

Hamiltonians and gauge-invariant Hilbert space for lattice Yang-Mills-like theories with finite gauge group

A. Mariani¹, S. Pradhan², and E. Ercolessi²

¹*Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland*

²*Dipartimento di Fisica, Università di Bologna and INFN, Via Irnerio 46, 40126 Bologna, Italy*



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Motivated by quantum simulation, we consider lattice Hamiltonians for Yang-Mills gauge theories with finite gauge group, for example a finite subgroup of a compact Lie group. We show that the electric Hamiltonian admits an interpretation as a certain natural, nonunique Laplacian operator on the finite Abelian or non-Abelian group and derive some consequences from this fact. Independent of the chosen Hamiltonian, we provide a full explicit description of the physical, gauge-invariant Hilbert space for pure gauge theories and derive a simple formula to compute its dimension. We illustrate the use of the gauge-invariant basis to diagonalize a dihedral gauge theory on a small periodic lattice.

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I. INTRODUCTION

Quantum simulation is a field of growing interest both experimentally and theoretically [1–4]. It holds the promise to overcome technical difficulties, such as the “sign problem” [5–7], which affect numerical Monte Carlo simulations in several interesting regimes, with applications to particle physics and condensed matter systems [6,8]. In recent years, advances in experimental techniques have allowed the proposal and realization of different setups to simulate physical theories via quantum-mechanical systems [9–20]. The long-term hope is that experimental and theoretical advances may one day allow the quantum simulation of some currently inaccessible nonperturbative aspects of strongly coupled theories.

A particularly interesting application is the quantum simulation of gauge theories [7,21–26], which are ubiquitous in both particle and condensed matter physics. From a theoretical standpoint, quantum simulation is most natural in a Hamiltonian formulation with a finite-dimensional Hilbert space. One possibility to achieve this is by replacing the gauge group, a Lie group, with a finite group, for example one of its finite subgroups [27–29]. Finite subgroups of Lie groups were already considered in the early days of lattice field theory [30–34]. It was found that, unlike Lie groups, finite groups undergo a “freezing” transition, beyond which they cease to approximate the physics of their parent Lie group. It was later pointed out, however, that as long as an appropriate comparison is made in the vicinity of the phase transition, finite-group gauge theory based on sufficiently large, but fixed, subgroups of the Lie group provides an effective description of the Lie group gauge theory [35]. Gauge theories with finite gauge

group have also found direct applications in condensed matter physics [36–40] and quantum gravity [41]. In the Hamiltonian formulation, the Abelian case was considered in [27,28,42], while the general formulation in the non-Abelian case was given in [29]. This formulation, however, lacked an explicit form of the electric term of the Hamiltonian. In [41] the full Hamiltonian for a finite-group gauge theory was derived via the transfer-matrix formulation from the Wilson action.

In the present work we consider a class of finite-group gauge theories in the Hamiltonian formulation, which mimic some aspects of Lie group Yang-Mills theories on the lattice. The Hamiltonian is based on the construction of a natural Laplacian operator on the finite group and is valid for any Abelian or non-Abelian finite group. As a special case, this includes both the finite-group Hamiltonian obtained via the transfer-matrix formulation from the finite-group Wilson action, but also a wider class of non-Lorentz-invariant theories. The characterization of the Hamiltonian using the finite-group Laplacian may be used to obtain nontrivial information about the theory. In particular, we obtain a simple criterion for degeneracy of the electric Hamiltonian, with implications for adiabatic quantum simulation and possibly the phase diagram of the theory. Irrespective of the choice of Hamiltonian, we show that spin-network states are particularly suitable to give a description of the physical, gauge-invariant Hilbert space for pure gauge theories, and, based on this, we derive a simple formula to compute the dimension of the physical Hilbert space. This is especially useful to quantify the amount of resources required in gauge-invariant quantum simulation schemes. Finally, we illustrate the use of the gauge-invariant basis by constructing the Hamiltonian for a

gauge theory based on the dihedral group D_4 and compute some quantities of interest via exact diagonalization.

II. FINITE-GROUP GAUGE THEORY

A. The Hilbert space

1. Basic construction

In the Hamiltonian formulation of lattice gauge theories [29,43,44], time is continuous while the d spatial dimensions are discretized into a hypercubic lattice. Classically, we assign a group element $g \in G$ to each spatial lattice link, where G is the gauge group. In the Lie group case, one would typically write $U_\mu(x) = \exp(iA_\mu(x)) \in G$ for the gauge field variable assigned to the lattice link between points x and $x + \hat{\mu}$, where $A_\mu(x)$ is the vector potential. Links are oriented, and if a link is traversed in the opposite orientation, then g is replaced with g^{-1} . Note that finite groups have no Lie algebras, so we work with group-valued quantities as far as possible. In what follows, we write $g \in G$ for a group element indifferently for both finite and Lie groups G .

Since a classical configuration is given by a choice of group element g on each lattice link, in the quantum theory the states in the Hilbert space of each link are generally given by a superposition [44]

$$|\psi\rangle = \int dg \psi(g) |g\rangle, \quad (1)$$

where $\{|g\rangle\}$ is the group element orthonormal basis, consisting of one state $|g\rangle$ per group element g ; it can be thought of as a “position basis” on the group. In the case of a Lie group, the wave function $\psi(g)$ is square integrable with respect to the Haar measure. Hence, the Hilbert space on each link can then be identified with $L^2(G)$, i.e. the space of square-integrable functions on G [44]. For a finite group, the Haar measure is replaced with a sum over the group elements, $\int dg = \sum_g$, and the Hilbert space is simply the “group algebra” $\mathbb{C}[G]$, which is the complex vector space spanned by the group element basis.

The overall Hilbert space is then given by the tensor product $\mathcal{H}_{\text{tot}} = \otimes_{\text{links}} L^2(G)$, or $\mathcal{H}_{\text{tot}} = \otimes_{\text{links}} \mathbb{C}[G]$. Note that, for a finite group, $\mathbb{C}[G]$ has finite dimension, because it is spanned by the finitely many group element states $\{|g\rangle\}$. Therefore, the Hilbert space on each link is finite dimensional and \mathcal{H} is finite dimensional on a finite lattice; the dimension is given by $|G|^L$ where L is the number of lattice links. For a Lie group, on the other hand, we have infinitely many basis states $\{|g\rangle\}$ and therefore the Hilbert space is infinite dimensional *on each link*.

In the Hamiltonian formulation of gauge theories, the statement that the theory is invariant under gauge transformations translates at the level of the Hilbert space by restricting the allowed states only to those which are gauge

invariant. In particular, on the single-link Hilbert space one can define left and right “translation” operators, in the analogy where $\{|g\rangle\}$ is a position basis in group space [29],

$$L_g |h\rangle = |gh\rangle, \quad R_g |h\rangle = |hg^{-1}\rangle. \quad (2)$$

A local gauge transformation is given by a choice of group element $g_x \in G$ at every site x of the lattice [44]. This acts on the overall Hilbert space \mathcal{H}_{tot} via the operator

$$\mathcal{G}(\{g_x\}) = \bigotimes_{l=\langle xy \rangle \in \text{links}} L_{g_x} R_{g_y}, \quad (3)$$

where $\{g_x\}$ is an arbitrary choice of group elements g_x at each lattice site x , and the link $l = \langle xy \rangle$ connects the points x and y . In other words, each link state $|g_l\rangle$ transforms as $|g_l\rangle \mapsto |g_x g_l g_y^{-1}\rangle$.

The only physical states are those which satisfy the so-called Gauss law constraint [43–45]

$$\mathcal{G}(\{g_x\}) |\psi\rangle = |\psi\rangle, \quad (4)$$

for any possible choice of local assignments $\{g_x\}$ of group variables to lattice sites. Gauss’s law (4) is an exponentiated version of the usual Gauss law formulated in terms of Lie algebra generators. The states which satisfy Eq. (4) form the physical, gauge-invariant Hilbert space $\mathcal{H}_{\text{phys}}$. Note that the condition (4) only involves group-valued quantities and is thus valid for both Lie groups and finite groups. In the case of finite groups, the condition simplifies because it is sufficient to impose invariance against a set of generators of the finite group.

One can also straightforwardly include matter fields such as fermion fields which live on each lattice site. Under a gauge transformation, they transform as $\Psi(x) \rightarrow R(g_x) \Psi(x)$, where R is some representation of the gauge group.

