



Gorenstein curve singularities of genus three

Luca Battistella¹

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Abstract

We classify the analytic germs of isolated Gorenstein curve singularities of genus three, and relate them to the connected components of strata of abelian differentials.

Keywords Curves of low genus · Gorenstein singularities · Strata of differentials

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Introduction

A Cohen-Macaulay variety is Gorenstein if its dualising sheaf is a line bundle. This is a local condition. For instance, ordinary double points are Gorenstein, whereas the union of the coordinate axes in \mathbb{A}^3 is not. The class of Gorenstein curves provides a very mild and useful generalisation of nodal curves from the moduli theoretic perspective.

- There are several compactifications of $\mathcal{M}_{g,n}$ parametrising only Gorenstein curves: the most notable one is the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$, and the alternatives serve to illustrate its birational geometry, see for instance [3, 6, 15, 27, 42]. Indeed, all singularities appearing in the Hassett–Keel program seem to be Gorenstein [2]. Typically, replacing nodal subcurves with few special points with a Gorenstein singularity produces a contraction or flip of $\overline{\mathcal{M}}_{g,n}$, thus wors-

✉ Luca Battistella
luca.battistella2@unibo.it

¹ Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, Bologna, Italy

ening the singularities of the moduli space, but otherwise reducing its complexity (the number of parameters).

- Gorenstein curves can be used to test the geometry of a target variety [12, 47]. They have been successfully employed to construct alternative compactifications and modular desingularisations of Kontsevich's moduli space of stable maps [8, 9, 16, 39, 40], thus defining simpler enumerative invariants.

Let us now consider the analytic germ of an isolated curve singularity (C, p) . The simplest of its invariants are the following:

- b , the number of branches (or preimages of p in the normalisation);
- δ , the number of independent conditions that a function on the normalisation must satisfy in order to descend to the singular curve. Out of these, $b - 1$ impose that the function attains the same value at the b preimages of p ;
- the remaining $g := \delta - b + 1$ form the local contribution of the singularity to the arithmetic genus of any projective curve containing it; g is called the *genus* of the singularity.

Several more sophisticated invariants of curve singularities have been introduced, see for instance the book [23], although most classical sources are only concerned with the singularities of curves on surfaces, see e.g. the book [14]. From the moduli-theoretic perspective, it is natural on the other hand to classify curve singularities by their genus, notwithstanding the fact that the embedding dimension grows linearly in b - and we have very little control on the deformation theory of singularities of codimension four or more. Isolated Gorenstein singularities of genus one have been classified by Smyth in [42] (with applications to moduli theory in [43, 44] and the other references above). Isolated Gorenstein singularities of genus two have been classified by the author in [6]; some non-isolated singularities, that we call “ribbons with tails”, appear in [9]. Smoothable hyperelliptic singularities have been classified in [7].

A theme has appeared in the author's works on hyperelliptic singularities: namely, the connection between smoothable Gorenstein singularities and logarithmic differentials, i.e. abelian (or more precisely multiscale [11]) differentials on nodal curves, together with their tropical incarnations: piecewise-linear functions on the dual graph belonging to the tropical canonical linear series. We outline this connection briefly here, and refer the reader to the next section for more details.

- Given a smoothable Gorenstein curve C of genus g with a unique singular point p of genus g , let η_0 be an abelian differential generating $\omega_{C,p}$. By smoothing (C, η_0) and then applying semistable reduction (i.e. the properness of the moduli space of generalised multiscale differentials), we may replace the central fibre with a reducible nodal curve C^{ss} and a multiscale differential η^{ss} . The preimage Z of p in C^{ss} , and the restriction of η^{ss} to it, give us a smoothable logarithmic differential.
- Given a multiscale differential, we can put it in a one-parameter smoothing, and, possibly after blowing up some points in the central fibre, conjecturally use its various level truncations to produce corresponding Gorenstein contractions. The multiscale differential should then descend to a generator of the dualising bundle of the Gorenstein curves.

In line with this program, the main result of this note is the following:

Theorem *There are seven families of isolated Gorenstein singularities of genus three, each corresponding to a connected component of a stratum of abelian differentials of genus three.*

Sect. 2 consists of the proof of this result; Sect. 1 describes the geometry of genus three curves, theta characteristics, Gorenstein singularities, and strata of differentials. In the following table we provide a list of the strata and the corresponding singularities, including the gap sequence and minimum number of branches. In the interest of legibility, we have simplified the generators by setting all crimping variables equal to 0 or 1 as appropriate. We work throughout over \mathbb{C} .

1 Geometric context

1.1 The canonical linear series on smooth curves

Let C be a smooth curve of genus three. The canonical linear series is a basepoint-free g_4^2 on C , producing it either as a smooth plane quartic, or as a double cover of the Veronese conic (if and only if C is hyperelliptic).

For a general point P of C , the orders of vanishing of $|K_C|$ at P are $0 < 1 < 2$. When the vanishing sequence is different from this, i.e. P is a ramification point of the canonical linear series, P is called a *Weierstrass point*. If the vanishing orders are denoted by $a_0 < a_1 < a_2$, the orders of ramification form the non-decreasing sequence $(\alpha_i = a_i - i)_{i=0,1,2}$. The weight of a Weierstrass point is defined as $\sum \alpha_i$. The classical Plücker formula states that the sum of the weights of all Weierstrass points is $(g - 1)g(g + 1) = 24$ (see e.g. [1, Chap. I, Exercise C]).

If C is hyperelliptic, the ramification sequence of a Weierstrass point is $(0, 1, 2)$, hence there are exactly 8 Weierstrass points.¹ Ignoring automorphisms, the hyperelliptic locus can be thus identified with $\mathcal{M}_{0,8}/\mathfrak{S}_8$. It forms a divisor in \mathcal{M}_3 whose class was computed by Mumford at the very beginning of the subject of intersection theory on moduli spaces [35] (the class of its closure in $\overline{\mathcal{M}}_3$ is also well-known, see [28, Chapter 3, §H] and [21]).

If C is a general plane quartic, it has exactly 24 flexes, i.e. simple Weierstrass points with ramification sequence $(0, 0, 1)$. Interestingly, the general plane quartic can be reconstructed from its inflection lines (even without knowing where they meet the curve!), see [38]. A hyperflex is a point of ramification sequence $(0, 0, 2)$, hence there are at most 12 of them. The locus of curves with a hyperflex is an irreducible divisor in the space of quartic plane curves, whose class was studied by Faber [22]. In fact, a smooth quartic with any number n of hyperflexes, $0 \leq n \leq 9$ or $n = 12$, can be found; see Vermeulen's thesis [46]. A stratification of plane quartics according to their singularity type can be found in [29].

¹ This can also be deduced from the Riemann–Hurwitz formula.

Table 1 List of Gorenstein singularities of genus three and corresponding strata of differentials

Diff. str.	Gap seq.	b_{min}	Simplified generators	$k : \tilde{m}^k \subseteq m$	Planar
$\mathcal{H}(1^4)$	[2, 1]	4	$A_3 = ct_1 \quad t_2 \quad t_3 \quad 0 \quad 0 \quad \dots \quad 0$	$(\text{mod } \tilde{m}^3)$	4-uple pt
			$A_4 = t_1 \quad ct_2 \quad 0 \quad t_4 \quad 0 \quad \dots \quad 0$		
			$A_5 = t_1^2 \quad 0 \quad 0 \quad 0 \quad t_5 \quad \dots \quad 0$		
			$w/\text{analytic invariant } \mathbf{c} \in \mathbb{C} \setminus \{\pm 1\}$		
			$A_b = t_1^2 \quad 0 \quad 0 \quad 0 \quad 0 \quad \dots \quad t_b$		
			$B_1 = t_1^2 \quad t_2^2 \quad 0 \quad 0 \quad 0 \quad \dots \quad 0$		
$\mathcal{H}(2, 1^2)$	[2, 0, 1]	3	$A_3 = 0 \quad t_2 \quad t_3 \quad 0 \quad \dots \quad 0$	$(\text{mod } \tilde{m}^4)$	
			$A_4 = t_1^3 \quad 0 \quad 0 \quad t_4 \quad \dots \quad 0$		
			...		
			$A_b = t_1^3 \quad 0 \quad 0 \quad 0 \quad \dots \quad t_b$		
			$B_1 = t_1^2 \quad 0 \quad 0 \quad 0 \quad \dots \quad 0$		
			$B_2 = t_1^3 \quad t_2^2 \quad 0 \quad 0 \quad \dots \quad 0$		
			$A_3 = t_1^3 \quad 0 \quad t_3 \quad \dots \quad 0$		
			...		
$\mathcal{H}(2^2)^{odd}$	[2, 0, 1]	2	$A_b = t_1^3 \quad 0 \quad 0 \quad \dots \quad t_b$	$(\text{mod } \tilde{m}^4)$	
			$B_1 = t_1^2 \quad 0 \quad 0 \quad \dots \quad 0$		
			$B_2 = 0 \quad t_2^2 \quad 0 \quad \dots \quad 0$		
			$C_2 = t_1^3 \quad t_2^3 \quad 0 \quad \dots \quad 0$		
			$A_2 = t_1 \quad t_2 \quad 0 \quad \dots \quad 0$		
$\mathcal{H}(2^2)^{ev}$	[1, 1, 1]	2	$A_3 = t_1^3 \quad 0 \quad t_3 \quad \dots \quad 0$	$(\text{mod } \tilde{m}^4)$	A_7, D_8
			...		
			$A_b = t_1^3 \quad 0 \quad 0 \quad \dots \quad t_b$		
			$D_1 = t_1^4 \quad -t_2^4 \quad 0 \quad \dots \quad 0$		
			$A_2 = t_1^2 \quad t_2 \quad 0 \quad \dots \quad 0$		
$\mathcal{H}(3, 1)$	[1, 1, 0, 1]	2	$A_3 = t_1^4 \quad 0 \quad t_3 \quad \dots \quad 0$	$(\text{mod } \tilde{m}^5)$	E_7
			...		
			$A_b = t_1^4 \quad 0 \quad 0 \quad \dots \quad t_b$		
			$C_1 = t_1^3 \quad 0 \quad 0 \quad \dots \quad 0$		
			$A_2 = t_1^5 \quad t_2 \quad 0 \quad \dots \quad 0$		
$\mathcal{H}(4)^{odd}$	[1, 1, 0, 0, 1]	1	$A_b = t_1^5 \quad 0 \quad 0 \quad \dots \quad t_b$	$(\text{mod } \tilde{m}^6)$	E_6
			$C_1 = t_1^3 \quad 0 \quad 0 \quad \dots \quad 0$		
			$D_1 = t_1^4 \quad 0 \quad 0 \quad \dots \quad 0$		
			$A_2 = t_1^5 \quad t_2 \quad 0 \quad \dots \quad 0$		
			...		
$\mathcal{H}(4)^{ev}$	[1, 0, 1, 0, 1]	1	$A_b = t_1^5 \quad 0 \quad 0 \quad \dots \quad t_b$	$(\text{mod } \tilde{m}^6)$	A_6, D_7
			$B_1 = t_1^2 \quad 0 \quad 0 \quad \dots \quad 0$		
			...		

