



Limit Theorems for the Cubic Mean-Field Ising Model

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Abstract. We study a mean-field spin model with three- and two-body interactions. The equilibrium measure for large volumes is shown to have three pure states, the phases of the model. They include the two with opposite magnetization and an unpolarized one with zero magnetization, merging at the critical point. We prove that the central limit theorem holds for a suitably rescaled magnetization, while its violation with the typical quartic behavior appears at the critical point.

1. Introduction

In this paper, we investigate the mean-field Ising spin model with quadratic and cubic interactions. The interest in such a model comes from two large fields of research. The first is condensed matter physics, where the three-body interaction plays a role in the description of the phase separation phenomena of some magnetic alloys [1] lacking spin-flip symmetry. Those physical systems cannot be described by the sole use of a two-body interaction, while a three-body term captures some features of their behavior [2]. This fact is well paralleled by the Ginibre theorem about functions of spin configurations that are fully classified by an orthonormal base of k -body interactions [3]. Those physical phenomena are well described by statistical mechanics models on regular lattices in finite ($d = 2, 3$) dimensions. While some of those models have an exact solution in very special cases [4, 5], it is well known that the mean-field approximation provides an analytically viable setting and a fair description of the phase separation. In those cases, the term mean-field approximation is understood in the sense of a special class of probability measure where the Boltzmann–Gibbs variational principle is optimised: instead of minimizing the free energy over all probability measures, one restricts it to product measures on single spins [6, 7].

The other field in which the three-body interactions came to play a role is that of the applications to complex systems, in particular those of socio-technical nature [8] where the social network structure with long-range interaction represents a realistic description of the phenomenon and not an approximation of its finite-dimensional version [9–12]. In this case, from a mathematical perspective, the introduction of the three-body interaction entails moving from a graph-theoretical environment of vertices and edges to a richer hypergraph setting where the three-body terms, representing the faces of the hypergraph, are also taken into account.

The presence of the cubic interactions brings technical difficulties in the analysis of the model. In particular, the non-convex energy contribution due to the cubic power prevents the use of the Hubbard–Stratonovich transform, which instead is very efficient in the case of quadratic interactions. More precisely, even if the thermodynamic limit of the free energy can be easily computed by large deviation arguments, the fluctuations of the order parameter cannot be analysed with the classical rigorous methods for a mean-field system with pairwise interaction [13–15]. In order to overcome this obstacle we need a fine control on the N -asymptotic behavior of the partition function that is obtained by a method similar to that recently introduced in [16].

This paper presents a rigorous analysis of the mean-field model with three- and two-body interactions in a zero magnetic field. We show that the infinite-volume properties of the model display new phenomena that are absent in the quadratic mean-field case. In particular, we prove that the equilibria of the system include not only positively and negatively polarized states but also an unpolarized stable state in the presence of a non-zero cubic term that breaks the spin-flip symmetry. Finally, we also study the fluctuation of the magnetization in the entire phase space, specifying the behavior at phase separation and at the critical point. The critical exponent for the magnetization, moreover, takes on a value of zero towards the unpolarized directions of the phase space, and phase transitions can occur in the antiferromagnetic region.

This paper is organised as follows: Sect. 2 contains the formal definition of the model as well as a statement of the main results. In Sect. 3, we study the properties of the consistency equation that describes the system in its stationary equilibrium state. These properties provide an analytical description of the system’s phase diagram and the magnetization’s limiting behavior, as well as the computation of the critical exponents. Finally, Sect. 4 contains conclusions and perspectives, and the Appendices A and B contain technical and concentration results used throughout the work.

2. Definitions and Main Results

Let us consider N spins $\sigma = (\sigma_i)_{i \leq N} \in \{-1, +1\}^N$ interacting through an Hamiltonian of the form

$$H_N(\sigma) = -\frac{K}{3N^2} \sum_{i,j,k=1}^N \sigma_i \sigma_j \sigma_k - \frac{J}{2N} \sum_{i,j=1}^N \sigma_i \sigma_j - h \sum_{i=1}^N \sigma_i, \quad (1)$$

where $(K, J, h) \in \mathbb{R}^3$, K and J tune the interactions among triples and pairs of spins, respectively, while h represents an external field acting on the system. When $K = 0$, the previous Hamiltonian reduces to the well-know Curie–Weiss case. In this work we will concentrate on the case $h = 0$ and use the parameter K as a spin-flip symmetry breaking term reducing (1) to an Hamiltonian that can be represented as

$$H_N(\sigma) = -N \left(\frac{K}{3} m_N^3(\sigma) + \frac{J}{2} m_N^2(\sigma) \right) \quad (2)$$

where m_N is the magnetization per particle:

$$m_N(\sigma) = \frac{1}{N} \sum_{i=1}^N \sigma_i. \quad (3)$$

The expression (2) highlights the mean-field nature of the model. The Boltzmann–Gibbs probability measure associated to H_N is

$$\mu_N(\sigma) = \frac{e^{-H_N(\sigma)}}{Z_N}, \quad (4)$$

where $Z_N = \sum_{\sigma \in \{-1, +1\}^N} \exp(-H_N(\sigma))$ is the partition function. In Eq. (4), we set the usual inverse temperature β to 1 without loss since it has been reabsorbed in the parameters of the model. Notice that since the Hamiltonian (2) is invariant under the transformation $K \mapsto -K$, and $\sigma_i \mapsto -\sigma_i$ for $i = 1, \dots, N$, one can study the model only for $K > 0$ without loss.

Our aim is to obtain a complete characterization of the model’s *phase diagram*, an analysis of the asymptotic distribution of the magnetization in the presence and absence of phase transitions, the *fluctuations* of the suitably rescaled magnetization (3) w.r.t. the Boltzmann–Gibbs measure (4) at and away from the critical point, and the computation of the *critical exponents*.

All the above properties are strictly related to the analytical properties of the *free energy* of the system, which is the starting point of our analysis. Let us define the thermodynamic pressure, i.e., the generating functional as:

$$p_N = \frac{1}{N} \log Z_N. \quad (5)$$

Notice that p_N equals the free energy up to a minus sign. The thermodynamic limit of (5) can be easily computed applying Varadhan’s integral lemma [13, 17], obtaining:

Proposition 2.1. *Given $(K, J) \in \mathbb{R}^2$ the limiting pressure of (5) admits the following variational representation:*

$$p := \lim_{N \rightarrow \infty} p_N = \sup_{m \in [-1, 1]} \phi(m), \quad (6)$$

where $\phi(m) = u(m) - I(m)$ with

$$u(m) = \frac{K}{3} m^3 + \frac{J}{2} m^2 \quad (7)$$

is the energy contribution and

$$I(m) = \frac{1-m}{2} \log\left(\frac{1-m}{2}\right) + \frac{1+m}{2} \log\left(\frac{1+m}{2}\right) \tag{8}$$

is the binary entropy contribution.

The critical points of (6) satisfy the consistency equation,

$$m = \tanh(Km^2 + Jm). \tag{9}$$

A careful analysis shows that, among the solutions of (9), the function $\phi(m)$ in (6) can have one or two global maximizers in the interval $(-1, 1)$ for fixed (K, J) (see Fig. 1).

In particular, we can divide the parameter space $(K, J) \in \mathbb{R}_+ \times \mathbb{R}$ accordingly to the following:

Proposition 2.2 (Phase diagram). *For any $K > 0$, there exists $J = \gamma(K)$ defined in Proposition 3.3 such that the function $m \mapsto \phi(m)$ has a unique maximum point m^* for $(K, J) \in (\mathbb{R}_+ \times \mathbb{R}) \setminus \gamma$. Moreover, on the curve γ there are two global maximizers, $0 = m_0 < m_1$ and the limit as $K \rightarrow 0$ of $\gamma(K)$ identifies the critical point $(K_c, J_c) = (0, 1)$ where the magnetization takes the value $m_c = 0$.*