2. The representation basis and the Peter-Weyl theorem

It turns out to be fruitful to introduce a different basis of the overall Hilbert space \mathcal{H} , “dual” to the group element basis. The operators L_g and R_g introduced in Eq. (2) are unitary representations of G , known as the left and right regular representations [46,47]. This is because $L_g L_h = L_{gh}$ and $(L_g)^{-1} = L_{g^{-1}} = (L_g)^\dagger$, as can be explicitly checked by acting on the group element basis, and the same holds for R . Their representation theory leads to the Peter-Weyl theorem [44,47,48], which states that, for a finite or compact Lie group G ,

$$L^2(G) = \bigoplus_{j \in \Sigma} V_j^* \otimes V_j, \quad (5)$$

where j is a label for the irreducible representations (irreps) of G , and Σ is the set of all irreps of G . For finite groups

$L^2(G)$ is replaced, as usual, with $\mathbb{C}[G]$. Here V_j is the representation vector space corresponding to representation j and V_j^* its dual vector space. For both compact Lie groups and finite groups the irreps are finite dimensional and can be chosen to be unitary. For a finite group, Σ is a finite set, while it is countably infinite for a compact Lie group [46,47]. In terms of the Peter-Weyl decomposition, the left and right regular representations take a particularly simple form [48],

$$L_g R_h = \bigoplus_j \rho_j(g)^* \otimes \rho_j(h), \quad (6)$$

where ρ_j is the matrix of the j th irrep of G . The individual action of either L_g or R_h may be obtained by setting either g or h to the identity. Equation (6) is especially useful because, as we will see in Sec. III, it simplifies the action of the Gauss law constraint (4).

The Peter-Weyl theorem provides an alternative basis for the single-link Hilbert space. For each irrep j one chooses appropriate bases for V_j^* and V_j , which we denote $\{|jm\rangle\}$ and $\{|jn\rangle\}$ respectively, where $1 \leq m, n \leq \dim j$. Here $\dim j \equiv \dim V_j$ is the dimension of the representation. On each representation subspace, we use the shorthand notation $|jmn\rangle \equiv |jm\rangle \otimes |jn\rangle$. Then the ‘‘representation basis’’ for $L^2(G)$ or $\mathbb{C}[G]$ is given by the set $\{|jmn\rangle\}$ for all $j \in \Sigma$ and $1 \leq m, n \leq \dim j$. In terms of the group element basis, one has [29]

$$\langle g|jmn\rangle = \sqrt{\frac{\dim(j)}{|G|}} [\rho_j(g)]_{mn}, \quad (7)$$

where the bases $\{|jm\rangle\}$, $\{|jn\rangle\}$ are chosen so that ρ_j is unitary. It should be emphasized that Eq. (7) is valid for both finite and compact Lie groups; $|G|$ is either the order of the finite group or the volume $|G| \equiv \int dU 1$ given by the possibly unnormalized Haar measure [44,48]. It is a basic result of the representation theory of finite groups that $\sum_j (\dim j)^2 = |G|$, which ensures that the group element basis and the representation basis have the same number of states [46].

Since every group admits a trivial, one-dimensional irrep with $\rho(g) \equiv 1$, we always have a singlet representation state $|0\rangle$, which may be extended to the whole lattice to form the ‘‘electric ground state’’ $|0_E\rangle$,

$$|0_E\rangle = \bigotimes_{l \in \text{links}} |0\rangle_l, \quad |0\rangle = \frac{1}{\sqrt{|G|}} \sum_g |g\rangle, \quad (8)$$

where we used Eq. (7) to express $|0\rangle$ in the group element basis. We have summarized the representation theory of some groups of interest in Appendix A.

In the specific case of the group \mathbb{Z}_N , the representations are all one dimensional because \mathbb{Z}_N is Abelian and

therefore $m = n = 1$ and can be omitted. If ξ is a generator of \mathbb{Z}_N , then the group elements are $\mathbb{Z}_N = \{1, \xi, \xi^2, \dots, \xi^{N-1}\}$ and the irreps are simply $\rho_j(\xi^k) = \omega_N^{kj}$ for $j = 0, 1, \dots, N-1$, with $\omega_N = e^{2\pi i/N}$. The bases $\{|\xi^k\rangle\}$ and $\{|j\rangle\}$ are related by

$$|j\rangle = \sum_{k=0}^{N-1} \langle \xi^k | j \rangle |\xi^k\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{kj} |\xi^k\rangle, \quad (9)$$

which is just the discrete Fourier transform. In the case of the dihedral group D_4 with eight elements, we have four one-dimensional representations, each of which spans a one-dimensional subspace of $\mathbb{C}[G]$. We then have a two-dimensional representation which spans a $2^2 = 4$ -dimensional subspace of $\mathbb{C}[G]$ through the four basis elements $|jmn\rangle$ for $1 \leq m, n \leq 2$.

B. The Hamiltonian

1. Basic construction

The Hamiltonian for a Yang-Mills gauge theory on the lattice takes the form [29,41,43,44,49]

$$H = \lambda_E \sum_{l \in \text{links}} h_E + \lambda_B \sum_{\square} h_B(g_{\square}), \quad (10)$$

where h_E depends only on each lattice link, while h_B depends on the lattice plaquettes \square and $g_{\square} = g_1 g_2 g_3^{-1} g_4^{-1}$ is the product of the four link variables in a lattice plaquette with the appropriate orientation. It is also possible to add matter fields, but we focus here on the pure gauge theory.

If the gauge group is a Lie group, each group element $g = e^{iX}$ can be written as the exponential of a Lie algebra element X . Then one also has infinitesimal generators of left translations $\hat{\mathcal{L}}_L^a$ such that $L_{e^{i\tau a}} = \exp(i\hat{\mathcal{L}}_L^a)$, where T^a is a Lie algebra basis and a is a color index [44]. In other words, $\hat{\mathcal{L}}_L^a$ is the Lie algebra representation corresponding to the group representation L and plays the role of the chromoelectric field.

The Lie group Hamiltonian, also known as the Kogut-Susskind Hamiltonian, is then given by [43,44]

$$h_E = \sum_a (\hat{\mathcal{L}}_L^a)^2, \quad h_B = 2(\dim \rho - \text{Re tr} \rho(g_{\square})), \quad (11)$$

where ρ is the fundamental representation of $\text{SU}(N)$, with couplings $\lambda_E = g^2/2$ and $\lambda_B = 1/g^2$ in terms of a coupling constant g (the lattice spacing is set to 1). As the group element basis may be thought of as a position basis in group space, the infinitesimal generators of translations $\hat{\mathcal{L}}_L^a$ may be thought of as ‘‘momentum’’ operators in group space. Then the electric Hamiltonian h_E , which is the sum of the squares of the momenta in all directions, is a Laplacian in

group space. Applying the Peter-Weyl decomposition (6) to $\hat{\mathcal{L}}_L^a$, one finds that [44,48]

$$h_E = \sum_a (\hat{\mathcal{L}}_L^a)^2 = \sum_{jmn} C(j) |jmn\rangle \langle jmn|, \quad (12)$$

where $C(j)$ is the quadratic Casimir operator, which depends only on the representation $C(j)$. For $U(1)$, for example $C(j) = j^2$, while for $SU(2)$ one finds $C(j) = j(j+1)$.

We note that the magnetic Hamiltonian depends only on group-valued quantities and is therefore well defined for both Lie groups and finite groups. On the other hand, the electric Hamiltonian depends on the infinitesimal Lie algebra through $\hat{\mathcal{L}}_L^a$ and therefore the definition does not extend to finite groups. The decomposition (12) is well defined also for finite groups, but one must leave the coefficients $C(j)$ unsatisfactorily unspecified because finite groups do not have a Casimir operator [29].

If one thinks of a finite group as a natural discretization of some parent Lie group, the natural choice of electric Hamiltonian is a discrete Laplacian on the finite group. The geometric structure of a finite group is that of a graph, with group elements as vertices and the group operation defining the edges. This is called a ‘‘Cayley graph.’’ The discrete Laplacian on the finite group is then naturally given by the graph Laplacian of the Cayley graph. This choice also preserves the interpretation of the electric Hamiltonian as a quantum-mechanical rotor in group space [43].

We explain the construction of the finite-group Laplacian in detail in Sec. II B 2, and the resulting Hamiltonian takes the form of (10) with

$$H = \lambda_E \sum_{l \in \text{links}} h_E + \lambda_B \sum_{\square} h_B(g_{\square}),$$

$$h_E = \sum_{g \in \Gamma} (1 - L_g), \quad h_B = h_B(g_{\square}), \quad (13)$$

where $\Gamma \subset G$ is a subset of the group (*not* a subgroup) such that

- (1) $1 \notin \Gamma$, i.e. Γ does not contain the identity element.
 - (2) $\Gamma^{-1} = \Gamma$, i.e. it is invariant under inversion of group elements. In other words, if $g \in \Gamma$, then $g^{-1} \in \Gamma$ also.
 - (3) $g\Gamma g^{-1} = \Gamma$, i.e. it is invariant under conjugation. In other words, Γ is a union of conjugacy classes of G .
- These conditions on Γ ensure that the electric Hamiltonian is gauge invariant. On the other hand, as usual, the magnetic term is gauge invariant as long as h_B is any real function such that $h_B(g_1 g_{\square} g_1^{-1}) = h_B(g_{\square})$ for any $g_1 \in G$. As explained in Sec. II B 3, the Hamiltonian (13) includes as a special case the transfer-matrix Hamiltonian obtained in [41] which consists of a certain specific choice of subset Γ . The choice of Γ is in fact not unique, a fact which we will also discuss in later sections.

