ON THE SURFACE AVERAGE FOR HARMONIC FUNCTIONS: A STABILITY INEQUALITY

SULLA MEDIA DI SUPERFICIE PER LE FUNZIONI ARMONICHE: UNA DISUGUAGLIANZA DI STABILITÀ

GIOVANNI CUPINI AND ERMANNO LANCONELLI

ABSTRACT. In this article we present some of the main aspects and the most recent results related to the following question: If the surface mean integral of every harmonic function on the boundary of an open set D is "almost" equal to the value of these functions at x_0 in D, then is D "almost" a ball with center x_0? This is the stability counterpart of the rigidity question (the statement above, without the two "almost") for which several positive answers are known in literature. A positive answer to the stability problem has been given in a paper by Preiss and Toro, by assuming a condition that turns out to be sufficient for ∂D to be geometrically close to a sphere. This condition, however, is not necessary, even for small Lipschitz perturbations of smooth domains, as shown in our recent paper, in which a stability inequality is obtained by assuming only a local regularity property of the boundary of D in at least one of its points closest to x_0.

SUNTO. In questo articolo presentiamo alcuni degli aspetti principali e i risultati più recenti relativi al seguente quesito: Se la media integrale di superficie di ogni funzione armonica sulla frontiera di un insieme aperto D è "quasi" uguale al valore di queste funzioni in x_0 in D, allora D è "quasi" una palla con centro x_0? Questa è la controparte di stabilità del quesito di rigidità (la frase sopra senza i due "quasi") per il quale diverse risposte affermative sono note in letteratura. Una risposta affermativa al problema di stabilità è stata data in un articolo di Preiss e Toro, assumendo una condizione che si rivela sufficiente per ∂D ad essere geometricamente vicino a una sfera. Questa condizione, tuttavia, non è necessaria, anche per piccole perturbazioni Lipschitziane di domini lisci, come dimostrato nel nostro recente articolo, in cui si ottiene una disuguaglianza di stabilità assumendo solo una proprietà di regolarità locale del bordo di D in almeno uno dei suoi punti più vicini a x_0.

2020 MSC. Primary 35B05; Secondary 31B05, 35B06.

KEYWORDS: Surface Gauss mean value formula, stability, harmonic functions, rigidity

Bruno Pini Mathematical Analysis Seminar, Vol. 14, No. 2 (2023) pp. 129–138

Dipartimento di Matematica, Università di Bologna

ISSN 2240-2829.

1. Introduction: past and recent history

Let $D \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded open set such that $|\partial D| < \infty$. We say that ∂D is a *harmonic pseudosphere* with center $x_0 \in D$ if

$$u(x_0) = \fint_{\partial D} u \, d\sigma \qquad \forall u \in \mathcal{H}(D) \cap C(\overline{D}),$$

where $\mathcal{H}(D)$ denotes the set of harmonic functions in D.

By the classical Gauss Theorem on the mean value property of the harmonic functions, every Euclidean sphere is a harmonic pseudosphere. In general the viceversa is not true. Indeed, in 1937, Keldysch and Lavrentieff proved the existence of a harmonic pseudosphere in $\mathbb{R}^2$ which is *not a circle*. Many years later - in 1991 - Lewis and Vogel proved that in every Euclidean space $\mathbb{R}^n$, $n \geq 3$, there exist harmonic pseudospheres which are not Euclidean spheres, see [6].

Then the following question arises: *when is a harmonic pseudosphere a Euclidean sphere*, or, equivalently, *is it possible to characterize the Euclidean spheres in terms of the surface mean value formula for harmonic functions?*

Several authors, under different hypotheses, have proved that the answer is yes (see [3], [2], [7]). The most general result, proved by Lewis and Vogel in 2002 (see [8]), is the following one:

Let $D \subseteq \mathbb{R}^n$ be open and bounded and let $|\partial D| < \infty$. Suppose ∂D is a harmonic pseudosphere centered at $x_0 \in D$. Then D is a Euclidean sphere centered at x_0 if D and ∂D have the following properties:

(i) *D is Dirichlet regular, i.e. the boundary value problem*

$$\begin{cases} \Delta u = 0 \quad \text{in } D \\ u|_{\partial D} = \varphi \end{cases}$$

has a classical solution $u \in \mathcal{H}(D) \cap C(\overline{D})$ for every $\varphi \in C(\partial D)$;