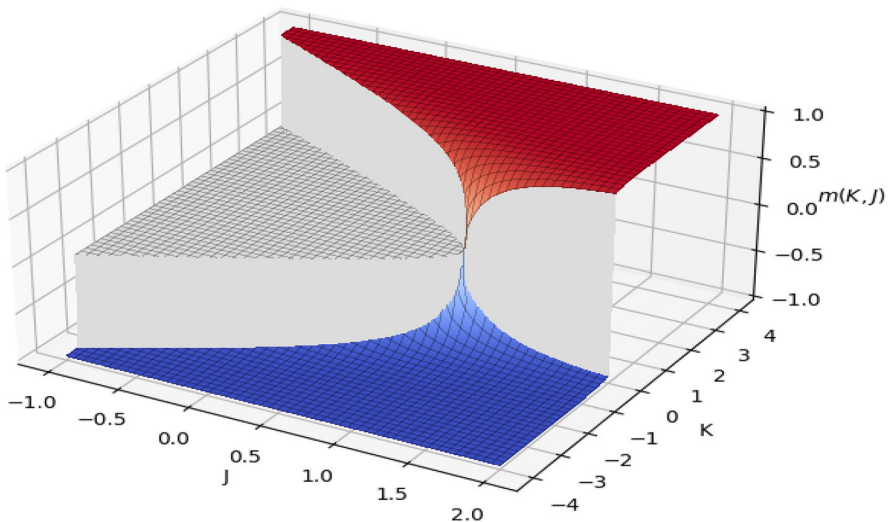


FIGURE 1. Stable solutions of the mean-field equation as a function of K and J . There are three stable phases presented here: the positive polarized phase depicted in red, the unpolarized phase given as the gray plateau, and the negative polarized phase denoted by the blue color. At the critical point, $(K, J) = (0, 1)$, the three phases of the cubic model as well as the two phases of the Curie–Weiss plane ($K = 0$) coalesce

In physical terms, the presence of two global maximizers corresponds to the existence of two different thermodynamic equilibrium phases, whereas the curve γ represents the coexistence curve. Let's note that m_0 and m_1 represent a stable paramagnetic state and a positively polarised state, respectively. The paramagnetic state is characterized by the absence of spontaneous magnetic order and the presence of symmetry between the up and down spin, with no preference for either direction. The jump from the paramagnetic state to the polarized state, namely when the magnetization jumps from m_0 to m_1 , represents a *first-order phase transition*, which is markedly different from the quadratic mean-field model ($K = 0$) having a second-order phase transition in J . More precisely if we denote by $m^*(K, J)$ the unique maximizer of ϕ , for any $\bar{K} > 0$ there exists $\bar{J} = \gamma(\bar{K}) \in (-\infty, 1)$ such that

$$0 = \lim_{J \rightarrow \bar{J}^-} m(J, \bar{K}) \neq \lim_{J \rightarrow \bar{J}^+} m(J, \bar{K}) > 0.$$

This behavior is somehow reminiscent of the Curie–Weiss Potts model analyzed in [18] where for any value of the parameter q a first order phase transition is observed. Numerical simulations of the phase diagram described in Proposition 2.2 can be seen in Fig. 2.

In the standard Curie–Weiss model, when $J > 0$ we know that as soon as $h > 0$ one obtains a positive magnetization. The reason is that the energy contribution due to h favors only spins aligned with $\text{sign}(h)$. On the contrary, in our system, $J, K > 0$, the energy contribution due to K can be minimized by configurations containing both up and down spin signs. This implies that the entropy contribution can dominate also for small but non-zero K , giving a zero magnetization.

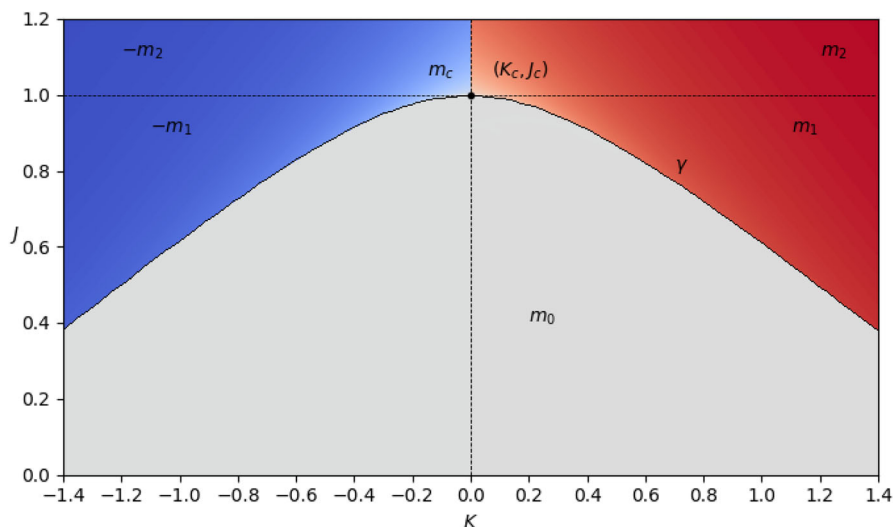


FIGURE 2. Phase diagram of the model with coexistence curve γ and the critical point (K_c, J_c) in the (K, J) plane

The next theorem contains the law of large numbers and the central limit theorem for the distribution of $m_N(\sigma)$ with respect to the Boltzmann–Gibbs measure.

Theorem 2.1 (Asymptotic distribution of the magnetization). *Consider the Hamiltonian in (2), then the following holds:*

1. For $(K, J) \in (\mathbb{R}_+ \times \mathbb{R}) \setminus (\gamma \cup (K_c, J_c))$ the function $\phi(m)$ in (6) has a unique global maximizer m^* such that $\phi''(m^*) < 0$ and

$$m_N \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \delta_{m^*}. \tag{10}$$

Moreover,

$$N^{\frac{1}{2}}(m_N - m^*) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}\left(0, -\frac{1}{\phi''(m^*)}\right). \tag{11}$$

2. Given $(K, J) \in \gamma$ we denote by $m_0 < m_1$ the two global maximizers of $\phi(m)$. For $i \in \{0, 1\}$ we define the quantity

$$\rho_i := \frac{[(m_i^2 - 1)\phi''(m_i)]^{-\frac{1}{2}}}{[(m_0^2 - 1)\phi''(m_0)]^{-\frac{1}{2}} + [(m_1^2 - 1)\phi''(m_1)]^{-\frac{1}{2}}}. \tag{12}$$

Then we have that

$$m_N \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \sum_{i \in \{0,1\}} \rho_i \delta_{m_i}. \tag{13}$$

Moreover let $A_i \subseteq [-1, 1]$ be an interval containing m_i in its interior such that $\phi(m_i) > \phi(m)$ for all $m \in \text{cl}(A_i) \setminus \{m_i\}$, then

$$N^{\frac{1}{2}}(m_N - m_i) | \{m_N \in A_i\} \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}\left(0, -\frac{1}{\phi''(m_i)}\right). \tag{14}$$

3. At the critical point (K_c, J_c) , we have that

$$m_N \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \delta_0. \tag{15}$$

Moreover,

$$N^{\frac{1}{4}} m_N \xrightarrow[N \rightarrow \infty]{\mathcal{D}} C \exp\left(\frac{\phi^{(4)}(0)}{24} x^4\right) dx = C \exp\left(\frac{-x^4}{12}\right) dx, \tag{16}$$

where $\phi^{(4)}(0) = -2$ denote the fourth derivative of $\phi(m)$ evaluated at $m = 0$ and $C^{-1} = \int_{-\infty}^{\infty} \exp\left(\frac{-x^4}{12}\right) dx = \frac{\sqrt[4]{3} \Gamma(\frac{1}{4})}{\sqrt{2}}$.

Finally, we study the behavior of the limiting value of the magnetization near the critical point $(K_c, J_c) = (0, 1)$ namely the critical exponents of the model. The average value of the magnetization is given by the LLN in Theorem 2.1 and will be denoted by $m^*(K, J)$. The following proposition describes the critical behavior of $m^*(K, J)$ when $(K, J) \rightarrow (K_c, J_c)$ from various directions.

Proposition 2.3. *Let $m^*(K, J)$ be the unique maximizer of $\phi(m)$ defined in Corollary 3.1. Given $\alpha \in \mathbb{R}$ consider the lines*

$$J(K) = 1 + \alpha K, \quad K > 0 \quad (17)$$

and the function $m^(K) \equiv m^*(K, J(K))$. Then, for $K \rightarrow 0^+$, the following holds*

$$m^*(K) \sim \begin{cases} \sqrt{3\alpha}\sqrt{K}, & \text{for } \alpha > 0 \\ 3K, & \text{for } \alpha = 0 \\ 0, & \text{for } \alpha < 0. \end{cases} \quad (18)$$

Remark 2.2. Notice that when $\alpha < 0$ the critical exponent is 0. The case $K = 0$ and $J \rightarrow 1^+$ corresponds to the classical Curie–Weiss model and is well known that

$$m^*(0, J) \sim \sqrt{\frac{3(J-1)}{J^3}}. \quad (19)$$

3. Proofs

This section contains the proofs of the above results and is organised as follows:

In Sect. 3.1, we prove Proposition 2.2 by studying the properties of the function $\phi(m)$ appearing in the variational problem (6). Section 3.2 contains the proof of Theorem 2.1 and is based on the asymptotic expansion given in Appendix B. Finally, in Sect. 3.3, we derive the critical exponents of the model.

3.1. Proof of Proposition 2.2

The complete proof of Proposition 2.2 follows from Propositions 3.1, 3.2, 3.3 and 3.4 below.