While the magnetic Hamiltonian h_B is diagonal in the group element basis, the electric Hamiltonian h_E is diagonal in the representation basis, and in fact [29]

$$h_E = \sum_{jmn} h_E(j) |jmn\rangle \langle jmn|, \quad (14)$$

$$h_E(j) = |\Gamma| - \frac{1}{\dim j} \sum_{g \in \Gamma} \chi_j(g), \quad (15)$$

where $|\Gamma|$ is the number of elements in Γ and χ_j is the character of the irrep labeled j . The electric Hamiltonian may be interpreted as an ‘‘on-link’’ hopping term within group space; in fact, up to a constant, it may be written as

$$h_E = - \sum_{g \in \Gamma} \sum_{h \in G} |gh\rangle \langle h|, \quad (16)$$

and it favors each link to sit in the electric ground state (8), which is fully delocalized in group space. On the other hand, the magnetic term is a plaquette-based potential which pushes plaquettes close to the identity. The competition between the two noncommuting terms gives rise to nontrivial dynamics.

We would like to emphasize that the description of the electric Hamiltonian h_E in Eq. (13) as the graph Laplacian of the Cayley graph associated with the group is not simply an interesting analogy, but also a tool which may be used to extract information on the Hamiltonian itself. As an example, we note the well-known fact that the smallest eigenvalue of a graph Laplacian is always zero [with at least one eigenvector given by the trivial representation state (8)] and its degeneracy equals the number of connected components of the graph [50]. Moreover, it is not hard to show that if Γ does not generate the group G , but rather only a subgroup $\langle \Gamma \rangle < G$, then the Cayley graph splits into connected components which are identified with the cosets of $\langle \Gamma \rangle$ in G . The number of such components, and therefore the degeneracy of the ground state of h_E on each link, is given by

$$\text{electric ground state degeneracy} = \frac{|G|}{|\langle \Gamma \rangle|}. \quad (17)$$

This is the degeneracy of h_E on each link; the degeneracy of the electric Hamiltonian $H_E = \sum_{\text{links}} h_E$ on the Hilbert space of the whole lattice is larger. If, instead, Γ generates the whole group, then the electric Hamiltonian is not degenerate. A detailed proof can be found in Appendix B 1.

The degeneracy of the electric ground state is not only an important feature of the theory, but is also technically important for methods such as adiabatic quantum simulation, which typically start from the electric ground state (8) and adiabatically switch on the magnetic Hamiltonian. If the ground state is degenerate, this becomes

more difficult. We also note that a different formulation with possibly degenerate electric Hamiltonian and a finite gauge group was already considered in [51].

As we will see at the end of Sec. II B 2, an electric Hamiltonian with degenerate ground state can be constructed in the simple case of the dihedral group D_4 . In general, the electric ground state degeneracy can also occur with the choice of Γ arising from the transfer-matrix formulation of the Wilson action, as described in Sec. II B 3. For example, consider the permutation group $G = S_5$. Starting from the Wilson action in the six-dimensional faithful irrep of S_5 , one finds h_E with Γ equal to the conjugacy class of the 5-cycles; then Γ generates the subgroup $\langle \Gamma \rangle = A_5$ and since $|S_5|/|A_5| = 2$, the electric Hamiltonian ground state is twofold degenerate, with the ground states spanned by the two representation states corresponding to the one-dimensional irreps.

2. The finite-group Laplacian

In this section we explain in detail the construction of the electric Hamiltonian on each link as the finite-group Laplacian, i.e. the graph Laplacian of the Cayley graph of the finite group. Given a finite group G , we choose a set of generators $\Gamma \subset G$, which we require to be invariant under inversion, that is $\Gamma^{-1} = \Gamma$, and moreover that it is the union of conjugacy classes, so that it is invariant under conjugation, $g\Gamma g^{-1} = \Gamma$ for any g in G [50]. We choose Γ not to include the identity element and we note that the choice of Γ is not unique. The Cayley graph has the group elements as vertices, and we place an edge between $g \in G$ and $h \in G$ if $hg^{-1} \in \Gamma$. The result is a simple undirected graph. Examples of Cayley graphs for the groups \mathbb{Z}_5 and D_4 are shown in Fig. 1. Given any graph, its Laplacian is defined as [50]

$$\Delta = D - A, \quad (18)$$

where A is the adjacency matrix and D is the degree matrix. Each of these matrices acts on the vector space of graph vertices, which in the case of a Cayley graph can be identified with the group algebra $\mathbb{C}[G]$. The degree matrix

is always diagonal, and in this case $D = |\Gamma|1$. The adjacency matrix A is given by

$$A_{gh} = \begin{cases} 1 & gh^{-1} \in \Gamma \\ 0 & \text{otherwise} \end{cases} \quad (19)$$

for group elements g, h . On a basis element, one has

$$A|g\rangle \equiv \sum_h A_{hg}|h\rangle = \sum_{k \in \Gamma} |gk\rangle = \sum_{k \in \Gamma} |gk^{-1}\rangle = \sum_{k \in \Gamma} R_k|g\rangle, \quad (20)$$

where R_k is the right regular representation, and we used the closure of Γ under inversion. Therefore as an operator on $\mathbb{C}[G]$,

$$A = \sum_{k \in \Gamma} R_k = \bigoplus_j \mathbb{1}_j \otimes \left(\sum_{k \in \Gamma} \rho_j(k) \right), \quad (21)$$

where we used the Peter-Weyl decomposition (6) for R_k . Then we see that

$$\begin{aligned} \left(\sum_{k \in \Gamma} \rho_j(k) \right) \rho_j(g) &= \sum_{k \in \Gamma} \rho_j(kg) = \sum_{k \in \Gamma} \rho_j(gkg^{-1}g) \\ &= \rho_j(g) \left(\sum_{k \in \Gamma} \rho_j(k) \right), \end{aligned} \quad (22)$$

where we used the closure of Γ under conjugation.

Hence the operator $(\sum_{k \in \Gamma} \rho_j(k))$ commutes with the irreducible representation ρ_j and as such is proportional to the identity by Schur's lemma [46]. The constant of proportionality can be readily computed by taking a trace. This therefore implies

$$A = \sum_j \lambda_j P_j, \quad \lambda_j = \frac{1}{\dim j} \sum_{k \in \Gamma} \chi_j(k), \quad (23)$$

where $P_j = \sum_{mn} |jmn\rangle \langle jmn|$ is the projector onto the j th representation subspace, and χ_j is the character of the irrep

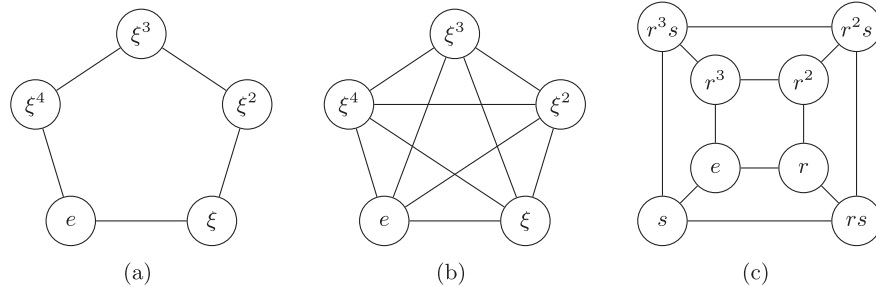


FIG. 1. Examples of Cayley graphs. (a),(b) \mathbb{Z}_5 with $\Gamma = \{\xi, \xi^{-1}\}$ and $\Gamma = \{\xi, \xi^2, \xi^{-1}, \xi^{-2}\}$, respectively. In both cases, ξ is a generator of the group. (c) D_4 with $\Gamma = \{r, r^{-1}, s\}$.

labeled j . Therefore the Laplacian of the Cayley graph is given by

$$\Delta = \sum_j f(j) P_j, \quad f(j) = |\Gamma| - \frac{1}{\dim(j)} \sum_{k \in \Gamma} \chi_j(k), \quad (24)$$

which is the same form as the electric Hamiltonian in the representation basis (14). For any finite group, this formula defines the electric energy $f(j)$ to be assigned to each irrep.

