PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 17 · 2024 · Pages 1–11

A compact algebraic representation of cardinal GB-splines

Lucia Romani[a] · Milvia Rossini[b] · Alberto Viscardi[c,*]

Communicated by Costanza Conti

Abstract

This work introduces a compact algebraic representation of generalized B-spline basis functions built upon uniform knot partitions (also known as cardinal GB-splines), that stands out for its simplicity with respect to the well-known integral formulation. Moreover, this result clarifies the relationship between cardinal GB-splines and classical polynomial B-splines, as it isolates the polynomial component of a GB-spline from the non-polynomial contribution brought by the two non-monomial generators of the function space.

Keywords: generalized B-splines (GB-splines); uniform knots; cardinal GB-splines; algebraic representation

1 Introduction and purpose of the work

Generalized B-splines (GB-splines for short), first introduced by Kvasov in [15, 16], are special types of Chebyshevian splines [1, 2, 4, 23]. Chebyshevian splines are a generalization of classical polynomial splines, where each spline piece belongs to the same extended Chebyshev space, rather than to a polynomial space. Extended Chebyshev spaces (see, e.g., [4, Definition 1]) allow the presence of transcendental functions in addition to polynomials, that can be exploited, e.g., to reproduce circles and other shapes that cannot be represented by polynomials only. Whenever the underlying extended Chebyshev space is spanned by all the monomials up to a certain degree and two other special functions, we are in the domain of GB-splines. GB-splines include cycloidal (or helix) splines [5], hyperbolic-polynomial splines [20], Unified-Extended splines (or shortly UE-splines) [30] and, of course, polynomial B-splines [24, 28]. In the past few years GB-splines received more and more attention due to their increasing number of applications which currently range from geometric design [7, 12, 14, 17, 19, 25–27] and numerical analysis (especially quadrature formulas, differentiation and numerical solutions of linear Fredholm integral equations) [11, 18] to isogeometric analysis [8, 13, 21, 22] and imaging [6, 9, 10]. Differently from [4, 29], where algorithms for evaluating general forms of Chebyshevian B-splines have been proposed, we here focus our attention on GB-splines built upon uniform knot partitions, and we address our efforts towards underlying their connection with cardinal polynomial B-splines. The goal of this paper is indeed to present an elegant algebraic expression of cardinal GB-splines that stands out for its compactness and its simplicity with respect to the well-known integral formulation. The proposed expression is intended for having an easy-to-manipulate general symbolic formulation for cardinal GB-splines and not for numerical evaluation, as it is well known that numerical instabilities might arise (see, e.g., [4, 29]). To pave the way for our result, we first recall from [15] some preliminary notions (Section 2) that are later exploited to get our novel algebraic representation of cardinal GB-splines (Section 3). Finally, we exploit the obtained expression of cardinal GB-spline basis functions of arbitrary order, to get back the algebraic expressions of some known instances and work out new non-trivial examples (Section 4). Section 5 concludes the paper with some final remarks.

2 Cardinal GB-splines: preliminary notions

For a fixed interval length $h > 0$, we consider the uniform partition of $\mathbb{R}$ given by the intervals $\{\mathcal{I}_k := [hk, h(k+1)]\}_{k\in\mathbb{Z}}$. Let $p \in \mathbb{N} \setminus \{1\}$ and $U, V : \mathcal{I}_0 \to \mathbb{R}$ be such that

(i) $U, V \in \mathcal{C}^{p-1}(\mathcal{I}_0)$;

(ii) the quantity

$$\Delta := U^{(p-1)}(0)\, V^{(p-1)}(h) - V^{(p-1)}(0)\, U^{(p-1)}(h),$$

[a]Dipartimento di Matematica, Alma Mater Studiorum Università di Bologna, Italy (lucia.romani@unibo.it)
[b]Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, Italy (milvia.rossini@unimib.it)
[c]Dipartimento di Matematica 'Giuseppe Peano', Università di Torino, Italy (alberto.viscardi@unito.it)

is non-zero and the functions

$$u(x) \;:=\; a\,U^{(p-1)}(x) \;+\; b\,V^{(p-1)}(x) \qquad \text{and} \qquad v(x) \;:=\; c\,U^{(p-1)}(x) \;+\; d\,V^{(p-1)}(x), \tag{1}$$

where

$$a := V^{(p-1)}(h)/\Delta, \qquad b := -\,U^{(p-1)}(h)/\Delta, \qquad c := -\,V^{(p-1)}(0)/\Delta, \qquad d := U^{(p-1)}(0)/\Delta,$$

are such that $\{u, v\}$ is a Chebyshev system on $\mathcal{I}_0$, i.e., any non-trivial element in $\operatorname{span}\{u, v\}$ has at most one zero in $\mathcal{I}_0$. In particular, due to (1), we have $u(0) = v(h) = 1$ and $u(h) = v(0) = 0$.

We are interested in the space of GB-splines of order $p+1$ given by

$$\mathcal{S}_p^{U,V} \;:=\; \left\{ f \in \mathcal{C}^{p-1}(\mathbb{R}) \;:\; f(\cdot + hk)|_{\mathcal{I}_0} \in \mathcal{P}_p^{U,V},\; k \in \mathbb{Z} \right\},$$

where

$$\mathcal{P}_p^{U,V} \;:=\; \operatorname{span}\left\{ \{(\cdot)^j\}_{j=0}^{p-2},\; U,\; V \right\}.$$

The goal of this work is to present a fully explicit and compact expression of the cardinal GB-spline basis function associated to $\mathcal{S}_p^{U,V}$, i.e., the function $\phi_p^{U,V} \in \mathcal{C}^{p-1}(\mathbb{R})$ such that

(a.1) $\phi_p^{U,V} \;\in\; \mathcal{S}_p^{U,V}$;

(b.1) $\operatorname{supp}(\phi_p^{U,V}) \;=\; [0, h(p+1)] = \bigcup_{k=0}^{p} \mathcal{I}_k$;

(c.1) for every $f \;\in\; \mathcal{S}_p^{U,V}$,

$$f(x) \;=\; \sum_{k\in\mathbb{Z}} f_k\, \phi_p^{U,V}(x - hk),$$

for some $\{f_k \in \mathbb{R}\}_{k\in\mathbb{Z}}$.

