



On the compensator of step processes in progressively enlarged filtrations and related control problems

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Abstract. For a step process X with respect to its natural filtration \mathbb{F} , we denote by \mathbb{G} the smallest right-continuous filtration containing \mathbb{F} and such that another step process H is adapted. We investigate some structural properties of the step process X in \mathbb{G} . We show that $Z = (X, H)$ possesses the weak representation property with respect to \mathbb{G} . Moreover, in the case $H = 1_{[\tau, +\infty)}$, where τ is a random time (but not an \mathbb{F} -stopping time) satisfying Jacod's absolute continuity hypothesis, we compute the \mathbb{G} -predictable compensator $\nu^{\mathbb{G}, X}$ of the jump measure of X . Thanks to our theoretical results on $\nu^{\mathbb{G}, X}$, we can consider stochastic control problems related to model uncertainty on the intensity measure of X , also in presence of an external risk source modeled by the random time τ .

1. Introduction

Let (X, \mathbb{F}) and (H, \mathbb{H}) denote two step processes with respect to their natural filtration, respectively. By \mathbb{G} we denote the smallest right-continuous filtration containing \mathbb{F} and \mathbb{H} . If τ is a random time but not an \mathbb{F} -stopping time and H is the associated default process, i.e., $H = 1_{[\tau, +\infty)}$, then the filtration \mathbb{G} is called *progressive enlargement* of \mathbb{F} by τ .

In the present paper we investigate some structural properties of the semimartingale (X, \mathbb{F}) in the filtration \mathbb{G} . For example, we show that the \mathbb{G} -semimartingale $Z = (X, H)$ possesses the weak representation property (from now on WRP) with respect to \mathbb{G} , see Theorem 3.1. This extends the results obtained in Di Tella and Jeanblanc (2021), in which X and H are *simple point processes*. The

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weak representation property for marked point processes has also been recently studied in the article [Calzolari and Torti \(2022\)](#). However, the results of [Calzolari and Torti \(2022\)](#) are very general and go beyond the semimartingale context, while here we give a concise independent proof of the WRP in the special case of step processes.

Furthermore, in the special case of the progressive enlargement by τ , we compute the \mathbb{G} -predictable compensator $\mathbf{v}^{\mathbb{G},X}$ of the jump measure μ^X of X under *Jacod's absolute continuity hypothesis*, see Section 4. These latter results extend those recently obtained by Gapeev, Jeanblanc and Wu in [Gapeev et al. \(2021\)](#), where the stronger *Jacod's equivalence hypothesis* is assumed. Thanks to our theoretical results, we can consider stochastic control problems under model uncertainty and in presence of an external risk source. The additional risk source is modeled here by the occurrence time τ of a completely external shock event, such as a default time or the death time of an agent, that cannot be inferred using the information available in \mathbb{F} . The model uncertainty affects the \mathbb{G} -predictable compensator $\mathbf{v}^{\mathbb{G},X,u}$ of μ^X , that is not a-priori fixed but it rather depends on the \mathbb{G} -predictable control processes u . To every \mathbb{G} -predictable compensator $\mathbf{v}^{\mathbb{G},X,u}$ corresponds a different probability measure \mathbb{P}_u .