1.2 Theta characteristics and singular curves

A theta characteristic L is a square-root of the canonical line bundle. The space of theta characteristics $S \subseteq \text{Pic}^{g-1}(C)$ is a torsor under the 2-torsion subgroup of the Jacobian $J_2(C)$; in particular, if C is smooth, S consists of 2^{2g} elements. It is a classical fact due to Riemann, Atiyah [5], and Mumford [34] that the *parity* of the dimension $h^0(L)$ of the space of regular sections of a theta characteristic is locally constant in families. The space S of theta characteristics can be divided into two subspaces S^+ and S^- accordingly. For a fixed smooth curve of genus g ,

$$\#S^+ = 2^{g-1}(2^g + 1), \quad \#S^- = 2^{g-1}(2^g - 1),$$

summing up to 2^{2g} .

1.2.1 Bitangents

Bitangents of a smooth quartic curve are odd theta characteristics, so there are 28 of them. In general, the 36 even theta characteristics are not effective. In the hyperelliptic case $\psi: C \rightarrow \mathbb{P}^1$, instead, there is an effective even theta characteristic corresponding to the class of $\psi^* \mathcal{O}_{\mathbb{P}^1}(1)$; the 28 odd theta characteristics can be written as $q_1 + q_2$, with q_1 and q_2 two distinct Weierstrass points. Any smooth quartic can be reconstructed from its bitangent lines; in fact from any of a number of special configurations of seven of them, known as Aronhold systems [20, 32].

1.2.2 Projective duality

Flexes and bitangents play an important role in projective duality. Let C be a reduced plane curve of degree d . The *projectively dual* curve C^* is the closure of the locus of tangent lines to smooth points of C in the dual projective plane \mathbb{P}^{2*} . If C is smooth, C^* has degree $d^* = d(d-1)$, but it is usually singular: indeed, a bitangent of C corresponds to a node of C^* , and an ordinary flex to a cusp. Let δ^* (resp. κ^*) denote the number of bitangents (resp. flexes) of C . If C is general, the Plücker formulae imply:

$$\begin{aligned} \delta^* &= \frac{1}{2}(d+3)d(d-2)(d-3), \\ \kappa^* &= 3d(d-2). \end{aligned}$$

For quartics we recover: 28 bitangents and 24 flexes. When C is singular, the formulae above need to be corrected by taking the singularities into account. See the work of Wall [48] on how singularities behave under projective duality; for instance, a hyperflex corresponds to an E_6 -singularity (of genus three, see below).

1.2.3 Harris

Theta characteristics can be defined as above on any Gorenstein curve [24] (more generally, one needs to consider rank-one torsion-free sheaves, because such is the dualising sheaf itself [37]). The parity of the space of global sections remains locally constant, and the space of theta characteristics remains a torsor under the 2-torsion subgroup of the Jacobian $J_2(C)$, but the cardinality of the latter may vary. Indeed, let $C^\nu \rightarrow C^{sn} \rightarrow C$ denote the normalisation (respectively, seminormalisation, i.e. the initial curve homeomorphic to C) of the reduced curve C . The kernel of $\text{Pic}(C^{sn}) \rightarrow \text{Pic}(C^\nu)$ is an algebraic torus of dimension β , so the cardinality of its 2-torsion is 2^β . On the other hand, the kernel of $\text{Pic}(C) \rightarrow \text{Pic}(C^{sn})$ is unipotent, i.e. an extension of copies of \mathbb{G}_a ; in particular it is divisible and it has trivial 2-torsion. So, $J_2(C)$ has cardinality $2^{2\tilde{g}+\beta}$, where \tilde{g} is the sum of the genera of the components of C^ν . See for instance [33, §7.5].

We offer a brief summary of (Mumford's and) Harris' work: given a theta characteristic L , the function

$$q_L(D) = h^0(L(D)) - h^0(L) \pmod{2}$$

is a quadratic form on the $\mathbb{Z}/2\mathbb{Z}$ -vector space $J_2(C)$; its associated bilinear form is the Weil pairing.

Let Γ_2 denote the kernel of the pullback map $J_2(C) \rightarrow J_2(C^\nu)$. The restriction of q_L to Γ_2 is linear and independent of L ; denote it by l . Interestingly, l is determined by local data. Indeed, we can describe a set of generators of Γ_2 as follows: we let e_i be the line bundle that is trivial on C^ν , and glued with -1 between branch i and any other branch, and 1 between any two branches other than the i -th. These line bundles descend to C , and they satisfy exactly one relation $\sum e_i = 0$. If D denotes an adjoint divisor, that is defined by the pullback of the conductor ideal to C^ν , Harris proves that

$$l(e_i) = \text{mult}_{p_i}(D),$$

where p_i are the points of C^ν over $p \in C$.

If l is not 0, then the set S of theta characteristics subdivides equally between even and odd thetas. If $l \equiv 0$, instead, q_L is induced by a quadratic form on $J_2(C^\nu)$, and there are two possibilities for the latter: in particular, the number of even - resp. odd - theta characteristics is $2^{\tilde{g}+\beta-1}(2^{\tilde{g}} + 1)$ - resp. $2^{\tilde{g}+\beta-1}(2^{\tilde{g}} - 1)$ -, or viceversa. Let us set $Q(C) = 0$ in the former case, and $Q(C) = 1$ in the latter. Once again, Harris proves that this dichotomy is determined only by local data around the singularities of C . Indeed, if $l \equiv 0$, the adjoint divisor D is even, i.e. $D = 2E$; let I' denote the ideal of \mathcal{O}_C that pulls back to $\mathcal{O}(-E)$ on C^ν , and let $\epsilon(p) = \text{len}(\mathcal{O}_C/I')$. Then

$$Q(C) = \sum_{p \in C^{\text{sing}}} \epsilon(p) \pmod{2}.$$

Notice that the space of theta characteristics is at any rate finite (albeit not étale) over the moduli stack of Gorenstein curves.

1.3 Isolated Gorenstein curve singularities

A Cohen–Macaulay variety is called Gorenstein if its dualising sheaf is a line bundle. This is a local condition that can be checked after taking completions. Here we shall study analytic germs of curve singularities. For example, plane curve (and more generally l.c.i.) singularities are Gorenstein. Gorenstein curves need not be reduced: being Cohen–Macaulay, they have no embedded points, but they may still contain multiple components; in this case, the generic slice² must itself be Gorenstein (for example, double structures always work, but, among the triple structures, only the curvilinear ones are allowed). However, we will focus on isolated singularities in this paper.