Let

$$\varphi_1^{u,v}(x) \;:=\; \delta_1^{u,v} \begin{cases} v(x), & \text{if } x \in \mathcal{I}_0 = [0,h], \\[2ex] u(x-h), & \text{if } x \in \mathcal{I}_1 = [h,2h], \\[2ex] 0, & \text{otherwise,} \end{cases} \qquad \text{with} \quad \delta_1^{u,v} \;:=\; \left(\int_{\mathcal{I}_0} v(y)dy \;+\; \int_{\mathcal{I}_1} u(y-h)dy \right)^{-1}. \tag{2}$$

Starting with $\varphi_1^{u,v}$ we can then proceed, as in the polynomial case, with the construction of $\phi_p^{U,V}$ via the following recursive relation

$$\begin{aligned} \varphi_{q+1}^{u,v}(x) \;&=\; \frac{1}{h}\,\varphi_q^{u,v} * \chi_{\mathcal{I}_0}(x) \;=\; \frac{1}{h}\int_{x-h}^{x} \varphi_q^{u,v}(y)\,dy \\ &=\; \frac{1}{h}\int_0^x \left(\varphi_q^{u,v}(y) \;-\; \varphi_q^{u,v}(y-h) \right) dy, \end{aligned} \qquad q \in \{1, \ldots, p-1\}, \tag{3}$$

which yields $\phi_p^{U,V}(x) = \varphi_p^{u,v}(x)$. In particular, defining for every $q \in \{1, \ldots, p-1\}$,

$$\mathcal{D}^q \mathcal{S}_p^{U,V} \;:=\; \left\{ f \in \mathcal{C}^{p-q-1}(\mathbb{R}) \;:\; f(\cdot + hk)|_{\mathcal{I}_0} \in \mathcal{D}^q \mathcal{P}_p^{U,V},\; k \in \mathbb{Z} \right\},$$

with

$$\mathcal{D}^q \mathcal{P}_p^{U,V} \;:=\; \begin{cases} \operatorname{span}\left\{ \{(\cdot)^j\}_{j=0}^{p-q-2},\; U^{(q)},\; V^{(q)} \right\}, & \text{if } q < p-1, \\[2ex] \operatorname{span}\{U^{(p-1)}, V^{(p-1)}\} \;=\; \operatorname{span}\{u, v\}, & \text{if } q = p-1, \end{cases}$$

we have that

(a.2) $\varphi_q^{u,v} \;\in\; \mathcal{D}^{p-q} \mathcal{S}_p^{U,V}$;

(b.2) $\operatorname{supp}(\varphi_q^{u,v}) \;=\; [0, h(q+1)] = \bigcup_{k=0}^{q} \mathcal{I}_k$;

(c.2) for every $f \;\in\; \mathcal{D}^{p-q} \mathcal{S}_p^{U,V}$,

$$f(x) \;=\; \sum_{k\in\mathbb{Z}} f_k\, \varphi_q^{u,v}(x - hk),$$

for some $\{f_k \in \mathbb{R}\}_{k\in\mathbb{Z}}$.

Remark 1. We emphasize that:

(I) If $U(x) = V(h-x)$, then $\phi_p^{U,V}$ is symmetric within its support.

(II) There is a trade off between normalization and partition of unity. Indeed,

$$\begin{aligned}\int_{\mathbb{R}} \phi_p^{U,V}(y)\,dy = \int_{\mathbb{R}} \varphi_p^{u,v}(y)\,dy &\underset{(3)}{=} \frac{1}{h}\int_{\mathbb{R}} \varphi_{p-1}^{u,v} * \chi_{\mathcal{I}_0}(y)\,dy = \frac{1}{h}\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi_{p-1}^{u,v}(y-z)\chi_{\mathcal{I}_0}(z)\,dz\,dy \\ &= \frac{1}{h}\int_{\mathbb{R}} \chi_{\mathcal{I}_0}(z)\,dz \int_{\mathbb{R}} \varphi_{p-1}^{u,v}(y)\,dy = \int_{\mathbb{R}} \varphi_{p-1}^{u,v}(y)\,dy \\ &\underset{(3)}{=} \cdots \underset{(3)}{=} \int_{\mathbb{R}} \varphi_1^{u,v}(y)\,dy \underset{(2)}{=} 1\end{aligned}$$

and

$$\sum_{k\in\mathbb{Z}} \phi_p^{U,V}(x-hk) = \frac{1}{h}\sum_{k\in\mathbb{Z}} \int_{x-h(k+1)}^{x-hk} \varphi_{p-1}^{u,v}(y)\,dy = \frac{1}{h}\int_{\mathbb{R}} \varphi_{p-1}^{u,v}(y)\,dy = \frac{1}{h}.$$

Thus, only for $h = 1$, it is possible to have

$$1 = \sum_{k\in\mathbb{Z}} \phi_p^{U,V}(x-k) = \int_{\mathbb{R}} \phi_p^{U,V}(y)\,dy, \qquad p \in \mathbb{N}\setminus\{1\}.$$

3 Cardinal GB-splines: a compact algebraic expression of the basis functions

This section contains the main result of this work. Precisely, the following theorem provides a fully explicit and compact representation of the cardinal GB-spline basis function $\phi_p^{U,V}(x)$, that isolates its polynomial component and expresses it in terms of linear combinations of classical polynomial B-splines of degree $s \le p-2$ built upon the uniform knot partition $h\mathbb{Z}$.

As it is well-known [24, 28], for $k \in \mathbb{Z}$ and $B_{k,0}(x) := \chi_{\mathcal{I}_k}(x)$, the degree-$s$ polynomial B-spline $B_{k,s}$, $s \in \mathbb{N}$, can be defined via the convolutional recursion

$$B_{k,s}(x) := \frac{1}{h} B_{k,s-1} * B_{k,0}(x), \tag{4}$$

or, more explicitly, via the recurrence relation

$$B_{k,s}(x) = \frac{x-hk}{hs} B_{k,s-1}(x) + \frac{h(k+1+s)-x}{hs} B_{k+1,s-1}(x). \tag{5}$$

Moreover,

$$B_{k,s}(x) = B_{k-1,s}(x-h) = B_{0,s}(x-hk),$$

and (4) can be rewritten as

$$B_{k,s}(x) = \frac{1}{h}\int_{hk}^{x} \left(B_{k,s-1}(y) - B_{k+1,s-1}(y) \right) dy. \tag{6}$$

For later use, it is convenient to introduce $S, T \in \operatorname{span}\{U, V\}$ such that

$$S^{(p-1)}(x) = u(x) \qquad \text{and} \qquad T^{(p-1)}(x) = v(x),$$

i.e., due to (ii),

$$S(x) = a\,U(x) + b\,V(x) \qquad \text{and} \qquad T(x) = c\,U(x) + d\,V(x). \tag{7}$$

Clearly, for any $p \in \mathbb{N}\setminus\{1\}$, $\mathcal{S}_p^{U,V} = \mathcal{S}_p^{S,T}$ and $\phi_p^{U,V}(x) = \phi_p^{S,T}(x)$.
Moreover, we point out that $\delta_1^{u,v}$ in (2) can be rewritten in terms of S and T as

$$\delta_1^{u,v} = D_{p-2}^{-1} \qquad \text{with} \qquad D_s := T^{(s)}(h) - T^{(s)}(0) + S^{(s)}(h) - S^{(s)}(0), \quad s \in \{0,\dots,p-2\}. \tag{8}$$