Denoting by $I^u(X) = (I_t^u(X))_{t \in [0, T]}$ a \mathbb{G} -measurable (that is, a *defaultable*) cost functional, where $T > 0$ is a fixed maturity and u an admissible control, we address the problem of optimizing the expected cost functional $J(u) = \mathbb{E}_u[I_T^u(X)]$, where \mathbb{E}_u denotes the expectation under \mathbb{P}_u . In our model the class of the admissible \mathbb{G} -predictable compensators of μ^X is dominated by $\mathbf{v}^{\mathbb{G},X}$ and the optimizer controls the \mathbb{G} -predictable density of $\mathbf{v}^{\mathbb{G},X,u}$. Notice that a special example of a \mathbb{G} -measurable cost functional is obtained if $I^u(X)$ itself is \mathbb{F} -measurable but the optimization problem is pursued only up to τ , that is, the expected cost functional to optimize becomes $J^\tau(u) := \mathbb{E}_u[I_{T \wedge \tau}^u(X)]$. Hence, in this article, we consider both the problems of optimizing the expected value of a defaultable cost functional at maturity $T > 0$ as well as the problem of optimizing the expected value of a non-defaultable (i.e., \mathbb{F} -measurable) cost functional up to the exogenous random time $T \wedge \tau$. This latter problem has the interpretation of the one of an agent who only disposes of the information available in the reference filtration \mathbb{F} but, for some reasons as her death (as it happens, for example, in *life insurance*) or the default of part of the market (as it happens, for example, in *credit risk*), she has only access to the market up to τ . We stress that our model for $I^u(X)$ is very general and covers classical cases as the exponential utility function. A possible frame of application of these control problems could be the one of an insurance company who, given a risk model represented by X , a \mathbb{G} -measurable (or \mathbb{F} -measurable) running and terminal costs, has to determine the best or worst case for the expected cost up to maturity (or up to default), see Section 6.

To represent the value function associated to these class of control problems, we follow a dynamical approach based on a class of BSDEs with a \mathbb{G} -predictable Lipschitz-continuous generator of sub-linear growth and driven by the jump measure μ^Z of the semimartingale $Z = (X, H)$. Existence and uniqueness of the solution of the involved BSDEs rely on the theory developed in [Confortola and Fuhrman \(2013\)](#). In [Confortola and Fuhrman \(2013\)](#) the authors assume that the driving marked point process is quasi-left-continuous, i.e., its compensator is continuous. This assumption is crucial: Indeed, it is shown in [Confortola et al. \(2016, Remark 10.1\)](#) that if the compensator is not continuous, one cannot expect that the corresponding BSDE admits a solution, in general. On the other side, in [Di Tella and Jeanblanc \(2021, Counterexample 4.7\)](#) the authors show that the \mathbb{F} -quasi-left continuity of the marked point process μ^X is not preserved in \mathbb{G} , that is, in particular, $\mathbf{v}^{\mathbb{G},X}$ need not be continuous. Therefore, the theoretical results in Section 4, together with the martingale representation theorem obtained in [Theorem 3.1](#), are crucial to ensure that these BSDEs admit a unique solution also with respect to the enlarged filtration \mathbb{G} .

We stress that the theory of BSDEs in progressively enlarged filtrations and the related dynamical approach to optimal control problems has known important developments in the last decade. For instance, in [Pham \(2010\)](#), Pham studied an optimal investment problem for an agent delivering the defaultable claim at maturity T . Similar problems have been addressed in [Lim and Quenez \(2011\)](#), [Ankirchner et al. \(2010\)](#) and [Jiao and Pham \(2011\)](#).

Further results concern expected utility optimization problems with random terminal time, where the progressively enlarged filtration is used to handle the time τ of a shock that affects the market or the agent. These problems can be solved by introducing a suitable BSDE over $[0, T \wedge \tau]$. In [Kharroubi and Lim \(2014\)](#) and [Kharroubi et al. \(2013\)](#) the authors consider optimization problems on $[0, T \wedge \tau]$ in a progressively enlarged Brownian filtration \mathbb{G} reducing the study of the BSDEs on $[0, T \wedge \tau]$ to the one of an associated BSDEs with deterministic horizon T in the reference Brownian filtration \mathbb{F} . This method is often called in the literature reduction method. Following the approach of [Kharroubi et al. \(2013\)](#), [Jeanblanc et al. \(2015\)](#) study an exponential utility maximization problem over $[0, T \wedge \tau]$ in a progressively enlarged Brownian filtration. In both [Kharroubi et al. \(2013\)](#) and [Jeanblanc et al. \(2015\)](#) the random time τ avoids stopping times and satisfies the immersion property. More general BSDEs over $[0, T \wedge \tau]$ have been considered in the recent paper [Aksamit et al. \(2023\)](#) where τ is a random time satisfying some mild conditions.

