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ORIGINAL PAPER

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Energy dissipation in viscoelastic Bessel media

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Abstract We investigate the specific attenuation factor for the Bessel models of viscoelasticity. We find that the quality factor for this class can be expressed in terms of Kelvin functions and that its asymptotic behaviours confirm the analytical results found in previous studies for the rheological properties of these models.

1 Introduction

Viscoelasticity represents one of the most compelling and vibrant research topics in continuum mechanics, both from the perspectives of applied sciences and mathematics. For comprehensive reviews of the history of this field and its modern developments, we refer the interested readers to, *e.g.*, [1–4]. In this context, it is of particular interest the role played by non-local operators, with particular regard for fractional ones [3–5]. More precisely, fractional derivatives are, loosely speaking, mathematical objects belonging to a subclass of *weakly singular Volterra-type convolution integro-differential operators*, for further details see, *e.g.*, [6–8].

In this work, we present a study of the processes of storage and dissipation of energy for a specific class of models of linear viscoelasticity, known as *Bessel models* [9]. To this end, we shall analytically compute the so-called *quality factor*, *i.e.* Q-factor, [4,10] starting from the Laplace representation of the creep compliance for a viscoelastic medium governed by the Bessel constitutive laws [9]. To carry out our analysis we will mostly follow [11, Sect. 2], that consists of a coherent summary of the arguments in [4,10].

The work is therefore organised as follows. In Sect. 2, we review the *creep representation* for the Bessel models of linear viscoelasticity and their generalities. In Sect. 3, we derive explicit expressions for the Q-factor for these models in terms of special functions of particular interest. Section 4 presents some numerical results and illuminating plots for the computed Q-factor. Lastly, in Sect. 5 we summarise the main results of the study and provide some concluding remarks.

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2 Bessel models in linear viscoelasticity

In linear viscoelasticity, the constitutive relation for a uniaxial homogeneous and isotropic viscoelastic body in the *creep representation* reads [4]

$$\varepsilon(t) = J_{\mathrm{g}}\,\sigma(t) + \int_0^t \dot{J}(t-t')\,\sigma(t')\,\mathrm{d}t', \tag{2.1}$$

where $\varepsilon(t)$ and $\sigma(t)$ are, respectively, the (uniaxial) stress and strain functions, $J(t)$ is the material function known as *creep compliance*, $J_{\mathrm{g}} := J(0^+) \geq 0$ is the glass compliance, and the dot denotes a derivative with respect to time. Note that $J(t)$ is a *causal function* and hence it vanishes for $t < 0$. Additionally, we define the *rate of creep (compliance)* for a viscoelastic system as

$$\Psi(t) := \frac{\dot{J}(t)}{J_{\mathrm{g}}}, \tag{2.2}$$

that keeps track of the memory effects in the model.

Under some loose regularity conditions, one can Laplace transform both sides of Eqs. (2.1) and (2.2), that yields

$$\widetilde{\varepsilon}(s) = s\widetilde{J}(s)\widetilde{\sigma}(s) \quad \text{and} \quad s\widetilde{J}(s) = 1 + \widetilde{\Psi}(s), \tag{2.3}$$

with $s \in \mathbb{C}$ the complex Laplace frequency and

$$L\{f(t);\, s\} \equiv \widetilde{f}(s) := \int_0^\infty \mathrm{e}^{-st}\, f(t)\,\mathrm{d}t, \tag{2.4}$$

denoting the Laplace transform of a sufficiently regular causal function $f(t)$.

The *Bessel models* [9] are a class of viscoelastic models characterised by a creep rate $\Psi(t;\nu)$ expressed, for $\nu > -1$, in terms of the Dirichlet series

$$\Psi(t;\nu) = 4(\nu+1)(\nu+2) + 4(\nu+1)\sum_{k=1}^{\infty} \exp\left(-j_{\nu+2,k}^2\, t\right), \tag{2.5}$$

where $j_{\nu+2,k}$ are the k-th positive real root of the Bessel function of the first kind $J_{\nu+2}(t)$ (see, *e.g.*, [12] for details). Note that this series is absolutely convergent for $t > 0$.

Taking the Laplace transform of Eq. (2.5), one finds

$$\widetilde{\Psi}(s;\nu) := \frac{2\,(\nu+1)}{\sqrt{s}}\,\frac{I_{\nu+1}(\sqrt{s})}{I_{\nu+2}(\sqrt{s})}, \tag{2.6}$$

where $I_\alpha(z)$ denotes the modified Bessel functions of the first kind [12]

$$I_\alpha(z) := \left(\frac{z}{2}\right)^{\alpha} \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)} \left(\frac{z}{2}\right)^{2m}, \tag{2.7}$$

with $\Gamma(z)$ representing the Euler Gamma function. Then, as showed in [9], one finds that the creep compliance for the Bessel models reads

$$s\widetilde{J}(s;\nu) = 1 + \widetilde{\Psi}(s;\nu) = 1 + \frac{2(\nu+1)}{\sqrt{s}}\,\frac{I_{\nu+1}(\sqrt{s})}{I_{\nu+2}(\sqrt{s})}, \tag{2.8}$$

that, taking advantage of the identity [12]

$$I_{\nu-1}(z) - I_{\nu+1}(z) = \frac{2\nu}{z} I_\nu(z),$$

can be recast as

$$s\widetilde{J}(s;\nu) = \frac{I_\nu(\sqrt{s})}{I_{\nu+2}(\sqrt{s})}. \tag{2.9}$$

This expression will be the starting point for computing the Q-factor for the Bessel models.