Theorem 3.1. *For $U, V : \mathcal{I}_0 \to \mathbb{R}$ satisfying (i) and (ii) and $p \in \mathbb{N}\setminus\{1\}$,*

$$\phi_p^{U,V}(x) = \frac{\delta_1^{u,v}}{h^{p-1}} \left[\sum_{k=0}^{p} \Big(A_{k,p} T(x-hk) + A_{k-1,p} S(x-hk) \Big) B_{k,0}(x) + \sum_{s=0}^{p-2} h^s \sum_{k=0}^{p-s} C_{k,p-s} B_{k,s}(x) \right], \tag{9}$$

where, for every $k \in \mathbb{Z}$,

$$\begin{cases} A_{k,1} := \delta_{k,0}, \ \text{(Kronecker delta)} \\ A_{k,s+1} := A_{k,s} - A_{k-1,s}, \qquad s \in \mathbb{N}, \end{cases} \tag{10}$$

and, for $s \in \{0,\dots,p-2\}$,

$$C_{k,p-s} := \sum_{j=0}^{k-2} A_{j,p-s}\, D_s + A_{k-1,p-s} \left(T^{(s)}(h) - T^{(s)}(0) - S^{(s)}(0) \right) - A_{k,p-s}\, T^{(s)}(0), \tag{11}$$

with D_s specified in (8).

Proof. Due to (3), we compute $\varphi_p^{u,v}(x) = \phi_p^{U,V}(x)$. We start rewriting (2) as

$$\begin{aligned}\varphi_1^{u,v}(x) &= \delta_1^{u,v}\left[\, v(x)B_{0,0}(x) + u(x-h)B_{1,0}(x)\,\right]\\ &= \delta_1^{u,v}\left[\, T^{(p-1)}(x)B_{0,0}(x) + S^{(p-1)}(x-h)B_{1,0}(x)\,\right]\\ &= \delta_1^{u,v}\sum_{k=0}^{1}\left(A_{k,1}\, T^{(p-1)}(x-hk)+A_{k-1,1}\, S^{(p-1)}(x-hk)\right)B_{k,0}(x),\end{aligned} \tag{12}$$

where $A_{k,1} = \delta_{k,0}$, $k\in\mathbb{Z}$.

Then, from (12) and (3), we get

$$\begin{aligned}\varphi_2^{u,v}(x) &= \frac{\delta_1^{u,v}}{h}\int_0^x \Bigg[\sum_{k=0}^{1}\left(A_{k,1}\, T^{(p-1)}(y-hk)+A_{k-1,1}\, S^{(p-1)}(y-hk)\right)B_{k,0}(y)\\ &\qquad -\sum_{k=0}^{1}\left(A_{k,1}\, T^{(p-1)}(y-h(k+1))+A_{k-1,1}\, S^{(p-1)}(y-h(k+1))\right)B_{k,0}(y-h)\Bigg]\, dy\\ &= \frac{\delta_1^{u,v}}{h}\int_0^x \Bigg[\sum_{k=0}^{1}\left(A_{k,1}\, T^{(p-1)}(y-hk)+A_{k-1,1}\, S^{(p-1)}(y-hk)\right)B_{k,0}(y)\\ &\qquad -\sum_{k=1}^{2}\left(A_{k-1,1}\, T^{(p-1)}(y-hk)+A_{k-2,1}\, S^{(p-1)}(y-hk)\right)B_{k-1,0}(y-h)\Bigg]\, dy\\ &= \frac{\delta_1^{u,v}}{h}\int_0^x \sum_{k=0}^{2}\left(\,(A_{k,1}-A_{k-1,1})\, T^{(p-1)}(y-hk)+(A_{k-1,1}-A_{k-2,1})\, S^{(p-1)}(y-hk)\,\right)B_{k,0}(y)\, dy\\ &= \frac{\delta_1^{u,v}}{h}\int_0^x \sum_{k=0}^{2}\left(\,A_{k,2}\, T^{(p-1)}(y-hk)+A_{k-1,2}\, S^{(p-1)}(y-hk)\,\right)B_{k,0}(y)\, dy,\end{aligned}$$

where $A_{k,2} = A_{k,1} - A_{k-1,1}$, $k\in\mathbb{Z}$. Now, for every $k\in\mathbb{Z}$,

$$\begin{aligned}\int_0^x T^{(p-1)}(y-hk)\, B_{k,0}(y)\, dy &= \int_{-hk}^{x-hk} T^{(p-1)}(y)\, B_{0,0}(y)\, dy = \int_0^{\min(h,x-hk)} T^{(p-1)}(y)\, dy\\ &= \begin{cases} 0, & \text{for } x\le hk,\\ T^{(p-2)}(x-hk) - T^{(p-2)}(0), & \text{for } x\in\mathcal{I}_k,\\ T^{(p-2)}(h) - T^{(p-2)}(0), & \text{for } x\ge h(k+1).\end{cases}\end{aligned}$$

A similar argument holds for $S^{(p-1)}$ and so

$$\begin{aligned}\frac{h}{\delta_1^{u,v}}\varphi_2^{u,v}(x)|_{\mathcal{I}_0} &= A_{0,2}\, T^{(p-2)}(x) + A_{-1,2}\, S^{(p-2)}(x) - \left(\, A_{0,2}\, T^{(p-2)}(0) + A_{-1,2}\, S^{(p-2)}(0)\,\right),\\ \frac{h}{\delta_1^{u,v}}\varphi_2^{u,v}(x)|_{\mathcal{I}_1} &= A_{1,2}\, T^{(p-2)}(x-h) + A_{0,2}\, S^{(p-2)}(x-h) - \left(\, A_{1,2}\, T^{(p-2)}(0) + A_{0,2}\, S^{(p-2)}(0)\,\right)\\ &\quad + A_{0,2}\left(\, T^{(p-2)}(h) - T^{(p-2)}(0)\,\right) + A_{-1,2}\left(\, S^{(p-2)}(h) - S^{(p-2)}(0)\,\right)\\ &= A_{1,2}\, T^{(p-2)}(x-h) + A_{0,2}\, S^{(p-2)}(x-h) + A_{0,2}\, T^{(p-2)}(h) + A_{-1,2}\, S^{(p-2)}(h)\\ &\quad - \left(\, A_{0,2} + A_{1,2}\,\right) T^{(p-2)}(0) - \left(\, A_{-1,2} + A_{0,2}\,\right) S^{(p-2)}(0),\end{aligned}$$