We give some examples of this construction. For the group \mathbb{Z}_N it is natural to construct the electric eigenvalues $f(j)$ with the generating set $\Gamma = \{\xi, \xi^{-1}\}$ where ξ is a generator of \mathbb{Z}_N , which results in

$$f(j) = f(N-j) = 4\sin^2\left(\frac{\pi j}{N}\right), \quad (25)$$

which is the same as in [27]. Moreover, for large N ,

$$f(j) \rightarrow \frac{4\pi^2}{N^2} j^2 \quad N \text{ large}, \quad (26)$$

which is proportional to the Casimir eigenvalues of U(1) gauge theory [28]. Thus both a truncation of U(1) theory and proper \mathbb{Z}_N theory naively approach U(1) theory for large N , albeit in different ways. One can however choose a different generating set, such as $\Gamma = \{\xi, \xi^{-1}, \xi^2, \xi^{-2}\}$ and the corresponding eigenvalues would be

$$f(j) = f(N-j) = 4\sin^2\left(\frac{\pi j}{N}\right) + 4\sin^2\left(\frac{2\pi j}{N}\right). \quad (27)$$

For the dihedral group D_4 we can choose, for example,

$$\Gamma = \Gamma_1 = \{r, r^3, s, r^2s\},$$

which gives rise to the eigenvalues $f(j)$ shown in Table I, where the representations are ordered like in the character table in Table III in Appendix A 2. Note that Γ_1 generates the whole group.

By looking at its character table, we may in fact classify all possible choices of Γ for D_4 . In fact, D_4 has five conjugacy classes,

TABLE I. Eigenvalues of the single-link electric Hamiltonian $f(j)$ for the finite group D_4 , with three choices of generating sets: Γ_1 , Γ_2 , and Γ_3 , respectively.

Γ	j	$f(j)$				
		0	1	2	3	4
$\Gamma_1 = \{r, r^3, s, r^2s\}$		0	4	4	8	4
$\Gamma_2 = \{r, r^3, s, rs, r^2s, r^3s\}$		0	8	8	8	6
$\Gamma_3 = \{r, r^2, r^3\}$		0	4	0	4	5

$$\begin{aligned} C_0 &= \{e\}, & C_1 &= \{r, r^3\}, & C_2 &= \{r^2\}, \\ C_3 &= \{s, r^2s\}, & C_4 &= \{rs, r^3s\}. \end{aligned}$$

One can check that, as is generally true, $\sum_i |C_i| = 8 = |G|$. In this case, all conjugacy classes are invariant under inversion, i.e. $C_i^{-1} = C_i$. Hence any union of the C_i 's, $i \neq 0$ is a valid choice for Γ . There are 2^4 such possibilities. Note that this is not true in general, in which case one must choose conjugacy classes to ensure that $\Gamma^{-1} = \Gamma$. Together with Γ_1 , in the next sections we will consider in more detail two other specific cases,

$$\begin{aligned} \Gamma_2 &= C_1 \cup C_3 \cup C_4 = \{r, r^3, s, rs, r^2s, r^3s\}, \\ \Gamma_3 &= C_1 \cup C_2 = \{r, r^2, r^3\}. \end{aligned}$$

The choice of Γ_2 is especially interesting, because it corresponds to the Hamiltonian arising from the transfer-matrix procedure when h_B is the real part of the trace of D_4 ; therefore, this choice gives rise to a manifestly Lorentz-invariant theory. Note that also Γ_2 generates the whole group. On the other hand, the set $\Gamma_3 = \{r, r^2, r^3\}$ does not generate the whole group, but only the subgroup of rotations; this is reflected in the electric eigenvalues in Table I, with the electric ground state being twofold degenerate on each link.

3. Action formulation and Lorentz invariance

The Kogut-Susskind Hamiltonian (11) may be obtained via the transfer-matrix formulation from the Euclidean Wilson action [52,53]

$$S = -\frac{2}{g^2} \sum_{\square} \text{Re tr} \rho(g_{\square}), \quad (28)$$

where g is the coupling. In the path-integral formulation, the lattice is fully discretized and thus plaquettes extend also in the time direction. The action (28) is also perfectly valid for finite groups, as one simply replaces the integration measure over the Lie group with a sum over the elements of a finite group. The representation ρ can be chosen to be any faithful representation of the finite group (not necessarily irreducible). One may then repeat the transfer-matrix formulation for an arbitrary finite group [41]. Starting from the action (28), the transfer-matrix procedure gives rise to a Hamiltonian of the form (13) that we have described, with

$$\begin{aligned} \Gamma &= \{g \in G, g \neq 1, \max[\text{Re tr} \rho(g)]\}, \\ h_B &= -2\text{Re tr} \rho. \end{aligned} \quad (29)$$

In other words, the magnetic Hamiltonian is directly inherited from the action, while the electric Hamiltonian takes the form of the finite-group Laplacian with a specific

choice of the generating set Γ for the Cayley graph. In the example of the gauge group D_4 , if we choose the faithful, two-dimensional irrep for h_B , then $\text{Re tr}\rho(g)$ can equal 2, 0, -2 on the different conjugacy classes (see the character table of D_4 in Table III). Since $\text{Re tr}\rho(1) = 2$, then Γ consists of all group elements g such that $\text{Re tr}\rho(g) = 0$. This gives rise to the generating set Γ_2 anticipated in Sec. II B 2.

These considerations are especially important for the Lorentz invariance of the theory. While the lattice discretization breaks the Lorentz symmetry to the subgroup of Euclidean cubic rotations, as long as this subgroup is preserved one expects to recover Lorentz invariance in the continuum limit. In particular, the action (28) is manifestly invariant under Euclidean cubic rotations and, therefore, one expects that it gives rise to a Hamiltonian which describes a Lorentz-invariant theory in the continuum. Intuitively, a Lorentz transformation can swap the electric and magnetic fields, and it is therefore not surprising that in a Lorentz-invariant theory the electric and magnetic Hamiltonians must satisfy specific relations with each other.

In particular, finite-group Hamiltonians of the form (13), which however do not respect the relations (29), cannot arise from an action of the form (28). For example, they could come from an action in which plaquettes extending in one direction (the “time” direction) are weighted differently. It is unclear whether such Hamiltonians describe a Lorentz-invariant theory. This includes in particular setting $h_E(j) = j^2$ for subgroups of $U(1)$ and $h_E(j) = j(j+1)$ for subgroups of $SU(2)$ in (14), while keeping h_B unchanged. In all such cases, the remnant Lorentz symmetry is explicitly broken. While Lorentz symmetry is required in particle physics applications, it might not be necessarily required in other cases, such as some condensed matter applications, and one may thus independently choose Γ and h_B .

4. Classification of the possible theories and other models

The construction of finite-group Yang-Mills gauge theories with Hamiltonian (13) involves a few arbitrary choices which can be classified. Since the Hilbert space is fixed to the physical, gauge-invariant Hilbert space $\mathcal{H}_{\text{phys}}$, the only possible choices involve the various terms in the Hamiltonian. Given a gauge group G in d spatial dimensions, one may arbitrarily choose:

- (1) A set Γ of group elements which does not contain the identity and is invariant under both inversion and conjugation $\Gamma^{-1} = \Gamma$ and $g\Gamma g^{-1}$.
- (2) A choice for the magnetic Hamiltonian $h_B = h_B(g_{\square})$. Since it must be real and satisfy $h_B(g_1 g_{\square} g_1^{-1}) = h_B(g_{\square})$, i.e. it is a “class function,” by a well-known result [46], it may be expanded as a sum of characters of irreducible representations, $h_B(g) = \sum_j c_j \chi_j(g)$ for coefficients c_j which may be chosen arbitrarily, while ensuring that $h_B(g)$ is real. Most

typically $h_B = -2\text{Re tr}\rho$, where ρ is some faithful representation (not necessarily an irrep).

- (3) If present, possible choices of representations and Hamiltonians in the matter sector.

Further considerations involve the Lorentz symmetry, as explained in Sec. II B 3.

We note that the above construction allows further generalizations. In particular, the discretized d -dimensional space does not have to take the form of a hypercubic lattice, but more generally can be a Bravais or non-Bravais lattice, or even a cell complex. No difference arises for the electric term, which is link based, and the plaquette variable in the magnetic term is replaced by an elementary closed loop in the lattice. Moreover, as we will see in Sec. III, the description of the gauge-invariant Hilbert space in terms of spin-network states is valid generally on an arbitrary graph discretizing spacetime.

III. THE PHYSICAL HILBERT SPACE

As we remarked in Sec. II A 1, while the overall Hilbert space of the pure gauge theory is $\mathcal{H}_{\text{tot}} = \otimes_{\text{links}} \mathbb{C}[G]$, only those states that satisfy the so-called Gauss law constraint (4) are to be considered physical [43–45]. For gauge theories based on most compact Lie groups, the Wilson loops (despite being overcomplete) span the space of gauge-invariant states [54,55]. This, however, is not necessarily true for finite groups [54,56]; this means that in some cases, it is possible to construct different gauge-invariant states, which nevertheless have identical Wilson loops. We mention that this difficulty does not arise for Abelian finite groups such as \mathbb{Z}_N , in which case the Wilson loops *do* span the physical Hilbert space $\mathcal{H}_{\text{phys}}$. The explicit description of the physical Hilbert space of gauge theories has a long history (see for example [57–60] for different approaches) and several works have attempted to quantify the size of the physical Hilbert space in various quantum simulation schemes (for example [61,62]). As will be discussed in Sec. III A, the gauge-invariant Hilbert space for pure gauge theories may be described in terms of “spin-network states.” This basis turns out to be particularly suitable for finite gauge groups and in Sec. III B we give a simple formula to compute the dimension of the physical Hilbert space for any finite gauge group.

