Monthly Notices of the ROYAL ASTRONOMICAL SOCIETY

MNRAS **525,** 2758–2764 (2023) https://doi.org/10.1093/mnras/stad2511
Advance Access publication 2023 August 21

The face-on projection of the Miyamoto & Nagai discs

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Accepted 2023 August 16. Received 2023 August 8; in original form 2023 July 13

ABSTRACT

The face-on projected density profile of the Miyamoto & Nagai discs of arbitrary flattening is obtained analytically in terms of incomplete elliptic integrals of first and second type, by using two complementary approaches, and then checked against the results of numerical integration. As computer algebra systems do not seem able to obtain the resulting formula in any straightforward way, the relevant mathematical steps are provided. During this study, three wrong identities in the Byrd & Friedman tables of elliptic integrals have been identified, and their correct expression is given.

Key words: methods: analytical – galaxies: elliptical and lenticular, cD – galaxies: kinematics and dynamics.

1 INTRODUCTION

The Miyamoto & Nagai discs (1975, hereafter MN; see also Nagai & Miyamoto 1976; Binney & Tremaine 2008; Ciotti 2021) albeit not very realistic, are among the most used disc models in numerical/theoretical works for their analytical simplicity, allowing to express in closed form several properties of interest (see e.g. Ciotti & Pellegrini 1996, hereafter CP96; Smet, Posacki & Ciotti 2015): in particular the MN edge-on projected density distribution can be written as a surprisingly simple algebraic formula (see e.g. Satoh 1980, and equation A3 in CP96). By contrast, the MN face-on projected density is not found in the literature, and the best known computer algebra systems seem unable to obtain in any straightforward way the closed-form solution of the projection integral, which is also missing from the most used tables of integrals such as Gradshteyn & Ryzik (2015, hereafter GR15), Prudnikov, Brychkov & Marichev (1990, hereafter P90), and Byrd & Friedman (1971, hereafter BF71).

Of course, for *all* practical purposes a numerical evaluation of the projected density is perfectly suitable (and strongly recommended), however, an inspection of the face-on projection integral reveals immediately its elliptic nature and so, motivated by curiosity, and by the desire to add the face-on formula to the already available edge-on formula, the explicit expression is found and reported here. For the reasons mentioned above, and in order to minimize the possibility of errors, all the integrations have been performed by paper and pencil, intermediate results double checked with available identities in the tables, and finally verified by comparison with results of numerical integrations. During the painstaking process, three wrong integrals have been detected in BF71 (which at the best of our knowledge are not reported in the published Errata), one of them unfortunately also used in other identities of the book.

The paper is organized as follows. In Section 2, the projection integral is reduced to a manageable form with the aid of the Poisson equation, and the final expression for the face-on projected density profile is given: readers just interested in the formula can stop here. In Sections 3 and 4, the formula is proved in two different ways. As the involved algebra is quite heavy (the fair copy summing up to more than 130 handwritten pages), only the main steps of the process are reported, however, all the important information and technical details are given in the Appendix, together with the derivation of the formulae correcting the three wrong identities discovered in the tables of BF71.

2 PROJECTION

The MN potential reads

$$\phi(R,z) = -\frac{GM}{\sqrt{R^2+(a+\zeta)^2}}, \quad \zeta = \sqrt{b^2+z^2}, \tag{1}$$

where a and b are two scale lengths, and M is the total mass of the system, so that the disc-like density distribution associated with equation (1) via the Poisson equation is

$$\rho(R,z) = \frac{\Delta\phi}{4\pi G} = \frac{Mb^2}{4\pi}\frac{aR^2+(a+3\zeta)(a+\zeta)^2}{\zeta^3\left[R^2+(a+\zeta)^2\right]^{5/2}}. \tag{2}$$

We recall that for $b = 0$ the MN model reduces to the razor-thin Kuzmin–Toomre disc (Kuzmin 1956; Toomre 1963), while for $a = 0$ it reduces to the Plummer (1911) sphere. The associated face-on projected density profile, the focus of the paper, is given by

$$\Sigma(R) = 2\int_0^\infty \rho\, dz = \frac{1}{2\pi G}\int_0^\infty \Delta\phi\, dz, \tag{3}$$

where the reflection property $\rho(R,-z) = \rho(R,z)$ has been taken into account. Due to the nested irrationalities appearing in the expression of the density ρ, the first integral in equation above looks quite formidable, and in fact symbolic integration is not performed by the most used computer algebra systems. Fortunately, the use of the Poisson equation in the projection integral allows for considerable preparatory simplification. In fact, the part of the density associated with the term $\partial^2\phi/\partial z^2$ does not contribute to the projection, a first simplification of the problem. Notice that this property is shared by

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all density distributions with vanishing force field for $|z| \to \infty$, and *regular*[1] at $z = 0$.

After normalization to physical scales M and b, we rewrite equation (3) as

$$\Sigma(R) = \frac{M}{2\pi b^2} I(R), \tag{4}$$

where from now on all the lengths are intended to be normalized to the scale b, the potential to GM/b, and the density to $M/(4\pi b^3)$. The dimensionless potential is

$$\phi = -\frac{1}{\sqrt{R^2 + (s+\zeta)^2}}, \quad \zeta = \sqrt{1+z^2}, \quad s = \frac{a}{b}, \tag{5}$$

and the dimensionless Poisson equation reads $\rho = \Delta\phi$. The evaluation of the resulting dimensionless integral

$$I(R) = \int_0^\infty \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \phi}{\partial R}\right) dz, \tag{6}$$

is our task, and in the following sections we give two alternative derivations of the main result of the paper, namely

$$I(R) = \left(U_0 + R^2 V_0\right) + \left(U_1 + R^2 V_1\right) \mathrm{F}(\varphi_1, k) + \left(U_2 + R^2 V_2\right) \mathrm{E}(\varphi_1, k). \tag{7}$$

In the expression above, F and E are the incomplete elliptic integrals of first and second kind[2] as given in equation (A1), and

$$\varphi_1 = \arccos\frac{\Delta_-}{\Delta_+}, \quad k^2 = \frac{\Delta_+^2 - 4}{\Delta_+^2 - \Delta_-^2}, \tag{8}$$

the quantities $\Delta_+(R, s)$ and $\Delta_-(R, s)$ are defined in equation (C1), and finally the functions $U_i(\Delta_+, \Delta_-)$ and $V_i(\Delta_+, \Delta_-)$ are given in equations (C7)–(C10).

