation and its links with *polygonal* spirals. Here, this transformation is achieved through an entirely new construction, that can be aided and eased by means of computer algebra and graphics tools.

1. Introduction

Inverse spiral similarity is a geometric transformation, that allows transforming a regular figure into a similar one, rotated around a center and by a certain angle, and contracted by a ratio k, compared to the originator [3, 2, 5].
In this work, this transformation is achieved through a new construction, whose properties are investigated, focusing, in particular, to links with polygonal spirals. Starting from the case of triangles, the application to polygons with more than three sides is studied, together with more general case of planar developments of geometric solids. The similarity demonstrations are illustrated step by step and by many figures; their construction can be lightened within software and computer algebra environments equipped with interactive graphics [4].
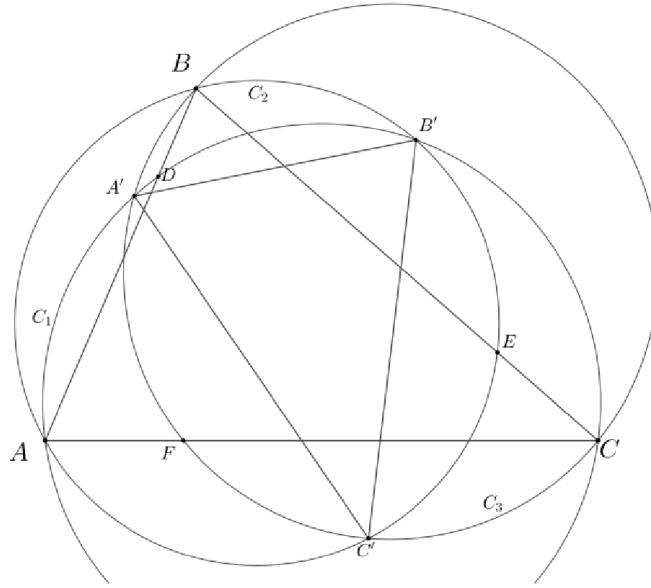
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FIGURE 1. Construction of the inverse spiral similarity transformation

2. THE CASE OF TRIANGLES

Let us start from the case of triangles since, for them, the construction we present always produces triangles similar to the originator, whether it is scalene or isosceles or equilateral or right–angled. In the case of polygons with more than three sides, our construction yields polygons similar to the initial one only if this is regular, as shown in § 5.

Consider triangle ABC in Figure 1, with sides $AB < BC < AC$, and let D, E, F be the points that divide AB, BC, AC, respectively, according to the ratio:

$$(1) \qquad k = \frac{m}{n}, \qquad \text{with} \quad (m, n) = 1, \quad m < n.$$

In other words:

$$(2) \qquad \frac{AD}{AB} = \frac{BE}{BC} = \frac{CF}{AC} = k.$$

Circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are then traced, respectively passing through the triplets of points:

$$A, C, D \in \mathcal{C}_1; \qquad A, B, E \in \mathcal{C}_2; \qquad B, C, F \in \mathcal{C}_3;$$

so that each pair of circles intersects at two points:

$$\mathcal{C}_1 \cap \mathcal{C}_2 = \{A, B'\}; \qquad \mathcal{C}_1 \cap \mathcal{C}_3 = \{C, A'\}; \qquad \mathcal{C}_2 \cap \mathcal{C}_3 = \{B, C'\}.$$

Look at triangle $A'B'C'$: it is rotated by an angle θ with respect to ABC, to which is similar with similarity ratio k given in (1); that is, the construction just described, and illustrated in Figure 1, implements an *inverse spiral similarity* of triangle ABC.

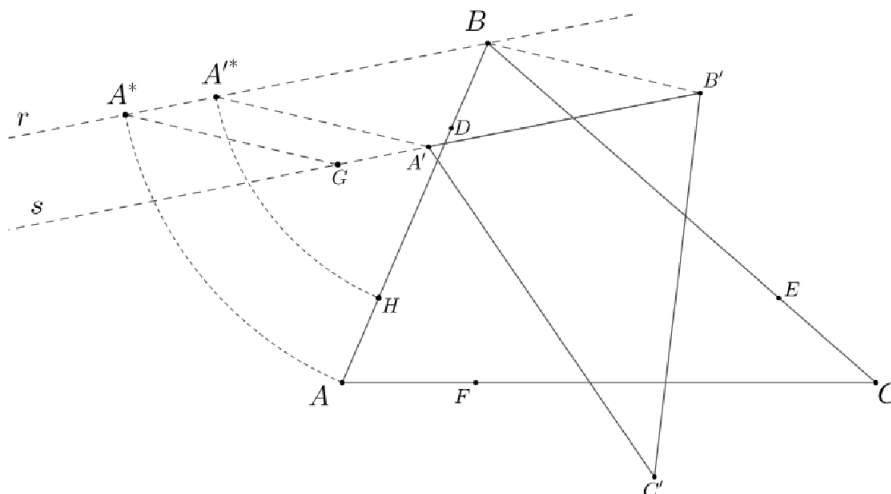Demonstration of the similiarity of ABC and $A'B'C'$ is now provided and graphed in Figure 2.

FIGURE 2. Demonstration of similiarity between triangles ABC, $A'B'C'$

Trace line r through B and such that $r\|s$, where s is the line through A', B'. Along r, determine segment A^*B congruent to AB. From A^*, draw the parallel A^*G to BB' to obtain parallelogram $A^*BB'G$. From A', conduct the parallel $A'A'^*$ to BB' so that $A'B' = A'^*B$. At this point, along AB, consider segment BH congruent to A'^*B: it is possible to prove that $BH = AD$. In fact:

$$(3) \qquad AB = AD + DB = AH + HB \quad\Longrightarrow\quad AD - BH = AH - DB.$$

By Thales' Theorem, the right–most relation in (3) also holds between abscissas and ordinates of A, H, D, B; referring triangle ABC to a Cartesian coordinate system and denoting $A(x_a, y_a)$, $H(x_h, y_h)$, $D(x_d, y_d)$, $B(x_b, y_b)$, it is:

$$(x_d - x_a) - (x_b - x_h) = 0 = (x_h - x_a) - (x_b - x_d),$$

and analogously for the corresponding ordinates. This means that (3) implies equalities $AD = BH$ and $AH = DB$. We have thus obtained the chain of equalities:

$$AD = BH = A'^*B = A'B',$$

which, since $AD = k\,AB$ by (2) and by construction and with k as in (1), implies:

$$A'B' = k\,AB.$$

With analogous reasoning, it can be proved that $B'C' = k\,BC$, $A'C' = k\,AC$. Hence, triangle $A'B'C'$ is similar to triangle ABC.