Before moving on to the explicit computation of the quality factor for these models, it is worth taking some time to highlight the origin and main results concerning this class of viscoelastic systems. To start off, it is worth mentioning that the Bessel viscoelastic class was formulated in [9] as a generalisation of a mathematical model for the propagation of blood pulses within large arteries [13]. The mathematical techniques developed in [9,13] were then used to provide an alternative derivation of the Rayleigh–Sneddon sum [14]. In [15], it was shown that the constitutive relations of the Bessel models are *ordinary infinite-order* differential equations, whereas the short time behaviour for these systems effectively reduces to a fractional Maxwell model of order $1/2$ and ν-dependent relaxation time. Lastly, taking advantage of the Buchen–Mainardi algorithm [16] (see [17] for a review on the subject) the propagation of transient waves in a semi-infinite Bessel medium was investigated, deriving the precise form of the wave-front expansion.

Remark 1 The time variable t, in this section and in the following, is effectively non-dimensional since, for the sake of convenience, we have set the relaxation time τ to unity.

3 Quality factor for the Bessel models

The specific attenuation factor or quality factor, often abbreviated as Q-factor, is a non-dimensional quantity that measures the dissipation of energy for sinusoidal excitations in stress or strain [4,10,11,18].

Following [4] and [11], given a complex creep compliance $\widetilde{J}(s)$, in the Laplace domain, one can obtain the corresponding Q-factor as [4]

$$Q^{-1}(\omega) = -\frac{\Im\left\{s\,\widetilde{J}(s)|_{s=\mathrm{i}\omega}\right\}}{\Re\left\{s\,\widetilde{J}(s)|_{s=\mathrm{i}\omega}\right\}}, \tag{3.1}$$

where $\omega > 0$ is the frequency of the harmonic excitations of the material.

First, let us define *Tricomi's uniform modified Bessel functions of the first kind* (in analogy with Tricomi's uniform Bessel functions discussed in [4]) as follows:

$$I_\alpha^{\mathrm{T}}(z) := \left(\frac{z}{2}\right)^{-\alpha} I_\alpha(z) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)} \left(\frac{z}{2}\right)^{2m}. \tag{3.2}$$

For these functions, one has that:

Lemma 1 *Let $\alpha \in \mathbb{R}$, $\alpha > -1$, and $z \in \mathbb{C}$. Then, $I_\alpha^{\mathrm{T}}(z)$ is an entire function and $I_\alpha^{\mathrm{T}}(\sqrt{z})$ is both single-valued and entire.*

Proof First, if $z = 0$ it is clear from Eq. (3.2) that $I_\alpha^{\mathrm{T}}(0) = 1/\Gamma(\alpha+1)$ which is surely well defined for $\alpha > -1$. If instead $z \in \mathbb{C} \setminus \{0\}$ and

$$a_k(z) = \frac{1}{k!\,\Gamma(k+\alpha+1)} \left(\frac{z}{2}\right)^{2k},$$

then $|a_{k+1}(z)/a_k(z)| = O(k^{-2})$ and $\left|a_{k+1}(\sqrt{z})/a_k(\sqrt{z})\right| = O(k^{-2})$ as $k \to +\infty$. Hence the series in $I_\alpha^{\mathrm{T}}(z)$ and $I_\alpha^{\mathrm{T}}(\sqrt{z})$ converge everywhere in $\mathbb{C}$ as a consequence of the ratio test. Second, from the definition (3.2) one has that

$$I_\alpha^{\mathrm{T}}(\sqrt{z}) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)} \left(\frac{z}{4}\right)^{m},$$

which is clearly a single-valued function on $\mathbb{C}$. □

Therefore, it follows that:

Proposition 1 *$s\widetilde{J}(s)$ as in Eq. (2.9) is single-valued and*

$$s\widetilde{J}(s;\nu) = \frac{4}{s}\frac{I_\nu^{\mathrm{T}}(\sqrt{s})}{I_{\nu+2}^{\mathrm{T}}(\sqrt{s})}. \tag{3.3}$$

Proof From Eqs. (2.9) and (3.2), one has that

$$s\widetilde{J}(s;\,\nu) = \frac{I_\nu(\sqrt{s})}{I_{\nu+2}(\sqrt{s})} = \frac{4}{s}\,\frac{I_\nu^{\mathrm{T}}(\sqrt{s})}{I_{\nu+2}^{\mathrm{T}}(\sqrt{s})}, \tag{3.4}$$

which is therefore single-valued as a consequence of Lemma 1. □

The last proposition allows one to compute $s\widetilde{J}(s;\,\nu)\big|_{s=\mathrm{i}\omega}$ without the risk of incurring in branch cuts and branch points for positive real frequencies ω.