1.3.1 Rosenlicht

Recall that the dualising bundle of a smooth variety is identified with the determinant of its cotangent bundle, also called the canonical bundle. Rosenlicht gave an explicit description of the dualising sheaf of a reduced curve C in terms of meromorphic differentials on its normalisation C^v (see for instance [4, Chapter VIII]): a rational differential (on the normalisation) $\eta \in \omega_{K(C)/\mathbb{C}}$ descends to a regular section $\eta \in \Gamma(C, \omega_C)$ if and only if, for every (germ of) regular function $f \in \mathcal{O}_{C,p}$, the sum of the *residues* of $f \cdot \eta$ at the points of C^v over p is zero:

$$\sum_{p_i \in v^{-1}(p)} \text{Res}_{p_i}(f \cdot \eta) = 0.$$

Let $\mathfrak{c} = \text{Ann}(v_*\mathcal{O}_{C^v}/\mathcal{O}_C)$ denote the conductor ideal of the singularity; it can be viewed both as an ideal of \mathcal{O}_C and as one of $v_*\mathcal{O}_{C^v}$. If D is the associated (adjoint) divisor on C^v , notice that $\text{mult}_{p_i}(D)$ bounds from above the possible orders of pole of η at p_i ; in particular, η must be regular at smooth points of C . Indeed, the residue pairing:

$$(v_*\mathcal{O}_{C^v}/\mathcal{O}_C) \times (\omega_C/v_*\omega_{C^v}) \rightarrow \mathbb{C} \tag{1}$$

is perfect, and when looking at germs it can be enhanced to a graded pairing.

Let $\delta = \text{len}(v_*\mathcal{O}_{C^v,p}/\mathcal{O}_{C,p})$; this is called the δ -invariant of (C, p) . It represents the number of conditions that a function on the normalisation must satisfy in order to descend to the singularity. If we factor the normalisation v through the seminormalisation, $C^v \rightarrow C^{sn} \rightarrow C$, and if b is the number of branches of C at p (i.e. the number of analytically connected components of the normalisation), then there are exactly $b - 1$ conditions for a function to descend to C^{sn} , expressing the requirement that functions attain the same value at the b preimages of p in C^v . Indeed, the germ of the seminormalisation is isomorphic to the union of the coordinate axes in \mathbb{A}^b . Dually under (1) we find the differentials $\frac{dt_1}{t_1} - \frac{dt_i}{t_i}$, $i = 2, \dots, b$, on the seminormalisation; in particular, the seminormalisation is never Gorenstein for $b > 2$.

² Locally at the generic point of any component, the underlying reduced curve is regular and defined by one parameter. By quotienting this out, we obtain a 0-dimensional Gorenstein ring.

The remaining

$$g = \delta - b + 1$$

conditions determine the so-called *genus* of the singularity. This is also the local contribution of the singularity to the arithmetic genus of a projective curve containing it. For instance, the cusp and the tacnode are singularities of genus 1; they appear naturally in families of plane cubics.

1.3.2 Serre

For graded finite-dimensional \mathbb{C} -algebras, the Gorenstein condition reduces to Poincaré duality. More generally, a ring is Gorenstein if it can be cut down to a Gorenstein Artinian ring by means of a regular sequence.

A monomial unibranch curve is Gorenstein if and only if its gap sequence is symmetric: if $J = \{j_1, \dots, j_g\}$ are the exponents of the missing monomials, then $n \in J$ if and only if $2g - 1 - n \notin J$ [30].

For reduced curves, Serre [41, §4, Proposition 7] shows that an equivalent condition to being Gorenstein is

$$\text{len}(\mathcal{O}_{C,p}/\mathfrak{c}) = \text{len}(v_*\mathcal{O}_{C^v,p}/\mathcal{O}_{C,p})(= \delta). \quad (2)$$

1.3.3 Stevens

A curve singularity (C, p) is *decomposable* if C is the union of two curves C_1, C_2 , living in smooth spaces that intersect transversely in p . Equivalently, there are two parameters t_1 and t_2 on the normalisation, and a partition $(x_1, \dots, x_a, y_1, \dots, y_b)$ of the generators of $\mathfrak{m}/\mathfrak{m}^2$, such that the \mathbf{x} do not involve t_2 , and the \mathbf{y} do not involve t_1 . A useful observation [2, Proposition 2.1] is that every Gorenstein curve singularity, except for the node, is indecomposable. For instance, the rational (or ordinary) b -fold point, i.e. the singularity of the seminormalisation, is not Gorenstein for $b \geq 3$.

1.3.4 Smyth

Let (R, \mathfrak{m}) denote the complete local ring of an isolated curve singularity, with normalisation $(\tilde{R}, \tilde{\mathfrak{m}}) \simeq (\mathbb{C}[[t_1]] \oplus \dots \oplus \mathbb{C}[[t_b]], (t_1, \dots, t_b))$. Following [42, Appendix A] and [6, §1], we consider \tilde{R}/R as a \mathbb{Z} -graded R -module (and finite \mathbb{C} -algebra) with:

$$(\tilde{R}/R)_i := \tilde{\mathfrak{m}}^i / (\tilde{\mathfrak{m}}^i \cap R) + \tilde{\mathfrak{m}}^{i+1}.$$

The following observations provide the starting point for the classification:

- (1) $b + 2 = \delta(p) = \sum_{i \geq 0} \dim_{\mathbb{C}}(\tilde{R}/R)_i$;
- (2) $3 = g = \sum_{i \geq 1} \dim_{\mathbb{C}}(\tilde{R}/R)_i$;
- (3) if $(\tilde{R}/R)_i = (\tilde{R}/R)_j = 0$ then $(\tilde{R}/R)_{i+j} = 0$;
- (4) $\sum_{i \geq j} (\tilde{R}/R)_i$ is a grading of $\tilde{\mathfrak{m}}^j / (\tilde{\mathfrak{m}}^j \cap R)$;

(5) there are short exact sequences of $R/\mathfrak{m} = \mathbb{C}$ -modules:

$$0 \rightarrow K_i := \frac{\tilde{\mathfrak{m}}^i \cap R}{\tilde{\mathfrak{m}}^{i+1} \cap R} \rightarrow \frac{\tilde{\mathfrak{m}}^i}{\tilde{\mathfrak{m}}^{i+1}} \rightarrow (\tilde{R}/R)_i \rightarrow 0.$$

Accordingly, the set

$$\{i \in \mathbb{N} \mid (\tilde{R}/R)_i = 0\} \cup \{0\}$$

is a submonoid; to the remaining integers we may associate the dimension of $(\tilde{R}/R)_i$: this forms a non-increasing sequence that we call the *gap sequence*.

1.4 Semistable tails and differentials

Suppose that a Gorenstein singularity (C, p) is smoothable. Consider it as a projective curve C by compactifying every branch to a distinct, smooth and disjoint away from p , copy of \mathbb{P}^1 . Then C is smoothable as well by [25, Example 29.10.2]. Consider a one-parameter smoothing $\mathcal{C} \rightarrow \Delta$, and apply semistable reduction to \mathcal{C} , in such a way to obtain a (minimal) contraction $\Phi: \mathcal{C}^{ss} \rightarrow \mathcal{C}$. We now focus on the special fibre. What we see is a nodal curve \mathcal{C}^{ss} of genus g with a morphism ϕ to C . Let Z denote the preimage of p . Then $C_0 = \overline{C} \setminus Z$ is the disjoint union of b copies of \mathbb{P}^1 , and ϕ presents it as the normalisation of C . The exceptional curve Z , instead, is a nodal curve of genus g , naturally marked by its intersection with the b branches of C_0 . It is called a *semistable tail* of the singularity. Although Z does depend on the chosen smoothing, we shall see that we can provide a uniform characterisation in terms of logarithmic differentials.

Let η be a local generator of the dualising sheaf $\omega_{C,p}$. Since the singularity is indecomposable, the multiplicity of the conductor ideal at every preimage p_i in the normalisation is at least two, and so is the order of pole of η . This means that, when compactifying the i -th branch to \mathbb{P}^1 via $s_i = 1/t_i$, the (meromorphic) differential $\eta_i = dt_i/t_i^{m_i} + h.o.t.$ (η restricted to the i -th branch C_i) extends to a holomorphic differential on the complement of p_i . If m_i is the order of pole at p_i , it will indeed have $m_i - 2$ zeroes on this branch. Thus $\eta \in \Gamma(C, \omega_C)$, and $\deg(\omega_C|_{C_i}) = m_i - 2$.

Going back to the smoothing family, consider the relative dualising bundle $\omega_{\mathcal{C}/\Delta}$. Its pullback $\Phi^*\omega_{\mathcal{C}/\Delta}$ to \mathcal{C}^{ss} :

- is trivial on the exceptional locus Z ;
- coincides with $\omega_{\mathcal{C}^{ss}/\Delta}$ outside Z , where Φ is an isomorphism.

Assuming regularity of \mathcal{C}^{ss} , then, we may write $\Phi^*\omega_{\mathcal{C}/\Delta} = \omega_{\mathcal{C}^{ss}/\Delta}(\tilde{Z})$, where \tilde{Z} is a (vertical) Cartier divisor supported on Z . On the other hand, the previous paragraph implies that the multiplicity of \tilde{Z} at the i -th component of C_0 is $m_i - 1$ (the 1 comes from Noether's formula: for a nodal curve X , and a closed subcurve $Y \subset X$, the restriction of ω_Y to X is $\omega_{X|Y} = \omega_Y(Y \cap \overline{X \setminus Y})$; more generally, the difference is an adjoint divisor, see [17, Proposition 1.2]). We conclude:

$$\omega_Z \left(\sum_{i=1}^b (2 - m_i) p_i \right) = \mathcal{O}_Z,$$

that is $\sum_{i=1}^b (m_i - 2) p_i$ is a canonical divisor on Z .³

The multiplicities $m_i - 2$ are determined by the singularity (C, p) via its conductor. This determines a stratum of holomorphic differentials. Based on the classification of genus three singularities, we conjecture that every topological type of Gorenstein singularity determines in fact a *connected component* of a stratum of holomorphic differentials. We wonder whether the converse is also true.