$$
\begin{aligned}
\frac{h}{\delta_1^{u,v}} \varphi_2^{u,v}(x)|_{\mathcal{I}_2} &= A_{2,2}\, T^{(p-2)}(x-2h) + A_{1,2}\, S^{(p-2)}(x-2h) - \left(A_{2,2}\, T^{(p-2)}(0) + A_{1,2}\, S^{(p-2)}(0) \right) \\
&\quad + A_{1,2} \left(T^{(p-2)}(h) - T^{(p-2)}(0) \right) + A_{0,2} \left(S^{(p-2)}(h) - S^{(p-2)}(0) \right) \\
&\quad + A_{0,2} \left(T^{(p-2)}(h) - T^{(p-2)}(0) \right) + A_{-1,2} \left(S^{(p-2)}(h) - S^{(p-2)}(0) \right) \\
&= A_{2,2}\, T^{(p-2)}(x-2h) + A_{1,2}\, S^{(p-2)}(x-2h) \\
&\quad + \left(A_{0,2} + A_{1,2} \right) T^{(p-2)}(h) + \left(A_{-1,2} + A_{0,2} \right) S^{(p-2)}(h) \\
&\quad - \left(A_{0,2} + A_{1,2} + A_{2,2} \right) T^{(p-2)}(0) - \left(A_{-1,2} + A_{0,2} + A_{1,2} \right) S^{(p-2)}(0),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{h}{\delta_1^{u,v}} \varphi_2^{u,v}(x)|_{[3h,\infty)} &= \left(A_{0,2} + A_{1,2} + A_{2,2} \right) (T^{(p-2)}(h) - T^{(p-2)}(0)) + \left(A_{-1,2} + A_{0,2} + A_{1,2} \right) (S^{(p-2)}(h) - S^{(p-2)}(0)) \\
&= (\, 1 - 1 + 0 \,)\, (T^{(p-2)}(h) - T^{(p-2)}(0)) + (\, 0 + 1 - 1 \,)\, (S^{(p-2)}(h) - S^{(p-2)}(0)) \\
&= 0.
\end{aligned}
$$

Equivalently, in a more compact form,

$$
\varphi_2^{u,v}(x) = \frac{\delta_1^{u,v}}{h} \left[\sum_{k=0}^{2} \Big(A_{k,2} T^{(p-2)}(x-hk) + A_{k-1,2} S^{(p-2)}(x-hk) \Big) B_{k,0}(x) + \sum_{k=0}^{2} C_{k,2} B_{k,0}(x) \right], \tag{13}
$$

where, for $k \in \mathbb{Z}$,

$$
\begin{aligned}
C_{k,2} &= \sum_{j=0}^{k-1} A_{j,2}\, T^{(p-2)}(h) + \sum_{j=0}^{k-1} A_{j-1,2}\, S^{(p-2)}(h) - \sum_{j=0}^{k} A_{j,2}\, T^{(p-2)}(0) - \sum_{j=0}^{k} A_{j-1,2}\, S^{(p-2)}(0) \\
&= \sum_{j=0}^{k-2} A_{j,2}\, D_{p-2} + A_{k-1,2} \left(T^{(p-2)}(h) - T^{(p-2)}(0) - S^{(p-2)}(0) \right) - A_{k,2}\, T^{(p-2)}(0).
\end{aligned}
$$

When applying (3) again to (13) in order to obtain $\varphi_3^{u,v}$, for the first sum the computations are analogous to the ones just shown, while the terms of the second sum simply follow (4). Iterating this process up to $\varphi_p^{u,v}$ completes the proof. □

Remark 2. We observe that, for $s \in \mathbb{N}$ and $k \in \{0, \ldots, s-1\}$,

$$
A_{k,s} = (-1)^k \binom{s-1}{k} \qquad \text{and} \qquad \sum_{j=0}^{k-2} A_{j,p-s} = A_{k-2,p-s-1}.
$$

Remark 3. If $U(x) = V(h-x)$, then $u(x) = v(h-x)$ and $S(x) = (-1)^{p-1} T(h-x)$. Therefore, for $s \in \{0, \ldots, p-1\}$, $S^{(s)}(0) = (-1)^{p-s-1}\, T^{(s)}(h)$ and $S^{(s)}(h) = (-1)^{p-s-1}\, T^{(s)}(0)$, and, from (11), we have

$$
C_{k,p-s} = \begin{cases} - \left(A_{k-1,p-s} + A_{k,p-s} \right) T^{(s)}(0), & \text{if } p-s \in 2\mathbb{Z}+1, \\[2ex] 2\, A_{k-2,p-s-1} \left(T^{(s)}(h) - T^{(s)}(0) \right) + A_{k-1,p-s} \left(2T^{(s)}(h) - T^{(s)}(0) \right) \\ \qquad - A_{k,p-s}\, T^{(s)}(0), & \text{if } p-s \in 2\mathbb{Z}. \end{cases}
$$

In particular, if $U(x) = e^{x-\frac{h}{2}}$ and $V(x) = e^{\frac{h}{2}-x}$,

$$
v(x) = \frac{\sinh(x)}{\sinh(h)} \longrightarrow T(x) = \begin{cases} \dfrac{\cosh(x)}{\sinh(h)}, & \text{if } p \in 2\mathbb{N}, \\[2ex] \dfrac{\sinh(x)}{\sinh(h)}, & \text{if } p \in 2\mathbb{N}+1, \end{cases} \longrightarrow T^{(s)}(0) = \begin{cases} 0, & \text{if } p-s \in 2\mathbb{Z}+1, \\[2ex] \dfrac{1}{\sinh(h)}, & \text{if } p-s \in 2\mathbb{Z}, \end{cases}
$$

and so, for every $j, k \in \mathbb{Z}$, $C_{k,2j+1} = 0$. In this case, Theorem 3.1 yields an algebraic expression that is analogous to the one found in [20, Theorem 2]. To the best of our knowledge, this is the only other attempt at separating the B-spline components in the expression of the cardinal GB-spline. The approach of Theorem 3.1, however, is more general. For example, one can treat in a similar way the trigonometric case using, e.g., $U(x) = \cos(\pi x/(2h))$ and $V(x) = \sin(\pi x/(2h))$.

4 Cardinal GB-splines: algebraic expressions of classical and unexplored examples

For the sake of shortness, in all the following examples we focus our attention on the case $p=3$. We point out that, to the best of our knowledge, all the basis functions U and V chosen in the following examples are new and never appeared in the literature before. In the first example we start discussing a one-parameter family of GB-splines that includes two classical cases, namely the case of hyperbolic polynomial B-splines and the one of trigonometric polynomial B-splines.

Example 4.1. For a fixed $\alpha \in [0,1]$ and a fixed interval length $h=\pi/2$, consider

$$U(x) = \cos\left(\frac{\alpha\pi}{2h}x\right) e^{(1-\alpha)(x-h/2)} = \cos(\alpha x)\, e^{(1-\alpha)(x-\pi/4)} \qquad \text{and} \qquad V(x) = U(h-x).$$

When $\alpha=0$ and $\alpha=1$, one respectively recovers the celebrated cases of hyperbolic and trigonometric polynomial B-splines built upon a uniform knot partition having interval length $\pi/2$ (see, e.g., [5, 20, 30]). For such a U and V we have

$$U^{(1)}(x) = (\,(1-\alpha)\cos(\alpha x) - \alpha\sin(\alpha x)\,)\, e^{(1-\alpha)(x-\pi/4)}, \qquad V^{(1)}(x) = -U^{(1)}(h-x),$$

$$U^{(2)}(x) = (\,(1-2\alpha)\cos(\alpha x) - 2\alpha(1-\alpha)\sin(\alpha x)\,)\, e^{(1-\alpha)(x-\pi/4)}, \qquad V^{(2)}(x) = U^{(2)}(h-x).$$