Let us continue with some preliminary results.

Lemma 2 *Let $\omega \in \mathbb{R}$, $\omega > 0$, and $m \in \mathbb{N}$. Then,*

$$(\mathrm{i}\omega)^m = \begin{cases} (-1)^n \omega^{2n}, & \textit{if } \; m = 2n,\; n \in \mathbb{N} \; \textit{(even)} \\ \mathrm{i}(-1)^n \omega^{2n+1}, & \textit{if } \; m = 2n+1,\; n \in \mathbb{N} \; \textit{(odd)} \end{cases} \tag{3.5}$$

Proof Trivial. □

Now, defining the two functions

$$f_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{2^{4n}(2n)!\,\Gamma(2n+\alpha+1)}, \tag{3.6}$$

$$g_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2^{4n+2}(2n+1)!\,\Gamma(2n+\alpha+2)}, \tag{3.7}$$

One can conclude that:

Lemma 3 *Let $\omega, \alpha \in \mathbb{R}$, $\omega > 0$ and $\alpha > -1$. Then, $f_\alpha(z)$ and $g_\alpha(z)$ are entire functions.*

Proof Again, for $z = 0$ one gets that both $f_\alpha(0)$ and $g_\alpha(0)$ are finite. Furthermore, employing again the ratio test it is easy to see that the series in Eqs. (3.6) and (3.7) converge for all $z \in \mathbb{C} \setminus \{0\}$. □

Proposition 2 *Let $\omega, \nu \in \mathbb{R}$, $\omega > 0$ and $\nu > -1$, then*

$$I_\nu^T(\sqrt{\mathrm{i}\omega}) = f_\nu(\omega) + \mathrm{i}\, g_\nu(\omega), \tag{3.8}$$

and

$$s\widetilde{J}(s;\,\nu)\Big|_{s=\mathrm{i}\omega} = \frac{4}{\mathrm{i}\omega}\,\frac{f_\nu(\omega) + \mathrm{i}g_\nu(\omega)}{f_{\nu+2}(\omega) + \mathrm{i}g_{\nu+2}(\omega)} \tag{3.9}$$

Proof Consider the series representation (3.2), *i.e.*

$$I_\nu^{\mathrm{T}}(\sqrt{\mathrm{i}\omega}) = \sum_{m=0}^{\infty} \frac{(\mathrm{i}\omega)^m}{2^{2m}\, m!\,\Gamma(m+\nu+1)},$$

then taking advantage of Lemmas 2 and 3 one can split the right-hand side of this last expression and recognise that Eq. (3.8) holds. For the second part of the proof, it suffices to replace Eq. (3.8) into Eq. (3.3) in Proposition 1. □

Theorem 1 *Let $\omega, \nu \in \mathbb{R}$, $\nu > -1$, and $\omega > 0$. Then, the Q-factor for the Bessel models reads:*

$$Q^{-1}(\omega;\,\nu) = \frac{f_\nu(\omega) f_{\nu+2}(\omega) + g_\nu(\omega) g_{\nu+2}(\omega)}{g_\nu(\omega) f_{\nu+2}(\omega) - f_\nu(\omega) g_{\nu+2}(\omega)}. \tag{3.10}$$

Proof From Proposition 2, it is easy to see that

$$
\begin{aligned}
s\widetilde{J}(s;\nu)\Big|_{s=\mathrm{i}\omega} &= \frac{4}{\mathrm{i}\omega}\frac{f_\nu(\omega)+\mathrm{i}g_\nu(\omega)}{f_{\nu+2}(\omega)+\mathrm{i}g_{\nu+2}(\omega)} \\
&= \frac{4}{\omega}\frac{g_\nu(\omega)f_{\nu+2}(\omega)-f_\nu(\omega)g_{\nu+2}(\omega)-\mathrm{i}\big(f_\nu(\omega)f_{\nu+2}(\omega)+g_\nu(\omega)g_{\nu+2}(\omega)\big)}{|f_{\nu+2}(\omega)|^2+|g_{\nu+2}(\omega)|^2},
\end{aligned}
$$

via a direct computation. This allows one to separate directly the real and imaginary parts of $s\widetilde{J}(s;\nu)\big|_{s=\mathrm{i}\omega}$. Indeed, one finds

$$
\begin{aligned}
\Re\left\{s\,\widetilde{J}(s)\big|_{s=\mathrm{i}\omega}\right\} &= \frac{4}{\omega}\frac{g_\nu(\omega)f_{\nu+2}(\omega)-f_\nu(\omega)g_{\nu+2}(\omega)}{|f_{\nu+2}(\omega)|^2+|g_{\nu+2}(\omega)|^2}, \\
\Im\left\{s\,\widetilde{J}(s)\big|_{s=\mathrm{i}\omega}\right\} &= -\frac{4}{\omega}\frac{f_\nu(\omega)f_{\nu+2}(\omega)+g_\nu(\omega)g_{\nu+2}(\omega)}{|f_{\nu+2}(\omega)|^2+|g_{\nu+2}(\omega)|^2}.
\end{aligned}
$$