Before discussing the proof of equation (7), we notice that two preliminary checks can be obtained by comparison with already known results, namely (1) $I(R)$ in the spherical limit $s = 0$, and (2) the central value $I(0)$ for arbitrary flattening s. In these two cases, the integral in equation (3) becomes elementary and no elliptic integrals are involved, as discussed in Appendices B and C. In the first case, $\Delta_+ = 2\sqrt{R^2+1}$ and $\Delta_- = 0$, while only $U_0 = 8/\Delta_+^2$ and $V_0 = -32/\Delta_+^4$ survive in equations (C7)–(C10), and the projected density of the Plummer (1911) sphere is recovered as $I(R) = U_0 + R^2 V_0$ (e.g. see equation 13.134 in Ciotti 2021). In the second case, the verification that for $R = 0$ the function $I(R)$ in equation (7) reduces to equations (A1) and (A2) in CP96, i.e. that

$$I(0) = \frac{2 + s^2 - 3s\,F(s)}{(1-s^2)^2}, \tag{9}$$

$$F(s) = \begin{cases} \dfrac{\arccos(s)}{\sqrt{1-s^2}}, & 0 \le s < 1, \\ 1, & s = 1, \\ \dfrac{\mathrm{arccosh}(s)}{\sqrt{s^2-1}}, & s > 1, \end{cases} \tag{10}$$

is more complicated, as the the elliptic integral decomposition in equation (7) cannot be directly evaluated at $R = 0$, due to the divergence of the functions U_i and V_i at the origin. However, as described in Appendix C, it can be shown that the limit for $R \to 0$ of $I(R)$ in equation (7) recovers equations (9) and (10).

[1]The importance of regularity for this simplification, in addition to the vanishing of the force field at infinity, is best illustrated by the case of the face-on projection of razor-thin discs with finite total mass.

[2]In Mathematica, the argument k in elliptic integrals enters as $\mathrm{F}(\varphi, k) =$ EllipticF$[\varphi, k^2]$, and so on.

3 METHOD I

From inspection of equation (6) a first natural idea to reduce the difficulty of integration is to exchange the radial derivatives with integration over z. Unfortunately the exchange is impossible, as the projection of the potential of any system of *finite* total mass diverges logarithmically, a consequence of the monopole behaviour of ϕ at large distances from the origin: the idea is not to be fully discarded though, and we will return on it in the next section. Here, we prove equation (7) by brute-force integration of equation (6), after evaluation of the radial part of the Laplacian under integral. From simple algebra

$$I(R) = 2I_1(R, \infty) - 3R^2 I_2(R, \infty), \tag{11}$$

where $I_1(R, \infty)$ and $I_2(R, \infty)$ are the (finite) limits for $z \to \infty$ of

$$I_n(R, z) = \int_0^z \frac{dz}{[R^2 + (s+\zeta)^2]^{n+1/2}} = \int_1^{\zeta=\sqrt{1+z^2}} \frac{[R^2 + (s+t)^2]^{-n}\, t\, dt}{\sqrt{(t^2-1)[R^2+(s+t)^2]}}, \tag{12}$$

with $n = 1, 2$, respectively, and the last expression is obtained with the substitution $\zeta = \sqrt{1+z^2}$. Notice that from equation (5)

$$I_0(R, z) = -\int_0^z \phi\, dz, \tag{13}$$

with the expected logarithmic divergence for $z \to \infty$.

Now, even if the Legendre reduction theorem (see Appendix A, and references therein) guarantees that for integer n the functions I_n can be expressed in general in terms of elliptic integrals (with the exceptions of the elementary cases corresponding to $s = 0$, and to $R = 0$), it should be recalled in practical applications the explicit factorization of the cubic/quartic polynomial under the square root often leads to different sub-cases (depending on the nature and relative positions of the zeroes), and to cumbersome expressions. Quite remarkably, in the MN case the quartic under the square root in equation (12) is already nicely factorized in two quadratics, one with two real zeroes ± 1, and the other with complex conjugate roots $-s \pm \mathrm{i}\,R$. This factorization is considered in BF71 (equation 260.00, and following), but, unfortunately, the two integrals I_1 and I_2 are not reported, nor they are found in other standard references such as GR15 and P90. Moreover, also the latest releases of the most used computer algebra systems seem unable to evaluate the definite symbolic integrals in equation (12), while in case of indefinite integration they produce unmanageable expressions involving functions of complex arguments. For all these reasons here we followed the approach: (1) we perform the whole integration by paper and pencil, (2) whenever possible we compare intermediate results with available identities in the tables, and with numerical integrations performed with Mathematica NIntegrate and Maple evalf(Int) arbitrary precision commands, and finally (3) we double check equation (7) against the projected density profile obtained by numerical integration of the first projection integral in equation (3) for different values of the disc flattening s.

We found that the most efficient approach to this plan is not to solve directly the integrals I_n, but to obtain the closed form expression of the more general functions $H_n(y)$ missing from BF71, GR15, and P90, and then specialize them to the present problem, as described in Appendix B. In practice, equation (7) is proved by first obtaining $I_1(R, \infty)$ and $I_2(R, \infty)$, respectively from $H_1(y)$ and $H_2(y)$ in equations (B8) and (B9) with $a = 1, b = -1, a_1 = R, b_1 = -s, y = \zeta \to \infty$, and then combining them in equation (11), so that the explicit expressions for the radial functions $U_i(R)$ and $V_i(R)$ functions in equations (C1)–

(C10) are finally established. In the process, we detected three wrong identities in BF71, apparently missing from the available Errata of the book (Fettis 1972, 1981). The correct formulae and their proofs are reported in Appendix B.