Let us start studying in detail the variational principle (6) and observe that the function $\phi(m)$ satisfies

$$\begin{aligned} \frac{\partial}{\partial m} \phi(m) &= Km^2 + Jm - \frac{1}{2} \log \left(\frac{1+m}{1-m} \right), \\ \frac{\partial^2}{\partial m^2} \phi(m) &= 2Km + J - \frac{1}{1-m^2}. \end{aligned} \quad (20)$$

Therefore the variational pressure $\phi(m)$ attains its maximum at at least one point $m = m(K, J) \in (-1, 1)$, which satisfy

$$\frac{\partial}{\partial m} \phi(m) = 0, \quad \text{i.e., } m = \tanh(Km^2 + Jm). \quad (21)$$

Indeed, from (20) $\lim_{m \rightarrow -1^+} \phi'(m) = +\infty$ and $\lim_{m \rightarrow -1^-} \phi'(m) = -\infty$. Therefore, there exists $\epsilon > 0$ such that $\phi(m)$ is strictly increasing on $[-1, -1 + \epsilon]$ and strictly decreasing on $[1 - \epsilon, 1]$. This implies that, the local maximizers of $\phi(m)$ does not include -1 and $+1$. Notice also that, since $K > 0$, if $\bar{m} > 0$ then $\phi(\bar{m}) > \phi(-\bar{m})$ therefore the supremum of $\phi(m)$ cannot be reached at negative values.

A complete classification of the critical points of $\phi(m)$ is contained in the following proposition:

Proposition 3.1 (Classification of critical points). *For all $K > 0$ and $J \in \mathbb{R}$, the solutions to equation (21) can be described as follow:*

Define the function

$$\Psi(K) := \min_{m \in [0,1]} \frac{g(m, K)}{m} < 1 \tag{22}$$

where $g(m, K) := \operatorname{arctanh}(m) - Km^2$ and set $J_c = 1$. Then:

- (a) for $J < \Psi(K)$, there exist a unique solution, $m_0 = 0$, and it is the maximum point of $\phi(m)$,
- (b) for $\Psi(K) < J < J_c$, Eq. (21) has three solutions i.e., $m_0, m_1 > m_3 > 0$. Furthermore, m_0, m_1 are local maximum points while m_3 is a local minimum point of $\phi(m)$,
- (c) for $J = \Psi(K)$, there exist two solutions, m_0 and $m_1 > 0$. Where m_0 is the maximum point of $\phi(m)$ and m_1 is an inflection point.
- (d) If $J \geq J_c$, there exist a unique positive solution m_2 which is the only maximum point of $\phi(m)$ in Eq. (6).

Proof. Let us start by noticing that $m = 0$ is always a solution of (21). Moreover,

$$\phi''(0) \begin{cases} < 0, & \text{if } J < 1 \\ > 0, & \text{if } J > 1. \end{cases}$$

Now, let's rewrite (21) as

$$mJ = \underbrace{\left[\operatorname{arctanh}(m) - Km^2 \right]}_{=g(m, K)}. \tag{23}$$

The solutions of (21) are the intersections between the line mJ and the function $g(m, K)$. Therefore the function $\Psi(K)$ in (22) is a benchmark to study the number of solutions of $\phi'(m) = 0$ when J varies. Indeed by definition, $\Psi(K)$ represents the smallest value of J in order to have a positive solution for (23). Let us start collecting some properties of the function $g(m, K)$. By definition we have that

$$\begin{aligned} g'(m, K) &= \left[\frac{1}{1 - m^2} - 2Km \right] \\ g''(m, K) &= \left[\frac{2m}{(1 - m^2)^2} - 2K \right]. \end{aligned} \tag{24}$$

This implies that,

$$\begin{cases} g'(0, K) = 1, \\ g''(0, K) = -2K < 0 \quad \text{for all } K > 0. \end{cases}$$

Since the function $m \mapsto \frac{2m}{(1-m^2)^2}$ is strictly increasing on $[0, 1)$, then $g''(m, K) = 0$ has only one solution, namely $g(m, K)$ has only one inflection point. Moreover, observe that, as $m \rightarrow 1^-$, $g(m, K) \rightarrow +\infty$.

- a. If $J < \Psi(K)$ then it's clear that (21) has a unique solution $m_0 = 0$ which is a maximum point since in this case $\phi''(0) < 0$.
- b. If $\Psi(K) < J < J_c$, continuity of g and the fact that for $m \rightarrow 1^-$, $g(m, K) \rightarrow +\infty$, imply that (21) has three solutions, m_0, m_1 and m_3 , where m_1 and m_3 are positive. It's also easy to check using the properties of the function $g(m, K)$ that m_0 and m_1 are local maxima while m_3 is a local minima.
- c. If $J = \Psi(K)$, then there is only one intersection point m_4 between the line mJ and the function $g(m, K)$. Standard reasoning allows to conclude that m_4 is an inflection point for ϕ .
- d. Finally suppose that $J \geq J_c$. The fact that $g'(0, K) = 1$ and $g''(0, K) = -2K < 0$ for $K > 0$, means that the line mJ starts above the function g . Now, since g has at most one inflection point and $g(m, K) \rightarrow +\infty$ as $m \rightarrow 1^-$, one can conclude that there exist a unique positive solution $m_2 \in (0, 1)$ of $\phi'(m) = 0$.

□

The solutions made mention in Proposition 3.1 are displayed in Fig. 3.

In the next proposition we obtain the differentiability of the solution(s) of the consistency equation (21) with respect to the parameters J and K .

Proposition 3.2 (Regularity properties). *Let m_0, m_1 and m_2 be the (local) maxima of ϕ described in Proposition 3.1. Then for $K > 0$, the following properties hold:*

- (a) m_1 is continuous in its domain namely $\Psi(K) \leq J < J_c$ and C^∞ in its interior, while m_2 is C^∞ in its domain, namely $J \geq J_c$.
- (b) $\phi''(m_0) = \phi''(0) < 0$, $\phi''(m_1) < 0$ for $\Psi(K) < J < J_c$, and $\phi''(m_2) < 0$ for $J \geq J_c$.

Moreover, for any $i \in \{0, 1, 2\}$ it holds that

$$\frac{\partial}{\partial J} \phi(m_i) = \frac{1}{2} m_i^2, \quad \frac{\partial}{\partial K} \phi(m_i) = \frac{1}{3} m_i^3 \quad (25)$$

$$\frac{\partial m_i}{\partial J} = -\frac{m_i}{\phi''(m_i)}, \quad \frac{\partial m_i}{\partial K} = -\frac{m_i^2}{\phi''(m_i)}. \quad (26)$$

Remark 3.1. Notice that (b) implies that there are no degenerate maximum points of $\phi(m)$ for $K > 0$. Therefore the only degenerate maximum is obtained for $(K, J) = (K_c, J_c) = (0, 1)$, that is the critical point of a Curie–Weiss model, here the magnetization takes the value $m_c = 0$.

Proof. (a) Let's start with m_1 and take (K, J) in its domain, namely $D := \{(K, J) | K > 0, \Psi(K) \leq J < J_c\}$. We define $\tau(K, J) = \left(\frac{1}{J} - 1\right) \frac{J}{K} > 0$ and

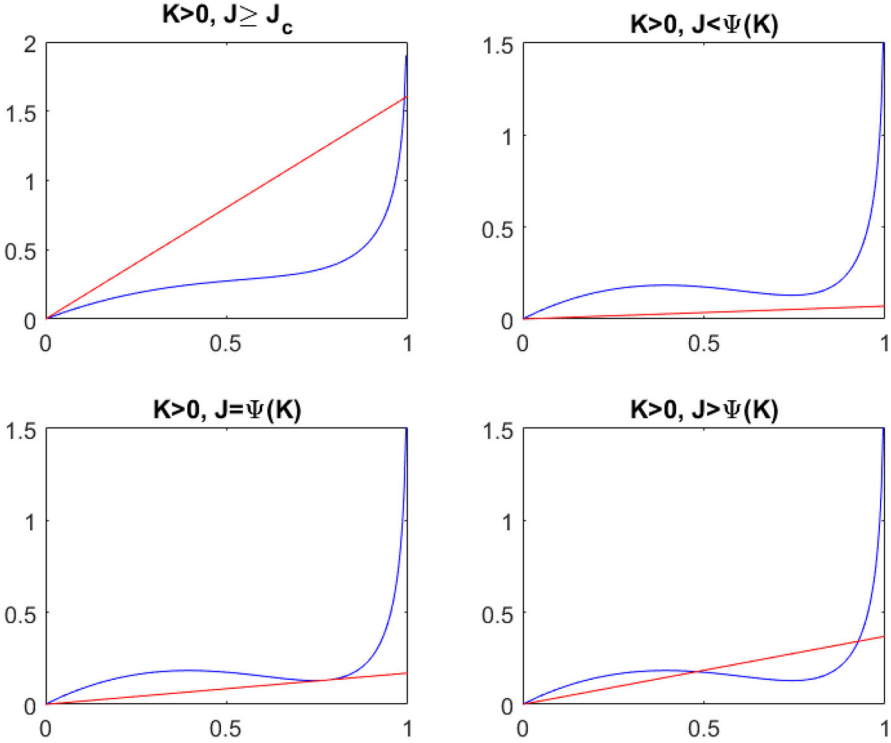


FIGURE 3. The points of intersection between the blue curve $g(m, K)$ as defined in (23) and red curve $f(m) = Jm$. The solution of the equation (21) are the points of intersection between $g(m, K)$ and $f(m)$