Inserting these expressions into Eq. (3.1) yields the final result. □

It is now interesting to introduce another couple of special functions. Specifically, let us consider the *Kelvin functions* [12] $\mathrm{ber}_\alpha(x)$ and $\mathrm{bei}_\alpha(x)$. These functions are, respectively, the real and imaginary parts of $J_\alpha\left(xe^{\mathrm{i}\frac{3}{4}\pi}\right)$, *i.e.*

$$
\mathrm{ber}_\alpha(z) := \left(\frac{z}{2}\right)^\alpha \sum_{k=0}^{\infty}\frac{\cos\left[\left(\frac{3\alpha}{4}+\frac{k}{2}\right)\pi\right]}{k!\Gamma(k+\alpha+1)}\left(\frac{z^2}{4}\right)^k, \tag{3.11}
$$

$$
\mathrm{bei}_\alpha(z) := \left(\frac{z}{2}\right)^\alpha \sum_{m=0}^{\infty}\frac{\sin\left[\left(\frac{3\alpha}{4}+\frac{m}{2}\right)\pi\right]}{m!\Gamma(m+\alpha+1)}\left(\frac{z^2}{4}\right)^m. \tag{3.12}
$$

Lemma 4 *Let $\alpha\in\mathbb{R}$ and $k\in\mathbb{Z}$. Then,*

$$
\begin{aligned}
\cos\left(\frac{3\pi}{4}\alpha+\frac{k\pi}{2}\right) &= \begin{cases}(-1)^n\cos(3\pi\alpha/4), & \text{if } m=2n,\ n\in\mathbb{N}\ \text{(even)}\\ (-1)^{n+1}\sin(3\pi\alpha/4), & \text{if } m=2n+1,\ n\in\mathbb{N}\ \text{(odd)}\end{cases}\\
\sin\left(\frac{3\pi}{4}\alpha+\frac{k\pi}{2}\right) &= \begin{cases}(-1)^n\sin(3\pi\alpha/4), & \text{if } m=2n,\ n\in\mathbb{N}\ \text{(even)}\\ (-1)^{n}\cos(3\pi\alpha/4), & \text{if } m=2n+1,\ n\in\mathbb{N}\ \text{(odd)}\end{cases}
\end{aligned}
$$

Proof Trivial. □

Proposition 3 *Let $\alpha,\omega\in\mathbb{R}$, $\alpha>-1$, and $\omega>0$. Then, one finds*

$$
\left(\frac{2}{\sqrt{\omega}}\right)^\alpha \mathit{ber}_\alpha\left(\sqrt{\omega}\right) = \cos\left(\frac{3\pi}{4}\alpha\right)f_\alpha(\omega)-\sin\left(\frac{3\pi}{4}\alpha\right)g_\alpha(\omega), \tag{3.13}
$$

$$
\left(\frac{2}{\sqrt{\omega}}\right)^\alpha \mathit{bei}_\alpha\left(\sqrt{\omega}\right) = \sin\left(\frac{3\pi}{4}\alpha\right)f_\alpha(\omega)+\cos\left(\frac{3\pi}{4}\alpha\right)g_\alpha(\omega). \tag{3.14}
$$

Proof It follows from Lemma 4 together with Eqs. (3.6), (3.6), (3.11), and (3.12). □

Now, it is fairly easy to see that the result in Proposition 3 can be rewritten as

$$
f_\alpha(\omega) = \left(\frac{2}{\sqrt{\omega}}\right)^\alpha\left[\cos\left(\frac{3\pi}{4}\alpha\right)\mathrm{ber}_\alpha\left(\sqrt{\omega}\right)+\sin\left(\frac{3\pi}{4}\alpha\right)\mathrm{bei}_\alpha\left(\sqrt{\omega}\right)\right], \tag{3.15}
$$

$$
g_\alpha(\omega) = \left(\frac{2}{\sqrt{\omega}}\right)^\alpha\left[-\sin\left(\frac{3\pi}{4}\alpha\right)\mathrm{ber}_\alpha\left(\sqrt{\omega}\right)+\cos\left(\frac{3\pi}{4}\alpha\right)\mathrm{bei}_\alpha\left(\sqrt{\omega}\right)\right], \tag{3.16}
$$

which means that one can recast the result in Theorem 1 as follows.