1.5 (Compact) strata of holomorphic differentials

The moduli space of holomorphic differentials \mathcal{H} on smooth projective curves is stratified by the multiplicities of the zeroes $\mu = (m_i) \vdash 2g - 2$. Every stratum consists of at most three connected components, depending on μ [31]:

- In case $\mu = (2g - 2)$ or $\mu = (g - 1, g - 1)$, there is a hyperelliptic stratum where the underlying curve is hyperelliptic, the differential is anti-invariant with respect to the hyperelliptic involution,⁴ and the zero(es) are located at a Weierstrass (resp. two conjugate) points.
- If every part m_i of μ is even, then $\Theta = \sum \frac{m_i}{2} p_i$ is an effective theta characteristic, and there are two connected components according to the parity of $h^0(\Theta)$ (the underlying curve is general).

Remark 1.1 In genus three, the presence of an effective even theta characteristic implies that the underlying curve is hyperelliptic. Hence there are always at most two connected component for every stratum when $g = 3$.

There are several ways of compactifying strata of holomorphic differentials. The underlying curve C is usually allowed to become nodal and reducible. If we consider differentials up to scaling, then:

- they are sections of the projectivisation of the Hodge bundle $\mathbb{E} = \pi_*(\omega_{\mathcal{C}/\mathcal{M}_g})$;
- they are determined by their vanishing locus, i.e. the associated divisor $\sum m_i p_i$.

Correspondingly, we can compactify $\mathcal{H}(\mu)$ by taking its closure in $\mathbb{P}(\mathbb{E})$ - notice that the Hodge bundle extends naturally to the Deligne–Mumford compactification -, resp. in $\overline{\mathcal{M}}_{g,b}$, where it has codimension $g - 1$. Unfortunately, taking closures is

³ If Z is reducible, these multiplicities determine further the shape of the dual graph: for instance, when $g = 1$, all the p_i must be at the same distance from the minimal genus one subcurve [42]. In general, these conditions can be phrased effectively in terms of logarithmic geometry, by requiring the existence of certain piecewise-affine functions on the dual graph, which turn out to be differentials in the sense of tropical geometry: see [6] and §1.5 below. It was pointed out to me by F. Viviani that this suggests the following statement: the variety of stable tails [26] of a Gorenstein singularity is the Deligne–Mumford compactification of a stratum of differentials.

⁴ A necessary condition for an anti-invariant differential with prescribed multiplicity μ to exist is that every odd part m_i appears an even number of times. Except for the two cases listed above, these hyperelliptic differentials arise as limits of differentials on smooth curves.

problematic from the point of view of moduli theory. The moduli space of multiscale differentials provides instead a modular compactification [11], admitting a log geometric interpretation [18, 19]; here is a naive description.

Consider a semistable one-parameter family $\mathcal{C} \rightarrow \Delta$ as above, and suppose the general fibre C_ξ is smooth, and η_ξ is a holomorphic differential on it. The central fibre C_0 may be nodal and reducible. The differential η_ξ can be extended to a differential η on the whole family, but the latter may vanish on a subcurve C'_0 of C_0 (equivalently, the associated divisor may have a non-trivial vertical part). Let $C_{0,0}$ be the subcurve on which η does not vanish, and let η_0 denote its restriction. Rescaling the differential by an appropriate (negative) power of the base parameter t , though, it is possible to extend η to a non-trivial differential on C'_0 ; this will have poles along $C_{0,0}$. Let η_1 denote the meromorphic restriction of η' to C'_0 , and $C_{0,-1} \subseteq C'_0$ the subcurve on which it is not trivial, and so on.

We end up with a collection $\{\eta_i\}$ of meromorphic differentials on subcurves $\{C_{0i}\}$ partitioning the central fibre. Moreover, the dual graph of the latter appears now to be ordered according to the order of vanishing of the differential η : we obtain a so-called level graph by putting C_{0i} at level i . This information can be enhanced to a compatible integral piecewise-affine function λ on the dual graph, the slopes recording the order of vanishing/poles of the differential η_i at the corresponding special point of the subcurve C_{0i} (as we have seen above, Noether's formula implies that the slope differs by the actual order by ± 1). In the logarithmic interpretation of multiscale differentials, λ is nothing but the tropicalisation of η . All of this is, roughly speaking, what is called a *generalised multiscale differential*.

Unfortunately, the space of generalised multiscale differentials is typically not a topological compactification. In fact, it is not irreducible, but the locus of smoothable differentials has been characterized in terms of a *global residue condition* [10]: a zero-sum condition on residues of η_i at poles belonging to possibly different irreducible components of the curve, which are however connected through components at higher levels. Here we reproduce a conjecture - originally due to D. Ranganathan and J. Wise, and spelled out in [7] - relating the smoothability of multiscale differentials and Gorenstein curves in purely algebraic (and characteristic-free) terms.

It turns out that the piecewise affine function λ above (or rather its negative) is indeed an element of the tropical canonical linear series. The same is true for every truncation of λ to the subcurve of C_0 lying at level i or below. Even for these purely combinatorial data, the moduli space is not in general irreducible (nor pure-dimensional); the locus of smoothable (*realisable*) tropical differentials has been described explicitly in [36]. We thus have the first part of:

Conjecture 1.2 (differential descent) *Let (C, η) be a generalised multiscale differential with tropicalisation λ . Then η is smoothable if and only if*

- (i) *for every level i , the truncation λ_i of λ is a realisable tropical differential;*
- (ii) *there exists a logarithmic modification $\tilde{C} \rightarrow C$, a natural extension $\tilde{\eta}$ of the pullback of η to \tilde{C} , and a reduced Gorenstein contraction $\sigma: \tilde{C} \rightarrow \bar{C}_i$ such that $\sigma^* \omega_{\bar{C}_i} = \omega_C(\lambda_i)$, and*
- (iii) *the differential $\tilde{\eta}_i$ at level i descends to a local generator of $\omega_{\bar{C}_i}$.*

We refer the reader to [7] for a more detailed discussion, particularly of the hyper-elliptic case.

1.6 Crimping spaces and moduli of singular curves

We think of a curve singularity as being obtained from the normalisation by pinching a finite subscheme [45, Lemma 1.2]. Algebraically, this presents the local ring as a fiber product, contained in the local ring of the normalisation. The *crimping space* (or moduli of attaching data) can be thought of as the moduli space of finite subschemes of C^v whose pinching produces a singularity with fixed analytic germ (or type τ), see for instance [43, §2.2]. Algebraically, it is the moduli space of subalgebras R of type τ in $\tilde{R} = \mathbb{C}[[t_1]] \oplus \dots \oplus \mathbb{C}[[t_b]]$, see [45, §1.3]. Since the conductor \mathfrak{c} is contained in R , and the numerics of the type determine a power k of $\tilde{\mathfrak{m}}$ that is contained in \mathfrak{c} , R is completely determined by its image in the Artinian quotient $\tilde{R}/\tilde{\mathfrak{m}}^k$. In this way we may view the crimping space as the quotient of $\text{Aut}_{\mathbb{C}}(\tilde{R}/\tilde{\mathfrak{m}}^k)$ by the stabiliser of $R/\tilde{\mathfrak{m}}^k$, and in particular, by a result of Rosenlicht, it is an extension of an affine space \mathbb{A}^h with a product of $\mathbb{A}^1 \setminus \{0\}$ [45, Theorem 1.74].

Here is some heuristic motivation of why crimping spaces are important. There are several examples of birational models of $\overline{\mathcal{M}}_{g,n}$ allowing for more singular curves in exchange for nodal subcurves with few special points, see the introduction for a sample. Crimping spaces and their compactifications appear as fibres in the flipping loci. From the above description we gather that they are rational, and that they can be described in terms of differential-geometric information on the normalisation; in particular, their classes are tautological. This is particularly relevant to the enumerative applications.

2 Algebraic classification

We organise the classification according to the finitely many possible gap sequences, see §1.3. If i is a gap of dimension a_i , we write $[a_i]_{i \geq 1}$ for the vanishing sequence, stopping when a_i becomes definitely 0. We look for generators of $\mathfrak{m}/\mathfrak{m}^2$. We write $A_i = \sum A_i(j, k)t_j^k$ for the linear generators (there exists a j such that $A(j, 1) \neq 0$), B_i for the quadratic ones, etc. If i is the last non-trivial gap, then $\tilde{\mathfrak{m}}^{i+1} \subseteq R$, so in particular $\tilde{\mathfrak{m}}^{i+1} \subseteq \mathfrak{c}$. This means that we only have to look for generators modulo $\tilde{\mathfrak{m}}^{i+1}$. We adopt two strategies simultaneously: studying the dimension of R/\mathfrak{c} as a \mathbb{C} -vector space, cf. Formula (2); and writing down the relations that a generator η of the dualising sheaf must satisfy, cf. Formula (1).