4 METHOD II

As anticipated in Section 3, the idea behind this second approach is to minimize the difficulties of integration of equation (6), by exchanging the operations of integration and radial derivatives. Unfortunately, $I_0(R, z)$ in equation (13) diverges logarithmically for $z \to \infty$, and so $I(R)$ cannot be obtained as the radial Laplacian of the (non-existent) $I_0(R, \infty)$: however, the idea is not to be dismissed. In fact, as suggested by equation (12), we first obtain the two putative functions

$$I_1(R, z) = -\frac{1}{R}\frac{\partial I_0(R, z)}{\partial R}, \tag{14}$$

$$I_2(R, z) = -\frac{1}{3R}\frac{\partial I_1(R, z)}{\partial R}, \tag{15}$$

where the two derivatives are taken at arbitrary but finite z, and then we consider their limit for $z \to \infty$, as required by equation (11). In practice, we evaluate the radial Laplacian of $I_0(R, z)$, and then we consider the limit for $z \to \infty$ of the resulting expression. That this approach is in fact legitimate can be shown by comparison of the functions $I_1(R, z)$ and $I_2(R, z)$ obtained from equations (14) and (15) with the functions H_1 and H_2 in equations (B8) and (B9), specialized to the MN case ($a = 1$, $b = -1$, $a_1 = R$, $b_1 = -s$, $y = \zeta = \sqrt{1 + z^2}$). From equation (14), we expect that the radial derivative of $I_0(R, z)$ cancels its logarithmic divergence for $z \to \infty$, being $I_1(R, \infty)$ in equation (12) convergent; notice also that, at variance with equation (14), the verification of equation (15) can be performed directly on $I_2(R, \infty)$, by evaluating the derivative of $I_1(R, \infty)$, since this latter function is convergent.

The evaluation of the derivative of $I_0(R, z)$ is not a trivial task, as one could have (naively) hoped. The starting point is $H_0(y)$ in equation (B7), specialized to the MN case, i.e.

$$I_0(R, z) = \frac{\Delta_-/\Delta_+}{\sqrt{AB}} \times [\Pi(\varphi_1, n, k) - \mathrm{F}(\varphi_1, k)] + \frac{1}{2}\ln\frac{\sqrt{AB}\,\mathrm{dn}(u_1) + \mathrm{sn}(u_1)}{\sqrt{AB}\,\mathrm{dn}(u_1) - \mathrm{sn}(u_1)}, \tag{16}$$

where all the quantities appearing in equation above are given in equations (C1) and (C2), and the Jacobian functions are computed from equation (A2). A careful analysis reveals that the logarithmic divergence of $I_0(R, z)$ for $z \to \infty$ (i.e. for $\zeta \to \infty$) is due *both* to the elliptic integral Π, and to the logarithmic term, so that some quite not trivial, exact cancellation is to be expected after taking the radial derivative of equation (16): that the cancellation is not trivial can be realized by inspection of the complicated expressions of the derivatives of elliptic integrals and Jacobian functions reported in equations (A3)–(A7), and in particular from the fact that the derivatives of the Π function with respect to the parameters are still expressed in terms of the (divergent) Π itself. A simplification of the (very) heavy algebra involved in the verification of equations (14) and (15) is obtained by noticing that from equations (C1) and (C2) the radial dependence of $I_0(R, z)$ and of $I_1(R, z)$ only occurs through the quantities $\Delta_+ = A(R) + B(R)$ and $\Delta_- = A(R) - B(R)$. In turn, from equation (C1) $\partial A/\partial R = R/A$ and $\partial B/\partial R = R/B$, so that for a generic function $f(\Delta_+, \Delta_-)$

$$\frac{\partial f}{\partial R} = \frac{4R}{\Delta_+^2 - \Delta_-^2}\left(\Delta_+\frac{\partial f}{\partial \Delta_+} - \Delta_-\frac{\partial f}{\partial \Delta_-}\right). \tag{17}$$

With this approach the equivalence for arbitrary z of $I_1(R, z)$ in equation (14) with the specialization of $H_1(y)$ to the MN case as required by equation (12), can be finally established. A somewhat simpler verification of equation (14) in the limit of large $|z|$ can be also carried out by considering the expansion of I_0 for $z \to \infty$ up to the order $\mathcal{O}(1/\zeta^2)$ with the aid of equation (C4), by evaluation of the radial derivative of the resulting expression, and finally by considering the limit for $\zeta \to \infty$. As noticed above, $I_2(R, \infty)$ can be obtained from equation (15) evaluated directly on $I_1(R, \infty)$ by using again the differentiation rule in equation (17), obtaining an expression in perfect agreement with $H_2(\infty)$ specialized to the MN case.

Overall, although the method described in this section is more elegant than the brute-force approach of Section 3, it actually requires an unexpected amount of work.

5 CONCLUSIONS

We have shown that the face-on projected density of the MN disc of arbitrary flattening can be expressed in closed form in terms of incomplete elliptic integrals of first and second kind, and the explicit formula is provided. The proof (based on two different but strictly related approaches) proceeds by first reducing the difficulty of integration thanks to the Poisson equation, and then by application of the Legendre reduction theorem of elliptic integrals. The resulting integrals are not evaluated symbolically by the best known computer algebra systems in any straightforward way, nor are they given in integral tables such as GR15, P90, and BF71. For these reasons, all the integrations have been performed by paper and pencil, intermediate results checked against available identities in the tables (that have been independently rederived as a sanity check), and the final formula verified by comparison with numerical integration of the original projection integral. During this study three wrong identities in BF71 have been identified and corrected, and are given together their proof in the Appendix.