All identities involved in the triangles similarity demonstration are gathered in (4)–(5), where k is given by (1), and they can be seen in Figure 1.

$$(4) \qquad \frac{AD}{AB} = \frac{A'B'}{AB} = k\,, \qquad \frac{BE}{BC} = \frac{B'C'}{BC} = k\,, \qquad \frac{CF}{AC} = \frac{A'C'}{AC} = k\,,$$

$$(5) \qquad\qquad A'B' = AD\,, \qquad B'C' = BE\,, \qquad A'C' = CF\,.$$
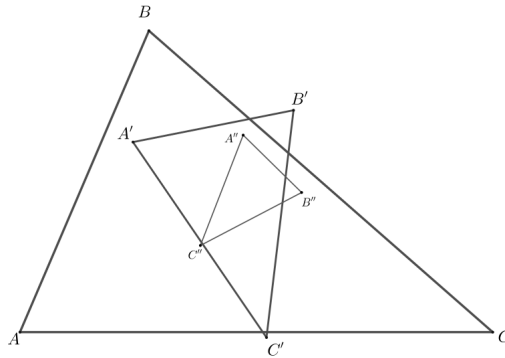
FIGURE 3. Succession of triangles, all similar to ABC

If $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{A}$, $\mathcal{A}'$ respectively are perimeters and areas of triangles ABC and $A'B'C'$, relations (4) imply:

$$\frac{\mathcal{P}'}{\mathcal{P}} = k \,, \qquad \frac{\mathcal{A}'}{\mathcal{A}} = k \,.$$

Figure 3 shows how, proceeding in the same way that led from ABC to $A'B'C'$, from the latter a new triangle $A''B''C''$ can be constructed, and so on, obtaining a sequence of similar triangles.

Once the similarity between the two triangles has been ascertained, it is necessary to determine the *transformation center*, that is the point O, in the plane of the figures, corresponding to which there is equality of the rotation angles leading A into A', B into B', C into C'. To this aim, Lemma 2.1 is employed [1, 6].

Lemma 2.1. Let ABC, $A'B'C'$ be triangles corresponding to each other through a spiral similarity. Let P be the intersection of lines AA' and BB', as shown in Figure 4. The two circles through A, B, P, and A', B', P,
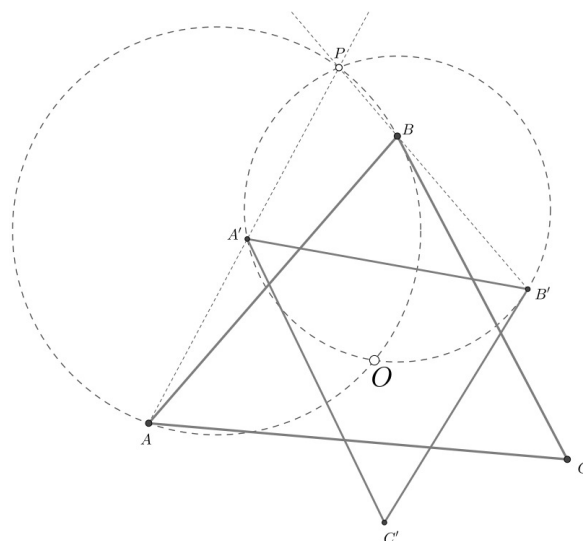

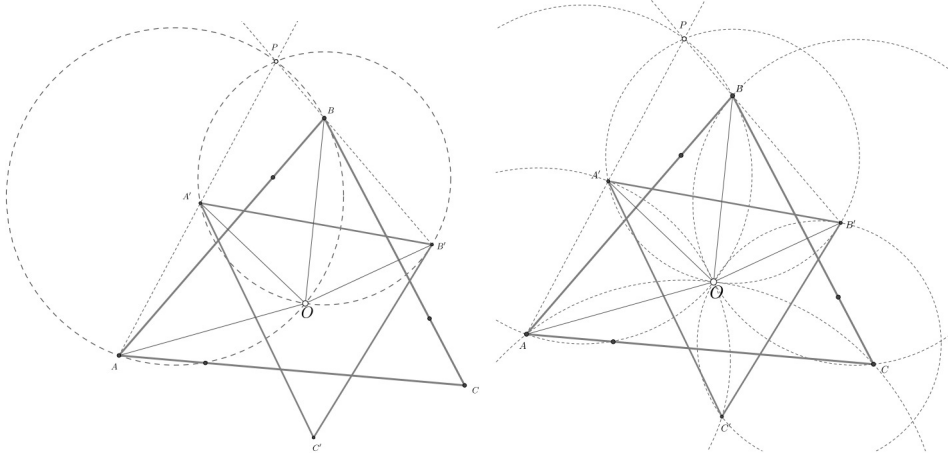FIGURE 4. Determining the center of spiral similarity

FIGURE 5. Cyclic quadrilaterals (left); complete determination
of spiral similarity center (right)

respectively, also meet at $O \neq P$. Then, O is the center of the spiral
similarity taking AB into $A'B'$. □

Lemma 2.1 is demonstrated here, for completeness, and illustrated in Figure 5 (left).