Theorem 2 *Let $\omega, \nu \in \mathbb{R}$, $\nu > -1$, and $\omega > 0$. Then, the Q-factor for the Bessel models reads:*

$$Q^{-1}(\omega; \nu) = \frac{bei_{\nu+2}\left(\sqrt{\omega}\right) ber_{\nu}\left(\sqrt{\omega}\right) - bei_{\nu}\left(\sqrt{\omega}\right) ber_{\nu+2}\left(\sqrt{\omega}\right)}{bei_{\nu}\left(\sqrt{\omega}\right) bei_{\nu+2}\left(\sqrt{\omega}\right) + ber_{\nu}\left(\sqrt{\omega}\right) ber_{\nu+2}\left(\sqrt{\omega}\right)}. \tag{3.17}$$

Proof See Appendix 6. □

Remark 2 Expressing $Q^{-1}(\omega; \nu)$ in terms of Kelvin functions turns out to be particularly useful for its numerical evaluation since these functions are already implemented in most of scientific softwares of common use.

3.1 Q-factor for the fractional Maxwell model and Bessel media

Let τ be a time scale, then the constitutive equation of the fractional Maxwell model of linear viscoelasticity reads [4]

$$\sigma(t) + \tau^{\beta}\, {}^{\mathrm{C}}D^{\beta} \sigma(t) = J_{\mathrm{g}}^{-1} \tau^{\beta}\, {}^{\mathrm{C}}D^{\beta} \epsilon(t), \quad 0 < \beta < 1, \tag{3.18}$$

where ${}^{\mathrm{C}}D^{\beta}$ denotes the Caputo derivative of order β with respect to t. Note that the case $\beta = 1$ corresponds to the (ordinary) Maxwell model [4]. Following a procedure akin to the one discussed in the present section, one can easily derive the specific dissipation function for this model (see [4] for details) which yields

$$Q_{\beta}^{-1}(\omega) = \frac{\sin(\pi\beta/2)}{(\omega\tau)^{\beta} + \cos(\pi\beta/2)}, \tag{3.19}$$

again, with $0 < \beta < 1$.

As shown in [9], the Bessel models approach the behaviour of a fractional Maxwell model of order $1/2$ for short times ($t \to 0^+$) and of an ordinary Maxwell model for long times ($t \to +\infty$). More precisely, we have the following results for the creep compliance of the Bessel models.

Lemma 5 (see [9]) *Consider the creep compliance for the Bessel models in the Laplace domain,* i.e. *Eq. (2.8). Then, one finds*

$$s\widetilde{J}(s; \nu) \sim \begin{cases} 1 + 2(\nu+1)s^{-1/2}, & \text{as } s \to \infty, \\ \dfrac{2(\nu+2)}{\nu+3} + \dfrac{4(\nu+1)(\nu+2)}{s}, & \text{as } s \to 0, \end{cases} \tag{3.20}$$

with $\nu > -1$.

Then, one can easily infer the following proposition.

Proposition 4 *Let $\nu > -1$. The asymptotic behaviour of the Q-factor of the Bessel models is given by*

$$Q^{-1}(\omega; \nu) \sim \begin{cases} \dfrac{\sqrt{2}(\nu+1)}{\omega^{1/2} + \sqrt{2}(\nu+1)}, & \text{as } \omega \to +\infty, \\ \dfrac{2(\nu+1)(\nu+3)}{\omega}, & \text{as } \omega \to 0^+. \end{cases} \tag{3.21}$$

Proof Set $s = \mathrm{i}\omega$, with $\omega \in \mathbb{R}$ and $\omega > 0$, in the results from Lemma 5. Note that the asymptotic expansion for $s \to \infty$ in Lemma 5 looks multivalued (although the full function is single-valued); hence, to perform the analysis one can simply choose the principal branch of $\sqrt{s}$. □

This result clearly shows that the high-frequency limit of $Q^{-1}(\omega; \nu)$ behaves as a fractional Maxwell model of order $1/2$, whereas in a similar fashion, the low-frequency behaviour of $Q^{-1}(\omega; \nu)$ approaches the one of a standard Maxwell body.