We conclude by noticing two consequences of the present study. The first is that not only the face-on projected density of MN discs can be obtained in closed form, but also their face-on projected (self-gravitating) velocity dispersion, given by

$$\Sigma(R)\sigma_{\mathrm{fo}}^2(R) = 2\int_0^\infty \rho\sigma_z^2 dz, \tag{18}$$

where the integrand is given in equation (16) of CP96. An inspection of the integrand shows that the quantity $\Sigma(R)\sigma_{\mathrm{fo}}^2(R)$ contains *elementary* functions only. However, after division by the face-on $\Sigma(R)$, we conclude that $\sigma_{\mathrm{fo}}^2(R)$ of the self-gravitating MN discs also involves incomplete elliptic integrals. Of course, as for $\Sigma(R)$, also for $\sigma_{\mathrm{fo}}^2(R)$, a numerical integration perfectly suffices for all practical purposes, and for this reason we do not discuss this problem any further. The second and final comment concerns the Satoh (1980) discs (e.g. see BT08, Ciotti 2021), a family of models strictly related to the MN discs, whose potential can be written as

$$\phi(R, z) = -\frac{GM}{\sqrt{R^2 - b^2 + (a + \zeta)^2}}, \quad \zeta = \sqrt{b^2 + z^2}. \tag{19}$$

Also for these models the edge-on projected density can be written in terms of elementary functions, while from the formula above an analysis similar to that of the MN discs shows that their face-on projected density contains incomplete elliptic integrals, in a formula analogous to equation (7).

ACKNOWLEDGEMENTS

The Referee, James Binney, and Alberto Parmeggiani of the Department of Mathematics of Bologna University are thanked for useful comments and suggestions.

DATA AVAILABILITY

No data sets were generated or analysed in support of this research.

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APPENDIX A: ELLIPTIC INTEGRALS AND ELLIPTIC FUNCTIONS

The literature on elliptic integrals and elliptic functions is immense, and here we just report the results strictly needed in this paper. The elliptic integrals of first, second and third kind in trigonometric form are given respectively by

$$\mathrm{F}(\varphi, k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \qquad \mathrm{E}(\varphi, k) = \int_0^{\varphi} \sqrt{1-k^2\sin^2\theta}\,d\theta, \qquad \Pi(\varphi, n, k) = \int_0^{\varphi} \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}, \tag{A1}$$

where φ is the *argument*, k the *modulus*, $k' = \sqrt{1-k^2}$ is the *complementary modulus*, and n the *parameter*: in this study $0 \le k \le 1$. A reduction theorem due to Legendre states that every integral of $Q[x, \sqrt{P(x)}]$, where $Q(x, y)$ is a generic rational function of two variables, and $P(x)$ is a polynomial of degree not higher than 4, can always be reduced to an integral of a rational function of x (and so in principle integrable by elementary methods), and to a linear combination of F, E, and Π (see e.g. BF71; Hancock 1958, Chapter 8). The standard change of variable $\sin\theta = t$ transforms the elliptic integrals in equation (A1) in the elliptic integrals in algebraic (or Jacobi) form. For assigned u, by inverting $\mathrm{F}(\varphi, k) = u$ with respect to φ, the *elliptic amplitude* $\mathrm{am}(u, k)$ is obtained, from which the Jacobian elliptic functions remain defined as follows:

$$u = \mathrm{F}[\mathrm{am}(u,k), k], \qquad \varphi = \mathrm{am}(u,k), \qquad \mathrm{sn}(u) = \sin\varphi, \qquad \mathrm{cn}(u) = \cos\varphi, \qquad \mathrm{dn}(u) = \sqrt{1-k^2\mathrm{sn}^2(u)}. \tag{A2}$$

In particular, $\mathrm{sn}(u)$ is the inverse function of the elliptic integral of first kind in algebraic form, and as usual in the Jacobian elliptic functions we do not indicate explicitly the dependence on k. From the inverse function theorem, one immediately proves that

$$\frac{\partial\,\mathrm{am}(u,k)}{\partial u} = \mathrm{dn}(u), \qquad \frac{\partial\,\mathrm{sn}(u)}{\partial u} = \mathrm{cn}(u)\mathrm{dn}(u), \qquad \frac{\partial\,\mathrm{cn}(u)}{\partial u} = -\mathrm{sn}(u)\mathrm{dn}(u), \qquad \frac{\partial\,\mathrm{dn}(u)}{\partial u} = -k^2\mathrm{sn}(u)\mathrm{cn}(u), \tag{A3}$$

while with some more work it can be proved that

$$\frac{\partial\,\mathrm{am}(u,k)}{\partial k} = \frac{\mathrm{dn}(u)}{kk'^2}\left[k'^2\mathrm{F}(\varphi,k) - \mathrm{E}(\varphi,k) + k^2\frac{\mathrm{sn}(u)\mathrm{cn}(u)}{\mathrm{dn}(u)}\right], \tag{A4}$$

and the partial derivatives of the Jacobian functions with respect to the modulus k are obtained from equations (A2) and (A4), in agreement with BF71 (710.50)–(710.53). The derivatives of $\mathrm{F}(\varphi, k)$, $\mathrm{E}(\varphi, k)$, and $\Pi(\varphi, n, k)$ with respect to the argument φ are elementary. For the other non-trivial identities we have

$$\frac{\partial\mathrm{F}(\varphi,k)}{\partial k} = \frac{\mathrm{E}(\varphi,k)}{kk'^2} - \frac{\mathrm{F}(\varphi,k)}{k} - \frac{k\sin\varphi\cos\varphi}{k'^2\sqrt{1-k^2\sin^2\varphi}}, \qquad \frac{\partial\mathrm{E}(\varphi,k)}{\partial k} = \frac{\mathrm{E}(\varphi,k)-\mathrm{F}(\varphi,k)}{k}, \tag{A5}$$

(e.g. BF71 710.07 and 710.09, or GR15 8.123.1 and 8.123.3),

$$\frac{\partial\Pi(\varphi,n,k)}{\partial k} = \frac{k\,\mathrm{E}(\varphi,k)}{k'^2(k^2-n)} - \frac{k\,\Pi(\varphi,n,k)}{k^2-n} - \frac{k^3\sin\varphi\cos\varphi}{k'^2(k^2-n)\sqrt{1-k^2\sin^2\varphi}}, \tag{A6}$$

(BF71 710.12), and

$$\frac{\partial\Pi(\varphi,n,k)}{\partial n} = \frac{\mathrm{F}(\varphi,k)}{2n(n-1)} + \frac{\mathrm{E}(\varphi,k)}{2(n-1)(k^2-n)} - \frac{\left(k^2-n^2\right)\,\Pi(\varphi,n,k)}{2n(n-1)(k^2-n)} - \frac{n\sin\varphi\cos\varphi\sqrt{1-k^2\sin^2\varphi}}{2(n-1)(k^2-n)(1-n\sin^2\varphi)}, \tag{A7}$$

(BF71 733.00).