Let P be the intersection of lines AA' and BB', consider the two circles
through A, B, P, and A', B', P, and let $O \neq P$ be their intersection.
Quadrilaterals $AOBP$ and $A'OB'P$ are *cyclic*, i.e. each of them is inscribed
in a circle. Thus, the following angles relations hold:

$$\widehat{OBP} = \pi - \widehat{OAP}, \qquad \widehat{PA'O} = \pi - \widehat{PB'O},$$

implying:

$$\widehat{OBB'} = \pi - \widehat{OAP}, \qquad \widehat{AA'O} = \pi - \widehat{PB'O}.$$

It follows that triangles OAA' and OBB' are similar, as they further verify
$\widehat{AOA'} = \widehat{BOB'}$, and O is their similarity center, since $\widehat{AOB} = \widehat{A'OB'}$.
In other words, AB is transformed into $A'B'$ through the clockwise $\widehat{AOB}$
angle rotation. The same construction can be used for the other two pairs
of corresponding sides of ABC, $A'B'C'$, as shown in Figure 5 (right); thus
O is the spiral similarity center for the two triangles (*qed*).

The transformation obtained is an *homothety*: referring to Figures 1 and 6
(left) as original and final configurations, if ABC is rotated around B so
that $AB \| A'B'$, and if analogous rotations are applied to each triangle in the
sequence of triangles similar to ABC, then the three lines joining triangles
correspondent vertices intersect at point O, which is the center of direct
homothety; since, by Thales' Theorem:

$$OB : OB' = A'B' : A''B'' = k,$$

the homothety is a *contraction* reducing the various triangles by $k < 1$
in (1).

Another demonstration is as follows. Consider Figures 1 and 6 (center) as
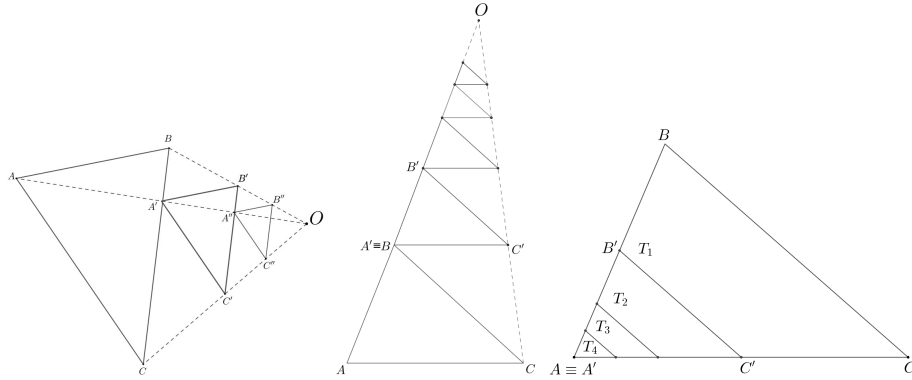initial and final configurations: triangle $A'B'C'$ is translated so that points

FIGURE 6. Direct homothety of triangles: three methods of verification

A' and B coincide, and rotated so that $A'C'\|AC$. If analogous operations are performed on each triangle in the sequence of triangles similar to ABC, then an homothety is obtained, with ratio k as in (1) and centered at point O.

A further way of considering this homothety is shown in Figure 6 (right): the latter is obtained, starting from Figure 1, by applying to $A'B'C'$ a translation whose effect is to make points A' and A coincide, and to make $A'C'$ and AC overlap. In this case, after performing analogous operations on each triangle in the sequence of triangles similar to ABC, the various homothetic triangles are enclosed one inside the other.

2.1. **Spiral similarity of particular triangles.** Special cases of this triangles transformation can be treated as the acute–angled triangle general case. Figure 7 illustrates how the various triangles get rotated and reduced according to an inverse similitude having coefficient k given in (1).

In particular, as displayed in Figure 8 (left) for $k = 1/2$, if the above transformation is performed on an equilateral triangle, then a sequence of equilateral intouch triangles emerges, and the spiral similarity center coincides with the initial triangle barycenter; the simple proof is not provided
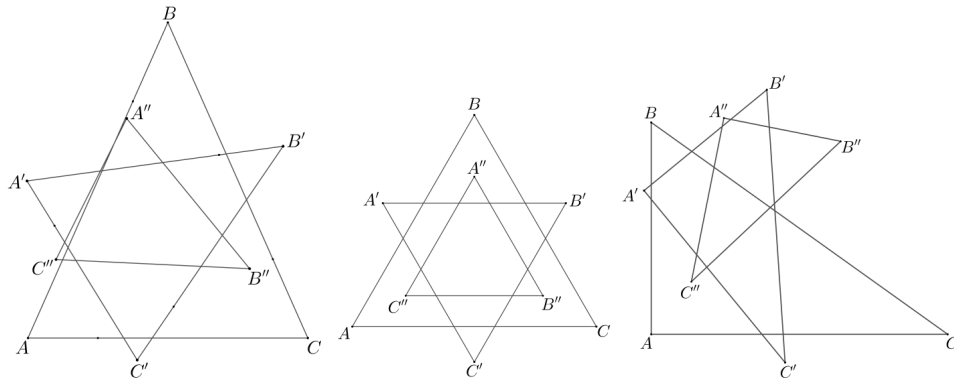

FIGURE 7. Spiral similarity for an isosceles, an equilateral and a right triangle
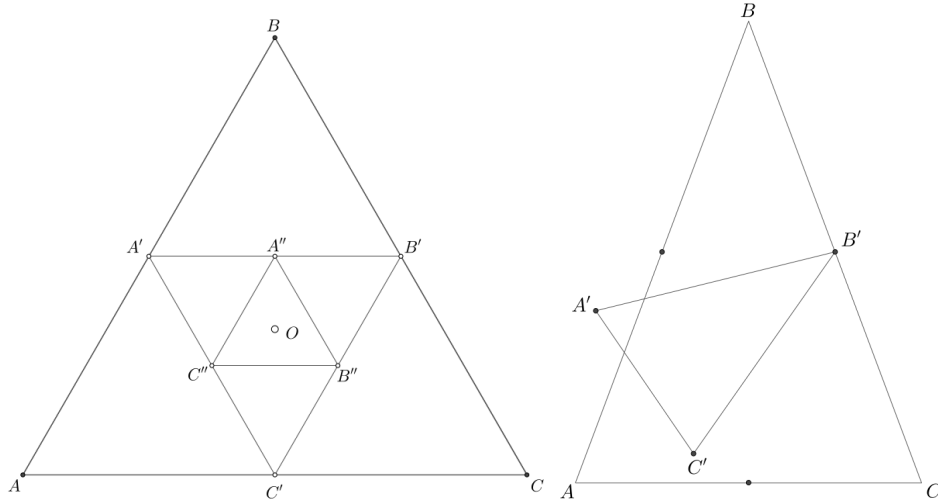
FIGURE 8. Cases of equilateral (left) and isosceles (right) triangles with $k = 1/2$

here. If the transformation is performed on a non–equilateral triangle, then a sequence of inscribed triangles is no longer be obtained. Figure 8 (right) shows it when the initial triangle is isosceles, and again with $k = 1/2$.