A. Spin-network states

The physical Hilbert space of pure gauge theories with either Lie or finite gauge group can be explicitly described in terms of spin-network states [58,59]. Spin-network states can be defined indifferently when the d -dimensional space is discretized as an arbitrary graph and are thus valid in arbitrary dimension with arbitrary lattices and boundary conditions.

The first key observation is that the action of the Gauss law operator (4) is block diagonal in the representation

basis, as can be seen from Eq. (6). Then starting from the Hilbert space in the representation basis (5), we can, as usual, permute the summation and product, obtaining

$$\mathcal{H}_{\text{tot}} = \bigotimes_{\text{links}} \bigoplus_{j \in \Sigma} V_j^* \otimes V_j = \bigoplus_{\{j\} \in \{\Sigma\}} \bigotimes_{l \in \text{links}} V_{j_l}^* \otimes V_{j_l}, \quad (30)$$

where now $\{j\}$ is an assignment of an irrep j_l to each lattice link l , and $\{\Sigma\}$ is the set of the possible assignments. The second key observation is that the gauge transformations (4) are given by an independent group-valued variable g_x at each site x of the lattice.

Moreover, due to (6) the gauge transformation associated with one site x acts at most on one of the spaces V_j or V_j^* associated with a link, but it cannot act on both. One can then split the two vector spaces V_j and V_j^* associated with each link and reorder the V 's in the tensor product over links so that V_j 's are now grouped together according to the *sites* and not the links,

$$\mathcal{H}_{\text{tot}} = \bigoplus_{\{j\} \in \{\Sigma\}} \bigotimes_{x \in \text{sites}} \left(\bigotimes_{l_- = x} V_{j_l}^* \right) \otimes \left(\bigotimes_{l_+ = x} V_{j_l} \right), \quad (31)$$

where by l_+ and l_- we denote respectively the target and source vertex of link l (see Fig. 2).

We can repeat the same set of operations for the gauge transformation operator (3), which is therefore given by

$$\mathcal{G}(\{g_x\}) = \bigoplus_{\{j\} \in \{\Sigma\}} \bigotimes_{x \in \text{sites}} \left(\bigotimes_{l_- = x} \rho_{j_l}^*(g_x) \right) \otimes \left(\bigotimes_{l_+ = x} \rho_{j_l}(g_x) \right). \quad (32)$$

In the above decomposition, the gauge transformations now act independently for each x and the Gauss law constraint (4) gives the physical Hilbert space

$$\mathcal{H}_{\text{phys}} = \bigoplus_{\{j\} \in \{\Sigma\}} \bigotimes_{x \in \text{sites}} \text{Inv} \left[\left(\bigotimes_{l_- = x} V_{j_l}^* \right) \otimes \left(\bigotimes_{l_+ = x} V_{j_l} \right) \right]. \quad (33)$$

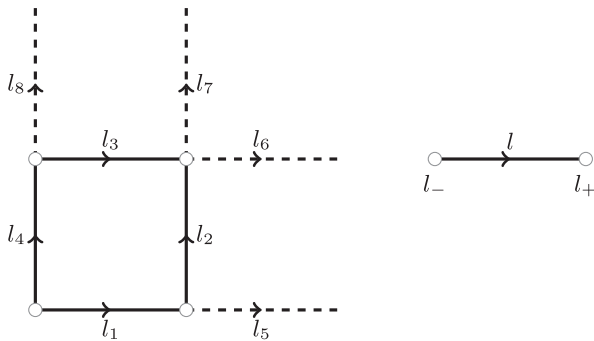


FIG. 2. Left: a 2×2 square lattice with periodic boundary conditions, showing the labels of the links. Right: labeling of sites attached to a link.

Given a representation ρ (not necessarily irreducible) with representation space V_ρ , the set of invariant vectors $\text{Inv}(V_\rho)$ is the set of vectors $v \in V_\rho$ such that $\rho(g)v = v$ for all $g \in G$. Note that this is a separate notion from that of an “invariant subspace.” The characterization of the Hilbert space (33) implies that any physical, gauge-invariant state $|\Psi\rangle$ [i.e. a state which satisfies the Gauss’s law (4)] may be expanded in a basis of spin-network states,

$$|\Psi\rangle = \sum_{\{j\}} \sum_A \Psi(\{j\}; A) |\{j\}, A\rangle, \quad (34)$$

$$|\{j\}, A\rangle = \bigotimes_{x \in \text{sites}} |\{j\}_x, a_x\rangle,$$

where $\{j\}$ is an assignment of irreps to lattice links and $A = (a_1, \dots, a_N)$ is a multi-index which labels the choice of a basis element of invariant states at each site. With $\{j\}_x$ we denote the irreps assigned to the links connected to site x .

For a hypercubic lattice in d dimensions with periodic boundary conditions, each site is connected to $2d$ links and therefore $2d$ terms appear in the tensor product within each Inv in Eq. (33). If instead we choose open boundary conditions, the sites in the bulk will again be connected to $2d$ links, but the sites on the boundary will be connected to fewer links and thus fewer terms will appear in the tensor product for those sites. In the general case, the number of terms in the tensor product within each Inv will thus depend on the site. We choose to work directly with the spaces of invariant vectors rather than with spaces of intertwiners more commonly employed in the literature on spin-network states [58,59]. We also would like to note that the physical Hilbert space (33) contains *all* gauge-invariant states, possibly also including states in sectors with a noncontractible spatial Wilson line. Instead noncontractible loops in the time direction require an extension of the total Hilbert space [41].

The calculation of a basis of invariant states (or, equivalently, of the intertwiners) can be difficult in the Lie group case, especially since they admit infinitely many irreps. On the other hand, since the number of links connected to each site is finite and independent of the lattice volume, one needs only compute the invariant states of a finite number of tensor product representations which does not scale with the lattice volume.

This can be achieved in practice by explicitly writing out the matrices of the tensor product representation $\rho(g) \equiv \left(\bigotimes_{l_- = x} \rho_{j_l}^* \right) \otimes \left(\bigotimes_{l_+ = x} \rho_{j_l} \right)$ and numerically solving the simultaneous equations $\rho(g)v = v$ for a set of generators of G . In a d -dimensional periodic hypercubic lattice, the number of terms in the tensor product equals $2d$ and the maximum dimension of the tensor product representation is bounded by $(\dim j)^{2d} \leq |G|^d$, owing to $\sum_j (\dim j)^2 = |G|$, independent from the lattice volume.

As an example, we work out explicitly the case of a 2×2 square lattice with periodic boundary conditions. As shown

in Fig. 2, this system has four vertices and eight links with appropriate labels. Expanding Eq. (33) explicitly, we see that in this case

$$\begin{aligned} \mathcal{H}_{\text{phys}} = & \bigoplus_{j_1, \dots, j_8} \text{Inv}[V_{j_1}^* \otimes V_{j_4}^* \otimes V_{j_5} \otimes V_{j_8}] \otimes \\ & \otimes \text{Inv}[V_{j_5}^* \otimes V_{j_2}^* \otimes V_{j_1} \otimes V_{j_7}] \otimes \\ & \otimes \text{Inv}[V_{j_6}^* \otimes V_{j_7}^* \otimes V_{j_3} \otimes V_{j_2}] \otimes \\ & \otimes \text{Inv}[V_{j_3}^* \otimes V_{j_8}^* \otimes V_{j_6} \otimes V_{j_4}]. \end{aligned} \quad (35)$$

Now consider a single invariant subspace $\text{Inv}[V_{j_1}^* \otimes V_{j_2}^* \otimes V_{j_3} \otimes V_{j_4}]$ with arbitrary assignment of irreps. This vector space admits an orthonormal basis $\{|j_1 j_2 j_3 j_4; a\rangle\}$ where $1 \leq a \leq \dim \text{Inv}[V_{j_1}^* \otimes V_{j_2}^* \otimes V_{j_3} \otimes V_{j_4}]$ indexes the basis vector. We can expand the basis vectors explicitly in terms of the bases of the V_j as [see also the discussion around Eq. (7)]

$$\begin{aligned} |j_1 j_2 j_3 j_4; a\rangle = & \sum_{m_1, m_2, m_3, m_4} \psi(j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4; a) \\ & \times |j_1 m_1\rangle \otimes |j_2 m_2\rangle \otimes |j_3 m_3\rangle \otimes |j_4 m_4\rangle. \end{aligned} \quad (36)$$