Therefore,

$$\begin{aligned}
\Delta &= U^{(2)}(0)V^{(2)}(h) - V^{(2)}(0)U^{(2)}(h) = \left(U^{(2)}(0)\right)^2 - \left(U^{(2)}(h)\right)^2 \\
&= (1-2\alpha)^2\, e^{-(1-\alpha)\pi/2} - (\,(1-2\alpha)\cos(\alpha\pi/2) - 2\alpha(1-\alpha)\sin(\alpha\pi/2)\,)^2\, e^{(1-\alpha)\pi/2},
\end{aligned}$$

$$a = d = U^{(2)}(0)/\Delta = \frac{(1-2\alpha)\, e^{-(1-\alpha)\pi/4}}{\Delta},$$

$$b = c = -U^{(2)}(h)/\Delta = -\frac{(\,(1-2\alpha)\cos(\alpha\pi/2) - 2\alpha(1-\alpha)\sin(\alpha\pi/2)\,)\, e^{(1-\alpha)\pi/4}}{\Delta},$$

the expression $\delta_1^{u,v}$ in (8) reads as

$$\begin{aligned}
\delta_1^{u,v} &= \left(a(U^{(1)}(h)-U^{(1)}(0)) + b(V^{(1)}(h)-V^{(1)}(0)) + c(U^{(1)}(h)-U^{(1)}(0)) + d(V^{(1)}(h)-V^{(1)}(0))\right)^{-1} \\
&= \left(\,2\,(a+b)\,(U^{(1)}(h)-U^{(1)}(0))\,\right)^{-1} \\
&= \left(\,2\,(a+b)\,\left(\,(\,(1-\alpha)\cos(\alpha\pi/2) - \alpha\sin(\alpha\pi/2)\,)\, e^{(1-\alpha)\pi/4} - (1-\alpha)\, e^{-(1-\alpha)\pi/4}\,\right)\right)^{-1},
\end{aligned}$$

and the functions u, v, S, T are of the form

$$u(x) = a\, U^{(2)}(x) + b\, V^{(2)}(x) = a\, U^{(2)}(x) + b\, U^{(2)}(h-x),$$

$$v(x) = c\, U^{(2)}(x) + d\, V^{(2)}(x) = b\, U^{(2)}(x) + a\, U^{(2)}(h-x) = u(h-x),$$

and

$$S(x) = a\, U(x) + b\, V(x) = a\, U(x) + b\, U(h-x),$$

$$T(x) = c\, U(x) + d\, V(x) = b\, U(x) + a\, U(h-x) = S(h-x).$$

Hence, using (2) and (9), we can easily compute the algebraic expression of $\varphi_1^{u,v}$ and $\phi_3^{U,V}$ for different values of $\alpha \in [0,1]$. For example, for $\alpha = 0$,

$$\varphi_1^{u,v}(x) = \frac{\cosh\left(\frac{\pi}{4}\right)}{2\sinh\left(\frac{\pi}{4}\right)\sinh\left(\frac{\pi}{2}\right)} \cdot \begin{cases} \sinh(x), & t \in \mathcal{I}_0, \\ \sinh(\pi - x), & t \in \mathcal{I}_1, \end{cases}$$

$$\phi_3^{U,V}(x) = \frac{2\cosh\left(\frac{\pi}{4}\right)}{\pi^2\sinh\left(\frac{\pi}{4}\right)\sinh\left(\frac{\pi}{2}\right)} \cdot \begin{cases} \sinh(x) - x, & t \in \mathcal{I}_0, \\ \sinh(\pi-x) + 2\sinh\left(\frac{\pi}{2}-x\right) + \left(1+2\cosh\left(\frac{\pi}{2}\right)\right)\left(x-\frac{\pi}{2}\right) - \frac{\pi}{2}, & t \in \mathcal{I}_1, \\ \sinh(x-\pi) + 2\sinh\left(x-\frac{3\pi}{2}\right) + \left(1+2\cosh\left(\frac{\pi}{2}\right)\right)\left(\frac{3\pi}{2}-x\right) - \frac{\pi}{2}, & t \in \mathcal{I}_2, \\ \sinh(2\pi-x) - (2\pi - x), & t \in \mathcal{I}_3, \end{cases}$$

and, for $\alpha = 1$,

$$\varphi_1^{u,v}(x) = \frac{1}{2} \cdot \begin{cases} \sin(x), & t \in \mathcal{I}_0, \\ \cos(x-\pi/2), & t \in \mathcal{I}_1, \end{cases}$$

$$\phi_3^{U,V}(x) = \frac{2}{\pi^2} \cdot \begin{cases} x - \sin(x), & t \in \mathcal{I}_0, \\ \pi - x - 2\cos(x) - \sin(x), & t \in \mathcal{I}_1, \\ x - \pi - 2\cos(x) + \sin(x), & t \in \mathcal{I}_2, \\ 2\pi - x + \sin(x), & t \in \mathcal{I}_3. \end{cases}$$

These functions are represented in Figure 1, together with the ones obtained by setting $\alpha = 1/3$ and $\alpha = 2/3$. For the sake of shortness, we decided to avoid including also their expressions.

Equation (9) can also be used to easily get the algebraic expressions of cardinal GB-splines having more exotic generators U and V, like in the following examples.

Example 4.2. Consider $h = \log(1+\sqrt{2})$ and $U(x) = \operatorname{sech}(x)$, $V(x) = \tanh(x)$. These two generators satisfy the relation $(U(x))^2 + (V(x))^2 = 1$, which is useful to describe circular arcs, but they are non-periodic and $U(x) \neq V(h-x)$. Therefore, starting from them, we will obtain a GB-spline $\phi_3^{U,V}$ that is non-symmetric. Indeed, for such a U and V we have

$$U^{(1)}(x) = -\tanh(x)\operatorname{sech}(x), \qquad V^{(1)}(x) = (\operatorname{sech}(x))^2,$$

$$U^{(2)}(x) = \operatorname{sech}(x)\left((\tanh(x))^2 - (\operatorname{sech}(x))^2\right), \qquad V^{(2)}(x) = -2\tanh(x)\,(\operatorname{sech}(x))^2,$$

$$\Delta = U^{(2)}(0)V^{(2)}(h) - V^{(2)}(0)U^{(2)}(h) = \frac{\sqrt{2}}{2},$$

$$a = \frac{V^{(2)}(h)}{\Delta} = -1, \quad b = -\frac{U^{(2)}(h)}{\Delta} = 0, \quad c = -\frac{V^{(2)}(0)}{\Delta} = 0, \quad d = \frac{U^{(2)}(0)}{\Delta} = -\sqrt{2},$$

and the expression $\delta_1^{u,v}$ in (8) reads as

$$\delta_1^{u,v} = \left((a+c)\left(U^{(1)}(h) - U^{(1)}(0)\right) + (b+d)\left(V^{(1)}(h) - V^{(1)}(0)\right)\right)^{-1} = 2(\sqrt{2}-1).$$