(ii) *the $(n-1)$-Hausdorff measure H^{n-1} restricted to ∂D has at most Euclidean growth, i.e.,*

$$\sup_{\substack{x \in \partial D \\ 0<r<1}} \frac{H^{n-1}(\partial D \cap B(x,r))}{r^{n-1}} < \infty.$$

In 2007 Preiss and Toro [9] proved a *stability result* for Lewis-Vogel's rigidity Theorem. Preiss and Toro's starting point is the following remark. Let D and x_0 as in Lewis-Vogel 2002's Theorem and let h be the Poisson kernel of D with pole at x_0, i.e.,

$$u(x_0) = \int_{\partial D} h(x)u(x)\, d\sigma(x) \qquad \forall u \in \mathcal{H}(D) \cap C(\overline{D}).$$

Then, obviously, ∂D is a harmonic pseudosphere centered at x_0 if

$$h(x) = \frac{1}{|\partial D|} \qquad \forall x \in \partial D.$$

Hence, Lewis-Vogel's Theorem can be rephrased as follows:

if D and ∂D satisfy (i) and (ii) then $|\partial D|h \equiv 1$.

Roughly speaking, Preiss and Toro proved that if

$|\partial D|h$ is close to 1

then *∂D is close to a Euclidean sphere centered at x_0.*

Precisely:

Theorem 1.1 (Preiss-Toro [9]). *Let D be a bounded open set satisfying (i) and (ii) and let $x_0 \in \partial D$. Then there exists $\varepsilon_0 > 0$ such that, if $0 < \varepsilon < \varepsilon_0$ and*

$$1 - \varepsilon < |\partial D|h < 1 + \varepsilon,$$

then

$$e^{-2\varepsilon} \le \frac{|\partial B|}{|\partial D|} \le \frac{|\partial B^*|}{|\partial D|} \le e^{2\varepsilon},$$

where B is the biggest Euclidean ball centered at x_0 and contained in D, while B^* is the smallest Euclidean ball centered at x_0 and containing D.

This theorem can be rephrased in terms of what we call *Gauss gap* of ∂D w.r.t. x_0, which we define as follows:

$$\begin{aligned}\mathcal{G}(\partial D, x_0) &= \text{Gauss gap of } \partial D \text{ w.r.t. } x_0\\ &:= \sup_{\substack{u \in \mathcal{H}(D)\cap C(\overline{D})\\ u \not\equiv 0}} \frac{\left| u(x_0) - \fint_{\partial D} u\, d\sigma \right|}{\fint_{\partial D} |u|\, d\sigma}.\end{aligned}$$

Then one has:

$$\mathcal{G}(\partial D, x_0) = \sup_{\partial D} ||\partial D|h - 1|,$$

(see (6.1) in the Appendix of [1]).

It can be easily proved that the condition of Preiss and Toro's stability Theorem is equivalent to the following statement:

If $\mathcal{G}(\partial D, x_0)$ is sufficiently small, then

$$\mathcal{G}(\partial D, x_0) \geq c\ \frac{|\partial D| - |\partial B|}{|\partial D|} \tag{1}$$

and

$$\mathcal{G}(\partial D, x_0) \geq c\ \frac{|\partial B^*| - |\partial D|}{|\partial D|}$$

where $c > 0$ is an absolute constant,

(see [1, Appendix] for the details).

It is noteworthy to remark that Preiss and Toro's stability result requires the *smallness* of the Gauss gap. On the other hand, in general, even for C^1-domains this gap is *not* finite. Then one can expect that the smallness of $\mathcal{G}(\partial D, x_0)$ implicitly implies some regularity properties of ∂D. As a matter of fact Preiss and Toro, in their paper [9], proved that for every $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that if $\mathcal{G}(\partial D, x_0) < \varepsilon$ then ∂D is *δ-Reifenberg flat*. Hence, by a quite classical Reifenberg's Theorem, ∂D is *homeomorphic to* $\partial B(0,1)$. Moreover, by a result by Kenig and Toro [4], one can define a generalized normal map

$$x \mapsto \nu(x), \qquad x \in \partial D,$$

which is VMO (Vanishing Mean value Oscillation).

Preiss and Toro's stability result gives a sufficient condition for the boundary of an open set to be close to a Euclidean sphere. This condition, however, is not necessary, even for small Lipschitz perturbation of smooth domains. Indeed, we have proved the following theorem.