APPENDIX B: THREE RELEVANT ELLIPTIC INTEGRALS

The integrals I_n in Sections 3 and 4 are special cases of the general identity 260.07 in BF71

$$H(y) = \int_a^y \frac{\mathcal{R}(t)\,dt}{\sqrt{(t-a)(t-b)[(t-b_1)^2+a_1^2]}} = \frac{1}{\sqrt{AB}}\int_0^{u_1} \mathcal{R}\left[\frac{m\,\mathrm{cn}(u)+p}{\Delta_+\mathrm{cn}(u)-\Delta_-}\right]du, \qquad a > b, \tag{B1}$$

where $\mathcal{R}(t)$ is a rational function, and

$$A^2 = (a-b_1)^2 + a_1^2, \quad B^2 = (b-b_1)^2 + a_1^2, \quad \Delta_+ = A + B \geq \Delta_- = A - B, \quad m = aB + bA, \quad p = aB - bA, \tag{B2}$$

$$k^2 = \frac{\Delta_+^2 - (a-b)^2}{4AB}, \quad k'^2 = 1 - k^2 = \frac{(a-b)^2 - \Delta_-^2}{4AB}, \quad \mathrm{am}(u_1, k) = \arccos\frac{\Delta_- y + p}{\Delta_+ y - m} = \varphi_1, \quad u_1 = F(\varphi_1, k); \tag{B3}$$

finally $4AB = \Delta_+^2 - \Delta_-^2$, and all the Jacobian elliptic functions of u_1 associated with $H(y)$ can be obtained from equations (A2)–(B3). In the special case $a_1 = 0$ the quartic under the square root in equation (B1) reduces to a quadratic, so that $H(y)$ is expressible in terms of elementary functions. $H(y)$ is also elementary when $b_1 = 0$, $a = -b$, and $\mathcal{R}(t)$ is an *odd* rational function,[3] so that equation (B1) can be integrated with the natural substitution $x = t^2$. Both these cases are relevant for the discussion at the end of Section 2 (see also the last paragraph in Appendix C).

The identity in equation (B1)-BF71 (260.07) can be established from the change of variable between t and u defined by the function inside the square parenthesis in the last integral in equation (B1): after some careful algebra

$$dt = \frac{2AB(a-b)\mathrm{sn}(u)\mathrm{dn}(u)}{[\Delta_+\mathrm{cn}(u)-\Delta_-]^2}du, \qquad (t-a)(t-b) = \frac{AB(a-b)^2\mathrm{sn}^2(u)}{[\Delta_+\mathrm{cn}(u)-\Delta_-]^2}, \qquad (t-b_1)^2 + a_1^2 = \frac{(2AB)^2\mathrm{dn}^2(u)}{[\Delta_+\mathrm{cn}(u)-\Delta_-]^2}, \tag{B4}$$

while the new extremes of integration derive immediately from the two last identities in equation (B3).

As apparent from the second integral in equation (12), in our problem $\mathcal{R}(t)$ in equation (B1) is

$$\mathcal{R}(t) = \frac{t}{[(t-b_1)^2 + a_1^2]^n}, \qquad n = 0, 1, 2, \tag{B5}$$

so that from the last expression in equation (B4) we must consider the integrals

$$H_n(y) = \frac{1}{4^n(AB)^{2n+1/2}}\int_0^{u_1} \frac{[m\,\mathrm{cn}(u)+p][\Delta_+\mathrm{cn}(u)-\Delta_-]^{2n-1}}{\mathrm{dn}^{2n}(u)}du, \tag{B6}$$

and then specialize them to $a = 1$, $b = -1$, $a_1 = R$, $b_1 = -s$, $y = \zeta = \sqrt{1+z^2}$.

As H_1 and H_2 are not explicitly given in BF71, GR15, and P90, and H_0 given in BF71 (361.54) is affected by a serious error, here we derive the expressions of these three functions by direct integration of equation (B6) with $n = 0, 1, 2$, adopting the same nomenclature of BF71 for ease of comparison. We have

$$H_0(y) = \frac{1}{\sqrt{AB}} \times \left[\frac{m}{\Delta_+}\mathrm{F}(\varphi_1, k) + \frac{\Delta_-(\Delta_+ p + \Delta_- m)}{\Delta_+(\Delta_+^2 - \Delta_-^2)}\Pi\left(\varphi_1, \frac{\Delta_+^2}{\Delta_+^2 - \Delta_-^2}, k\right)\right]$$
$$+ \frac{\Delta_+ p + \Delta_- m}{(\Delta_+^2 - \Delta_-^2)(a-b)}\ln\frac{\sqrt{AB}\,\mathrm{dn}(u_1) + (a-b)\mathrm{sn}(u_1)/2}{\sqrt{AB}\,\mathrm{dn}(u_1) - (a-b)\mathrm{sn}(u_1)/2}, \tag{B7}$$