Gap sequence [3]

Cannot possibly be Gorenstein, since $\tilde{\mathfrak{m}}^2 \subseteq \mathfrak{c}$, so:

$$\dim_{\mathbb{C}}(R/\mathfrak{c}) \leq \dim_{\mathbb{C}}(R/\tilde{\mathfrak{m}}^2) = \dim_{\mathbb{C}}(K_0 + K_1) = 1 + (b - 3) < \delta.$$

Gap sequence [21]

Note that $\tilde{m}^3 \subseteq R$. The Gorenstein condition implies $b \geq 4$: indeed, if $b = 3$, R/c would be generated by 1, at most one linear generator A , and two quadratic ones B_1 and B_2 , hence $5 = \delta = \dim R/c \leq 4$, a contradiction (similarly for $b = 1, 2$).

Up to relabelling, assume that t_1^2 does not belong to R . Then we have quadratic elements

$$B_i = t_i^2 + B_i(1, 2)t_1^2$$

in R for $i = 2, \dots, b$. Then t_1 does not belong to R either; there is another linear monomial missing, which, up to relabelling, we may assume is t_2 . Then we have linear generators

$$A_i = t_i + A_i(1, 1)t_1 + A_i(1, 2)t_1^2 + A_i(2, 1)t_2$$

for $i = 3, \dots, b$ (we may reduce to this form by summing appropriate scalar combinations of the B_i). By comparing quadratic terms in the A_i with the B_j , and since we assumed $t_1^2 \notin R$, we find the following relations must hold:

$$A_i^2 = B_i + A_i(2, 1)^2 B_2 \Leftrightarrow A_i(1, 1)^2 = B_i(1, 2) + A_i(2, 1)^2 B_2(1, 2), \tag{3a}$$

$$A_i A_j = A_i(2, 1)A_j(2, 1)B_2 \Leftrightarrow A_i(1, 1)A_j(1, 1) = A_i(2, 1)A_j(2, 1)B_2(1, 2). \tag{3b}$$

Suppose that $B_2(1, 2) = 0$, or that $A_i(2, 1) = 0$ for all i . Then, by (3b), there exists at most one index (say $i = 3$) such that $A_i(1, 1) \neq 0$. Then, $B_j = A_j^2 = t_j^2$ for $j = 4, \dots, b$, hence $R/c \subseteq \langle 1, A_3, \dots, A_b, B_2, A_3^2, B_3 \rangle$, with the last three linearly dependent by (3a), so R/c has dimension $\leq b + 1 < \delta$, a contradiction. So we may assume $A_3(2, 1) \neq 0$ and $B_2(1, 2) \neq 0$. In particular, t_2^2 does not belong to R .

Now suppose that $A_j(1, 1) = 0$ for all $j = 4, \dots, b$. Then, by setting $i = 3$ in (3b), we see that $A_j(2, 1) = 0$ for all $j = 4, \dots, b$ as well, so $B_j = A_j^2 = t_j^2$ which gives a contradiction as above. So assume $A_4(1, 1) \neq 0$.

From (3b) we find

$$A_4(2, 1) = \frac{A_3(1, 1)}{B_2(1, 2)A_3(2, 1)} A_4(1, 1), \tag{and}$$

$$A_j(2, 1) = \frac{A_3(1, 1)}{B_2(1, 2)A_3(2, 1)} A_j(1, 1) = \frac{A_3(1, 1)A_4(2, 1)}{A_3(2, 1)A_4(1, 1)} A_j(2, 1)$$

$$= \frac{A_3(1, 1)^2}{A_3(2, 1)^2 B_2(1, 2)} A_j(2, 1), \tag{j = 5, \dots, b.}$$

Let

$$\Delta := \frac{A_3(1, 1)A_4(2, 1)}{A_3(2, 1)A_4(1, 1)} = \frac{A_3(1, 1)^2}{A_3(2, 1)^2 B_2(1, 2)}. \tag{4}$$

If $\Delta = 1$, in particular $A_3(1, 1) \neq 0$ then $A_j(2, 1) = \frac{A_3(2,1)}{A_3(1,1)}A_j(1, 1)$, and

$$A_j^2 - \frac{A_j(1, 1)^2}{B_2(1, 2)}B_2 = t_j^2$$

belongs to R , therefore to \mathfrak{c} , for all $j = 4, \dots, b$. This is a contradiction as above.

So $\Delta \neq 1$, which implies that $A_j(1, 1) = A_j(2, 1) = 0$ for all $j = 5, \dots, b$. Finally, $A_j(1, 2) \neq 0$ for $j = 5, \dots, b$, otherwise the singularity would be decomposable.

Summing up, modulo \tilde{m}^3 , the generators of $\mathfrak{m}/\mathfrak{m}^2$ can be written as:

$$\begin{array}{llllllll} A_3 = & A_3(1, 1)t_1 + A_3(1, 2)t_1^2 & A_3(2, 1)t_2 & t_3 & 0 & 0 & \dots & 0 \\ A_4 = & A_4(1, 1)t_1 + A_4(1, 2)t_1^2 & A_4(2, 1)t_2 & 0 & t_4 & 0 & \dots & 0 \\ A_5 = & & A_5(1, 2)t_1^2 & 0 & 0 & 0 & t_5 & \dots & 0 \\ \dots & & & & & & & & \\ A_b = & & A_b(1, 2)t_1^2 & 0 & 0 & 0 & 0 & \dots & t_b \\ B_2 = & & B_2(1, 2)t_1^2 & t_2^2 & 0 & 0 & 0 & \dots & 0 \end{array}$$

with $A_3(2, 1), A_4(1, 1), A_5(1, 2), \dots, A_b(1, 2), B_2(1, 2) \in \mathbb{G}_m$ and $A_3(1, 1), A_4(2, 1), A_3(1, 2), A_4(1, 2) \in \mathbb{G}_a$, such that

$$A_4(2, 1) = \frac{A_4(1, 1)}{A_3(2, 1)B_2(1, 2)}A_3(1, 1), \quad \text{and} \quad \Delta = \frac{A_3(1, 1)A_4(2, 1)}{A_3(2, 1)A_4(1, 1)} \neq 1.$$

R/\mathfrak{c} is spanned by $1, A_3, \dots, A_b, B_2, A_3^2, A_4^2$, with dimension $b + 2 = \delta$.

The dualising line bundle is generated by the meromorphic differential:

$$\begin{aligned} & \frac{dt_1}{t_1^3} - B_2(1, 2)\frac{dt_2}{t_2^3} + A_3(2, 1)^2B_2(1, 2)(1 - \Delta)\frac{dt_3}{t_3^3} + A_4(1, 1)^2(\Delta - 1)\frac{dt_4}{t_4^3} \\ & - \sum_{i=3}^m A_i(1, 2)\frac{dt_i}{t_i^2}. \end{aligned}$$

It follows that, if p_i denotes the attaching point of the branch with coordinate t_i to the semistable tail Z (cf. §1.4), then we can write:

$$\omega_Z = \mathcal{O}_Z(p_1 + p_2 + p_3 + p_4).$$

Thus, the pointed semistable tail $(Z; p_1, \dots, p_4)$ belongs to the generic (open) stratum of differentials $\mathcal{H}(1, 1, 1, 1)$.

Remark 2.1 In case $A_3(1, 1) \neq 0$, the element B_2 is a non-zero multiple of A_3A_4 . For $b = 4$, set $x = A_3, y = A_4$, and let a_1, a_2 and b_1, b_2 denote their linear coefficients.

The equation of this singularity is $xy(b_1x - a_1y)(b_2x - a_2y)$. The conditions that all the coefficients are non-zero and $\Delta \neq 1$ ensure that the four lines are distinct, so that the resulting singularity is reduced. The planar quadruple point is the singularity of smallest genus and number of branches that has analytic moduli, in the sense that the cross-ratio of the four lines is invariant under analytic isomorphisms. See [23, Example 3.43.2]. This family is a generalisation of the planar quadruple point to an arbitrary number of branches/embedding dimension.

Remark 2.2 Once the invariant (4), i.e. the analytic type of the singularity, is fixed, the remaining coefficients are parametrised by $\mathbb{G}_m^{b-1} \times \mathbb{G}_a^2$ (as a variety, not as a group). This is the crimping space (see §1.6). It can be described more geometrically in terms of pinching data as follows (set $b = 4$ for simplicity). Start from the rational 4-fold point. Firstly, collapse a generic 2-plane in the tangent space at the origin, i.e. one that does not intersect any of the coordinate 2-planes. We can represent it as a map:

$$\mathbb{C}[[t_1, \dots, t_4]]/(t_i t_j) \rightarrow \mathbb{C}[\epsilon_1, \epsilon_2]/(\epsilon_1, \epsilon_2)^2.$$

By genericity, and up to the GL_2 -action, the map is represented by the matrix:

$$\begin{pmatrix} 1 & 0 & -a_1 & -a_2 \\ 0 & 1 & -b_1 & -b_2 \end{pmatrix}$$

Algebraically, pinching corresponds to the fibre product, with generators:

$$\begin{array}{rccccc} x' = & a_1 t_1 & a_2 t_2 & t_3 & 0 \\ y' = & b_1 t_1 & b_2 t_2 & 0 & t_4 \\ z' = & t_1^2 & 0 & 0 & 0 \end{array}$$

Secondly, collapse a tangent line not contained in the (x', y') -plane; we can write the map $\mathbb{C}[[x', y', z']]/(\dots) \rightarrow \mathbb{C}[\epsilon']/(\epsilon')^2$ as:

$$x' \mapsto \alpha_1 \epsilon', \quad y' \mapsto \beta_1 \epsilon', \quad z' \mapsto \epsilon'.$$

Generators of the fibre product are given by $x = x' - \alpha_1 z'$, $y = y' - \beta_1 z'$ as above.