Theorem 1.2 ([1], Theorem 1.2)**.** *There exists a family $(D_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ of equi-Lipschitz domains, small perturbations of the Euclidean ball $B(0,1)$, such that, for every ε, $0 < \varepsilon < \varepsilon_0$,*

(i) $B(0,1) \subseteq D_\varepsilon \subseteq B(0, 1+\varepsilon)$

(ii) $\frac{1}{c}\varepsilon^{n-1} \leq |\partial D_\varepsilon| - |\partial B(0,1)| \leq c\,\varepsilon^{n-1}$

(iii) $\liminf_{\varepsilon\to 0} \mathcal{G}(\partial D_\varepsilon, 0) > 0$,

where c is an absolute constant. Moreover $B(0,1)$ is the biggest ball centered at 0 and contained in D_ε.

In [1] we have also proved that a stability inequality like (1) can be obtained *only* assuming a regularity property of the ∂D *in at least one of its points closest to* x_0. In particular, our result applies to the domains of the family $(D_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ of the previous theorem.

Our method is direct and does not require the profound *real* and *harmonic analysis* techniques used by Preiss and Toro and by Lewis and Vogel in the papers quoted above. Our idea is to measure the gap between

$$u(x_0) \quad \text{and} \quad \fint_{\partial D} u\, d\sigma$$

for a particular family of $u \in \mathcal{H}(D) \cap C(\overline{D})$ constructed with the Poisson kernel of balls close to the biggest one centered at x_0 and contained in D.

Acknowledgment: The first author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

2. The Kuran's function

For $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$, we define

$$h_\alpha(x) := |\alpha|^{n-2} \frac{|x|^2 - |\alpha|^2}{|x-\alpha|^n}, \qquad x \neq \alpha$$

and

$$k_\alpha(x) = 1 + h_\alpha(x), \qquad x \in \mathbb{R}^n \setminus \{\alpha\}.$$

We call k_α the *α-Kuran's function*: it was introduced by Kuran in [5]. The function $h_\alpha|_{B(0,|\alpha|)}$ is, up to a multiplicative constant, the Poisson kernel of the Euclidean ball $B(0,|\alpha|)$. In particular, it is harmonic in $B(0,|\alpha|)$. As a consequence, being h_α real analytic in $\mathbb{R}^n \setminus \{\alpha\}$, one has

$$h_\alpha \in \mathcal{H}(\mathbb{R}^n \setminus \{\alpha\}).$$

Since $h_\alpha(0) = -1$ the Kuran function k_α has the following properties which are crucial for our aims:

$$k_\alpha \in \mathcal{H}(\mathbb{R}^n \setminus \{\alpha\}), \qquad k_\alpha(0) = 0.$$

Then, if D is a pseudosphere centered at 0, by the Gauss Mean Value Theorem,

$$0 = k_\alpha(0) = \fint_{\partial D} k_\alpha \, d\sigma \qquad \forall \alpha \notin \overline{D}.$$

Hence, if

$$\fint_{\partial D} k_\alpha \, d\sigma \neq 0 \qquad \text{for some } \alpha \notin \overline{D},$$

then ∂D is not a pseudosphere.

3. The Kuran gap

Let D be a bounded open set with $|\partial D| < \infty$ and let $x_0 \in D$. Our aim in this section is to introduce a new parameter, that we call *Kuran gap of* ∂D *w.r.t.* x_0. Basically, when $x_0 = 0$, our new parameter is defined similarly to the Gauss gap, by using instead of the general family $\mathcal{H}(D) \cap C(\overline{D})$ its subclass $\{k_\alpha \,:\, \alpha \notin \overline{D}\}$.

To begin with, denoting by B the *biggest ball centered at* x_0 and *contained in* D, we define

$$\begin{aligned} T(\partial D, x_0) =& \textit{regular touching set of } \partial D \textit{ w.r.t. } x_0 \\ :=& \{z \in \partial D \cap \partial B \,:\, \partial D \text{ is Lyapunov-Dini regular at } z\}. \end{aligned}$$

When we say that ∂D is Lyapunov-Dini regular at $z \in \partial D$, we mean that there exists a neighborhood V of z s.t.

$$\partial D \cap V \text{ is a } C^1\text{-manifold}$$

and the outward normal map

$$\partial D \cap V \ni x \mapsto \nu(x)$$

is Dini-continuous at z.