$$H_1(y) = \frac{1}{4(AB)^{5/2}} \times \left[\frac{\Delta_+ m}{k^2}\mathrm{F}(\varphi_1, k) - \left(\frac{\Delta_+ m}{k^2} + \frac{\Delta_- p}{k'^2}\right)\mathrm{E}(\varphi_1, k) + (\Delta_+ p - \Delta_- m)\frac{\mathrm{sn}(u_1)}{\mathrm{dn}(u_1)}\right.$$
$$\left. + k^2\left(\frac{\Delta_+ m}{k^2} + \frac{\Delta_- p}{k'^2}\right)\frac{\mathrm{sn}(u_1)\mathrm{cn}(u_1)}{\mathrm{dn}(u_1)}\right], \tag{B8}$$

$$H_2(y) = \frac{1}{48(AB)^{9/2}} \times \left\{\left[\frac{2+k^2}{k^4}\Delta_+^3 m + \frac{\Delta_-^3 p}{k'^2} - 3\frac{\Delta_+\Delta_-(\Delta_+ p - \Delta_- m)}{k^2}\right]\mathrm{F}(\varphi_1, k)\right.$$
$$-2\left[\frac{1+k^2}{k^4}\Delta_+^3 m + \frac{2-k^2}{k'^4}\Delta_-^3 p - \frac{3(1-2k^2)}{2k^2k'^2}\Delta_+\Delta_-(\Delta_+ p - \Delta_- m)\right]\mathrm{E}(\varphi_1, k)$$
$$+3\left[\Delta_+^2(\Delta_+ p - 3\Delta_- m) + \Delta_-^2(3\Delta_+ p - \Delta_- m)\right]\frac{\mathrm{sn}(u_1)}{\mathrm{dn}(u_1)}$$
$$+\left[k^2\Delta_-^2(3\Delta_+ p - \Delta_- m) - k'^2\Delta_+^2(\Delta_+ p - 3\Delta_- m)\right]\frac{\mathrm{sn}^3(u_1)}{\mathrm{dn}^3(u_1)}$$
$$+2\left[\frac{k^2(2-k^2)}{k'^4}\Delta_-^3 p + \frac{1+k^2}{k^2}\Delta_-^3 m - \frac{3(1-2k^2)}{2k'^2}\Delta_+\Delta_-(\Delta_+ p - \Delta_- m)\right]\frac{\mathrm{sn}(u_1)\mathrm{cn}(u_1)}{\mathrm{dn}(u_1)}$$
$$\left. + \left[\frac{k^2}{k'^2}\Delta_-^3 p - \frac{k'^2}{k^2}\Delta_+^3 m - 3\Delta_+\Delta_-(\Delta_+ p - \Delta_- m)\right]\frac{\mathrm{sn}(u_1)\mathrm{cn}(u_1)}{\mathrm{dn}^3(u_1)}\right\}. \tag{B9}$$

Proof: the expression of H_0 can be obtained from BF71 (260.03) with $m = 1$, followed by (341.02-03), and finally from (361.54). In the case of MN discs, when $a = 1 > b = -1$, it is easy to show that the formula of interest is the third expression in BF71 (361.54): *unfortunately, the*

[3] A generic rational function $\mathcal{R}(t)$ can be always written as the sum of an even and an odd rational function, with $\mathcal{R}(t) = \mathcal{R}_1(t^2) + t\,\mathcal{R}_2(t^2)$.

given expression is wrong, as can be seen by numerical integration, or by differentiation with respect to the argument. The error was first spotted in the process of recovering H_0 from integration of equation (B6) with $n = 0$, as briefly illustrated. From a partial fraction decomposition of the integrand in equation (B6) with $n = 0$, we arrive at

$$H_0(y) = \frac{1}{\sqrt{AB}} \times \left[\frac{m\,u_1}{\Delta_+} - \frac{\Delta_+ p + \Delta_- m}{\Delta_+\Delta_-}\int_0^{u_1}\frac{du}{1+\alpha\,\mathrm{cn}(u)}\right], \qquad \alpha = -\frac{\Delta_+}{\Delta_-} \le -1, \tag{B10}$$

where the last integral above is BF71 (361.54), and whose correct expression is obtained in Appendix B1.2. Notice that from equation (B14) we have $n = \Delta_+^2/(\Delta_+^2 - \Delta_-^2)$, so that $n{-}k^2 = (a{-}b)^2/(4AB) > 0$, and the third case of equation (B15) applies. Simple algebra then proves equation (B7).

The functions H_1 and H_2, after the expansion of the integrand in equation (B6), reduce respectively to a linear combination of the integrals

$$H_{1n} = \int_0^{u_1}\frac{\mathrm{cn}^n(u)}{\mathrm{dn}^2(u)}du, \qquad H_{2n} = \int_0^{u_1}\frac{\mathrm{cn}^n(u)}{\mathrm{dn}^4(u)}du. \tag{B11}$$

About the components of the H_1 function, H_{10} is obtained from BF71 (315.02), H_{11} from BF71 (352.51) corrected for a typo as given in equation (B13) evaluated for $m = 1$, or from GR15 (5.137.4). Finally, H_{12} can be obtained from the last of BF71 (355.01) for $n = 0$, $m = 1$, $p = 2$, or from BF71 (320.02).

About the components of the H_2 function, H_{20} is obtained from BF71 (315.04), and H_{21} again from equation (B13) evaluated for $m = 2$. H_{22} can be obtained from the last of BF71 (355.01) for $n = 0$, $m = 2$, $p = 2$, and the result is a linear combination of H_{10} and H_{20}. Analogously, H_{23} can be obtained from the last of BF71 (355.01) with $n = 0$, $m = 2$, $p = 3$, and the result is a linear combination of H_{11} and H_{21}. Finally, H_{24} can be obtained from the last of BF71 (355.01) for $n = 0$, $m = 2$, $p = 4$, or from BF71 (320.04).

B1 Three wrong integrals in BF71

In the process of verification/calculation of the integrals needed in this work, two typos and a seriously wrong integral were discovered in the magnificent book BF71. A search in the available Errata (Fettis 1972, 1981) indicates that these corrections are almost surely unknown/unpublished, and so we report them here.