Remark 2.3 If $A_3(1, 1) = 0$ (so also $A_4(2, 1) = \Delta = 0$), the resulting singularity is hyperelliptic. Indeed, consider the involution ι of $\mathbb{C}[[t_1]] \oplus \dots \oplus \mathbb{C}[[t_b]]$ defined by:

$$\begin{cases} t_4 \leftrightarrow A_4(1, 1)t_1 + A_4(1, 2)t_1^2 \\ t_3 \leftrightarrow A_3(2, 1)t_2 - \frac{A_3(1, 2)}{B_2(1, 2)}t_2^2 \\ t_i \leftrightarrow -\frac{A_i(1, 2)}{A_{i+1}(1, 2)}t_{i+1} & i = 5, 7, \dots, b - 1 \end{cases}$$

if b is even; otherwise extend to b by setting $t_{b+1} = t_b$ and $A_{b+1} = A_b$ formally, so to send t_b to $-t_b$. This defines an automorphism of the normalisation since every term in the expression above is a parameter for the corresponding branch. Observe then that ι preserves the subalgebra R and makes the differential η anti-invariant.

We may also check that these singularities are hyperelliptic by matching the local generators with the expressions given in [7, §3]. For simplicity, set all crimping parameters to 1 or 0 as appropriate. Consider the change of coordinates:

$$\begin{cases} s_1 = A_4 & = t_1 + t_4, \\ s_2 = A_3 & = t_2 + t_3, \\ u_2 = 2B_2 - A_3^2 - A_4^2 & = t_1^2 + t_2^2 - t_3^2 - t_4^2, \\ s_i = A_{2i} - A_{2i+1} & = t_{2i} - t_{2i+1}, \\ u_i = A_{2i} + A_{2i+1} - A_4^2 & = t_1^2 - t_4^2 + t_{2i} + t_{2i+1}, \quad i = 3, 4, \dots, \lfloor \frac{b}{2} \rfloor \end{cases}$$

if b is even, and adjoin

$$\begin{cases} s_{b'} = A_b^2 - A_4^4 & = -t_4^4 + t_b^2, \\ u_{b'} = A_b & = t_1^2 + t_b, \end{cases}$$

in case b is odd (where $b' = \frac{b+1}{2}$). The following vanish: $u_2^2 - s_1^4 - s_2^4, u_i^2 - s_1^4 - s_i^2, s_1(u_i - u_j), s_i u_j, u_i u_j = s_1^4$, and $u_{b'}^2 - s_1^4 - s_{b'}$ in case b is odd. The corresponding tropical differential has slope 2 along the edges e_1, e_4 (conjugate), e_2, e_3 (conjugate), and slope 1 along all the other edges.

Note that for $b = 4$ the embedding dimension is 3 in this case.

Gap sequence [2 0 1]

In this case $\tilde{m}^4 \subseteq R$ and $b \geq 2$. Let us assume that t_1^3 does not belong to R , then neither does t_1 ; another linear monomial is missing, which we assume to be t_2 . We can then find generators:

$$\begin{aligned} C_i &= t_i^3 + C_i(1, 3)t_1^3, & i &= 2, \dots, b; \\ B_i &= t_i^2 + B_i(1, 3)t_1^3, & i &= 1, \dots, b; \\ A_i &= t_i + A_i(1, 1)t_1 + A_i(1, 3)t_1^3 + A_i(2, 1)t_2, & i &= 3, \dots, b. \end{aligned}$$

Observe that $A_i B_1 = A_i(1, 1)t_1^3$, hence $A_i(1, 1) = 0$ for all i . Then $A_i B_2 = A_i(2, 1)t_2^3$ must be a multiple of C_2 , so $C_2(1, 3) = 0$ or $A_i(2, 1) = 0$ for all i .

- If $C_2(1, 3) = 0$, then

$$A_i^3 = C_i + A_i(2, 1)^3 C_2 \Leftrightarrow C_i(1, 3) = 0.$$

So $C_i = t_i^3$ for all $i = 2, \dots, b$. Suppose that $A_i(2, 1) = 0$ for all $i = 3, \dots, b$. Then, $B_i - A_i^2 = B_i(1, 3)t_1^3 \pmod{\tilde{m}^4}$ implies $B_i(1, 3) = 0$ for all $i = 3, \dots, b$. Hence R/\mathfrak{c} would be generated by $1, A_3, \dots, A_b, B_1, B_2$ and of dimension $\leq b + 1$, which is too small. We may therefore assume $b \geq 3$ and $A_3(2, 1) \neq 0$. If there was another index j (say $j = 4$) such that $A_4(2, 1) \neq 0$, then A_3A_4 would be a non-zero multiple of t_2^2 . Subtracting a multiple of t_2^2 from A_i^2 we would find $t_i^2 \in R$ for all $i = 3, \dots, b$ as well. Hence $R/\mathfrak{c} = \langle 1, A_3, \dots, A_b, B_1 \rangle$, which is again too small. Summing up, the generators of $\mathfrak{m}/\mathfrak{m}^2$ up to \tilde{m}^4 are:

$$\begin{array}{rcccccccc}
 A_3 = & & A_3(1, 3)t_1^3 & & A_3(2, 1)t_2 & & t_3 & & 0 & & \dots & & 0 \\
 A_4 = & & A_4(1, 3)t_1^3 & & & & 0 & & 0 & & t_4 & & \dots & & 0 \\
 \dots & & & & & & & & & & & & & & \\
 A_b = & & A_b(1, 3)t_1^3 & & & & 0 & & 0 & & 0 & & \dots & & t_b \\
 B_1 = & & t_1^2 + B_1(1, 3)t_1^3 & & & & 0 & & 0 & & 0 & & \dots & & 0 \\
 B_2 = & & B_2(1, 3)t_1^3 & & & & t_2^2 & & 0 & & 0 & & \dots & & 0
 \end{array}$$

with $A_3(2, 1), B_2(1, 3)$ and $A_i(1, 3), i = 4, \dots, b$, invertible (by indecomposability) and $A_3(1, 3), B_1(1, 3)$ arbitrary. The crimping space is isomorphic to $\mathbb{G}_m^{b-1} \times \mathbb{G}_a^2$. We verify that $R/\mathfrak{c} = \langle 1, A_3, \dots, A_b, B_1, B_2, A_3^2 \rangle$. The dualising line bundle is generated by the meromorphic differential:

$$(1 - B_1(1, 3)t_1) \frac{dt_1}{t_1^4} - B_2(1, 3) \frac{dt_2}{t_2^2} + B_2(1, 3)A_3(2, 1)^2 \frac{dt_3}{t_3^3} - \sum_{i=3}^b A_i(1, 3) \frac{dt_i}{t_i^2}.$$

It follows that in the semistable tail:

$$\omega_Z = \mathcal{O}_Z(2p_1 + p_2 + p_3),$$

so this singularity corresponds to the stratum $\mathcal{H}(2, 1, 1)$.

Remark 2.4 In case $B_1(1, 3) = 0$, these singularities are hyperelliptic. Indeed, consider the involution ι of $\mathbb{C}[[t_1]] \oplus \dots \oplus \mathbb{C}[[t_b]]$ defined by:

$$\left\{ \begin{array}{l} t_1 \leftrightarrow -t_1 \\ t_3 \leftrightarrow A_3(2, 1)t_2 - \frac{A_3(1, 2)}{B_2(1, 3)}t_2^2 \\ t_i \leftrightarrow -\frac{A_i(1, 2)}{A_{i+1}(1, 2)}t_{i+1} \quad i = 5, 7, \dots, b - 1 \end{array} \right.$$

if b is even; otherwise extend to b by setting $t_{b+1} = t_b$ and $A_{b+1} = A_b$ formally, so to send t_b to $-t_b$. This defines an automorphism of the normalisation since every term in the expression above is a parameter for the corresponding branch. Observe

then that ι preserves the subalgebra R and makes the differential η anti-invariant. The corresponding tropical differential has slope 3 along the edge e_1 (Weierstrass), 2 along e_2, e_3 (conjugate), and slope 1 along all the other edges.