$\tilde{\phi}(m) := \phi(m)|_{[\tau(K, J), 1]}$. Observe from (21) that,

$$m_1 = \frac{1}{J} \left[\underbrace{\operatorname{arctanh}(m_1)}_{\geq m_1} - Km_1^2 \right]$$

$$\implies m_1 \geq \left(\frac{1}{J} - 1 \right) \frac{J}{K} = \tau(K, J).$$

Hence, m_1 is the unique maximum point of $\tilde{\phi}(m)$, then by the Berge’s maximum Theorem A.1 (see [19, 20]), m_1 is continuous for $(K, J) \in D$. To prove the smoothness of m_1 on the interior of its domain it’s enough to show that $\phi''(m_1) < 0$ and then apply the implicit function Theorem A.2 (see [20, 21]). Let $G(m) := \phi''(m)$ then,

$$\frac{\partial G}{\partial m}(m) = 2K - \frac{2m}{(1 - m^2)^2}$$

$$\frac{\partial^2 G}{\partial m^2}(m) = -\frac{2(3m^2 + 1)}{(1 - m^2)^3} < 0 \quad \forall m \in [0, 1)$$

and hence,

$$\begin{cases} G(0) = J - 1 < 0, & \forall J < J_c \\ G'(0) = 2K > 0, & \forall K > 0 \\ G''(0) = -2 \\ \lim_{m \rightarrow 1^-} G(m) = -\infty & \forall K > 0 \text{ and } J < J_c. \end{cases} \quad (27)$$

We want to prove that $G(m_1) < 0$ if $\Psi(K) < J < J_c$. Clearly since m_1 is a local maximizer it's enough to show that $G(m_1) \neq 0$. Recall that m_1 is the biggest positive solution of $\phi'(m) = 0$. It's easy to check that $G(m) = 0$ has at most two solutions. Assume by contradiction that $G(m_1) = 0$ if $\Psi(K) < J < J_c$, then $G(m) < 0$ or $G(m) > 0$ in a left neighbourhood of m_1 .

- Suppose that $G(m) < 0$ in a left neighbourhood of m_1 then $G(m)$ cannot be always negative, otherwise $\phi'(m)$ is decreasing and, since $\phi'(0) = 0$ then $\phi'(m) = 0$ can not have more than one solution. This contradicts point b) of Proposition 3.1. Therefore there exist an interval where $G(m) > 0$ but keeping in mind the properties of G in (27) and the fact that G is continuous, this implies that there are at least three solutions for $G(m) = 0$, but this is impossible because we already observed that $G(m) = 0$ has at most two solutions.
- Suppose that $G(m) > 0$ in a left neighbourhood of m_1 , then $G(m) = 0$ has in addition to m_1 another solution that we denote by \tilde{m} . Clearly $\tilde{m} < m_3$ otherwise m_3 cannot satisfies $\phi'(m_3) = 0$. Therefore $G(m) \equiv \phi''(m) > 0$ if $m_3 < m < m_1$ and this contradicts the fact that $\phi'(m_3) = \phi'(m_1) = 0$.

Let's focus on m_2 . Since for $K > 0$ and $J \geq J_c$, m_2 is the only maximizer of $\phi(m)$ it's enough to show that $\phi''(m_2) < 0$ to get smoothness of m_2 by using the implicit function theorem. Let's note that if $J \geq J_c$ then $\phi''(0) \geq 0$ and $\phi''(m) = 0$ has a unique positive solution. Furthermore, $\phi(m)$ has a unique maximum point, $m_2 \in (0, 1)$ and $\phi'(m_2) = 0$. It is easy to show that $\phi''(m_2) \neq 0$ by contradiction. Let's assume that $\phi''(m_2) = 0$ then $\phi''(m) > 0$ for $m < m_2$, therefore, using the Taylor's series expansion of $\phi(m)$ around m_2 one gets $\phi(m) > \phi(m_2)$ which contradicts the fact that m_2 is the global maximum.

Therefore by the implicit function Theorem A.2, since $\phi''(m) \neq 0$ on the interior of the domains of m_1 and m_2 , we can conclude that m_1 and m_2 are C^∞ .

(b) We already proved that for any $i \in \{0, 1, 2\}$, $\phi''(m_i) < 0$ for suitable K, J . For the second part a direct computation shows that:

$$\begin{aligned} \frac{\partial}{\partial J} \phi(m_i) &= \frac{\partial}{\partial m} \phi(m) \Big|_{m=m_i} \frac{\partial m_i}{\partial J} + \frac{m_i^2}{2} \\ &= \frac{m_i^2}{2} \end{aligned} \quad (28)$$

and similarly,

$$\begin{aligned} \frac{\partial}{\partial K} \phi(m_i) &= \frac{\partial}{\partial m} \phi(m) \Big|_{m=m_i} \frac{\partial m_i}{\partial K} + \frac{m_i^3}{3} \\ &= \frac{m_i^3}{3}. \end{aligned} \tag{29}$$

Using the fact that $m_i, i = \{0, 1, 2\}$ are the stationary points of $\phi(\cdot)$, we have that $\frac{\partial m_i}{\partial K}$ satisfies

$$\begin{aligned} \frac{1}{1 - m_i^2} \frac{\partial m_i}{\partial K} - m_i^2 - 2K m_i \frac{\partial m_i}{\partial K} - J \frac{\partial m_i}{\partial K} &= 0 \\ \frac{\partial m_i}{\partial K} \left[\frac{1}{1 - m_i^2} - 2K m_i - J \right] &= m_i^2 \\ \frac{\partial m_i}{\partial K} &= -\frac{m_i^2}{\phi''(m_i)} \end{aligned} \tag{30}$$

and similarly for $\frac{\partial m_i}{\partial J}$ one obtains

$$\begin{aligned} \frac{1}{1 - m_i^2} \frac{\partial m_i}{\partial J} - 2K m_i \frac{\partial m_i}{\partial J} - m_i - J \frac{\partial m_i}{\partial J} &= 0 \\ \frac{\partial m_i}{\partial J} \left[\frac{1}{1 - m_i^2} - 2K m_i - J \right] &= m_i \\ \frac{\partial m_i}{\partial J} &= -\frac{m_i}{\phi''(m_i)} \end{aligned} \tag{31}$$

and this concludes the proof. □

Now we study which of the stationary points described by Proposition 3.1 are global maximizers of $\phi(m)$ and show the existence of a phase transition. These stationary points are: m_0, m_1 , and m_2 . Let us start by recalling the result of Proposition 3.1:

- if $J < \Psi(K)$, then m_0 is the only global maximum point of ϕ
- if $\Psi(K) < J < J_c$ then $\phi(m)$ has two local maximizers m_0 and m_1
- if $J \geq J_c$ then m_2 is the only the global maximum point of $\phi(m)$

To identify the coexistence of two global maximum points of $\phi(m)$ when $\Psi(K) < J < J_c$, consider the following function:

$$\Delta(K, J) = \phi(m_1, K, J) - \phi(m_0, K, J). \tag{32}$$

Notice that $\Delta(K, J)$ can be extended by continuity at $J = \Psi(K)$ and $J = J_c$. In the above equation we use $\phi(\cdot, K, J)$ to emphasis the dependence of ϕ on the parameters.

Proposition 3.3 (*Existence and uniqueness*). *For all $K > 0$ there exists a unique $J = \gamma(K) \in (\Psi(K), J_c)$ such that $\Delta(K, J) = 0$. Furthermore,*

$$\Delta(K, J) \begin{cases} < 0, & \text{if } \Psi(K) \leq J < \gamma(K) \\ > 0, & \text{if } \gamma(K) < J \leq J_c. \end{cases} \quad (33)$$

Proof. Let us start by observing that

- $\Delta(K, \Psi(K)) < 0$, since for $J = \Psi(K)$, m_0 is the only maximum point of $\phi(m, K, J)$.
- $\Delta(K, J_c) > 0$, since $\lim_{J \rightarrow 1^-} m_1(K, J) = m_2(K, 1)$ and $m_2(K, 1)$ is the only global maximum for $\phi(m, K, J)$.