B1.1 The integrals BF71 (352.01) and (352.51)

Two typos appear in the indefinite integrals BF71 (352.01) and (352.51). For BF71 (352.01) the correct expression reads

$$\int \frac{\mathrm{sn}(u)}{\mathrm{dn}^{2m}(u)}du = \frac{1}{k'^{2m}}\sum_{j=0}^{m-1}\frac{(-1)^{j+1}k^{2j}}{2j+1}\binom{m-1}{j}\left[\frac{\mathrm{cn}(u)}{\mathrm{dn}(u)}\right]^{2j+1}. \tag{B12}$$

Proof: replace $\mathrm{sn}(u)$ at numerator with the aid of the third identity in equation (A3), write the resulting $\mathrm{dn}(u)^{2m+1}$ at the denominator in terms of $\mathrm{cn}(u)$, and reduce the integral to algebraic form by first setting $t = k\,\mathrm{cn}(u)$, and then $z = t/\sqrt{k'^2 + t^2}$, corresponding to the substitution $z = k\,\mathrm{cn}(u)/\mathrm{dn}(u)$ in the original integral. For integer $m = 1, 2, \ldots$ a binomial expansion proves equation (B12).

For BF71 (352.51) the correct expression reads

$$\int \frac{\mathrm{cn}(u)}{\mathrm{dn}^{2m}(u)}du = \sum_{j=0}^{m-1}\frac{k^{2j}}{2j+1}\binom{m-1}{j}\left[\frac{\mathrm{sn}(u)}{\mathrm{dn}(u)}\right]^{2j+1}. \tag{B13}$$

Proof: replace $\mathrm{cn}(u)$ at numerator by using the second identity in equation (A3), reduce the integral to algebraic form by first setting $t = k\,\mathrm{sn}(u)$, and then $z = t/\sqrt{1+t^2}$, corresponding to the substitution $z = k\,\mathrm{sn}(u)/\mathrm{dn}(u)$ in the original integral. For integer $m = 1, 2, \ldots$, a binomial expansion finally proves equation (B13).

B1.2 The integral BF71 (361.54)

The integral is involved in the evaluation of the function H_0 in equation (B7). The identity

$$\int_0^u \frac{dv}{1+\alpha\,\mathrm{cn}(v)} = \frac{\Pi(\varphi, n, k) - \alpha f_1(u)}{1-\alpha^2}, \qquad \alpha^2 \ne 1, \qquad n = \frac{\alpha^2}{\alpha^2 - 1}, \qquad \varphi = \mathrm{am}(u), \tag{B14}$$

reported both in BF71 (41.03) and (361.54) is correct. Unfortunately, the third case of f_1 reported in (361.54) is wrong, and this error propagates in other identities in BF71 that are expressed in terms of f_1. The correct expression reads

$$f_1(u) = \int_0^u \frac{\mathrm{cn}(v)\,dv}{1 - n\,\mathrm{sn}^2(v)} = \int_0^{\frac{\mathrm{sn}(u)}{\mathrm{dn}(u)}}\frac{dz}{1+(k^2-n)z^2} = \begin{cases} \dfrac{1}{\sqrt{k^2-n}}\arctan\dfrac{\sqrt{k^2-n}\,\mathrm{sn}(u)}{\mathrm{dn}(u)}, & n < k^2,\\ \dfrac{\mathrm{sn}(u)}{\mathrm{dn}(u)}, & n = k^2,\\ \dfrac{1}{2\sqrt{n-k^2}}\ln\dfrac{\mathrm{dn}(u)+\sqrt{n-k^2}\,\mathrm{sn}(u)}{\mathrm{dn}(u)-\sqrt{n-k^2}\,\mathrm{sn}(u)}, & n > k^2,\end{cases}$$

where from equation (B14) $|n - k^2| = (k^2 + k'^2\alpha^2)/|\alpha^2 - 1|$. *Proof*: multiply the numerator and denominator of the integrand in equation (B14) by $1 - \alpha\,\mathrm{cn}(v)$, and use the identity $\mathrm{cn}^2(v) = 1 - \mathrm{sn}^2(v)$; for $\alpha^2 \neq 1$ factor out the quantity $1 - \alpha^2$ at the denominator, and write the resulting integral as the sum of two integrals. The first, from the change of variable $\vartheta = \mathrm{am}(v)$ and the use of the first of equation (A3), or from BF71 (110.04), is immediately recognized as the elliptic integral of third kind, while the second is the f_1 function, thus proving identity in equation (B14). We now focus on the f_1 function. The $\mathrm{cn}(v)$ function in the integrand of equation (B15) is expressed by using the second of equation (A3), followed by the substitution $t = \mathrm{sn}(v)$, and finally by $z = t/\sqrt{1 - k^2 t^2}$, corresponding to the single change of variable $z = \mathrm{sn}(v)/\mathrm{dn}(v)$. This leads to the second integral in equation (B15), and a last elementary integration proves the three cases of f_1.

APPENDIX C: EXPLICIT FORMULAE FOR THE FACE-ON PROJECTION OF THE MN DISC

The functions I_n in Section 3, pertinent to the MN disc face-on projection, are obtained from the functions $H_n(y)$ in equation (B6) for $a = 1$, $b = -1$, $a_1 = R$, $b_1 = -2$, $y = \zeta = \sqrt{1 + z^2}$, when equations (B2) and (B3) reduce to

$$A(R) = \sqrt{R^2 + (1+s)^2}, \quad B(R) = \sqrt{R^2 + (1-s)^2}, \quad \Delta_+ = A + B, \quad \Delta_- = A - B, \quad m = -\Delta_-, \quad p = \Delta_+, \tag{C1}$$

$$k^2 = \frac{\Delta_+^2 - 4}{4AB}, \quad k'^2 = 1 - k^2 = \frac{4 - \Delta_-^2}{4AB}, \quad n = \frac{\Delta_+^2}{4AB}, \quad n - k^2 = \frac{1}{AB}, \quad \mathrm{am}(u_1, k) = \arccos\frac{\Delta_-\zeta + \Delta_+}{\Delta_+\zeta + \Delta_-} = \varphi_1, \tag{C2}$$