2.2. **Links between triangles, rotation angle and k–ratio.** We examine, here, some dependencies that exist between the angle θ of rotation, the original and transformed triangles and the similarity $k-$ratio. In particular, we show that θ depends on the given triangle but not on k; moreover, observing that $0 < k < 1$ by definition (1), we analyse the limit of the image triangle as k tends to 1.

Angle θ depends on the triangle considered. This is shown in Figure 9, where left and right images illustrate, respectively, the transformation $A'B'C'$ of an acute or obtuse triangle ABC. In both cases, the rotation is centered at point O and the similarity ratio is $k = A'B'/AB = 3/4$. The rotation
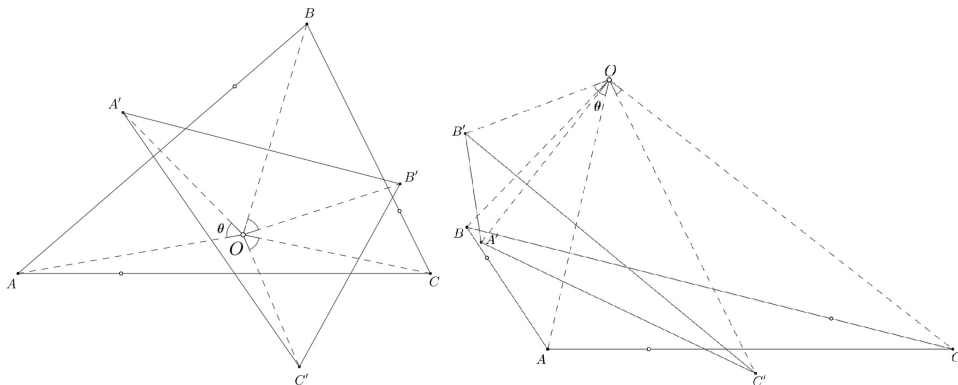

FIGURE 9. The rotation angle depends on the triangle considered. The similarity ratio is $k = A'B'/AB = 3/4$ in both acute/obtuse triangles (left/right).
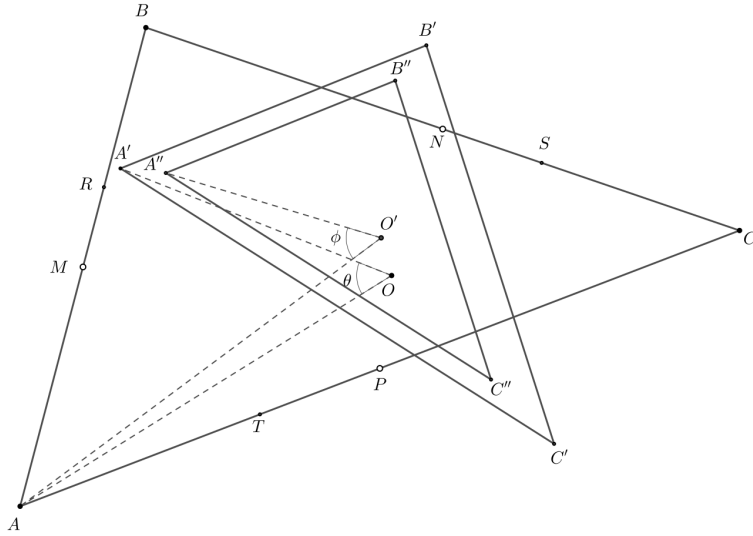
FIGURE 10. For a given triangle, the rotation angle does not depend on the k-ratio. ABC is transformed into $A'B'C'$ via a similarity of ratio $k_1 = A'B'/AB = 2/3$ and angle θ; a second similarity takes ABC into $A''B''C''$ via angle ϕ and ratio $k_2 = A''B''/AB = 1/2$. It can be proved that $\theta = \phi$.

angle differs in the two cases considered: it is $\theta = 55.11^o$ in the acute triangle transformation, while $\theta = 25.42^o$ in the obtuse triangle similarity. Angle θ does not depend on the k-ratio. To see this, let us apply two transformations, with different k-ratios, to the same triangle ABC. In Figure 10, ABC is transformed into $A'B'C'$ via a similarity that is centered at point O and that has angle θ and ratio k_1, while a second similarity, centered in O', takes ABC into $A''B''C''$ via an angle ϕ and ratio k_2, where:

$$k_1 = \frac{A'B'}{AB} = \frac{AR}{AB} = \frac{2}{3}, \qquad k_2 = \frac{A''B''}{AB} = \frac{AM}{AB} = \frac{1}{2}.$$