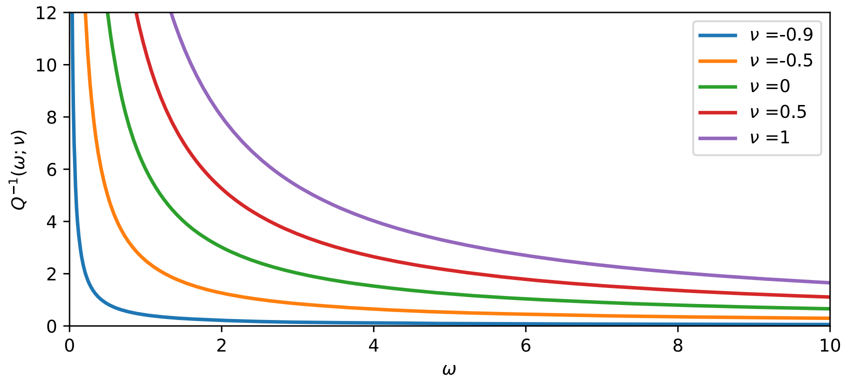

Fig. 1 $Q^{-1}(\omega;\nu)$ for different values of ν, in linear scale

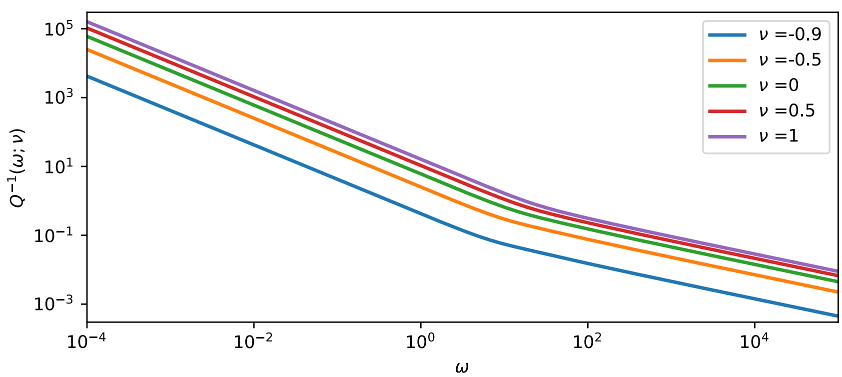

Fig. 2 $Q^{-1}(\omega;\nu)$ for different values of ν, in logarithmic scale

4 Numerical results

We shall now provide some numerical results and plots to elucidate the behaviour of Q-factor for the Bessel models governed by the analytical expression in Eq. (3.17). Specifically, we will provide the numerical evaluation of Eq. (3.17), against the frequency ω, as for different values of the parameter $\nu > -1$.

In Figs. 1 and 2, we provide the plots of the quality factors, for different values of $\nu > -1$, in both linear and in logarithmic scales. From Fig. 1, one can appreciate that $Q^{-1}(\omega;\nu)$ is overall a decreasing function, with a steep behaviour at low frequencies and a softer one at high frequencies. The plot in logarithmic scale, Fig. 2, shows the behaviour of the quality factor for an interval of values of the frequency larger than that of Fig. 1, ranging from 10^{-4} to 10^5. From Fig. 2, one can immediately identify two regions where $Q^{-1}(\omega;\nu)$ presents nearly constant slopes (in log–log scale), while the transition from one region to the other appears to be sharper for lower values of the parameter ν.

In Fig. 3, we show the numerical matching between the analytic expression of the full $Q^{-1}(\omega;\nu)$, Eq. (3.17), and its estimated asymptotic behaviour at high frequencies ($\omega \to \infty$) provided in Eq. $(3.21)_1$. Similarly, in Fig. 4 we show matching between Eq. (3.17) and its low-frequency asymptotic expansion provided in Eq. $(3.21)_2$. These plots further highlight the fact that at short times (high frequencies) Q-factor of the Bessel models approaches the one of a fractional Maxwell model of order $1/2$, whereas at late times (low frequencies) the model relaxes to a standard Maxwell model, in accordance with the results in [9] concerning the material and memory functions.

5 Discussion and conclusions

The Bessel models are a class of models of linear viscoelasticity that was originally derived in the context of hemodynamics [13]. The constitutive laws for these models are infinite-order ordinary differential equations [15] leading to material functions that, in the time domain, are expressed in terms of Dirichlet series, whereas in the Laplace domain they given by suitable ratios of modified Bessel functions of contiguous order [9].

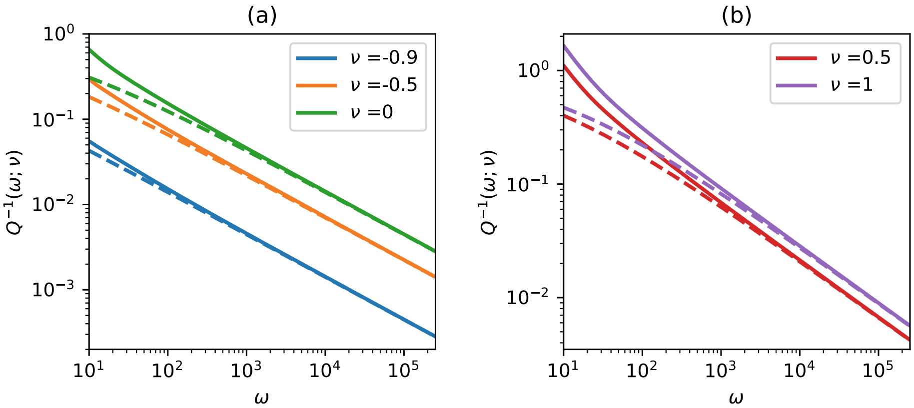

Fig. 3 Comparison between $Q^{-1}(\omega;\nu)$ (continuous line) and its asymptotic behaviour (dashed line) for $\omega \to \infty$, in logarithmic scale