Now, by continuity of $\phi(m)$ and m_1 , we have that $J \mapsto \Delta(K, J)$ is a continuous function, and then the existence of the wall $J = \gamma(K)$ follows from the application of the intermediate value theorem. For the uniqueness part we observe that $J \mapsto \Delta(K, J)$ is strictly increasing. Indeed from Proposition 3.2 we know that $\phi(m_1), m_1$ are smooth functions and

$$\begin{aligned} \frac{\partial \Delta}{\partial J}(K, J) &= \frac{\partial}{\partial J} \phi(m_1) - \frac{\partial}{\partial J} \phi(m_0) \\ &= \frac{1}{2} m_1^2 - \frac{1}{2} m_0^2 \\ &= \frac{1}{2} m_1^2 > 0 \end{aligned} \quad (34)$$

for $J \in (\Psi(K), J_c)$. □

Corollary 3.1. *The function $\phi(m)$ has a unique global maximum point $m^*(K, J)$ given by:*

$$m^*(K, J) := \begin{cases} m_0 = 0, & \text{if } J < \gamma(K) \\ m_1(K, J), & \text{if } \gamma(K) < J < J_c, \\ m_2(K, J), & \text{if } J \geq J_c, \end{cases} \quad (35)$$

where the function $\gamma(K)$ is defined by Proposition 3.3 and $\phi''(m^*) < 0$.

Note that on the curve γ there are two global maximum points of $\phi(m)$. Let us define

$$\bar{\gamma}(K) := \begin{cases} \gamma(K), & \text{if } K > 0 \\ J_c, & \text{if } K = K_c = 0. \end{cases} \quad (36)$$

Therefore by Proposition 3.2 one can conclude that $m^*(K, J)$ is continuous on its domain $(\mathbb{R}_+ \times \mathbb{R}) \setminus \gamma$ and it is C^∞ on $(\mathbb{R}_+ \times \mathbb{R}) \setminus \bar{\gamma}$. Moreover the following holds:

Proposition 3.4 (*Regularity properties*). *The function $\bar{\gamma}(K)$ is $C^\infty(\mathbb{R}_+ \setminus \{0\})$ and at least C^1 for $K = 0$. In particular,*

$$\bar{\gamma}'(K) := -\frac{2}{3} m_1(K, \gamma(K)) \quad \forall K > 0 \quad (37)$$

and

$$\bar{\gamma}'(K_c) := -\frac{2}{3}m_c. \tag{38}$$

Proof. i. We begin by showing that $\gamma(K) \in C^\infty(\mathbb{R}_+)$. By Proposition 3.3, $J = \gamma(K)$ is a unique solution of the equation

$$\Delta(K, J) = 0,$$

where Δ is defined by Eq. (32) for $\Psi(K) \leq J < J_c$ and $K > 0$. Furthermore, observe that Δ is C^∞ in its domain by the smoothness of ϕ and m_1 . Recall from the proof of Proposition 3.3 that

$$\frac{\partial}{\partial J}\Delta(K, J) \neq 0 \quad \forall (K, J) \text{ s.t. } J = \gamma(K), \tag{39}$$

hence, by the implicit function Theorem A.3 $\gamma(K) \in C^\infty(\mathbb{R}_+)$. Therefore

$$\begin{aligned} \Delta(K, \gamma(K)) \equiv 0 &= \frac{d}{dK}\Delta(K, \gamma(K)) \\ &= \frac{\partial \Delta}{\partial J}(K, \gamma(K))\gamma'(K) + \frac{\partial \Delta}{\partial K}(K, \gamma(K)) \\ \implies \gamma'(K) &= -\frac{\partial \Delta}{\partial K} / \frac{\partial \Delta}{\partial J}(K, \gamma(K)) \end{aligned} \tag{40}$$

From Eqs. (28) and (29), we have that,

$$\frac{\partial \Delta}{\partial K} = \frac{m_1^3}{3} - \frac{m_0^3}{3} \quad \text{and} \quad \frac{\partial \Delta}{\partial J} = \frac{m_1^2}{2} - \frac{m_0^2}{2},$$

hence

$$\gamma'(K) = -\frac{2}{3}m_1(K, \gamma(K)) \tag{41}$$

since $m_0(K, \gamma(K)) = 0, \forall K > 0$. Notice that, by (9), $m_1(K, \gamma(K)) \xrightarrow{K \rightarrow \infty} 1$ which implies that

$$\lim_{K \rightarrow \infty} \gamma'(K) = -\frac{2}{3}.$$

A consequence of this property is that also when $J < 0$ (antiferromagnetic case) and very large there is always going to be phase transition between a polarized and unpolarized state.

ii. Now we prove that the extended function $\bar{\gamma} \in C^1(\mathbb{R}_+)$. Recall that $\gamma(K) \in [\Psi(K), J_c]$ and observe that $\lim_{K \rightarrow K_c^+} \Psi(K) = J_c$ then

$$\lim_{K \rightarrow K_c^+} \gamma(K) = J_c$$

which implies that $\bar{\gamma}$ is continuous at K_c . Now we have that

$$\gamma'(K) = -\frac{2}{3}m_1(K, \gamma(K)) \xrightarrow{K \rightarrow K_c^+} -\frac{2}{3}m_c = 0 \tag{42}$$

which implies that $\bar{\gamma}'(K_c) = -\frac{2}{3}m_c = 0$ by the application of mean value theorem. \square

3.2. Proof of Theorem 2.1

In this section we provide the details of the proof for Theorem 2.1 following closely the argument in [16].

Proof. 1. By proposition 2.2 we know that if $(K, J) \in (\mathbb{R}_+ \times \mathbb{R}) \setminus (\gamma \cup (K_c, J_c))$ then $\phi(m)$ has a unique global maximizer m^* with $\phi''(m^*) < 0$. It's easy to check that $\phi(m)$ satisfies the hypothesis of Lemma B.1, therefore (64) gives concentration inequality for m_N in a suitable neighbourhood of m^* under the probability measure (4). More precisely, for any $\alpha \in (0, \frac{1}{6}]$ and N large enough one has

$$\mu_N(m_N \in B_{N,\alpha}^c(m^*)) = \exp \left\{ \frac{1}{2} N^{2\alpha} \phi''(m^*) \right\} \mathcal{O}(N^{\frac{3}{2}}) \quad (43)$$

where $B_{N,\alpha}^c(m^*) = \{m \in \mathbb{R} : |m - m^*| \leq N^{-\frac{1}{2}+\alpha}\}$. Therefore the convergence in distribution (10) follows from (43) by standard approximation arguments.

To obtain the central limit for m_N , it is enough to compute the limit of the moment generating function of the rescaled random variable $N^{\frac{1}{2}}(m_N - m^*)$. For a fixed $t \in \mathbb{R}$, the moment generating function of $N^{\frac{1}{2}}(m_N - m^*)$ can be expressed as

$$\mathbb{E} \left[e^{tN^{\frac{1}{2}}(m_N - m^*)} \right] = e^{-tN^{\frac{1}{2}}m^*} \frac{\bar{Z}_N(t)}{Z_N}. \quad (44)$$

where $\bar{Z}_N(t)$ is a perturbed partition function associated to an Hamiltonian

$$\bar{H}_N(\sigma) = H_N(\sigma) + \sqrt{N}t m_N(\sigma). \quad (45)$$

We start by noticing that $\bar{H}_N(\sigma) = -Nf_N(m_N(\sigma))$ where $f_N(x) = \frac{K}{3}x^3 + \frac{J}{2}x^2 + \frac{1}{\sqrt{N}}tx$ and then f_N together with all its derivatives tends uniformly to $f(x) = \frac{K}{3}x^3 + \frac{J}{2}x^2$. Therefore one can use Lemma B.1 to obtain an asymptotic expansion for both Z_N and $\bar{Z}_N(t)$. More precisely one gets

$$\frac{\bar{Z}_N(t)}{Z_N} = e^{N(\phi_N(m_N^*(t)) - \phi(m^*))} (1 + \mathcal{O}(N^{-\frac{1}{2}+3\alpha})), \quad (46)$$

where $\phi_N(x) = f_N(x) - I(x)$ and for N large enough $m_N^*(t)$ is its unique maximizer. Now, let's observe that $m_N^*(0) = m^*$ and $m_N^*(t)$ satisfies the equation

$$m_N^*(t) = \tanh \left(Km_N^*(t)^2 + Jm_N^*(t) + \frac{t}{\sqrt{N}} \right). \quad (47)$$