Remark 2.5 We sketch an alternative proof that this is the only family of singularities corresponding to the stratum $\mathcal{H}(2, 1, 1)$. We want $\omega_{C,p}$ to be generated by a differential of the form:

$$\eta = \frac{dt_1}{t_1^4}(1 + a_1t_1 + a_2t_1^2) + \frac{dt_2}{t_2^3}(u + b_1t_2) + \frac{dt_3}{t_3^3}(v + c_1t_3) + \sum_{i=4}^b u_i \frac{dt_i}{t_i^2},$$

with u, v, u_i invertible. We know then that

$$c = \langle t_1^4, t_2^3, t_3^3, t_4^2, \dots, t_b^2 \rangle.$$

Let S denote the local ring of the seminormalisation, $S = \mathbb{C}\llbracket t_1, \dots, t_b \rrbracket / (t_i t_j)$. The following diagram is Cartesian and displays the singularity as the result of pinching the seminormalisation along a finite subscheme:

$$\begin{array}{ccc} R & \hookrightarrow & S \\ \downarrow & \lrcorner & \downarrow \\ R/c & \hookrightarrow & S/c. \end{array}$$

Therefore, we have to look for subalgebras of

$$S/c = \mathbb{C}\llbracket t_1, \dots, t_b \rrbracket / (t_1^4, t_2^3, t_3^3, t_4^2, \dots, t_b^2, t_i t_j)$$

of codimension g . Moreover, they have to be isotropic with respect to the residue pairing $\langle f, g \rangle = \text{Res}(fg \eta)$. Suppose that we had a generator L_1 linear in t_1 . Then $L_1^3 = t_1^3$ would have residue 1 with η . For the same reason, t_2 and t_3 may only appear in combination $vt_2 - ut_3$. Finally, t_1^3 may not appear: this exhausts the 3-dimensional gap. Notice that the correct multiple of t_1^3 must be appended to every generator in order to ensure that the residue vanishes. We recover the same generators as above.

- Assume instead $C_2(1, 3) \neq 0$ and $A_i(2, 1) = 0$ for all i . In this case, the generators of $\mathfrak{m}/\mathfrak{m}^2$ up to $\tilde{\mathfrak{m}}^4$ are:

$$\begin{array}{cccccc} A_3 = & A_3(1, 3)t_1^3 & 0 & t_3 & \dots & 0 \\ \dots & & & & & \\ A_b = & A_b(1, 3)t_1^3 & 0 & 0 & \dots & t_b \\ B_1 = & t_1^2 + B_1(1, 3)t_1^3 & 0 & 0 & \dots & 0 \\ B_2 = & B_2(1, 3)t_1^3 & t_2^2 & 0 & \dots & 0 \\ C_2 = & C_2(1, 3)t_1^3 & t_2^3 & 0 & \dots & 0 \end{array}$$

with $C_2(1, 3)$ and $A_i(1, 3)$ invertible for $i = 3, \dots, b$, and $B_1(1, 3), B_2(1, 3)$ arbitrary. The crimping space is isomorphic to $\mathbb{G}_m^{b-1} \times \mathbb{G}_a^2$. We verify that $R/c = \langle 1, A_3, \dots, A_b, B_1, B_2, C_2 \rangle$.

Remark 2.6 For $b = 2$, the singularity consists of two cusps glued together non-transversely in \mathbb{A}^3 .

The dualising line bundle is generated by the meromorphic differential:

$$(1 - B_1(1, 3)t_1) \frac{dt_1}{t_1^4} - (C_2(1, 3) + B_2(1, 3)t_2) \frac{dt_2}{t_2^4} - \sum_{i=3}^b A_i(1, 3) \frac{dt_i}{t_i^2}.$$

It follows that in the semistable tail:

$$\omega_Z = \mathcal{O}_Z(2p_1 + 2p_2).$$

Lemma 2.7 $p_1 + p_2$ is an odd theta characteristic on Z .

We consider a rational compactification of (C, p) , i.e. a projective curve C of arithmetic genus three having p as its only singularity. We study theta characteristics on C as in [24]. We refer to §1.2.3 for the notation. Since the adjoint divisor is even, the linear form l vanishes identically. Let I' be the ideal of \mathcal{O}_C that pulls back to $\mathcal{O}(-p_1 - p_2)$ on the normalisation. In our case $I' = \mathfrak{m}$. The number of even and odd theta characteristics is determined by $\text{len}(\mathcal{O}_C/I') = 1$. In particular, there is exactly one odd theta characteristic θ (the sum of the two points at infinity on the cuspidal branches), and no even ones. Consider a smoothing of (C, θ) . By semistable reduction, we can substitute the central fibre with a nodal curve C^{ss} ; we may assume that there exists a contraction $C^{ss} \rightarrow C$. By collapsing the rational tails of C^{ss} , we see $p_1 + p_2$ as the limit of θ on the genus three semistable tail Z . Summing up, this family corresponds to $\mathcal{H}(2, 2)^{\text{odd}}$.

Gap sequence [1 1 1]

In this case too $\tilde{\mathfrak{m}}^4 \subseteq R$ and $b \geq 2$. Let us say that t_1^3 is missing; then so is t_1 . If the missing quadratic term is t_1^2 , then we can write our linear generators as $A_i = t_i + \sum_{j=1}^3 A_i(1, j)t_1^j$ for $i = 2, \dots, b$. Suppose instead that we do not have t_2^2 . We have though a linear generator of the form $L_2 = t_2 + L_2(1, 1)t_1 + h.o.t.$ with $L_2(1, 1) \neq 0$ (otherwise $L_2^2 = t_2^2 + h.o.t.$); but then we can add a multiple of L_2^2 to any generator of R in order to change their t_2^2 term into a t_1^2 term. Hence we can indeed write the linear generators A_i as above.

If $b \geq 3$, by pairing A_i with A_j for $i \neq j$ we see that there is at most one i such that $A_i(1, 1) \neq 0$. Indeed, there must be exactly one, otherwise $B_i = t_i^2$ and $C_i = t_i^3$, so $R/c = \langle 1, A_2, \dots, A_b \rangle$ would be too small.

Ultimately, the generators of m/m^2 are:

$$\begin{array}{rccccccc}
 A_2 = & A_2(1, 1)t_1 + A_2(1, 2)t_1^2 + A_2(1, 3)t_1^3 & t_2 & 0 & \dots & 0 \\
 A_3 = & & A_3(1, 3)t_1^3 & 0 & t_3 & \dots & 0 \\
 \dots & & & & & & \\
 A_b = & & A_b(1, 3)t_1^3 & 0 & 0 & \dots & t_b
 \end{array}$$

with $A_2(1, 1)$ and $A_i(1, 3)$ invertible for $i = 3, \dots, b$, and $A_2(1, 2), A_2(1, 3)$ arbitrary. The crimping space is isomorphic to $\mathbb{G}_m^{b-1} \times \mathbb{G}_a^2$. We verify that $R/\mathfrak{c} = \langle 1, A_2, \dots, A_b, A_2^2, A_2^3 \rangle$.

Remark 2.8 For $b = 2$, we need one more generator, for instance $D = t_1^4 - t_2^4 \in \tilde{m}^4$. This is the A_7 -singularity $\mathbb{C}[[A, D]]/(D^2 - A^8)$. It is hyperelliptic.

More generally, these singularities are hyperelliptic in the sense of [7]. In order to see it, set all crimping parameters to either 1 or 0 for simplicity. The corresponding tropical differential has slope 3 on the branches corresponding to t_1 and t_2 , and 1 on all other branches: if we have an even number of branches $b = 2b'$, then we can assume that all branches come in conjugate pairs; otherwise, we assume that the branch corresponding to t_{2b+1} is Weierstrass. In the former case we can choose new coordinates:

$$\begin{cases}
 s_1 = A_2 & = t_1 + t_2, \\
 s_i = A_{2i-1} - A_{2i} & = t_{2i-1} - t_{2i}, \\
 u_i = A_{2i-1} + A_{2i} - A_2^3 & = t_1^3 - t_2^3 + t_{2i-1} + t_{2i}, \quad i = 2, \dots, b',
 \end{cases}$$

satisfying $u_i^2 - s_1^6 - s_i^2, s_1(u_i - u_j), s_i u_j$, and $u_i u_j = s_1^6$, see [7, §3]. In the latter case, we adjoin

$$\begin{cases}
 s_{b'+1} = A_b^2 & = t_1^6 + t_b^2, \\
 u_{b'+1} = A_b - A_2^3 & = -t_2^3 + t_b,
 \end{cases}$$

satisfying $u_{b'+1}^2 + s_1^6 - s_{b'+1}$ and all the other equations as above.

The dualising line bundle is generated by the meromorphic differential η :

$$\begin{aligned}
 & \left(1 - 2 \frac{A_2(1, 2)}{A_2(1, 1)} t_1 - \left(\frac{A_2(1, 3)}{A_2(1, 1)} - 2 \frac{A_2(1, 2)^2}{A_2(1, 1)^2} \right) t_1^2 \right) \frac{dt_1}{t_1^4} \\
 & - A_2(1, 1)^3 \frac{dt_2}{t_2^4} - \sum_{i=3}^b A_i(1, 3) \frac{dt_i}{t_i^2}.
 \end{aligned}$$

Table 2 Left: rank function of an even theta. Right: odd theta

$H^0(\omega_Z(-))$	0	$-p_1$	$-2p_1$	$H^0(\omega_Z(-))$	0	$-p_1$	$-2p_1$
0	3	2	1	0	3	2	1
$-p_2$	2	2	1	$-p_2$	2	1	1
$-2p_2$	1	1	1	$-2p_2$	1	1	1

It follows that in the semistable tail the attaching points of the two osculating branches on the core Z are such that:

$$\omega_Z = \mathcal{O}_Z(2p_1 + 2p_2).$$

Lemma 2.9 $p_1 + p_2$ is an even theta characteristic on Z .