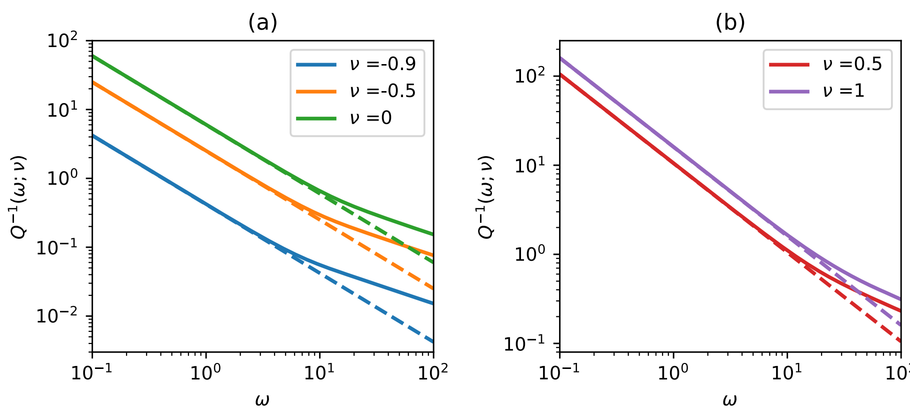

Fig. 4 Comparison between $Q^{-1}(\omega;\nu)$ (continuous line) and its asymptotic behaviour (dashed line) for $\omega \to 0$, in logarithmic scale

The specific attenuation factor, or Q-factor, is an important quantity in viscoelasticity that provides a quantitative estimate of the dissipation of energy for sinusoidal excitations in stress or strain due to the properties of the material [4,10,18].

In this work, we have provided the full analytic derivation of the Q-factor for the Bessel models. Specifically, Theorem 2 provides a precise expression for the Q-factor of these models in terms of a rate of Kelvin functions. Furthermore, in Proposition 4, we provided a precise characterisation of the asymptotic behaviour of the Q-factor at both low frequencies and high frequencies. This asymptotic analysis agrees with previous findings concerning the matching between this class of models and the fractional Maxwell model of order $1/2$, at short times, and the standard Maxwell model, at long times [9]. In other words, these models feature a continuous transition from a fractional-like behaviour to an ordinary one. Additionally, in Sect. 4 we provided some numerical evaluations of the quantities computed in Sect. 3 in order to elucidate on their full behaviour, that might not be apparent from the analytical results.

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Appendix A. Proof of Theorem 2

From Eqs. (3.15) and (3.16), one finds:

$$
\begin{aligned}
(i)\ \ f_\nu(\omega) f_{\nu+2}(\omega) = \left(\frac{2}{\sqrt{\omega}}\right)^{2\nu+2} \Bigg[& \cos\left(\frac{3\pi}{4}\nu\right) \cos\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{ber}_\nu\left(\sqrt{\omega}\right) \mathrm{ber}_{\nu+2}\left(\sqrt{\omega}\right) \\
& + \cos\left(\frac{3\pi}{4}\nu\right) \sin\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{ber}_\nu\left(\sqrt{\omega}\right) \mathrm{bei}_{\nu+2}\left(\sqrt{\omega}\right) \\
& + \sin\left(\frac{3\pi}{4}\nu\right) \cos\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{bei}_\nu\left(\sqrt{\omega}\right) \mathrm{ber}_{\nu+2}\left(\sqrt{\omega}\right) \\
& + \sin\left(\frac{3\pi}{4}\nu\right) \sin\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{bei}_\nu\left(\sqrt{\omega}\right) \mathrm{bei}_{\nu+2}\left(\sqrt{\omega}\right) \Bigg];\\
(ii)\ \ g_\nu(\omega) g_{\nu+2}(\omega) = \left(\frac{2}{\sqrt{\omega}}\right)^{2\nu+2} \Bigg[& \sin\left(\frac{3\pi}{4}\nu\right) \sin\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{ber}_\nu\left(\sqrt{\omega}\right) \mathrm{ber}_{\nu+2}\left(\sqrt{\omega}\right) \\
& - \sin\left(\frac{3\pi}{4}\nu\right) \cos\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{ber}_\nu\left(\sqrt{\omega}\right) \mathrm{bei}_{\nu+2}\left(\sqrt{\omega}\right) \\
& - \cos\left(\frac{3\pi}{4}\nu\right) \sin\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{bei}_\nu\left(\sqrt{\omega}\right) \mathrm{ber}_{\nu+2}\left(\sqrt{\omega}\right) \\
& + \cos\left(\frac{3\pi}{4}\nu\right) \cos\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{bei}_\nu\left(\sqrt{\omega}\right) \mathrm{bei}_{\nu+2}\left(\sqrt{\omega}\right) \Bigg];\\
(iii)\ \ g_\nu(\omega) f_{\nu+2}(\omega) = \left(\frac{2}{\sqrt{\omega}}\right)^{2\nu+2} \Bigg[& -\sin\left(\frac{3\pi}{4}\nu\right) \cos\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{ber}_\nu\left(\sqrt{\omega}\right) \mathrm{ber}_{\nu+2}\left(\sqrt{\omega}\right) \\
& - \sin\left(\frac{3\pi}{4}\nu\right) \sin\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{ber}_\nu\left(\sqrt{\omega}\right) \mathrm{bei}_{\nu+2}\left(\sqrt{\omega}\right) \\
& + \cos\left(\frac{3\pi}{4}\nu\right) \cos\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{bei}_\nu\left(\sqrt{\omega}\right) \mathrm{ber}_{\nu+2}\left(\sqrt{\omega}\right) \\
& + \cos\left(\frac{3\pi}{4}\nu\right) \sin\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{bei}_\nu\left(\sqrt{\omega}\right) \mathrm{bei}_{\nu+2}\left(\sqrt{\omega}\right) \Bigg];
\end{aligned}
$$