In the present paper, relying on the martingale representation theorem in \mathbb{G} and following the approach in [Di Tella \(2020\)](#), we show that the problem up to default (i.e., up to $T \wedge \tau$) can be solved as a problem with a \mathbb{G} -measurable cost functional up to maturity (i.e., up to T). Furthermore, we prove that the value function of the problem up to default coincides with the one of the problem up to maturity. Additionally, we establish an explicit relationship between the two optimal control processes (see [Theorem 5.17](#)). We also notice that the control problems from the literature mentioned before are different from those considered here. Indeed, the agent here controls the probability measure and the \mathbb{G} -predictable compensator of μ^X with the goal of determining the worst/best case for the expected cost functional rather than the optimal strategy.

The present paper has the following structure: In [Section 2](#) we recall some basic notions. In [Section 3](#) we obtain a martingale representation theorem in the enlarged filtration of a step process. [Section 4](#) is devoted to the study of the \mathbb{G} -quasi-left continuity of the step process $Z = (X, H)$, if $H = 1_{[\tau, +\infty)}$, and to the computation of $v^{\mathbb{G}, Z}$. The applications to different kinds of control problems both on $[0, T]$ and $[0, T \wedge \tau]$ are presented in [Section 5](#). In [Section 6](#) we provide an example on optimization of the expected exponential utility of the terminal wealth under the worst-case scenario. Finally, the proofs of technical results of [Sections 4](#) and [5](#) are postponed to the [Appendices A](#) and [B](#), respectively.

2. Basic Notions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ a right-continuous filtration of subsets of \mathcal{F} and by $\mathcal{O}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{F})$) the \mathbb{F} -optional (resp. \mathbb{F} -predictable) σ -algebra on $\Omega \times \mathbb{R}_+$. We define $\mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t$.

Let X be a stochastic process. We sometimes use the notation (X, \mathbb{F}) to mean that X is \mathbb{F} -adapted. By \mathbb{F}^X we denote the smallest right-continuous filtration such that X is adapted. If X is càdlàg, we denote by ΔX the jump process and use the convention $\Delta X_0 = 0$.

We say that an \mathbb{F} -adapted càdlàg process X is \mathbb{F} -quasi-left-continuous if $\Delta X_\sigma = 0$ a.s. for every finite-valued \mathbb{F} -predictable stopping time σ .

Random measures. For a Borel subset E of \mathbb{R}^d , we introduce $\tilde{\Omega} := \Omega \times \mathbb{R}_+ \times E$ and the product σ -algebra $\tilde{\mathcal{O}}(\mathbb{F}) := \mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(E)$ and $\tilde{\mathcal{P}}(\mathbb{F}) := \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(E)$. If W is an $\tilde{\mathcal{O}}(\mathbb{F})$ -measurable (resp. $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable) mapping from $\tilde{\Omega}$ into \mathbb{R} , it is called an \mathbb{F} -optional (resp. \mathbb{F} -predictable) function.

Let μ be a random measure on $\mathbb{R}_+ \times E$ (see Jacod and Shiryaev (2003, Definition II.1.3)).

For a nonnegative \mathbb{F} -optional function W , we write $W * \mu = (W * \mu_t)_{t \geq 0}$, where $W * \mu_t(\omega) := \int_{(0,t] \times E} W(\omega, s, x) \mu(\omega, ds, dx)$ is the process defined by the (Lebesgue–Stieltjes) integral of W with respect to μ (see Jacod and Shiryaev (2003, II.1.5) for details). If $W * \mu$ is \mathbb{F} -optional (resp. \mathbb{F} -predictable), for every optional (resp. \mathbb{F} -predictable) function W , then μ is called \mathbb{F} -optional (resp. \mathbb{F} -predictable).