It is possibile to prove the identity $\theta = \phi$, from which it follows that the rotation angle is independent of the similarity ratio. The proof relies on analytical geometry and is rather long, therefore we do not include it here. To analyse the limit of the image triangle as k tends to 1, consider for a given triangle its transformations according to similarity ratios $k \lessgtr 1$. With reference to Figure 11, we can consider, for example, the acute triangle ABC and its transformations obtained with ratios:

$$k_n = \frac{2^n - 1}{2^n}.$$

Then, $k_n \longrightarrow 1-$ for $n \longrightarrow \infty$, and we obtain a sequence of triangles enclosed one inside the other, and with centers of rotation $O, O', O'', \dots, O^n, \dots$, corresponding to each of the prescribed ratio k_n. These rotation centers translate in such a way that the angle θ of rotation of the different transformations remains unchanged. Furthermore, since the various transformed triangles are homothetic, they have parallel corresponding sides. In other
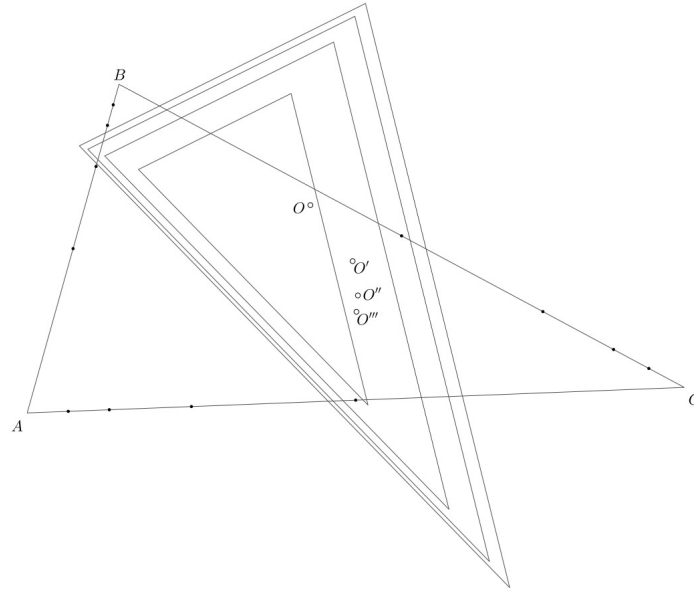
FIGURE 11. Study of the limit of the image triangle, for a given ABC, as $k \longrightarrow 1$

words, all image triangles (with $k_n \longrightarrow 1-$) tend to a triangle congruent to the original one rotated by the angle θ.

3. SPIRAL SIMILARITY OF TWO CONSECUTIVE SEGMENTS

Construction of the spiral similarity of a triangle, according to a proportionality factor k, also allows obtaining the spiral similarity of a pair of consecutive segments.

Consider AB and BC in Figure 12; they are consecutive segments with:
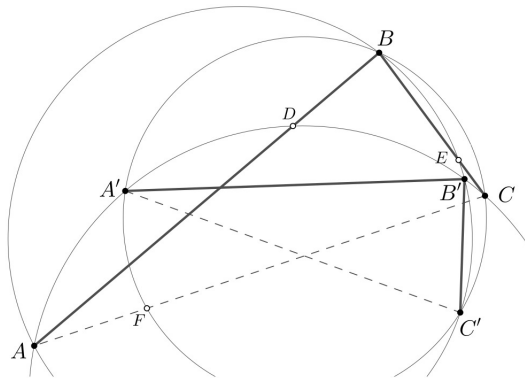
$$\frac{AB}{AD} = \frac{BC}{BE} = k.$$
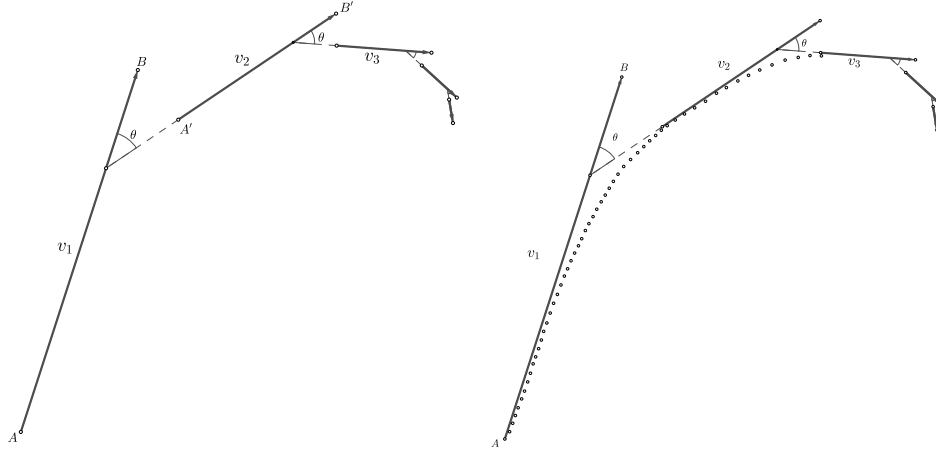

FIGURE 12. Spiral similarity of two consecutive segments

FIGURE 13. Physical interpretation associated to our spiral similarity construction

Spiral similarity of triangle ABC is constructed via the auxiliary segment AC, with:

$$\frac{AC}{FC} = k.$$

Then, segments $A'B'$ and $B'C'$ are the transforms of AB and BC.

3.1. **An heuristic application.** As seen in the previous section, our construction of the spiral similarity allows to tranform a segment by regarding it as a triangle side. Taking this idea further, we can consider a vector whose modulus is given by the length of the tranformed segment. In a sequence of transformed segments, the corresponding vectors turn out to be tangent to the spiral considered (see Figure 13), thus they may be thought of as velocities.

This suggests that a physical interpretation may be associated to our spiral similarity. Figure 13 (left) illustrates the effect of applying a spiral similarity of ratio $k = m/n$ and angle θ to vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \dots$, and so on: the modulus of each $\mathbf{v_k}$ becomes m/n of the modulus of the previous vector $\mathbf{v_{k-1}}$, after reaching a rotation of θ.

In other words, vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \dots$, can be interpreted as related to a particle of mass m, which follows a spiral trajectory, as shown in Figure 13 (right), and whose speed gradually decreases, until when, at each θ rotation, its modulus becomes m/n of that of the initial velocity. Hence, it is a decelerated motion of an elementary particle, which could be what happens in a bubble chamber.