and where of course $4AB = \Delta_+^2 - \Delta_-^2$. Moreover,

$$\mathrm{cn}(u_1) = \frac{\Delta_-\zeta + \Delta_+}{\Delta_+\zeta + \Delta_-}, \quad \mathrm{sn}(u_1) = \frac{\sqrt{(\Delta_+^2 - \Delta_-^2)(\zeta^2 - 1)}}{\Delta_+\zeta + \Delta_-}, \quad \mathrm{dn}(u_1) = \sqrt{1 - \frac{(\Delta_+^2 - 4)(\zeta^2 - 1)}{(\Delta_+\zeta + \Delta_-)^2}}, \tag{C3}$$

where the asymptotic behaviour for $\zeta \to \infty$ at the order $\mathcal{O}(1/\zeta^2)$ can be easily obtained:

$$\mathrm{cn}(u_1) \sim \frac{\Delta_-}{\Delta_+} + \frac{\Delta_+^2 - \Delta_-^2}{\Delta_+^2\zeta}, \quad \mathrm{sn}(u_1) \sim \frac{\sqrt{\Delta_+^2 - \Delta_-^2}}{\Delta_+}\left(1 - \frac{\Delta_-}{\Delta_+\zeta}\right), \quad \mathrm{dn}(u_1) \sim \frac{2}{\Delta_+} + \frac{(\Delta_+^2 - 4)\Delta_-}{2\Delta_+^2\zeta}. \tag{C4}$$

Finally, the following identities used in Section 4 can be established with simple algebra by using equation (17)

$$\frac{\partial\varphi_1}{\partial R} = 8R\frac{\Delta_+\Delta_-}{(\Delta_+^2 - \Delta_-^2)^2}\mathrm{sn}(u_1), \qquad \frac{\partial n}{\partial R} = -16R\frac{\Delta_+^2\Delta_-^2}{(\Delta_+^2 - \Delta_-^2)^3}, \qquad \frac{\partial k}{\partial R} = 4R\frac{k'^2\Delta_+^2 - k^2\Delta_-^2}{k(\Delta_+^2 - \Delta_-^2)^2}. \tag{C5}$$

From the general formulae (B8) and (B9) we have

$$I_1(R, \infty) = H_1(\infty) = \frac{U_0 + U_1\mathrm{F}(\varphi_1, k) + U_2\mathrm{E}(\varphi_1, k)}{2}, \qquad I_2(R, \infty) = H_2(\infty) = -\frac{V_0 + V_1\mathrm{F}(\varphi_1, k) + V_2\mathrm{E}(\varphi_1, k)}{3}, \tag{C6}$$

to be used in equations (11) and (7), and after some careful algebra we finally obtain

$$U_0 = \frac{32}{(\Delta_+^2 - \Delta_-^2)(4 - \Delta_-^2)}, \quad U_1 = -\frac{16\Delta_+\Delta_-}{(\Delta_+^2 - \Delta_-^2)^{3/2}(\Delta_+^2 - 4)}, \quad U_2 = -\frac{16\Delta_+\Delta_-(\Delta_+^2 + \Delta_-^2 - 8)}{(\Delta_+^2 - \Delta_-^2)^{3/2}(\Delta_+^2 - 4)(4 - \Delta_-^2)}. \tag{C7}$$

$$V_0 = 64\frac{\Delta_-^6 + \Delta_-^4(7\Delta_+^2 - 36) - 4\Delta_-^2(5\Delta_+^2 - 24) - 8\Delta_+^2(\Delta_+^2 - 4)}{(\Delta_+^2 - \Delta_-^2)^3(\Delta_+^2 - 4)(4 - \Delta_-^2)^2}, \tag{C8}$$

$$V_1 = -32\Delta_+\Delta_-\frac{\Delta_-^4(\Delta_+^2 - 12) + \Delta_-^2(7\Delta_+^4 - 28\Delta_+^2 + 64) - 8\Delta_+^2(3\Delta_+^2 - 8)}{(\Delta_+^2 - \Delta_-^2)^{7/2}(\Delta_+^2 - 4)^2(4 - \Delta_-^2)}, \tag{C9}$$

$$V_2 = -32\Delta_+\Delta_-\frac{\Delta_-^6(\Delta_+^2 - 12) + 2\Delta_-^4(7\Delta_+^4 - 42\Delta_+^2 + 104) + \Delta_-^2(\Delta_+^6 - 84\Delta_+^4 + 352\Delta_+^2 - 512) - 4\Delta_+^2(3\Delta_+^4 - 52\Delta_+^2 + 128)}{(\Delta_+^2 - \Delta_-^2)^{7/2}(\Delta_+^2 - 4)^2(4 - \Delta_-^2)^2}. \tag{C10}$$

Concerning the discussion at the end of Section 2 about the central value of the face-on MN projected density, it is important to note that equations (C7)–(C10) cannot be evaluated at $R = 0$ by direct substitution, because $\Delta_+(0) = 2$ for $0 \le s \le 1$, and $\Delta_-(0) = 2$ for $s \ge 1$, and so the associated denominators in the formulae above vanish at the origin. However, it is possible to show that the limit for $R \to 0$ of the three functions $U_i + R^2V_i$ in equation (7) exists $\forall s \ge 0$. In particular, from equation (C2) it follows that at the centre $k \to 0$ for $0 \le s < 1$, and $k \to 1$ for $s > 1$. In the same intervals of s, and for $\zeta \to \infty$ and $R \to 0$, we have $\varphi_1 \to \arccos(s)$ and $\varphi_1 \to \arccos(1/s)$, respectively. Therefore, for $R \to 0$ the elliptic integrals in equation (7) can be expressed from BF71 (111.01) and (111.04) in terms of elementary functions, in accordance with the first special case $a_1 = 0$ discussed after equation (B3): the final expression is in perfect agreement with equations (9) and (10), and the case $s = 1$ is then obtained as a limit.

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