Then, we define $\mathcal{K}(\partial D, x_0)$, the Kuran gap of ∂D w.r.t. x_0, as follows.
If $x_0 \neq 0$, we simply let

$$\mathcal{K}(\partial D, x_0) := \mathcal{K}(\partial(-x_0 + D), 0).$$

If $x_0 = 0$ and $T(\partial D, 0) = \emptyset$, we put

$$\mathcal{K}(\partial D, 0) = \infty,$$

while, if $T(\partial D, 0) \neq \emptyset$, we define

$$\mathcal{K}(\partial D, 0) := \inf_{z \in T(\partial D, 0)} L(z)$$

where

$$\begin{aligned} L(z) :=& \liminf_{\substack{\alpha \searrow z \\ \alpha \notin \overline{D}}} \left| k_\alpha(0) - \fint_{\partial D} k_\alpha \, d\sigma \right| \\ =& \liminf_{\substack{\alpha \searrow z \\ \alpha \notin \overline{D}}} \left| \fint_{\partial D} k_\alpha \, d\sigma \right|. \end{aligned}$$

Here, with the notation $\alpha \searrow z$ we mean that α *radially goes to* z *from* $\mathbb{R}^n \setminus \overline{D}$, i.e.,

$$\alpha = tz, \quad t > 1, \quad t \searrow 1.$$

The Kuran gap has the following properties:

$\mathcal{K}(\partial D, x_0) < \infty$ if $T(\partial D, x_0) \neq \emptyset$,

$\mathcal{K}(\partial D, x_0)$ is invariant w.r.t. Euclidean *translations, rotations* and *dilations.*

$\mathcal{K}(\partial D, x_0) = 0$ if $T(\partial D, x_0) \neq \emptyset$, and ∂D is a *harmonic pseudosphere centered at* x_0.

Moreover, if $(D_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is the family of domains in Theorem 1.2, there exists an absolute constant $c > 0$ such that

$$\frac{1}{c}\varepsilon^{n-1} \leq \mathcal{K}(\partial D_\varepsilon, 0) \leq c\,\varepsilon^{n-1} \tag{2}$$

for every $\varepsilon \in]0, \varepsilon_0[$.

4. Main Theorem and some of its consequences

The main result proved in our paper [1] is the following one.

Theorem 4.1. *Let* $D \subseteq \mathbb{R}^n$ *be a bounded open set such that* $|\partial D| < \infty$ *and let* $x_0 \in D$.

Then, if B denotes the biggest Euclidean ball centered at x_0 and contained in D, we have

$$\mathcal{K}(\partial D, x_0) \geq \frac{|\partial D| - |\partial B|}{|\partial D|}. \tag{3}$$

We immediately remark that inequality (3) is sharp, in the following sense.

Let $(D_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ be the family of domains in Theorem 1.2. Then, by (2),

$$\frac{1}{c} \leq \frac{\mathcal{K}(\partial D_\varepsilon, 0)}{\varepsilon^{n-1}} \leq c \qquad \forall \varepsilon \in]0, \varepsilon_0[,$$

where $c > 0$ is an absolute constant, independent of $\varepsilon \in]0, \varepsilon_0[$. Moreover by (i) and (ii) in Theorem 1.2,

$$\frac{1}{c'} \leq \frac{1}{\varepsilon^{n-1}} \frac{|\partial D_\varepsilon| - |\partial B|}{|\partial D_\varepsilon|} \leq c'$$

where $c' > 0$ only depends on n. Then, on the family $(D_\varepsilon)_{0<\varepsilon<\varepsilon_0}$, inequality (3) is basically an equality.

By using the isoperimetric inequality the right hand side of (3) can be estimated from below as follows

$$\frac{|\partial D| - |\partial B|}{|\partial D|} \geq c(n) \frac{|D \setminus B|}{|D|^{\frac{1}{n}} ||\partial D|}$$

where $c(n)$ denotes the *isoperimetric constant* in $\mathbb{R}^n$.

Then we have the following corollary of Theorem 4.1.

Corollary 4.1. *Let $D \subseteq \mathbb{R}^n$ be a bounded open set with $|\partial D| < \infty$ and let $x_0 \in D$. Then, if B denotes the biggest Euclidean ball centered at x_0 and contained in D we have*

$$\mathcal{K}(\partial D, x_0) \geq c(n) \frac{|D \setminus B|}{|D|^{\frac{1}{n}} |\partial D|}. \tag{4}$$

Thanks to Theorem 4.1 and Corollary 4.1 we can therefore say that if

$$\mathcal{K}(\partial D, x_0) \text{ is small}$$

then

$$D \text{ is close to } B$$

and

$$|\partial D| \text{ is close to } |\partial B|.$$

From Corollary 4.1 one obtains a new sufficient condition for a harmonic pseudosphere to be a Euclidean sphere.