The basis vectors can be chosen to be orthonormal. By virtue of spanning the space $\text{Inv}[V_{j_1}^* \otimes V_{j_2}^* \otimes V_{j_3} \otimes V_{j_4}]$, they are invariant vectors of the tensor product representation $\rho \equiv \rho_{j_1}^* \otimes \rho_{j_2}^* \otimes \rho_{j_3} \otimes \rho_{j_4}$; as such, they satisfy $\rho(g)|j_1 j_2 j_3 j_4; a\rangle = |j_1 j_2 j_3 j_4; a\rangle$ for all $g \in G$. The coefficients of the expansion $\psi(j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4; a)$ may be easily computed, for example by writing the tensor product representation matrices $\rho(g)$ explicitly and then solving the simultaneous equations $\rho(g)v = v$. The dimension of the space of invariant vectors depends on the four representations assigned to the relevant site. Now let $A = (a_1, a_2, a_3, a_4)$, which implicitly depends on $\{j\}$ (because the range of each a_x depends on the irreps assigned to site x). Given any assignment of irreps $\{j\}$, A is a choice of a basis vector of invariant states at the four sites. Therefore an orthonormal basis for the gauge-invariant Hilbert space is given by

$$\begin{aligned} |\{j\}; A\rangle = & |j_1 j_4 j_5 j_8; a_1\rangle \otimes |j_5 j_2 j_1 j_7; a_2\rangle \otimes \\ & \otimes |j_6 j_7 j_3 j_2; a_3\rangle \otimes |j_3 j_8 j_6 j_4; a_4\rangle, \end{aligned} \quad (37)$$

for any possible assignment $\{j\}$ of irreps to links and then all possible choices A of an invariant vector at each of the four sites. The spin-network states $|\{j\}; A\rangle$ then form a basis of the gauge-invariant Hilbert space $\mathcal{H}_{\text{phys}}$. Expanding the tensor product, we find an explicit expression for these states in terms of the representation basis,

$$\begin{aligned} |\{j\}; A\rangle = & \sum_{n_1, \dots, n_8} \sum_{m_1, \dots, m_8} \psi(j_1 m_1 j_4 m_4 j_5 n_5 j_8 n_8 | a_1) \\ & \times \psi(j_5 m_5 j_2 m_2 j_1 n_1 j_7 n_7 | a_2) \\ & \times \psi(j_6 m_6 j_7 m_7 j_3 n_3 j_2 n_2 | a_3) \\ & \times \psi(j_3 m_3 j_8 m_8 j_6 n_6 j_4 n_4 | a_4) \\ & \times |j_1 m_1 n_1\rangle \otimes |j_2 m_2 n_2\rangle \otimes \dots \otimes |j_8 m_8 n_8\rangle, \end{aligned} \quad (38)$$

where we have restored the ordering of the vector spaces V_{j_s} and used again the shorthand $|jmn\rangle = |jm\rangle \otimes |jn\rangle$. We note in particular that, despite having introduced a splitting of the variables at each link, in the final answer this splitting disappears and the spin-network states can be entirely expressed in terms of the representation basis $|jmn\rangle$.

B. The dimension of the physical Hilbert space

As we have seen in the previous section, spin-network states give an explicit description of the physical Hilbert space $\mathcal{H}_{\text{phys}}$ as

$$\mathcal{H}_{\text{phys}} = \bigoplus_{\{\rho\} \in \{\Sigma\}} \bigotimes_{v \in \text{sites}} \text{Inv} \left[\left(\bigotimes_{l=v} V_{\rho_l}^* \right) \otimes \left(\bigotimes_{l=v} V_{\rho_l} \right) \right], \quad (39)$$

where $\text{Inv}(\rho)$ is the space of invariant vectors of the representation ρ , $\{\rho\}$ is an assignment of irreps to links and $\{\Sigma\}$ is the set of such possible assignments. For a finite group,

$$\dim \text{Inv}(\rho) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g), \quad (40)$$

where χ_ρ is the character of ρ . A proof of this result can be found in Appendix B 2. This fact can be used to obtain a general formula for the dimension of $\mathcal{H}_{\text{phys}}$, which is valid for any lattice in any dimension with any boundary conditions. On a connected lattice with L links and V sites, we will show that

$$\dim \mathcal{H}_{\text{phys}} = \sum_C \left(\frac{|G|}{|C|} \right)^{L-V}, \quad (41)$$

where the sum runs over all conjugacy classes C of the group, and $|C|$ is the size of C . The ratio $|G|/|C|$ is always an integer by the orbit-stabilizer theorem [46]. Since for a connected graph $L - V \geq -1$, the $\dim \mathcal{H}_{\text{phys}}$ in Eq. (41) is always an integer. This is clear for $L - V \geq 0$; when $L - V = -1$ the graph is a tree and since $\sum_C |C| = |G|$, we find $\dim \mathcal{H}_{\text{phys}} = 1$; this is to be expected because, on a tree, all the physical degrees of freedom can be rotated away by gauge transformations.

Using Eq. (40), together with the fact that the character of a tensor product is given by the product of the characters,

we may readily prove Eq. (41). From the general formula for the gauge-invariant Hilbert space, we have

$$\begin{aligned} \dim \mathcal{H}_{\text{phys}} &= \sum_{j_1 j_2 \dots j_L} \prod_{x \in \text{sites}} \dim \text{Inv} \left[\left(\bigotimes_{l=x} V_{\rho_l}^* \right) \otimes \left(\bigotimes_{l=x} V_{\rho_l} \right) \right] \\ &= \frac{1}{|G|^V} \sum_{j_1 j_2 \dots j_L} \sum_{g_1 g_2 \dots g_V} \prod_{x \in \text{sites}} \left(\prod_{l=x} \chi_{j_l}^*(g_x) \right) \\ &\quad \times \left(\prod_{l=x} \chi_{j_l}(g_x) \right). \end{aligned} \quad (42)$$

Within the product over all sites, there are exactly $2L$ factors of characters χ , as each link contributes two representation spaces V and each representation space gives rise to a character. Thus grouping characters by link, we obtain

$$\dim \mathcal{H}_{\text{phys}} = \frac{1}{|G|^V} \sum_{g_1 g_2 \dots g_V} \prod_{l=\langle xx' \rangle \in \text{links}} \langle g_x, g_{x'} \rangle, \quad (43)$$

where we denoted $\langle g, h \rangle = \sum_j \chi_j(g) \chi_j^*(h)$. It is a well-known result that $\langle g, h \rangle$ is zero unless g and h belong to the same conjugacy class, in which case $\langle g, h \rangle = |G|/|C|$ where C is the conjugacy class of both g and h [46]. If any two adjacent sites x and x' have g_x and $g_{x'}$ in different conjugacy classes, then $\langle g_x, g_{x'} \rangle = 0$ and the corresponding term in the sum is zero. Assuming that the lattice is connected, this implies that the product over all links is zero unless all the g_x at each site x belong to the same conjugacy class. Then, since $\langle g_x, g_{x'} \rangle$ is constant on conjugacy classes, we can write

$$\dim \mathcal{H}_{\text{phys}} = \frac{1}{|G|^V} \sum_C \sum_{g_1 g_2 \dots g_V \in C} \frac{|G|^L}{|C|^L} = \sum_C \left(\frac{|G|}{|C|} \right)^{L-V}, \quad (44)$$

which concludes the proof. In the Abelian case the above formula simplifies as all conjugacy classes are singlets and therefore $\dim \mathcal{H}_{\text{phys}} = |G|^{L-V+1}$. Thus finite Abelian groups have the largest physical Hilbert space among all groups of the same order. For periodic boundary conditions in a hypercubic lattice, $L = Vd$ and as such $\dim \mathcal{H} = |G|^{Vd}$, while $\dim \mathcal{H}_{\text{phys}} \approx |G|^{V(d-1)}$, so that the physical Hilbert space has roughly the same size as the overall Hilbert space in one lower dimension. Nonetheless, both spaces grow exponentially with the lattice size.

As a further example, we consider the dimension of the Hilbert space for pure D_4 gauge theory. Using Eq. (41), we find for $G = D_4$ on a lattice with L links and V sites (see also [63]),

$$\dim \mathcal{H}_{\text{phys}} = 8^{L-V} \left(2 + \frac{3}{2^{L-V}} \right). \quad (45)$$

TABLE II. Dimension of the physical subspace of D_4 gauge theory on some small lattices in $2 + 1$ dimensions with different boundary conditions (BCs), compared to the dimension of the total Hilbert space. L is the number of links and V is the number of vertices. Numbers above 10^6 are approximated values.

Size	BCs	L	V	$L - V$	$\dim \mathcal{H}_{\text{phys}}$	$\dim \mathcal{H}$
2×2	Open	4	4	0	5	4096
	Periodic	8	4	4	8960	1.6×10^7
2×3	Open	7	6	1	28	2.1×10^6
	Periodic	12	6	6	536576	6.9×10^{10}
3×3	Open	12	9	3	1216	6.9×10^{10}
	Periodic	18	9	9	2.7×10^8	1.8×10^{16}

The dimension of the physical Hilbert space for some two-dimensional finite square lattices in $2 + 1$ dimensions is shown in Table II. We see that its size grows quickly with the lattice size. We point out that even for a 2×2 periodic lattice with a small group such as D_4 it is not practical to write down all possible gauge-invariant states. Unless the structure happens to be very sparse, writing down the 8960 physical basis elements in terms of the $|G|^L = 8^8$ basis elements in the representation basis using 4B floating point numbers would require roughly 600 GB of memory. For a 3×3 periodic lattice this number rises to 20YB or 2×10^{16} GB.