4. POLYGONAL SPIRALS

Consider again the sequence of triangles ABC, $A'B'C'$, $A''B''C''$, and so on, generated by spiral similarity. Join all points A, A', A'', etcetera; then, join points $B, B', B'', \dots$; finally, join points C, C', C'', etc. In this way, as visualized in Figure 14, three inverse and similar polygonal spirals are
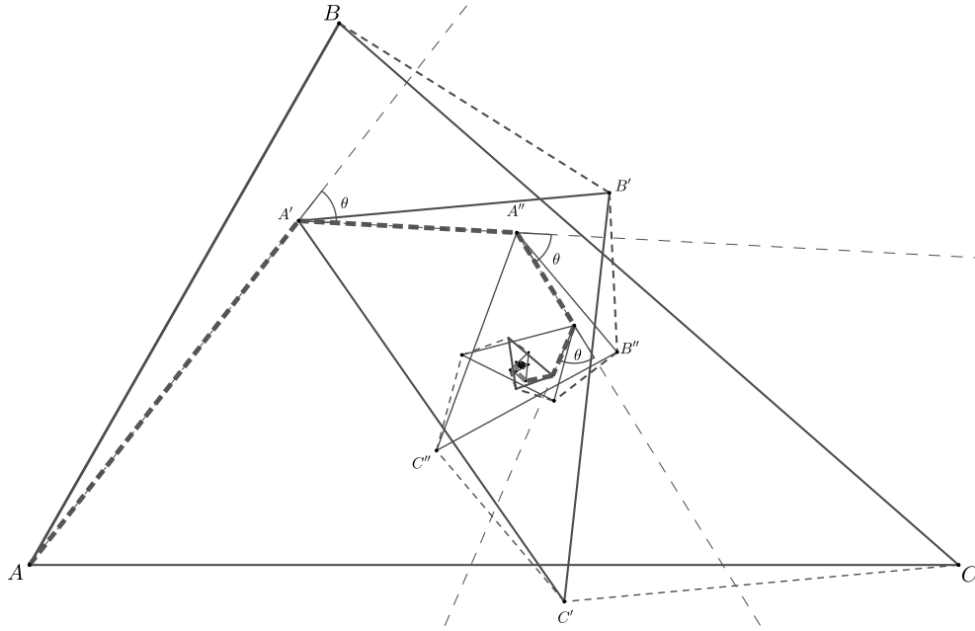
FIGURE 14. Polygonal spirals

obtained, in which the various sides verify (with respect to each other) the
ratio k, and follow each other with all congruent angles.

In fact, it is sufficient to observe that the various homothetic triangles are
all rotated, with respect to each other, by the same angle, so that vertex
A' is arranged, with respect to A, in the same way in which vertex A'' is
arranged with respect to A'; thus, AA' and $A'A''$ are always homothetic
with respect to spiral similarity; analogous considerations apply to each
triangle in the sequence of triangles similar to ABC.

Spirals consist of segments whose lengths are in geometric progression of the
ratio k; hence, they are *logarithmic polygonal spirals*, and the length of each
of them is:

$$\sum_{i=1}^{\infty} k^i \, a \, ;$$

a is the length of the first segment of a chosen polygonal spiral, and k is
as in (1).

In this way, polygonal spirals of infinite length are obtained, since $k < 1$
implies that successive segments forming each spiral tend to zero.

Given any of these spirals, its *pole* is the point (in the plane) with respect
to which each segment (of the spiral) is subtended by an angle congruent to
the one formed by each side (in the spiral) with the previous side extension;
refer to Figure 14.

In other words, each logarithmic polygonal spiral, with first segment of
length a, rotation angle θ, and reduction factor k, converges to the vertex
opposite to the first a–length segment (on which a triangle is constructed),
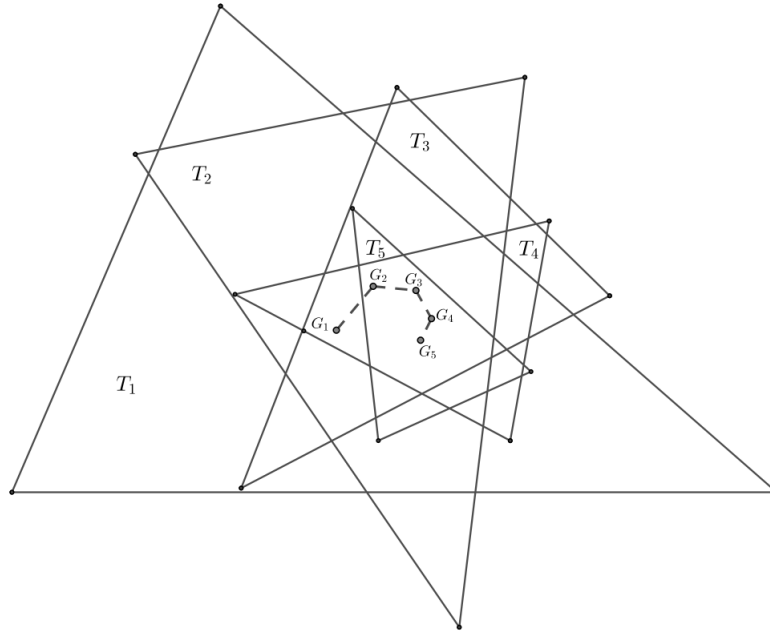so that the angle opposite to this segment is θ, and so that the other two

FIGURE 15. Spiral of barycenters

triangle sides, of lenght b, c, respectively, are in the relation prescribed by k, i.e. $c/b = k$, with $k = m/n$ as defined in (1); refer again to Figure 14. Now, by Carnot's Theorem:

$$(6) \qquad\qquad a^2 = b^2 + c^2 + 2\,b\,c\,\cos(\theta)\,.$$

Since a, θ are known, equalities (6) and $c = b\,k = b\,m/n$ allow to determine b, c :

(7)
$$b = n\ \mathcal{K}\,, \qquad\qquad c = m\ \mathcal{K}\,, \qquad\qquad \mathcal{K} = \frac{a}{\sqrt{m^2 + n^2 + 2\,m\,n\,\cos(\theta)}}\,.$$

For $k = m/n = 1/2$:

$$b = 2\ \mathcal{K}\,, \qquad\qquad c = \mathcal{K}\,, \qquad\qquad \mathcal{K} = \frac{a}{\sqrt{5 + 4\,\cos(\theta)}}\,.$$

This allows to construct the spiral pole (refer, once more, to Figure 14), which is in common to the other two spirals, due to the above–mentioned reasons. Furthermore, if the spiral similarity center is constructed as in Figure 5 (right), it is possible to verify that it coincides with the pole of the various polygonal spirals.