Moreover,

$$u(x) = \operatorname{sech}(x)\left((\operatorname{sech}(x))^2 - (\tanh(x))^2\right), \qquad v(x) = 2\sqrt{2}\tanh(x)\,(\operatorname{sech}(x))^2,$$

and

$$S(x) = -\operatorname{sech}(x), \qquad T(x) = -\sqrt{2}\tanh(x).$$

Using (2) and (9), we can thus compute the algebraic expression of $\varphi_1^{u,v}$ and $\phi_3^{U,V}$. In particular,

$$\varphi_1^{u,v}(x) = 2(\sqrt{2}-1) \cdot \begin{cases} 2\sqrt{2}\tanh(x)\,(\operatorname{sech}(x))^2, & t \in \mathcal{I}_0, \\ \operatorname{sech}(x-h)\left((\operatorname{sech}(x-h))^2 - (\tanh(x-h))^2\right), & t \in \mathcal{I}_1, \end{cases}$$

$$\phi_3^{U,V}(x) = \frac{2(\sqrt{2}-1)}{\left(\log(\sqrt{2}+1)\right)^2} \cdot \begin{cases} \sqrt{2}(x - \tanh(x)), & t \in \mathcal{I}_0, \\ \dfrac{\lambda_1(x,h)\cosh(x) + \lambda_2(x,h)\sinh(x) - \sqrt{2}}{2\cosh(x) - \sqrt{2}\sinh(x)}, & t \in \mathcal{I}_1, \\ \dfrac{\lambda_3(x,h)\cosh(x-2h) + \lambda_4(h)\sinh(x-2h) + 16\sinh(h)}{2\sqrt{2}(\cosh(h))^3\cosh(x-2h)}, & t \in \mathcal{I}_2, \\ \dfrac{\lambda_5(x,h)\cosh(x-3h) + \lambda_6(h)\sinh(x-3h) - 4\sinh(h)}{20\sqrt{2}\cosh(x) - 28\sinh(x)}, & t \in \mathcal{I}_3, \end{cases}$$

where

$$\lambda_1(x,h) = 5\sqrt{2}h - 4 - 3\sqrt{2}x, \qquad \lambda_2(x,h) = 3x - 5h + 4\sqrt{2},$$

$$\lambda_3(x,h) = 2(x-3h)(\cosh(h))^2(\cosh(h)-2) + \sinh(3h) - 2\sinh(2h) - 7\sinh(h) + 4(3x-8h),$$

$$\lambda_4(h) = 4\left(2(\cosh(h))^2 - (\cosh(h))^3 - 4\right), \qquad \lambda_5(x,h) = \sinh(2h) + 8h - 2x, \qquad \lambda_6(h) = 2\left(1 - (\sinh(h))^2\right).$$

These functions are represented in Figure 2.

Finally, in the last example, we consider a space of GB-splines with rational derivatives.

Example 4.3. Consider $h = e-1$ and $V(x) = \log(x+1)$, $U(x) = V(h-x)$. Then, for such a U and V, we have

$$V^{(1)}(x) = \frac{1}{x+1}, \qquad U^{(1)}(x) = -V^{(1)}(h-x),$$
$$V^{(2)}(x) = -\frac{1}{(x+1)^2}, \qquad U^{(2)}(x) = V^{(2)}(h-x),$$
$$\Delta = U^{(2)}(0)\,V^{(2)}(h) - V^{(2)}(0)\,U^{(2)}(h) = \left(V^{(2)}(h)\right)^2 - \left(V^{(2)}(0)\right)^2 = e^{-4} - 1,$$
$$a = d = \frac{V^{(2)}(h)}{\Delta} = \frac{1}{2\sinh(2)}, \qquad b = c = -\frac{V^{(2)}(0)}{\Delta} = -\frac{e^2}{2\sinh(2)},$$

and the expression $\delta_1^{u,v}$ in (8) reads as

$$\delta_1^{u,v} = \Big((a+c)\,(\,U^{(1)}(h) - U^{(1)}(0)\,) + (b+d)\,(\,V^{(1)}(h) - V^{(1)}(0)\,)\,\Big)^{-1} = \frac{\cosh(1)}{e-1}.$$

Moreover,

$$v(x) = \frac{1}{2\sinh(2)}\left(\frac{e^2}{(e-x)^2} - \frac{1}{(x+1)^2}\right), \qquad u(x) = v(h-x),$$

and

$$T(x) = \frac{\log(x+1) - e^2\log(e-x)}{2\sinh(2)}, \qquad S(x) = T(h-x).$$

Thus, using (2) and (9), we can compute the algebraic expression of $\varphi_1^{u,v}$ and $\phi_3^{U,V}$, later illustrated in Figure 3. In particular,

$$\varphi_1^{u,v}(x) = \frac{1}{4(e-1)\sinh(1)} \cdot \begin{cases} \dfrac{e^2}{(e-x)^2} - \dfrac{1}{(x+1)^2}, & t \in \mathcal{I}_0, \\[2ex] \dfrac{e^2}{(2-e+x)^2} - \dfrac{1}{(2e-1-x)^2}, & t \in \mathcal{I}_1, \end{cases}$$

$$\phi_3^{U,V}(x) = \begin{cases} \dfrac{-e^2\left((e+1)x - \log(x+1) + e^2\log(e-x) - e^2\right)}{(e-1)^3(e^4-1)\mathrm{sech}(1)}, & t \in \mathcal{I}_0, \\[2ex] \dfrac{\lambda_1 x + \lambda_2\log(2e-x-1) - \lambda_3\log(x-e+2) - 2e^5 - e^4 + 2e}{(e-1)^3(e^4-1)\mathrm{sech}(1)}, & t \in \mathcal{I}_1, \\[2ex] \dfrac{-\lambda_1 x + \lambda_2\log(x-2e+3) - \lambda_3\log(3e-x-2) + 6e^5 - 5e^4 + 4e^2 - 6e}{(e-1)^3(e^4-1)\mathrm{sech}(1)}, & t \in \mathcal{I}_2, \\[2ex] \dfrac{e^2\left((e+1)x + \log(4e-x-3) - e^2\log(x-3e+4) - 3e^2 + 4\right)}{(e-1)^3(e^4-1)\mathrm{sech}(1)}, & t \in \mathcal{I}_3, \end{cases}$$

where

$$\lambda_1 = e(2e^3 + e^2 + e + 2), \qquad \lambda_2 = e^2(2e^2 + 1), \qquad \lambda_3 = e^2(e^2 + 2).$$

5 Closing remark

In this paper we have presented a new easy-to-manipulate algebraic formulation of cardinal GB-splines that isolates the polynomial component from the non-polynomial contribution brought by the functions U and V. The polynomial component is expressed in terms of linear combinations of uniform polynomial B-splines of degree up to $p-2$, whereas the non-polynomial component is specified by a piecewise function whose pieces are explicitly written in terms of U and V.