Finally, we remark that since matter fields are site based, the spin-network states may be extended to this case as well; the physical Hilbert space would then be given again by Eq. (33) with an extra factor of the matter Hilbert space at each site within each Inv. The detailed description of the gauge-invariant Hilbert space with matter fields will be given in a future publication.

IV. DIHEDRAL GAUGE THEORY ON A SMALL PERIODIC LATTICE

In this section we consider pure gauge theory with gauge group $G = D_4$, the dihedral group with eight elements, on a small 2×2 periodic lattice (see Fig. 2). We compute the Hamiltonian in the gauge-invariant spin-network basis and diagonalize it exactly. As remarked in Sec. III B, the physical Hilbert space of this theory has dimension equal to 8960 and it is not practical to store the gauge-invariant states directly. Instead, we first compute numerically the basis of invariant states at a site for all possible combinations of irreps assigned to the four links attached to the site [i.e. we compute the coefficients ψ in Eq. (36)]. Using these coefficients we then compute the matrix elements of the electric and magnetic Hamiltonians separately in the spin-network basis (38). The electric Hamiltonian is diagonal, and the magnetic Hamiltonian is off diagonal. In units of $\lambda_E + \lambda_B$ the Hamiltonian can be written as $H = (1 - \lambda)H_E + \lambda H_B$ for $\lambda \in [0, 1]$, where $\lambda = \lambda_B / (\lambda_E + \lambda_B)$.

In practice, for each λ , H is a 8960×8960 matrix. As expected for spin-network states [59], we find H to be very sparse: less than 1% of the elements are nonzero.

The electric and magnetic Hamiltonians were chosen as in Eq. (13). In particular, we chose $h_B = -2\text{tr}\rho_4$ where ρ_4 is the two-dimensional irrep of D_4 and considered the three different choices of the set Γ for h_E described in Sec. II B 2. These are $\Gamma_1 = \{r, r^3, s, r^2s\}$, $\Gamma_2 = \{r, r^3, s, rs, r^2s, r^3s\}$, and $\Gamma_3 = \{r, r^2, r^3\}$. We recall that the electric Hamiltonian is twofold degenerate on each link with the choice of Γ_3 but is not degenerate with Γ_1 or Γ_2 . The choice of Γ_2 , unlike the other two, gives rise to a Lorentz-invariant theory.

The results of the exact diagonalization are shown in Fig. 3. The ground state energy and the ground state expectation values of the electric and magnetic

Hamiltonians are shown in Figs. 3(a)–3(d), respectively. The qualitative picture is rather similar in all three cases, with the electric energy increasing with λ and the magnetic energy decreasing with λ . The ground state energies coincide at $\lambda = 0$, because h_E always has a zero eigenvalue, and at $\lambda = 1$ where the Hamiltonian reduces to H_B , which is the same in the three cases. The plaquette Wilson loop is equal to H_B apart from an overall prefactor, and therefore its behavior is also given by Fig. 3(d). We note that our data for the ground state energies agree with that obtained in [63] with a different method.

The difference in energy between the first two states $E_1 - E_0$ is shown in Figs. 3(e) and 3(f). The difference $E_1 - E_0$ coincides with the energy gap (in finite volume) except where there are degenerate ground states, in which

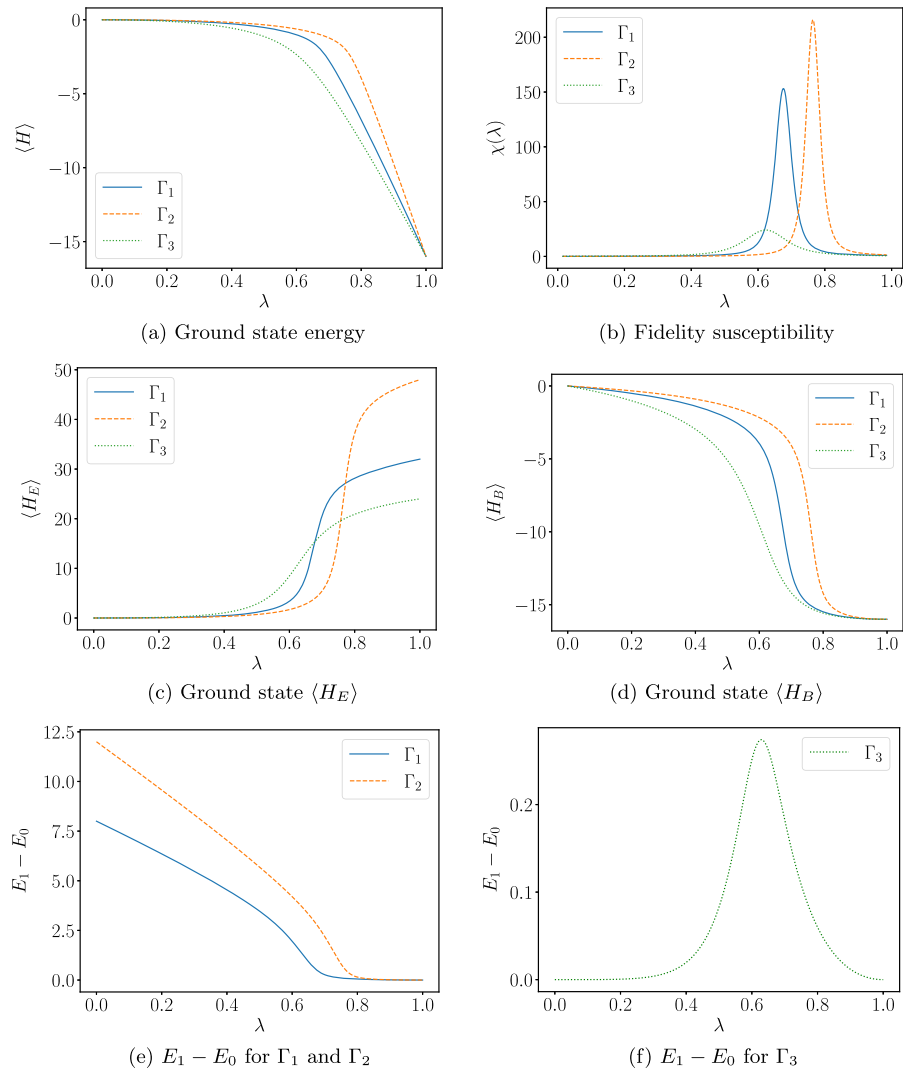


FIG. 3. Exact diagonalization results for D_4 gauge theory on a 2×2 periodic lattice in the gauge-invariant basis, for three different choices of generating set Γ , i.e. $\Gamma_1 = \{r, r^3, s, r^2s\}$ (solid blue), $\Gamma_2 = \{r, r^3, s, rs, r^2s, r^3s\}$ (relativistic, dashed orange), and $\Gamma_3 = \{r, r^2, r^3\}$ (degenerate, dotted green). (a) Ground state energy. (b) Fidelity susceptibility. (c),(d) Ground state expectation values of the electric and magnetic Hamiltonians. (e),(f) Energy difference between the first two energy levels.

case $E_1 - E_0 = 0$. Ground state degeneracy occurs in all three cases at $\lambda = 1$ and, for Γ_3 , also at $\lambda = 0$. For Γ_3 , the ground state degeneracy at $\lambda = 0$ is lifted at any $\lambda > 0$ but the scale of the gap remains much smaller than in the other two cases. In all three cases we recognize a transition region at $\lambda \sim 0.6\text{--}0.8$ (depending on Γ), in which the ground state quickly evolves from minimizing the electric energy to minimizing the magnetic energy. For Γ_1 and Γ_2 , at the transition the gap rapidly becomes very small, while for Γ_3 it is always close to zero except in the transition region, where it peaks.

One possible way to locate the transition point is to identify it as the point of sharpest variation of $\langle H_E \rangle$ and/or $\langle H_B \rangle$ (i.e. the maximum of the absolute value of their derivative with respect to λ). With this identification, the transition points given by either $\langle H_E \rangle$ or $\langle H_B \rangle$ coincide at $\lambda_1^* = 0.67(1)$ and $\lambda_2^* = 0.76(1)$ for Γ_1 and Γ_2 , but show a small difference for Γ_3 , at $\lambda_{3,E}^* = 0.63(1)$ and $\lambda_{3,B}^* = 0.61(1)$. Alternatively, calling $|\psi_0(\lambda)\rangle$ the ground state of $H(\lambda)$, one can look at the fidelity susceptibility [64]

$$\chi(\lambda) = -\frac{\partial^2}{\partial \epsilon^2} \log |\langle \psi_0(\lambda) | \psi_0(\lambda + \epsilon) \rangle|^2 |_{\epsilon=0}, \quad (46)$$

which is expected to peak at the transition [65]. Figure 3(b) shows $\chi(\lambda)$ for the three cases and its peak identifies the transition point as $\lambda_{1,\chi}^* = 0.67(1)$, $\lambda_{2,\chi}^* = 0.76(1)$, and $\lambda_{3,\chi}^* = 0.62(1)$, in agreement with the previous method.