$$
\begin{aligned}
(iv)\ \ f_\nu(\omega) g_{\nu+2}(\omega) = \left(\frac{2}{\sqrt{\omega}}\right)^{2\nu+2} \Bigg[& -\cos\left(\frac{3\pi}{4}\nu\right) \sin\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{ber}_\nu\left(\sqrt{\omega}\right) \mathrm{ber}_{\nu+2}\left(\sqrt{\omega}\right) \\
& + \cos\left(\frac{3\pi}{4}\nu\right) \cos\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{ber}_\nu\left(\sqrt{\omega}\right) \mathrm{bei}_{\nu+2}\left(\sqrt{\omega}\right) \\
& - \sin\left(\frac{3\pi}{4}\nu\right) \sin\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{bei}_\nu\left(\sqrt{\omega}\right) \mathrm{ber}_{\nu+2}\left(\sqrt{\omega}\right) \\
& + \sin\left(\frac{3\pi}{4}\nu\right) \cos\left(\frac{3\pi}{4}(\nu+2)\right) \mathrm{bei}_\nu\left(\sqrt{\omega}\right) \mathrm{bei}_{\nu+2}\left(\sqrt{\omega}\right) \Bigg].
\end{aligned}
$$

Then, plugging (i)–(iv) into Eq. (3.10) in Theorem 1 one immediately infers Eq. (3.17).

References

1. Pipkin, A.C.: Lectures on viscoelasticity theory, vol. 7. Springer Science & Business Media, (2012)
2. Rogosin, S., Mainardi, F.: George William Scott Blair - the pioneer of factional calculus in rheology. Commun. Appl. Ind. Math. **6**(1), e681 (2014)
3. Mainardi, F., Spada, G.: Creep, relaxation and viscosity properties for basic fractional models in rheology. Eur. Phys. J. Special Topics **193**, 133–160 (2011)
4. Mainardi, F.: Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models. World Scientific, 2nd edn., (2022)
5. Giusti, A., Colombaro, I., Garra, R., Garrappa, R., Polito, F., Popolizio, M., Mainardi, F.: A practical guide to Prabhakar fractional calculus. Fract. Calc. Appl. Anal. **23**(1), 9–54 (2020)
6. Gorenflo, R., Mainardi, F.: Fractional calculus: integral and differential equations of fractional order. In: Carpinteri, A., Mainardi, F. (eds.) Fractals and Fractional Calculus in Continuum Mechanics, pp. 223–276. Springer Verlag, Wien (1997)
7. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional integrals and derivatives: theory and applications. Taylor and Francis, (1993)
8. Diethelm, K., Garrappa, R., Giusti, A., Stynes, M.: Why fractional derivatives with nonsingular Kernels should not be used. Fract. Calc. Appl. Anal. **23**, 610–634 (2020)
9. Colombaro, I., Giusti, A., Mainardi, F.: A class of linear viscoelastic models based on Bessel functions. Meccanica **52**(4–5), 825–832 (2017)
10. Borcherdt, R.D.: Viscoelastic waves in layered media. Cambridge University Press, (2009)
11. Colombaro, I., Giusti, A., Vitali, S.: Storage and dissipation of energy in Prabhakar viscoelasticity. Mathematics **6**, 15 (2018)
12. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions. Dover, New York (1965)
13. Giusti, A., Mainardi, F.: A dynamic viscoelastic analogy for fluid-filled elastic tubes. Meccanica **51**, 2321 (2016)
14. Giusti, A., Mainardi, F.: On infinite series concerning zeros of Bessel functions of the first kind. Eur. Phys. J. Plus **131**, 206 (2016)
15. Giusti, A.: On infinite order differential operators in fractional viscoelasticity. Fract. Calc. Appl. Anal. **20**(4), 854 (2017)
16. Buchen, P.W., Mainardi, F.: Asymptotic expansions for transient viscoelastic waves. J. de Mec. **14**(4), 597–608 (1975)
17. Colombaro, I., Giusti, A., Mainardi, F.: On transient waves in linear viscoelasticity. Wave Motion **74**, 191–212 (2017)
18. Mainardi, F., Masina, E., Spada, G.: A generalization of the Becker model in linear viscoelasticity: creep, relaxation and internal friction. Mech. Time-Depend. Mater. **23**, 283 (2019)