Acknowledgements: This research has been accomplished within the Research ITalian network on Approximation (RITA) and UMI-TAA. The authors are members of the INdAM Research group GNCS, which has partially supported this work. The first author is also member of the Alma Mater research center on Applied Mathematics (AM^2).

Declarations: All authors have contributed equally to this work. The authors also declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] C.V. Beccari, G. Casciola, M.-L. Mazure. Piecewise extended Chebyshev spaces: a numerical test for design. *Appl. Math. Comput.*, 296:239-256, 2017.

[2] C.V. Beccari, G. Casciola, M.-L. Mazure. Design or not design: a numerical characterization for piecewise Chebyshevian splines. *Numer. Algo.*, 81:1-31, 2019.

[3] C.V. Beccari, G. Casciola, M.-L. Mazure. Critical length: An alternative approach. *J. Comput. Appl. Math.*, Vol. 370, 2020, 112603.

[4] C.V. Beccari, G. Casciola, L. Romani. A practical method for computing with piecewise Chebyshevian splines. *J. Comput. Appl. Math.*, Vol. 406, 2022, 114051.

[5] T. Bosner, M. Rogina. Numerically stable algorithm for cycloidal splines. *Ann. Univ. Ferrara*, 53:189-197, 2007.

[6] T. Bosner, B. Crnković, J. Škifić. Tension splines with application on image resampling. *Mathematical Communications*, 19:517-529, 2014.

[7] I. Cattiaux-Huillard, L. Saini. Characterization and extensive study of cubic and quintic algebraic trigonometric planar PH curves. *Adv. Comput. Math.*, Vol. 46, 2020, pp. 1-28.

[8] D. Cho. Overlapping Schwarz methods for isogeometric analysis based on generalized B-splines. *Comput. Methods Appl. Mech. Engrg.*, Vol. 372, 2020, 113430.

[9] C. Conti, L. Romani, D. Schenone. Semi-automatic spline fitting of planar curvilinear profiles in digital images using the Hough transform. *Pattern Recognition*, 74:64-76, 2018.

[10] C. Conti, L. Romani, M. Unser. Ellipse-preserving Hermite interpolation and subdivision. *J. Math. Anal. Appl.*, 426:211-227, 2015.

[11] S. Eddargani, A. Lamnii, M. Lamnii, D. Sbibih, A. Zidna. Algebraic hyperbolic spline quasi-interpolants and applications. *J. Comput. Appl. Math.*, 347:196-209, 2019.

[12] L. Fang, Y. Li. Algebraic and geometric characterizations of a class of Algebraic-Hyperbolic Pythagorean-Hodograph curves. *Comput. Aided Geom. Des.*, Vol. 97, 2022, 102121.

[13] M. Fang, W. Ma. Isogeometric boundary element analysis based on UE-splines. *J. Comput. Appl. Math.*, Vol. 398, 2021, 113700.

[14] Q. Guo. Cubic GB-spline curves. *J. Information and Computational Science*, 2:465-471, 2005.

[15] B.I. Kvasov. GB-splines and their properties. *Ann. Numer. Math.*, 3:139-149, 1996.

[16] B.I. Kvasov, P. Sattayatham. GB-splines of arbitrary order. *J. Comput. Appl. Math.*, 104:63-88, 1999.

[17] B.I. Kvasov. Algorithms for shape preserving local approximation with automatic selection of tension parameters. *Comput. Aided Geom. Des.*, 17:17-37, 2000.

[18] A. Lamnii, M.Y. Nour, D. Sbibih, A. Zidna. Generalized spline quasi-interpolants and applications to numerical analysis. *J. Comput. Appl. Math.*, Vol. 408, 2022, 114100.

[19] Y. Li, M. Hoffmann, G. Wang. On the shape parameter and constrained modification of GB-spline curves. *Ann. Mathematicae et Informaticae*, 34: 51-59, 2007.

[20] Y. Lü, G. Wang, X. Yang. Uniform hyperbolic polynomial B-spline curves. *Comput. Aided Geom. Des.*, 19:379-393, 2002.

[21] C. Manni, F. Pelosi, M.L. Sampoli. Generalized B-splines as a tool in isogeometric analysis. *Comput. Methods Appl. Mech. Engrg.*, 200:867-881, 2011.

[22] C. Manni, F. Roman, H. Speleers. Generalized B-splines in isogeometric analysis. In: G.E. Fasshauer, L.L. Schumaker (eds.) Approximation Theory XV, 239-267, 2017.

[23] H. Prautzsch. B-Splines with arbitrary connection matrices. *Constr. Approx.*, 20:191-205, 2004.

[24] H. Prautzsch, W. Boehm, M. Paluszny. Bézier and B-spline techniques. Springer-Verlag Berlin Heidelberg 2002.

[25] X. Qin, G. Hu, Y. Yang, G. Wei. Construction of PH splines based on H-Bézier curves. *Appl. Math. Comput.*, 238:460-467, 2014.

[26] L. Romani, L. Saini, G. Albrecht. Algebraic-trigonometric Pythagorean-Hodograph curves and their use for Hermite interpolation. *Adv. Comput. Math.*, 40:977-1010, 2014.

[27] L. Romani, F. Montagner. Algebraic-trigonometric Pythagorean-Hodograph space curves. *Adv. Comput. Math.*, 45:75-98, 2019.

[28] L.L. Schumaker. Spline functions: basic theory. Third edition, Cambridge U.P., 2007.

[29] H. Speleers Algorithm 1020: Computation of Multi-Degree Tchebycheffian B-Splines *ACM TOMS*, 48:1-31, 2021.

[30] G. Wang, M. Fang. Unified and extended form of three types of splines. *J. Comput. Appl. Math.*, 216:498-508, 2008.