Hence, it's easy to check that $\frac{\partial m_N^*(t)}{\partial t} \Big|_{t=0} = -\frac{1}{\sqrt{N}\phi''(m^*)}$ and $\frac{\partial^2 m_N^*(t)}{\partial t^2} = \mathcal{O}(N^{-1})$. Therefore the Taylor's expansion of $m_N^*(t)$ around $t = 0$ is

$$m_N^*(t) = m_N^*(0) - \frac{t}{\sqrt{N}\phi''(m^*)} + \mathcal{O}(N^{-1}). \quad (48)$$

Moreover one can easily check that $\phi_N(m_N^*(t)) = \phi(m_N^*(t)) + \frac{t}{\sqrt{N}}m_N^*(t)$. Hence the Taylor expansion of $\phi(m_N^*(t))$ around m^* is

$$N \left(\phi_N(m_N^*(t)) - \phi(m^*) \right) = \frac{N}{2} \left[(m_N^*(t)) - (m^*) \right]^2 \phi''(m^*) + \sqrt{N}tm_N^*(t) + o(1). \tag{49}$$

Finally using Eqs. (48) and (49) in the above, one gets

$$N \left(\phi_N(m_N^*(t)) - \phi(m^*) \right) = t\sqrt{N}m^* - \frac{t^2}{2\phi''(m^*)} + o(1) \tag{50}$$

and by (46) the limiting moment generating function is given as

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[e^{tN^{\frac{1}{2}}(m_N(\sigma) - m^*)} \right] = \exp \left\{ - \frac{t^2}{2\phi''(m^*)} \right\}, \tag{51}$$

which implies (11).

2. Let's recall that by Proposition 2.2 there exist two global maximizers m_i of $\phi(m)$ for $i \in \{0, 1\}$ on γ . Moreover by point b) of Proposition 3.2 we know that $\phi''(m_i) < 0$ for $i \in \{0, 1\}$. Now, following the same argument as before, formula (73) in Lemma B.2 gives the concentration inequality for m_N within a suitable neighbourhood of m_i with respect to the Gibbs measure (4). Therefore the convergence in distribution (13) and (12) follows the asymptotic expansions of the (restricted) partition function in Lemma B.2.

To obtain the local central limit theorem for m_N around the global maximizers m_i , we will show that the moment generating function of $N^{\frac{1}{2}}(m_N - m_i) | \{m_N \in A_i\}$ with respect to the measure μ_N converges pointwise in distribution to the moment generating function of $\mathcal{N}\left(0, -\frac{1}{\phi''(m_i)}\right)$. Here $A_i \subset [-1, 1]$ is such that m_i is the unique maximizer of $\phi(m)$ on its interior. The moment generating function of $N^{\frac{1}{2}}(m_N - m_i) | \{m_N \in A_i\}$ at a fixed $t \in \mathbb{R}$ is

$$\mathbb{E} \left[e^{tN^{\frac{1}{2}}(m_N - m_i)} | \{m_N \in A_i\} \right] = e^{-tN^{\frac{1}{2}}m_i} \frac{\bar{Z}_N(t) |_{A_i}}{Z_N |_{A_i}}. \tag{52}$$

Following the asymptotic expansion of the partition function in (74) (see Lemma B.2), the fraction on the right side of Eq. (52) reduces to

$$\frac{\bar{Z}_N(t) |_{A_i}}{Z_N |_{A_i}} \sim e^{N(\phi_N(m_{i,N}(t)) - \phi(m_i))}. \tag{53}$$

Now, taking Taylor's expansion of $\phi_N(m_{i,N}(t))$ at m_i up to the second order, one can repeat the same arguments as in the unique maximum case, obtaining

$$\mathbb{E} \left[e^{tN^{\frac{1}{2}}(m_N - m_i)} | \{m_N \in A_i\} \right] \xrightarrow{N \rightarrow \infty} \exp \left\{ - \frac{t^2}{2\phi''(m_i)} \right\}. \tag{54}$$

This completes the proof of (14).

3. Notice that the critical point $(K_c, J_c) = (0, 1)$ is a degenerate maximum point for $\phi(m)$ in the sense that $\phi''(m^*(K, J)) |_{(K,J)=(0,1)} = 0$. This does not

allow the use of the asymptotic expansions in Lemma B.1. However, one can simply notice that the Hamiltonian H_N of the model at the critical point $(K_c, J_c) = (0, 1)$ coincides at any $N \in \mathbb{N}$ with the Hamiltonian function of the standard Curie–Weiss model at the critical temperature $J = 1$ and zero external field. Therefore (15) and (16) are a well known results and their proof can be found in [14]. \square

3.3. Proof of Proposition 2.3

Proof. Let us start with the case $\alpha \geq 0$. This implies from Eq. (17) that $J(K) \geq J_c = 1$ and then $m^*(K) \equiv m_2(K, J(K))$ where m_2 is the only positive solution of the consistency equation (21).

Clearly $m^*(K) \rightarrow 0$ as $K \rightarrow 0^+$, hence by Taylor’s expansion we have that

$$\begin{aligned} m^*(K) &= J(K)m^*(K) + Km^*(K)^2 - \frac{J(K)^3 m^*(K)^3}{3} + \mathcal{O}(m^*(K)^4) \\ &= (1 + \alpha K)m^*(K) + Km^*(K)^2 - \frac{(1 + \alpha K)^3 m^*(K)^3}{3} + \mathcal{O}(m^*(K)^4). \end{aligned} \quad (55)$$

Hence

$$\frac{(1 + \alpha^3 K^3 + 3\alpha^2 K^2 + 3\alpha K)m^*(K)^2}{3} - Km^*(K) - \alpha K = \mathcal{O}(m^*(K)^3).$$

From the above equation, neglecting higher order corrections we have

$$m^*(K) \sim \frac{3}{2} \left(K + \sqrt{K^2 + \frac{4}{3}\alpha K + 4\alpha^2 K^2} \right). \quad (56)$$

Now, if $\alpha > 0$ then

$$m^*(K) \sim \sqrt{3\alpha K}. \quad (57)$$

Otherwise if $\alpha = 0$, then

$$m^*(K) \sim \frac{3}{2} \left(K + \sqrt{K^2} \right) \sim 3K. \quad (58)$$

Let’s turn on the case $\alpha < 0$. From Proposition 3.4 we know that $\gamma(K)$ is at least C^1 at $K = 0$. Since $\lim_{K \rightarrow 0^+} \gamma'(K) = 0$ we know that if $J(K) < \gamma(K)$ for K small enough, then $m^*(K) \equiv m_0(K, J) = 0$. \square

4. Conclusion and Perspectives

In this work, we have studied how the three-body interaction, which provides a spin-flip symmetry-breaking parameter, induces phase transitions with novel properties in the mean-field setting. In particular, we derived all the critical exponents and the limiting distribution of a suitably rescaled magnetization in the entire phase space. The presence of a stable paramagnetic phase and the fact that, also in the antiferromagnetic regime, the model presents phase transitions and phase coexistence are interesting for applications in socio-technical environments [8] and possibly in other fields [22, 23].

A possible research development will be to extend the results of the present work to multi-populated models [8, 24–32]. In these models, the invariance of the Hamiltonian with respect to permutations among sites is replaced by a weaker one that takes into account the existence of different species of spins. This setting is particularly useful in social science applications [8, 27, 30, 31]. Moreover, as mentioned in the introduction, the mean-field approximation involved in the study of some finite-dimensional lattices provides a natural emergence of the multi-populated models. It is well known, for instance, that a system on a simple cubic lattice [33, 34] with ferromagnetic and antiferromagnetic couplings has a factorized equilibrium measure that corresponds to a two-populated mean-field model. Similarly, it has been shown in [7] that on a regular square lattice, a system with cubic interaction has a product state equilibrium described by a two-populated mean-field model, while on a regular triangular lattice [6], by a three-populated mean-field model.

We also mention that in the case of quadratic interaction, Stein’s method provides stronger results (Berry-Esseen type bounds) on the rate at which the convergence to the normal distribution takes place (see [35, 36]). The extension of the above method to our model and more generally to higher order interaction is an interesting open problem. We plan to develop those research directions in the future.

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A. Technical Results

This section of the appendix presents some useful technical results applied in the work. We begin by stating the Berge's maximum theorem in the following Proposition.

Proposition A.1. *Let $f : [-1, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $c : \mathbb{R}^m \rightarrow [-1, 1]$ be continuous functions.*

(a) *The following function is continuous:*

$$F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}, \quad F(x, y) = \max_{v \in [-1, c(y)]} f(v, x).$$

(b) *Suppose that for all $x, y \in \mathbb{R}^n$ the function $v \mapsto f(v, x)$ achieves its maximum on $[-1, c(y)]$ in a unique point. Then also the following function is continuous:*

$$V : \mathbb{R}^n \times \mathbb{R}^m \rightarrow [-1, 1], \quad V(x, y) = \operatorname{argmax}_{v \in [-1, c(y)]} f(v, x).$$

The following proposition partially states Dini's implicit function theorem. Then we provide two simple corollaries that are used in the paper.