Overall, these results point toward the expected picture of a two-phase structure for all three cases. The data for Γ_1 and Γ_2 are consistent with the usual picture of a confining phase at small λ and a deconfined phase at large λ . For Γ_3 , the structure of the first few states is rather different and it is not clear whether it would be consistent with the interpretation of this phase as confining. Of course, due to the small volume these results are only qualitative and preliminary; a study with larger volumes would be required in order to properly establish the phase structure. However, they point to the possibility that theories with electric degeneracy may display different behavior and phase structure compared to those with no electric degeneracy.

V. CONCLUSIONS

In this work we considered Hamiltonians for gauge theories with a finite gauge group and we have shown that the electric term may be interpreted as a natural Laplacian operator on the finite group, constructed as the graph Laplacian of its Cayley graph. The choice of generating set of the Cayley graph has a simple relation with the ground state degeneracy of the electric Hamiltonian. We have also given careful consideration to the various choices involved in constructing a finite-group gauge theory and their consequences. Independent from the choice of Hamiltonian, we have shown that the physical,

gauge-invariant Hilbert space of pure gauge theories may be explicitly described in terms of spin-network states, which are particularly suitable for finite groups. This also allows us to derive a simple formula to compute the dimension of the physical Hilbert space on an arbitrary lattice. This is especially useful to quantify how much resources can be saved by working directly in the physical Hilbert space. Using the spin-network basis, we diagonalized D_4 gauge theory on a small periodic lattice with different Hamiltonians. Due to the small system size, these results are only suggestive, but they point to the possibility that theories with a degenerate electric Hamiltonian may have a different phase structure than commonly expected.

The methods employed in this work may be extended in several directions. The graph Laplacian construction may be adapted to those approaches where a Lie group is discretized to a finite subset, not necessarily a subgroup [66]. In that case the finite subset may be seen as a weighted graph, with the edge weights representing the distance between group elements in the parent Lie group. Another aspect which would be interesting to explore is the possibility of extending the Cayley graph construction to the case of improved electric Hamiltonians [67] and investigating their possible degeneracy.

As we have seen, working directly in the gauge-invariant basis reduces the size of the Hilbert space and implements Gauss's law exactly, at the expense of higher complexity of the Hamiltonian. It would be worthwhile to explore whether the gauge-invariant basis can be efficiently implemented, for example in a quantum circuit. It is also possible to extend the spin-network basis to gauge theories coupled to matter fields, and we will treat this case in a future publication.

Finally, it would be very interesting to explore the possibility of a nonstandard mechanism to obtain a continuum limit for finite-group gauge theories; for example, this is possible in quantum link models via the D-theory formulation [68,69].

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APPENDIX A: SOME GROUPS OF INTEREST

1. The cyclic groups \mathbb{Z}_N gauge theory

The cyclic group \mathbb{Z}_N is an Abelian group of order N . It is generated by one element ξ , which thus satisfies $\xi^N = 1$. Thus $\mathbb{Z}_N = \{1, \xi, \xi^2, \dots, \xi^{N-1}\}$. Since \mathbb{Z}_N is Abelian, its

TABLE III. Character table of D_4 .

	$\{1\}$	$\{r, r^3\}$	$\{r^2\}$	$\{s, r^2s\}$	$\{rs, r^3s\}$
χ_0	+1	+1	+1	+1	+1
χ_1	+1	-1	+1	+1	-1
χ_2	+1	+1	+1	-1	-1
χ_3	+1	-1	+1	-1	+1
χ_4	+2	0	-2	0	0

conjugacy classes are singlets, i.e. it has N conjugacy classes of one element; moreover, it has N inequivalent irreps, all of which are one dimensional,

$$\rho_j(\xi^k) = \omega_N^{kj}, \quad j = 0, 1, \dots, N-1, \quad (\text{A1})$$

with $\omega_N = e^{2\pi i/N}$. The bases $\{|\xi^k\rangle\}$ and $\{|j\rangle\}$ are related by

$$|j\rangle = \sum_{k=0}^{N-1} \langle \xi^k | j \rangle |\xi^k\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{kj} |\xi^k\rangle, \quad (\text{A2})$$

which is just the discrete Fourier transform.

2. The dihedral groups D_N

The dihedral groups D_N are non-Abelian groups of order $2N$. They are subgroups of $O(2)$, and they are generated by a rotation r and a reflection s , which satisfy $r^N = s^2 = 1$ and $srs = r^{-1}$. We describe in more detail the dihedral group D_4 of order 8. Its elements are $D_4 = \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$ and it has five conjugacy classes, $\{1\}, \{r, r^3\}, \{r^2\}, \{s, r^2s\}, \{rs, r^3s\}$. It has five irreducible representations, which we number from $j = 0$ (trivial representation) to $j = 4$. All the irreps are one dimensional except $j = 4$, which is two dimensional and faithful. The character table is shown in Table III.

As the $j = 4$ is the only faithful irrep, it is a natural choice for the magnetic Hamiltonian.

APPENDIX B: PROOFS

1. Degeneracy of electric Hamiltonian

As discussed in Sec. II, the degeneracy of the electric Hamiltonian given by the finite-group Laplacian Δ is directly related to the structure of the Cayley graph. In particular, it is a standard result that the graph Laplacian always has a zero mode and its degeneracy equals the number of connected components of the graph [50]. Here we show that the Cayley graph is connected if and only if its generating set Γ generates the whole group. If instead $\langle \Gamma \rangle \neq G$, then the Cayley graph splits into connected components identified with the cosets of $\langle \Gamma \rangle$ in G ; thus the degeneracy of the finite-group Laplacian Δ equals $|G|/|\langle \Gamma \rangle|$.

Any subset $\Gamma \subset G$ generates a subgroup $\langle \Gamma \rangle < G$. The right cosets of $\langle \Gamma \rangle$ are of the form $\langle \Gamma \rangle h$ for h in G . Since cosets partition the group, any two group elements g_1 and g_2 will belong to some coset, say $g_1 \in \langle \Gamma \rangle h_1$ and $g_2 \in \langle \Gamma \rangle h_2$. We want to show that there is an edge in the Cayley graph between group elements g_1 and g_2 if and only if they are in the same coset, i.e. $\langle \Gamma \rangle h_1 = \langle \Gamma \rangle h_2$. The fact that $g_i \in \langle \Gamma \rangle h_i$ means that $g_i = k_i h_i$ for some $k_i \in \langle \Gamma \rangle$. Moreover, there is an edge between g_1 and g_2 if and only if $g_1 g_2^{-1} = k_1 h_1 h_2^{-1} k_2 \in \Gamma$. But since $k_i \in \langle \Gamma \rangle$ this is equivalent to saying that $h_1 h_2^{-1} \in \langle \Gamma \rangle$, which is equivalent to $\langle \Gamma \rangle h_1 = \langle \Gamma \rangle h_2$. This concludes the proof.

2. Counting of invariant states

In deriving Eq. (41) we used the fact that for a generic representation ρ , the dimension of the space of invariant vectors is given by

$$\dim \text{Inv}(\rho) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g). \quad (\text{B1})$$

If ρ is a nontrivial irreducible then the corresponding character sums to zero [46] and there are no invariant states. This is to be expected since, by definition, irreducible representations have no nontrivial invariant subspaces, but any invariant vector would span an invariant subspace.

Here we provide a proof of the above formula. If v is an invariant vector for the representation ρ , by definition it satisfies $\rho(g)v = v$ for all $g \in G$. Now we construct a projector onto the subspace of invariant vectors. We define the averaging map $\text{Av}: V_\rho \rightarrow V_\rho$,

$$\text{Av}(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)v. \quad (\text{B2})$$

The averaging map is the projector onto the subspace of invariant vectors. In fact, given an arbitrary vector v , we see that $\text{Av}(v)$ is invariant because

$$\rho(g)\text{Av}(v) = \frac{1}{|G|} \sum_{h \in G} \rho(gh)v = \frac{1}{|G|} \sum_{h \in G} \rho(h)v = \text{Av}(v). \quad (\text{B3})$$

Therefore, Av maps the representation space to the subspace of invariant vectors $\text{Av}: V_\rho \rightarrow \text{Inv}(V_\rho)$. Moreover, if v is invariant, then $\text{Av}(v) = v$, and more generally, $\text{Av}^2 = \text{Av}$ by a similar calculation. This means that Av is a projector onto the subspace of invariant vectors. As usual, the size of projected subspace is given by the trace of the projector, $\dim \text{Inv}(\rho) = \text{tr} \text{Av}$, which reproduces the above formula.

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