Semimartingales. When we say that X is a semimartingale, we always assume that it is càdlàg. For an \mathbb{R}^d -valued \mathbb{F} -semimartingale X , we denote by μ^X the jump measure of X , that is, $\mu^X(\omega, dt, dx) = \sum_{s > 0} 1_{\{\Delta X_s(\omega) \neq 0\}} \delta_{(s, \Delta X_s(\omega))}(dt, dx)$, where, here and in the whole paper, δ_a denotes the Dirac measure at point a . From Jacod and Shiryaev (2003, Theorem II.1.16), μ^X is an *integer-valued random measure* on $\mathbb{R}_+ \times \mathbb{R}^d$ with respect to \mathbb{F} , see Jacod and Shiryaev (2003, Definition II.1.13). Thus, μ^X is, in particular, an \mathbb{F} -optional random measure. According to Jacod and Shiryaev (2003, Definition III.1.23), μ^X is called an \mathbb{R}^d -valued *marked point process* (with respect to \mathbb{F}) if $\mu^X(\omega; [0, t] \times \mathbb{R}^d) < +\infty$, for every $\omega \in \Omega$ and $t \in \mathbb{R}_+$. By ν^X we denote the \mathbb{F} -predictable compensator of μ^X , see Jacod and Shiryaev (2003, Definition II.2.6). We recall that ν^X is a predictable random measure characterized by the following properties: For any \mathbb{F} -predictable function W such that $|W| * \mu^X \in \mathcal{A}_{\text{loc}}^+(\mathbb{F})$, we have $|W| * \nu^X \in \mathcal{A}_{\text{loc}}^+(\mathbb{F})$ and $W * \mu^X - W * \nu^X \in \mathcal{H}_{\text{loc}}^1(\mathbb{F})$, $\mathcal{H}_{\text{loc}}^1(\mathbb{F})$ denoting the space of \mathbb{F} -local martingales and $\mathcal{A}_{\text{loc}}^+(\mathbb{F})$ the space of \mathbb{F} -adapted locally integrable càdlàg increasing processes starting at zero. We recall that X is quasi-left continuous if and only if there exists a version of ν^X that satisfies identically $\nu^X(\omega, \{t\} \times \mathbb{R}^d) = 0$, $t \geq 0$, see Jacod and Shiryaev (2003, Corollary II.1.19).

If $X = Y - Z$ with $Y, Z \in \mathcal{A}_{\text{loc}}^+(\mathbb{F})$, we then write $X \in \mathcal{A}_{\text{loc}}(\mathbb{F})$. For $X \in \mathcal{A}_{\text{loc}}(\mathbb{F})$ we denote by $X^{p, \mathbb{F}}$ the \mathbb{F} -dual predictable projection of X , that is the unique \mathbb{F} -predictable process in $\mathcal{A}_{\text{loc}}(\mathbb{F})$ such that $X - X^{p, \mathbb{F}} \in \mathcal{H}_{\text{loc}}^1(\mathbb{F})$.

An \mathbb{R}^d -valued semimartingale X is a step process with respect to \mathbb{F} if it can be represented in the form $X = \sum_{n=1}^{\infty} \xi_n 1_{[\tau_n, +\infty)}$, where $(\tau_n)_n$ is a sequence of \mathbb{F} -stopping times such that $\tau_n \uparrow +\infty$, $\tau_n < \tau_{n+1}$ on $\{\tau_n < +\infty\}$ and $(\xi_n)_{n \geq 1}$ is a sequence of \mathbb{R}^d -valued random variables such that ξ_n is \mathcal{F}_{τ_n} -measurable and $\xi_n \neq 0$ if and only if $\tau_n < +\infty$ (see He et al. (1992, Definition 11.48)). The process $N^X = \sum_{n=1}^{\infty} 1_{[\tau_n, +\infty)}$ is called the *point process associated to X* . If X is a step process with respect to \mathbb{F} , we then obviously have $\tau_n = \inf\{t > \tau_{n-1} : X_t \neq X_{\tau_{n-1}}\}$ ($\tau_0 := 0$), $\xi_n = \Delta X_{\tau_n} 1_{\{\tau_n < +\infty\}}$ and

$$\mu^X(dt, dx) = \sum_{n=1}^{\infty} 1_{\{\tau_n < +\infty\}} \delta_{(\tau_n, \xi_n)}(dt, dx). \quad (2.1)$$