Proposition A.2. *Let $F : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be a C^∞ function. Let $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}$ such that $F(x_0, y_0) = 0$ and $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$. Then there exist $\delta > 0, \epsilon > 0$ and a C^∞ function $f : B(x_0, \delta) \rightarrow B(y_0, \epsilon)$ such that for all $(x, y) \in B(x_0, \delta) \times B(y_0, \epsilon)$*

$$F(x, y) = 0 \quad \iff \quad y = f(x)$$

Corollary A.3. *Let $F : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be a C^∞ function. Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function such that for all $x \in \mathbb{R}^n$ such that $F(x, \varphi(x)) = 0$ and $\frac{\partial F}{\partial y}(x, \varphi(x)) \neq 0$, then $\varphi(x) \in C^\infty(\mathbb{R}^n)$.*

Corollary A.4. *Let $F : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be a C^∞ function. Let $a, b : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function such that for all $a < b$. Suppose that for all $x \in \mathbb{R}^n$ there exists a unique $y = \varphi(x) \in (a(x), b(x))$ such that $F(x, \varphi(x)) = 0$. Moreover, suppose that for all $x \in \mathbb{R}^n$, $\frac{\partial F}{\partial y}(x, \varphi(x)) \neq 0$, then $\varphi(x) \in C^\infty(\mathbb{R}^n)$.*

B. Concentration Results and Asymptotic Expansions

In this section of the appendix, we state concentration properties of the magnetization and asymptotic expansions of the partition function for a large class of Ising mean-field models and give proofs using the same methods and arguments recently introduced in [16].

Consider a mean-field spin model with energy density f_N , namely

$$H_N(\sigma) = -N f_N(m_N(\sigma)), \quad \sigma \in \{-1, 1\}^N \quad (59)$$

where $m_N = \frac{1}{N} \sum_{i \leq N} \sigma_i$ is the magnetization density. We assume that (f_N) is a sequence of continuous functions $f_N : [-1, 1]^N \rightarrow \mathbb{R}$ converging uniformly to f . We assume also that f_N has bounded derivatives up to order 4

converging uniformly to f', f'', f''', f'''' . We denote the law of the magnetization under the Gibbs measure by

$$\mu_N(\sigma) = \frac{e^{-H_N(\sigma)}}{Z_N}. \tag{60}$$

The partition function Z_N can be written as

$$Z_N = \sum_{x \in R_N} A_N(x) e^{Nf_N(x)}, \tag{61}$$

where $R_N = \{-1 + \frac{2k}{N}, k = 0, \dots, N\}$ and $A_N(x) = \text{card}\{\sigma \in \{-1, 1\}^N : m_N(\sigma) = x\}$. Now, it follows from [37] that, for some universal constant L

$$\frac{1}{L\sqrt{N}} e^{-NI(x)} \leq A_N \leq e^{-NI(x)} \tag{62}$$

where $I(x)$ is defined in (8). Define the sequence ϕ_N as

$$\phi_N(x) = f_N(x) - I(x). \tag{63}$$

Notice the assumption on (f_N) that $\phi_N \rightarrow \phi = f - I$ uniformly on $(-1, 1)$, as well as its derivatives up to order 4 on $(-1, 1)$.

The following lemmata contains concentration properties of the magnetization m_N w.r.t. the Gibbs measure μ_N and asymptotic expansions of the partition function Z_N . For any $\alpha > 0$ and $y \in \mathbb{R}$ we denote by $B_{N,\alpha}(y)$ the open ball with center y and radius $N^{-1/2+\alpha}$ and by $B_{N,\alpha}^c(y)$ its complement.

Lemma B.1. *Assume that $\phi(x)$ has a unique global maximizer $x^* \in (-1, 1)$ such that $\phi''(x^*) < 0$. Then for N large enough ϕ_N has a unique maximizer $x_N^* \rightarrow x^*$ such that $\phi_N''(x_N^*) < 0$. Moreover for $\alpha \in (0, \frac{1}{6}]$ and N large enough we have that*

$$\mu_N(m_N \in B_{N,\alpha}^c(x_N^*)) = \exp \left\{ \frac{1}{2} N^{2\alpha} \phi_N''(x_N^*) \right\} \mathcal{O}(N^{\frac{3}{2}}) \tag{64}$$

and the partition function (61) can be expanded as,

$$Z_N = \frac{e^{N\phi_N(x_N^*)}}{\sqrt{(x_N^{*2} - 1)\phi_N''(x_N^*)}} \left(1 + \mathcal{O}\left(N^{-\frac{1}{2}+\alpha}\right) \right), \tag{65}$$

Proof. Let x_N^* be any maximizer of ϕ_N which exists since $[-1, 1]$ is compact. Then there exist a subsequence $\{N_l\}_{l \geq 1}$ such that $x_{N_l}^*$ converges to some y . We know that $\phi_{N_l}(x_{N_l}^*) \geq \phi_{N_l}(x)$ for all $x \in [-1, 1]$, therefore by uniform convergence and taking $l \rightarrow \infty$ we obtain $\phi(y) \geq \phi(x)$ for all $x \in [-1, 1]$ and this implies that y is a global maximizer of $\phi(x)$. But x^* is the unique global maximizer of $\phi(x)$, hence $y = x^*$.

Since $\phi''(x^*) < 0$ one has, for ϵ small enough, $\phi(x) < 0$ for any $x \in [x^* - \epsilon, x^* + \epsilon]$. Let x_N and y_N be two global maximizers of ϕ_N . We already know that $x_N \rightarrow x^*$ and $y_N \rightarrow x^*$. Therefore for N large enough $x_N, y_N \in [x^* - \epsilon, x^* + \epsilon]$. Using the fact that ϕ_N'' converges uniformly to ϕ'' one can show that for N

large enough ϕ_N it is strongly convex on $[x^* - \epsilon, x^* + \epsilon]$ and therefore has unique maximizer which implies that $x_N = y_N$.

In order to lighten the notation set $B_{N,\alpha} = B_{N,\alpha}(x_N^*)$. From Eqs. (60), (61) and (62) we have that,

$$\begin{aligned} \mu_N(m_N \in B_{N,\alpha}^c) &= \frac{\sum_{x \in R_N \cap B_{N,\alpha}^c} A_N \exp \{N(f_N(x))\}}{\sum_{x \in R_N} A_N \exp \{N(f_N(x))\}} \\ &\leq \frac{LN^{\frac{1}{2}}(N+1) \sup_{x \in B_{N,\alpha}^c} e^{N\phi_N(x)}}{\sup_{x \in [-1,1]} e^{N\phi_N(x)}} \\ &= \exp \left\{ N \left(\sup_{x \in B_{N,\alpha}^c} \phi_N(x) - \phi_N(x_N^*) \right) \right\} \mathcal{O}(N^{\frac{3}{2}}). \end{aligned} \quad (66)$$

Now by Lemma B.11 in [16] we know that for $x \in B_{N,\alpha}^c$ the maximizer of $\phi_N(x)$ is, for N large enough, either $x_N^* - N^{-\frac{1}{2}+\alpha}$ or $x_N^* + N^{-\frac{1}{2}+\alpha}$. This implies that $\sup_{x \in B_{N,\alpha}^c} \phi_N(x)$ is either $\phi_N(x_N^* - N^{-\frac{1}{2}+\alpha})$ or $\phi_N(x_N^* + N^{-\frac{1}{2}+\alpha})$. Note that $\phi_N'(x_N^*) = 0$ since x_N^* is the maximizer and $\phi_N^{(3)}(x_N^*)$ is uniformly bounded on any closed interval in $(-1, 1)$. Hence by a second-order Taylor expansion of $\phi_N(x_N^* \pm N^{-\frac{1}{2}+\alpha})$ at the point x_N^* , we have that

$$\phi(x_N^* \pm N^{-\frac{1}{2}+\alpha}) = \phi_N(x_N^*) + \frac{1}{2}N^{-1+2\alpha}\phi_N''(x_N^*) + \mathcal{O}(N^{-\frac{3}{2}+3\alpha}), \quad (67)$$

where $\phi_N''(x_N^*) < 0$. This completes the proof of equation (64) following from Eq. (66).

To complete the proof of Lemma B.1, let's start by observing that almost all the contribution to Z_N comes from spin configurations having magnetization in a vanishing neighbourhood of the maximizer x_N^* , i.e., $\mu_N(m_N(\sigma) \in B_{N,\alpha}) = 1 - \mathcal{O}(e^{-N^\alpha})$. Hence,

$$Z_N = (1 + \mathcal{O}(e^{-N^\alpha})) \sum_{x \in R_N \cap B_{N,\alpha}} \underbrace{\left(\frac{N}{\frac{N(1+x)}{2}} \right) \exp \{N(f_N(x))\}}_{=\zeta(x)}. \quad (68)$$

where $\zeta : [-1, 1] \rightarrow \mathbb{R}$. With this, one can accurately approximate the partition function over all configurations σ whose mean lies within a vanishing neighbourhood of x^* using standard approximation techniques.