Before leaving this section, let us make an interesting observation using Figure 15. Consider barycenters G_1, G_2, G_3, etc., of the various triangles $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, ..., that correspond to each other in the spiral similarity: barycenters G_k are vertices of a polygonal spiral, whose sides verify, with respect to each other, ratio k of the spiral similarity between triangles. $\mathcal{T}_k$.
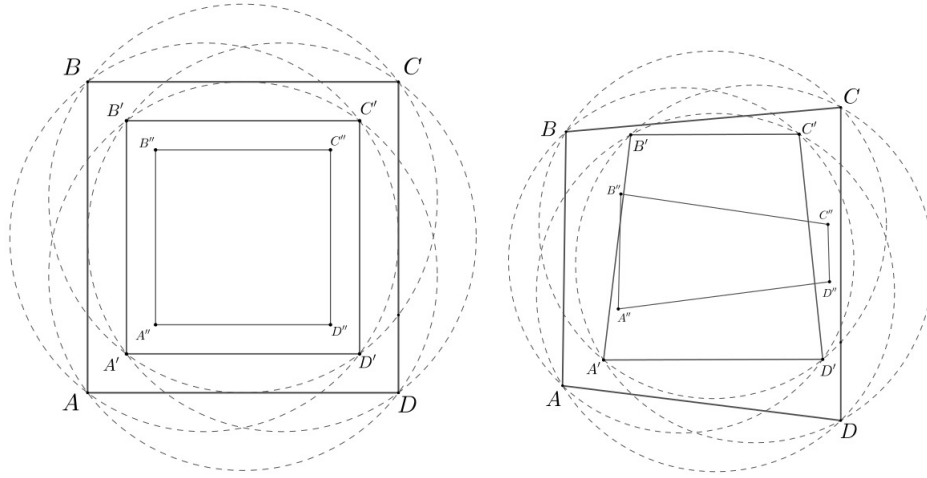
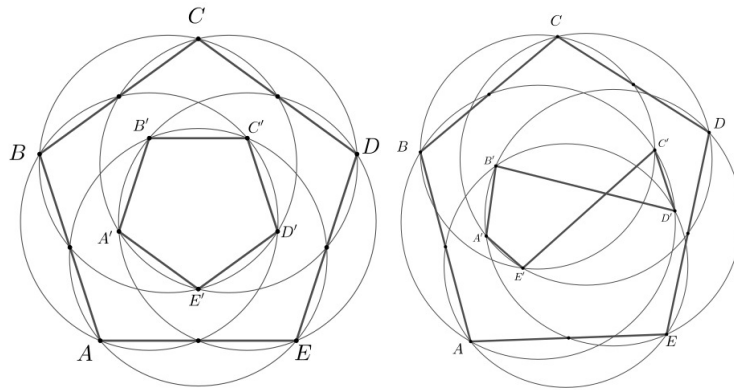FIGURE 16. Cases of square (left) and scalene (right) quadrilaterals

FIGURE 17. Cases of regular (left) and irregular (right) pentagons

5. THE CASE OF POLYGONS WITH MORE THAN THREE SIDES

If the same construction described for triangles is applied to a polygon with more than three side, then a k–ratio inverse similarity is obtained only if the starting polygon is regular. Figures 16 and 17 illustrates the cases of regular/irregular quadrilaterals and pentagons. Moreover, applied to a polygon with t sides, the presented construction creates $t-2$ polygons, among which one (only) turns out to be similar to the originator; the remaining $t-3$ generated polygons are still similar to each other, but no longer according to ratio k.

In other words, starting from a triangle ($t=3$), each iteration provides a single triangle similar to the initial one; application to a square ($t=4$) yields two squares, one of which is similar to the original by spiral similarity of ratio k; from a pentagon ($t=5$) three pentagons are created, with one k–ratio spirally similar to the originator; and so on. In Figure 18, only square $A'B'C'D'$ corresponds by k–ratio spiral similarity to the original
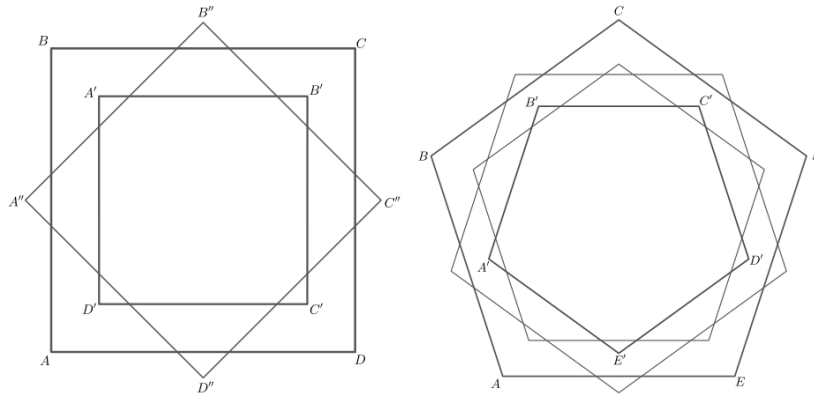
FIGURE 18. Cases of square (left) and of regular pentagon (right)

square $ABCD$; and analogously, only pentagon $A'B'C'D'E'$ corresponds to the initial pentagon $ABCDE$.