Theorem 4.2. *Let ∂D be a harmonic pseudosphere centered at x_0. If*

$$T(\partial D, x_0) \neq \emptyset,$$

then D is a Euclidean sphere centered at x_0.

Proof. The assumption implies $\mathcal{K}(\partial D, x_0) = 0$. Then, if B is the biggest ball contained in D and centered at x_0, by Corollary 4.1 we have

$$0 \leq |D \setminus B| \leq \frac{|D|^{\frac{1}{n}}|\partial D|}{c(n)}\mathcal{K}(\partial D, x_0) = 0,$$

so that $D = B$ and $\partial D = \partial B$. □

From our stability result we obtain a new answer to a rigidity problem raised by the following property of the spheres.

Let $B \subseteq \mathbb{R}^n$ be a Euclidean ball centered at x_0 and let Γ be the fundamental solution of the Laplacian Δ. Since $y \mapsto \Gamma(y - x)$ is a harmonic function in $\mathbb{R}^n \setminus \{x\}$, by the surface Gauss mean value theorem one has

$$\Gamma(x_0 - x) = \fint_{\partial B} \Gamma(y - x)\, d\sigma(y) \qquad \forall x \in \mathbb{R}^n \setminus \overline{B}.$$

Then, the following question naturally arises: is this property characteristic of the Euclidean spheres?

Several positive answers to this question are present in literature: to the best of our knowledge the most general one is implicitly contained in the paper [2] by G. Fichera, which reads as follows:

Let $D \subseteq \mathbb{R}^n$ be a C^1-bounded open set. Assume that

$$\Gamma(x_0 - x) = \fint_{\partial D} \Gamma(y - x)\, d\sigma(y) \qquad \forall x \in \mathbb{R}^n \setminus \overline{D}.$$

Then D is a Euclidean ball B centered at x_0, so that

$$\partial D = \partial B.$$

From our main Theorem 4.1 we obtain the following result which partially improves Fichera's Theorem.

Theorem 4.3. *Let $D \subseteq \mathbb{R}^n$ be a bounded open set such that $|\partial D| < \infty$ and let $x_0 \in D$. Assume that*

$$\Gamma(x_0 - x) = \fint_{\partial D} \Gamma(y - x)\, d\sigma(y) \qquad \forall x \in \mathbb{R}^n \setminus \overline{D}.$$

If $T(\partial D, x_0) \neq \emptyset$, then D is a Euclidean ball B centered at x_0 and, consequently,

$$\partial D = \partial B.$$

References

[1] G. Cupini, E. Lanconelli: *On the harmonic characterization of the spheres: a sharp stability inequality and some of its consequences,* submitted (2023), https://arxiv.org/abs/2309.11846

[2] G. Fichera: *A characteristic property of the sphere in $\mathbb{R}^n$*, (Italian) Atti Accad. Sci. Istit. Bologna Cl. Sci. Fis. Rend. (14) 2 (1984/85), 235-244 (1986).

[3] A. Friedman, W. Littman: *Bodies for which harmonic functions satisfy the mean value property,* Trans. Amer. Math. Soc. 102 (1962), 147-166.

[4] C.E. Kenig, T. Toro: *Poisson kernel characterization of Reifenberg flat chord arc domains,* Ann. Sci. École Norm. Sup. (4) 36 (2003), 323-401.

[5] Ü. Kuran: *On the mean-value property of harmonic functions,* Bull. London Math. Soc. 4 (1972), 311-312.

[6] J.L. Lewis, A. Vogel: *On pseudospheres*, Rev. Mat. Iberoamericana 7 (1991), no. 1, 25-54.

[7] J.L. Lewis, A. Vogel: *On some almost everywhere symmetry theorems*, Nonlinear diffusion equations and their equilibrium states, 3 (Gregynog, 1989), 347-374, Progr. Nonlinear Differential Equations Appl., 7, Birkhäuser Boston, Boston, MA, 1992.

[8] J.L. Lewis, A. Vogel: *A symmetry theorem revisited*, Proc. Amer. Math. Soc. 130 (2002), no. 2, 443-451.

[9] D. Preiss, T. Toro: *Stability of Lewis and Vogel's result,* Rev. Mat. Iberoam. 23 (2007), no. 1, 17-55.

Dipartimento di Matematica, Università di Bologna, Piazza di Porta san Donato 5, 40126 Bologna, Italy

Email address: giovanni.cupini@unibo.it

Email address: ermanno.lanconelli@unibo.it