We begin by applying the Laplace approximation of an integral over a shrinking interval $B_{N,\alpha}$ via the Riemann approximation of the sum in Eq. (68) with an integral and the binomial coefficient can be approximated by the Stirling's approximation method. Notice that by the Riemann approximation (see Appendix Lemma A.2 and B.7 of [16]) of the sum, we have that

$$\begin{aligned} \left| \int_{B_{N,\alpha}} \zeta(x) dx - \frac{2}{N} \sum_{x \in R_N \cap B_{N,\alpha}} \zeta(x) \right| &\leq \frac{1}{2}(N^{-\frac{1}{2}+\alpha}) \cdot N^{-1} \sup_{x \in B_{N,\alpha}} |\zeta'(x)| \\ &= \mathcal{O}(N^{-\frac{1}{2}+\alpha} \cdot N^{-1} \cdot N^{\frac{1}{2}+\alpha}) \zeta(x_N^*) \\ &= \mathcal{O}(N^{-1+2\alpha}) \zeta(x_N^*) \end{aligned} \quad (69)$$

and the binomial coefficient in (68) can be approximated as

$$\binom{N}{\frac{N(1+x)}{2}} = \sqrt{\frac{2}{\pi N(1-x^2)}} e^{-NI(x)} (1 + \mathcal{O}(N^{-1})). \quad (70)$$

It follows from Eqs. (69) and (70) and the Laplace approximation (see Appendix Lemma A.3 of [16]) of an integral over a shrinking interval $B_{N,\alpha}$ that:

$$\begin{aligned} \sum_{x \in R_N \cap B_{N,\alpha}} \zeta(m) &= \frac{N}{2} \int_{B_{N,\alpha}} \zeta(x) dx + \mathcal{O}(N^{2\alpha}) \zeta(x_N^*) \\ &= \frac{N}{2} \int_{B_{N,\alpha}} \sqrt{\frac{2}{\pi N(1-x^2)}} e^{N\phi_N(x)} (1 + \mathcal{O}(N^{-1})) dx \\ &\quad + \mathcal{O}(N^{2\alpha}) \cdot \left[\sqrt{\frac{2}{\pi N(1-x_N^{*2})}} e^{N\phi_N(x_N^*)} (1 + \mathcal{O}(N^{-1})) \right] \\ &= \frac{\sqrt{N}}{2} \sqrt{\frac{2\pi}{N|\phi_N''(x_N^*)|}} \sqrt{\frac{2}{\pi(1-x_N^{*2})}} e^{N\phi_N(x_N^*)} (1 + \mathcal{O}(N^{-\frac{1}{2}+3\alpha})) \\ &= \frac{e^{N\phi_N(x_N^*)}}{\sqrt{(x_N^{*2}-1)\phi_N''(x_N^*)}} \cdot (1 + \mathcal{O}(N^{-\frac{1}{2}+3\alpha})). \end{aligned} \quad (71)$$

Therefore,

$$Z_N = \frac{e^{N\phi_N(x_N^*)}}{\sqrt{(x_N^{*2}-1)\phi_N''(x_N^*)}} \cdot (1 + \mathcal{O}(N^{-\frac{1}{2}+3\alpha})). \quad (72)$$

This completes the proof of Lemma B.1. \square

Lemma B.2. *Suppose $\phi(x)$ has $S \in \mathbb{N}$ global maximizers x_i such that $\phi''(x_i) < 0$. For $i \leq S$, let $A_i \subset [-1, 1]$ be an interval such that $x_i \in \text{int}(A_i)$ is the unique maximizer of ϕ on $\text{cl}(A_i)$. Then for N large enough ϕ_N has a unique global maximizer $x_{i,N} \rightarrow x_i$ on A_i with $\phi_N''(x_{i,N}) < 0$ and for $\alpha \in (0, \frac{1}{6}]$, one has*

$$\mu_N(m_N \in B_{N,\alpha,S}^c) = \exp \left\{ \frac{1}{2} N^{2\alpha} \max_{i \leq S} \phi_N''(x_{i,N}) \right\} \mathcal{O}(N^{\frac{3}{2}}) \quad (73)$$

where $B_{N,\alpha,S} = \bigcup_{i \leq S} B_{N,\alpha}(x_{i,N})$, moreover the restricted partition function on A_i can be expanded as,

$$Z_N|_{A_i} = \frac{e^{N\phi_N(x_{i,N})}}{\sqrt{(x_{i,N}^2-1)\phi_N''(x_{i,N})}} \left(1 + \mathcal{O}\left(N^{-\frac{1}{2}+\alpha}\right) \right) \quad (74)$$

and the unrestricted partition function can be expanded as,

$$Z_N = \sum_{i \leq S} \frac{e^{N\phi_N(x_{i,N})}}{\sqrt{(x_{i,N}^2-1)\phi_N''(x_{i,N})}} \left(1 + \mathcal{O}\left(N^{-\frac{1}{2}+\alpha}\right) \right). \quad (75)$$

Note that, here, $\text{int}(A_i)$ and $\text{cl}(A_i)$ denote the interior and closure of A_i , respectively.

Proof. The fact that for N large enough ϕ_N has a unique maximizer $x_{i,N} \rightarrow x_i$ with $\phi''(x_{i,N}) < 0$ can be proved applying to the function ϕ_N restricted to $cl(A_i)$ and using the same argument of Lemma B.1.

Clearly, for N large enough, $B_{N,\alpha}(x_{i,N}) \subset A_i$ and

$$\mu_N(m_N(\sigma) \in B_{N,\alpha}^c(x_{i,N}) | m_N(\sigma) \in A_i) = \exp \left\{ \frac{1}{2} N^{2\alpha} \phi_N''(x_{i,N}) \right\} \mathcal{O}(N^{\frac{3}{2}}) \tag{76}$$

following a step-by-step argument used to prove equation (64).

Now, for $i \leq S$ and N large enough, one has that $A_i \setminus B_{N,\alpha}(x_{i,N}) = A_i \setminus B_{N,\alpha,S}$ and then $\mu_N(m_N(\sigma) \in B_{N,\alpha}^c(x_{i,N}) | m_N(\sigma) \in A_i) = \mu_N(m_N(\sigma) \in B_{N,\alpha,S}^c | m_N(\sigma) \in A_i)$. Therefore,

$$\begin{aligned} \mu_N(m_N(\sigma) \in B_{N,\alpha,S}^c) &= \sum_{1 \leq i \leq S} \mu_N(m_N(\sigma) \in B_{N,\alpha,S}^c | m_N(\sigma) \in A_i) \mu_N(m_N(\sigma) \in A_i) \\ &\leq \exp \left\{ \frac{1}{2} N^{2\alpha} \max_{1 \leq i \leq S} \phi''(x_i) \right\} \mathcal{O}(N^{\frac{3}{2}}) \sum_{1 \leq i \leq S} \mu_N(m_N(\sigma) \in A_i) \\ &= \exp \left\{ \frac{1}{2} N^{2\alpha} \max_{1 \leq i \leq S} \phi''(x_i) \right\} \mathcal{O}(N^{\frac{3}{2}}). \end{aligned} \tag{77}$$

This completes the proof of equation (73) following from Eq. (77).

The proof for the asymptotic expansion of the partition function when there are multiple global maximizers of ϕ follows exactly the same argument for the case with unique global maximizer. Note that for fixed $i \leq S$ and N large, $m_N(\sigma)$ concentrates around $x_i \in A_i$ as it was shown in Eq. (76). Hence,

$$\mu_N(m_N(\sigma) \in B_{N,\alpha} | m_N(\sigma) \in A_i) = \frac{1}{Z_N |_{A_i}} \sum_{x \in R_N \cap B_{N,\alpha}} \left(\frac{N}{N(1+x)} \right) \exp \{ N(f_N(x)) \}. \tag{78}$$

Now, following the exact computation and argument in Lemma B.1, we have that the restricted partition function for each of the global maximizers x_i can be expanded as

$$Z_N |_{A_i} = \frac{e^{N\phi_N(x_{i,N})}}{\sqrt{(x_{i,N}^2 - 1)\phi''(x_{i,N})}} \cdot (1 + \mathcal{O}(N^{-\frac{1}{2} + 3\alpha})). \tag{79}$$

Assuming that $m_N(\sigma)$ concentrates around S global maximizers $x_{i,N}$ for $i \leq S$ then, Eq. (75) follows from (79). Hence, we have

$$Z_N = \sum_{i \leq S} Z_N |_{A_i}. \tag{80}$$

□

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