Following Harris, we see once again that $l \equiv 0$, but in this case $R/I' = \mathbb{C}\langle 1, x \rangle$, so there is exactly one even theta characteristic θ , and no odd ones on C . We conclude by semistable reduction of (C, θ) as above.

We offer an alternative argument. Let C be the genus three compactification as above. Reasoning as in §1.4, we observe that $\{\eta, A_2\eta, A_2^2\eta\}$ extends to a basis of $H^0(C, \omega_C)$. Applying semistable reduction to $A_2\eta$ (resp. $A_2^2\eta$), we find a differential on the semistable tail Z with multiplicity $(1, 1)$ (resp. $(0, 0)$) at (p_1, p_2) . So, p_1 and p_2 do not impose independent conditions on $H^0(\omega_Z)$. This is the case when $p_1 + p_2$ is an even theta characteristic (two conjugate points on a hyperelliptic curve), but not when it is odd (bitangent of a quartic, or, specialising, sum of two distinct Weierstrass points on a hyperelliptic curve). See Table 2 for the corresponding rank functions.

Summing up, these singularities correspond to $\mathcal{H}(2, 2)^{\text{even}}$.

Gap sequence [1 1 0 1]

In this case $\tilde{m}^5 \subseteq R$. The monomial unibranch singularity $\mathbb{C}\llbracket t^3, t^5, t^7 \rrbracket$ is not Gorenstein: for instance, the associated monoid is not symmetric. So assume $b \geq 2$. The arguments are similar to the previous case: if we miss the monomial t_1^4 , then we also miss t_1 and t_1^2 . We have linear generators $A_i = t_i + \sum_{j=1,2,4} A_i(1, j)t_1^j$, $i = 2, \dots, b$, and a cubic generator $C_1 = t_1^3 + C_1(1, 4)t_1^4$. In order for $A_i C_1$ not to start with t_1^4 , we need $A_i(1, 1) = 0$ for all i . For the same reason, there is at most one index i such that $A_i(1, 2) \neq 0$. In fact there is exactly one, otherwise $R/c = \langle 1, A_2, \dots, A_b, C_1 \rangle$, which is too small. So we have generators:

$$\begin{array}{rccccccc}
 A_2 = & A_2(1, 2)t_1^2 + A_2(1, 4)t_1^4 & t_2 & 0 & \dots & 0 \\
 A_3 = & & A_3(1, 4)t_1^4 & 0 & t_3 & \dots & 0 \\
 \dots & & & & & & \\
 A_b = & & A_b(1, 4)t_1^4 & 0 & 0 & \dots & t_b
 \end{array}$$

$$C_1 = \quad \quad \quad t_1^3 + C_1(1, 4)t_1^4 \quad \quad 0 \quad \quad 0 \quad \quad \dots \quad \quad 0$$

with $A_2(1, 2)$ and $A_i(1, 4)$, $i = 3, \dots, b$, invertible, and $A_2(1, 4), C_1(1, 4)$ arbitrary. The crimping space is isomorphic to $\mathbb{G}_m^{b-1} \times \mathbb{G}_a^2$. We verify that $R/c = \langle 1, A_2, \dots, A_b, A_2^2, C_1 \rangle$.

Remark 2.10 For $b = 2$, setting $x = A_2, y = C_1$, we find local equations $y(y^2 - x^3)$. This is the E_7 -singularity.

The dualising line bundle is generated by the meromorphic differential:

$$\left(1 - C_1(1, 4)t_1 - \frac{A_2(1, 4)}{A_2(1, 2)}t_1^2\right) \frac{dt_1}{t_1^5} - A_2(1, 2)^2 \frac{dt_2}{t_2^3} - \sum_{i=3}^b A_i(1, 4) \frac{dt_i}{t_i^2}.$$

We can thus write:

$$\omega_Z = \mathcal{O}_Z(3p_1 + p_2),$$

so this singularity corresponds to the stratum $\mathcal{H}(3, 1)$.

Gap sequence [1 1 0 0 1]

In this case $\tilde{m}^6 \subseteq R$. We encounter the monomial unibranch singularity $\mathbb{C}[[t^3, t^4]]$, classically known as E_6 . In general, we have linear generators $A_i = t_i + \sum_{j=1,2,5} A_i(1, j)t_1^j$, $i = 2, \dots, b$, but also a cubic $C_1 = t_1^3 + C_1(1, 5)t_1^5$ and a quartic $D_1 = t_1^4 + D_1(1, 5)t_1^5$. Multiplying A_i with D_1 and C_1 , we see that both $A_i(1, 1)$ and $A_i(1, 2)$ must vanish. So:

$$\begin{array}{rcccccc} A_2 = & A_2(1, 5)t_1^5 & t_2 & 0 & \dots & 0 \\ \dots & & & & & \\ A_b = & A_b(1, 5)t_1^5 & 0 & 0 & \dots & t_b \\ C_1 = & t_1^3 + C_1(1, 5)t_1^5 & 0 & 0 & \dots & 0 \\ D_1 = & t_1^4 + D_1(1, 5)t_1^5 & 0 & 0 & \dots & 0 \end{array}$$

with $A_i(1, 5) \in \mathbb{G}_m$, $i = 2, \dots, b$, and $C_1(1, 5), D_1(1, 5) \in \mathbb{G}_a$. We check: $R/c = \langle 1, A_2, \dots, A_b, C_1, C_2 \rangle$.

The dualising line bundle is generated by the meromorphic differential:

$$\left(1 - D_1(1, 5)t_1 - C_1(1, 5)t_1^2\right) \frac{dt_1}{t_1^6} - \sum_{i=2}^b A_i(1, 5) \frac{dt_i}{t_i^2}.$$

It follows that we can write:

$$\omega_Z = \mathcal{O}_Z(4p_1).$$

Lemma 2.11 *2p is an odd theta characteristic.*

Similar to Lemma 2.7. So this singularity corresponds to $\mathcal{H}(4)^{\text{odd}}$.

Gap sequence [1 0 1 0 1]

In this case $\tilde{m}^6 \subseteq R$. We have a monomial unibranch singularity $\mathbb{C}[[t^2, t^7]]$, classically known as A_6 , which is hyperelliptic. For $b \geq 2$, there are linear generators $A_i = t_i + \sum_{j=1,3,5} A_i(1, j)t_1^j$, $i = 2, \dots, b$, and a quadratic one $B_1 = t_1^2 + B_1(1, 3)t_1^3 + B_1(1, 5)t_1^5$. Multiplying A_i with B_1^2 and with B_1 , we see that both $A_i(1, 1)$ and $A_i(1, 3)$ must vanish. So:

$$\begin{array}{rcccccc} A_2 = & & A_2(1, 5)t_1^5 & t_2 & 0 & \dots & 0 \\ & \dots & & & & & \\ A_b = & & A_b(1, 5)t_1^5 & 0 & 0 & \dots & t_b \\ B_1 = & t_1^2 + B_1(1, 3)t_1^3 + B_1(1, 5)t_1^5 & & 0 & 0 & \dots & 0 \end{array}$$

with $A_i(1, 5) \in \mathbb{G}_m$, $i = 2, \dots, b$, and $B_1(1, 3), B_1(1, 5) \in \mathbb{G}_a$. We check: $R/c = \langle 1, A_2, \dots, A_b, B_1, B_1^2 \rangle$.

The dualising line bundle is generated by the meromorphic differential:

$$\left(1 - 2B_1(1, 3)t_1 - B_1(1, 5)t_1^3\right) \frac{dt_1}{t_1^6} - \sum_{i=2}^b A_i(1, 5) \frac{dt_i}{t_i^2}.$$

It follows that we can write:

$$\omega_Z = \mathcal{O}_Z(4p).$$

Lemma 2.12 *2p is an even theta characteristic.*

Similar to Lemma 2.9. So this singularity corresponds to $\mathcal{H}(4)^{\text{even}}$.

Remark 2.13 These singularities are hyperelliptic in the sense of [7]. For simplicity, set all crimping parameters equal to 1 or 0, and assume that $b = 2b' + 1$. The associated tropical differential has slope 5 along the edge corresponding to p_1 , which is

Weierstrass. With coordinates:

$$\begin{cases} s_1 = B_1 & = t_1^2, \\ s_{i+1} = \frac{1}{2}(A_{2i+1} - A_{2i}) & = \frac{1}{2}(t_{2i+1} - t_{2i}), \\ u_{i+1} = \frac{1}{2}(A_{2i+1} + A_{2i}) & = t_1^5 + \frac{1}{2}(t_{2i+1} + t_{2i}), \quad i = 1, \dots, b', \end{cases}$$

we find the local equations $u_i^2 - s_1^5 - s_i^2$, $s_1(u_i - u_j)$, $s_i u_j$, and $u_i u_j = s_1^6$, see [7, §3]. The case b even can be addressed by adding another Weierstrass branch as in Remark 2.8.

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Declarations

Conflict of interest The author has no conflicts of interest to declare that are relevant to the content of this article.

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