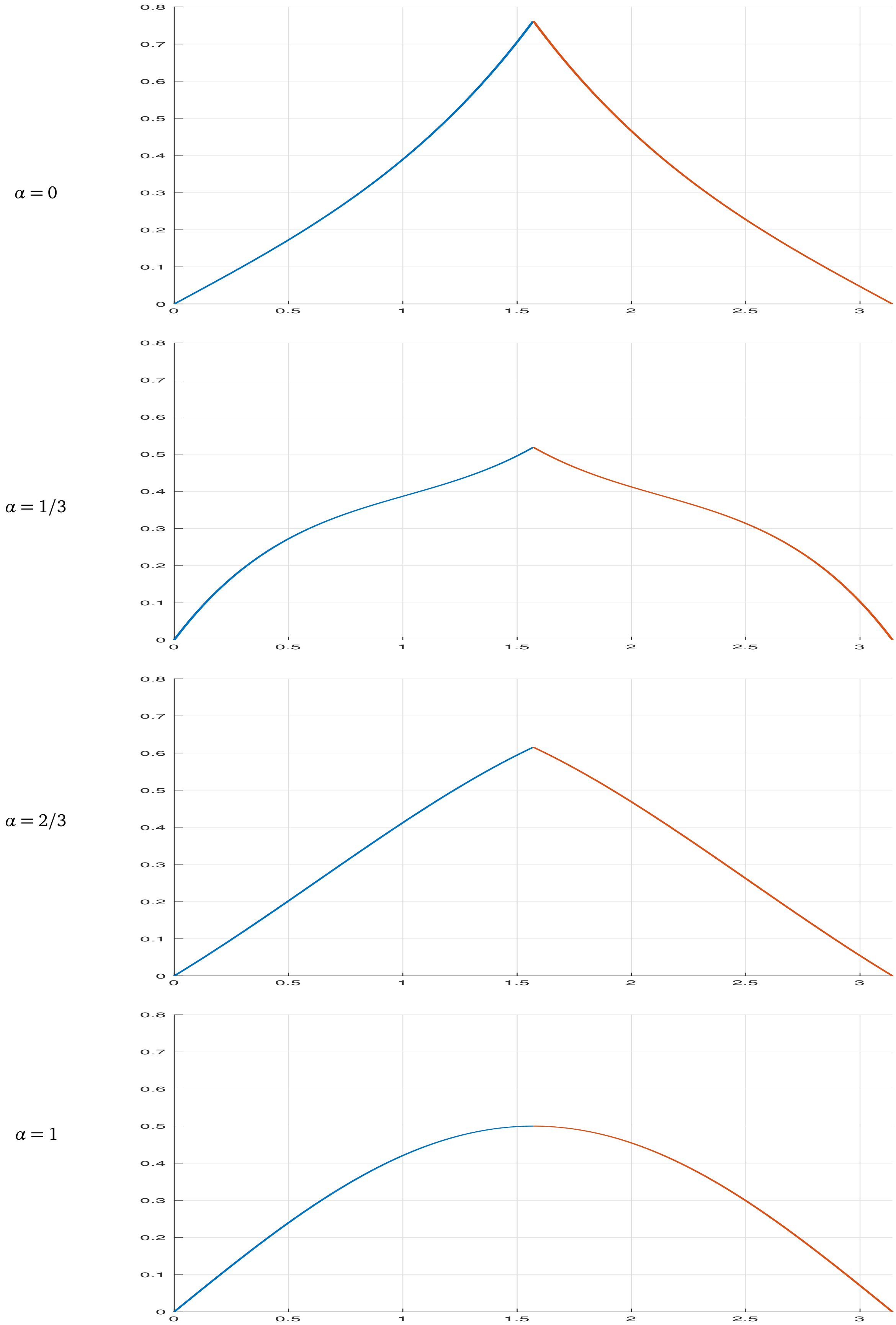

Figure 1: The functions $\varphi_1^{u,v}$ (first column) and $\phi_3^{U,V}$ (second column) obtained with the choice of h, U and V specified in Example 4.1 for values of $\alpha \in \{0, 1/3, 2/3, 1\}$.

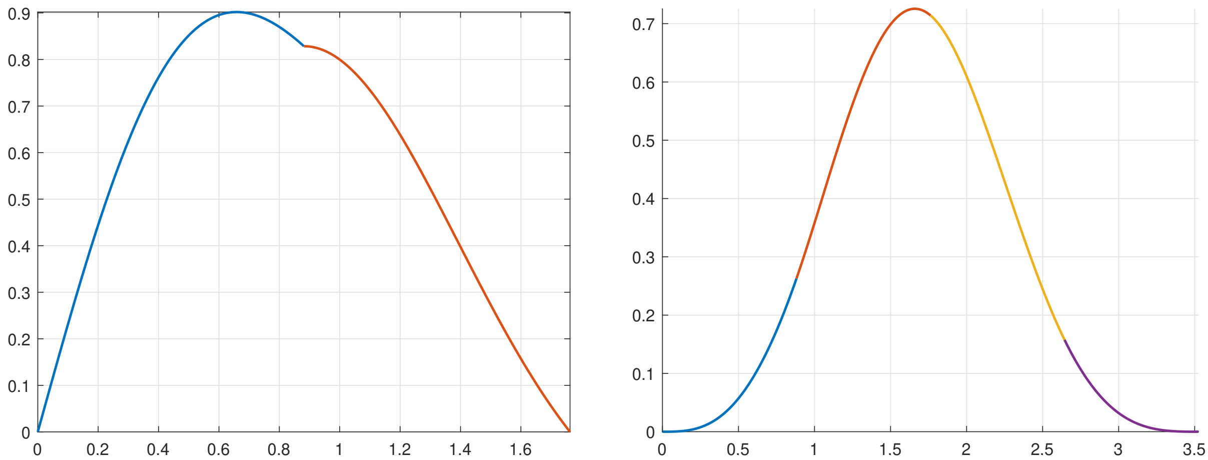

Figure 2: The functions $\varphi_1^{u,v}$ (left) and $\phi_3^{U,V}$ (right) obtained with the choice of h, U and V specified in Example 4.2.

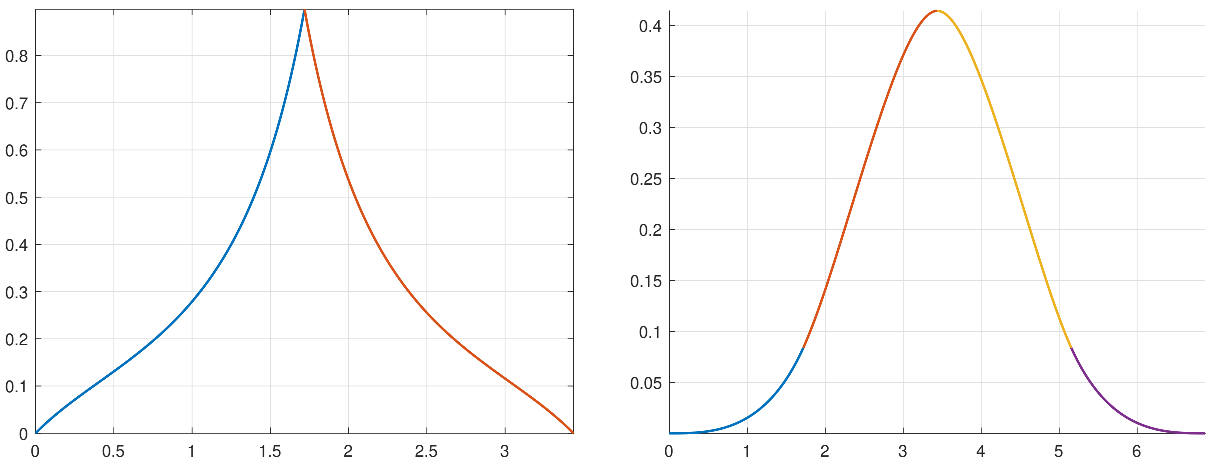

Figure 3: The functions $\varphi_1^{u,v}$ (left) and $\phi_3^{U,V}$ (right) obtained with the choice of h, U and V specified in Example 4.3.