6. The three–dimensional case

Results of the transformation of planar figures by spiral similarity can be interpreted in three–dimensional space. Let us see it employing, as an example, triangle ABC of Figure 1, thought of as base of a right–angled tetrahedron or prism. Assume the case of a tetrahedron $ABCD$ with triangular base ABC, as in Figure 19, and define:

$$k = \frac{AB}{AM} = \frac{BC}{BM'} = \frac{AC}{CM''}$$

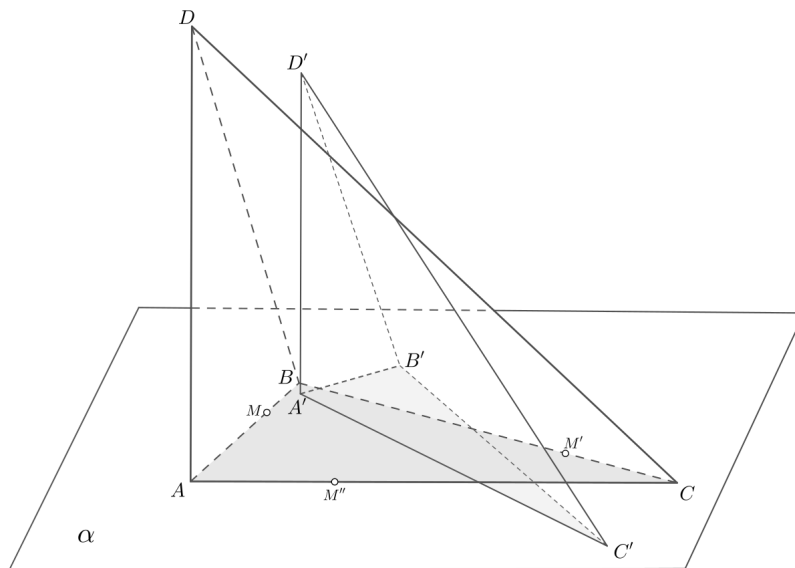to be the ratio according to which sides of ABC (belonging to plane α) are divided.

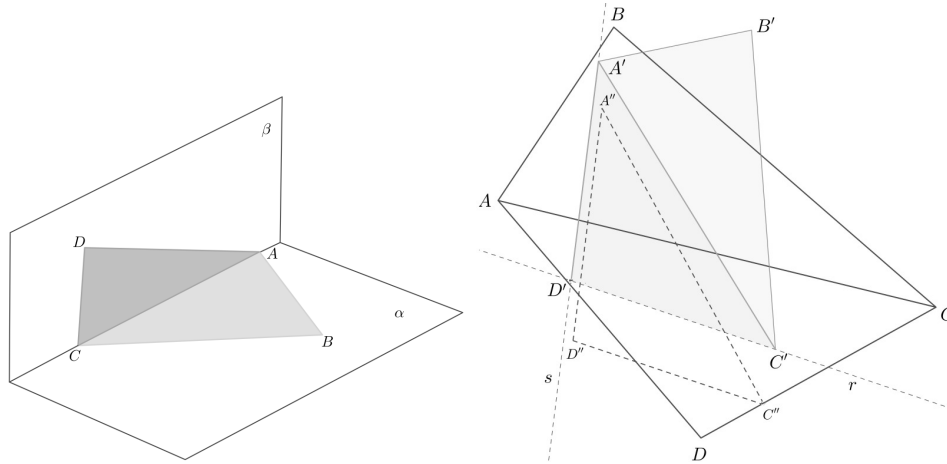FIGURE 19. Transformation of the tetrahedron

FIGURE 20. Dihedral (left) and spiral similarity of its two triangles (right)

Now, consider the planar *development* (or planar *net*) of $ABCD$ on the α–plane, and let $A'B'C'$ be the transformation of base ABC by k–ratio spiral similarity. Applying the same spiral similarity to all other faces of the development of $ABCD$, and recomposing the solid, the transformed tetrahedron $A'B'C'D'$ is obtained; compared to the initial tetrahedron, $A'B'C'D'$ is contracted by k.

As a general illustrative example, we perform the spiral similarity transformation of a simple solid figure: Figure 20 (left) displays a *dihedral* formed by two triangles, ABC and ACD, belonging to two different planes α and β, in three–dimensional space.

Let $A'B'C'$ and $A''C''D''$ be k–ratio transformations of triangles ABC, ACD, respectively. Now, determine line r through C' and parallel to side $C''D''$, and line s through A' and parallel to side $A''D''$: these two lines meet at D', as shown in Figure 20 (right). We have thus obtained the transformed $A'B'C'D'$ planar development of $ABCD$ by spiral similarity, from which the corresponding spatial figure can be recovered.

In other words, the construction we have presented can be applied to planar developments of geometric solids. As a further example, Figure 21 depicts the spiral similarity, of ratio $k = 3/2$, of the planar development $\mathcal{C}'$ of a cube $\mathcal{C}$.
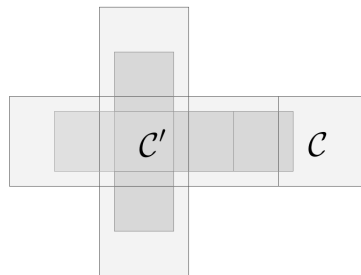

FIGURE 21. Spiral similarity of the planar development of a cube

7. Conclusion

Spiral similarity derives its name by its two components, the first of which is a rotation, defined by a center and an angle, that causes a spiral movement of the figure to which it is applied. In the direct transformation, rotation is composed with a dilation by a constant ratio k; in inverse spiral similarity, rotation is followed by a k–ratio contraction. In both cases, the rotated transformed figure is k–similar to the originator.

The inventor of spiral similarity seems to be unknown, in the history oh mathematics; its first documentation can be found in [3]. Nowadays, this transformation is commonly used in solving problems of Euclidean geometry, in particular during mathematical competitions. In general, it is a useful geometric tool that deserves to be investigated, which is what we meant to do in this work.

References

[1] Baca, J., *On a special center of spiral similarity*, Mathematical Reflections, **1 (2019)**, 1–9.

[2] Chen, E., *Euclidean Geometry in Mathematical Olympiads*, American Mathematical Society Press, Providence, RI, USA, 2016.

[3] Coxeter, H.S.M. and Greitzer, S.L., *Geometry Revisited*, Mathematical Association of America, Washington DC, USA, 1967.

[4] Weisstein, E.W., *Spiral Similarity*, MathWorld–A Wolfram Web Resource, Wolfram Research, Champaign, IL, USA, © 1999–2022.

[5] Wells, D., *The Penguin Dictionary of Curious and Interesting Geometry*, Penguin, London, UK, 1991.

[6] Zhao, Y., *Three Lemmas in Geometry*, Math Olympiad training handouts, MIT, Cambridge, MA, USA, 2010.

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