We say that a semimartingale X is a sum of jumps with respect to \mathbb{F} if X is \mathbb{F} adapted, of finite variation and $X = \sum_{0 \leq s \leq \cdot} \Delta X_s$. If X is a sum of jumps, then $X = \text{Id} * \mu^X$ holds, where $\text{Id}(x) := x$. Furthermore, it is evident, that μ^X is a marked step process if and only if X is a step process (see Jacod and Shiryaev (2003, III.1.21 and Proposition II.1.14)), that is, if X has finitely many jumps over compact time intervals.

3. Martingale Representation

For an \mathbb{R}^d -valued step process (X, \mathbb{F}^X) and a σ -field \mathcal{R}^X , called the *initial σ -field*, we denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the filtration \mathbb{F}^X initially enlarged by \mathcal{R}^X , thus $\mathcal{F}_t := \mathcal{R}^X \vee \mathcal{F}_t^X$. It is well-known that \mathbb{F} is right-continuous and clearly, (X, \mathbb{F}) is a step process. We stress that non-trivial initial σ -field

\mathcal{R}^X allows to include in the theory developed in the present paper, without any additional effort, also step processes with \mathcal{F}_0 -measurable semimartingale characteristics, that is, step-processes with conditionally independent increments with respect to \mathbb{F} given \mathcal{F}_0 .

We now consider an \mathbb{R}^ℓ -valued step process H and introduce $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ by $\mathcal{H}_t := \mathcal{R}^H \vee \mathcal{F}_t^H$, $t \geq 0$, where \mathcal{R}^H denotes a σ -field.

We denote by $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ the *progressive enlargement* of \mathbb{F} by \mathbb{H} , where

$$\mathcal{G}_t := \bigcap_{s > t} \mathcal{F}_s \vee \mathcal{H}_s \quad t \geq 0.$$

It is evident that \mathbb{G} is the *smallest* right-continuous filtration containing \mathbb{F}^X , \mathbb{F}^H , \mathcal{R}^X and \mathcal{R}^H (i.e., \mathbb{F} and \mathbb{H}).

As a special example of H , one can take the default process associated with a random time τ , i.e., $H_t(\omega) := 1_{[\tau, +\infty)}(\omega, t)$, where τ is a $(0, +\infty)$ -valued random variable. In this case (H, \mathbb{H}) is a point process. If \mathcal{R}^H is trivial, \mathbb{G} is called the progressive enlargement of \mathbb{F} by τ and it is the smallest right-continuous filtration containing \mathbb{F} and such that τ is a \mathbb{G} -stopping time.

We now introduce the $\mathbb{R}^d \times \mathbb{R}^\ell$ -valued \mathbb{G} -semimartingale $Z = (X, H)^\top$. Clearly, Z is a sum of jumps with respect to \mathbb{G} , hence it is a \mathbb{G} -semimartingale. The jump measure μ^Z of Z is an integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^\ell$ and satisfies

$$\mu^Z(\omega, dt, dx_1, dx_2) = \sum_{s > 0} 1_{\{\Delta Z_s(\omega) \neq 0\}} \delta_{(s, \Delta Z_s(\omega))}(dt, dx_1, dx_2).$$

Theorem 3.1. *Let (X, \mathbb{F}^X) and (H, \mathbb{F}^H) be step processes taking values in \mathbb{R}^d and \mathbb{R}^ℓ respectively and consider two initial σ -fields \mathcal{R}^X and \mathcal{R}^H . We define the filtrations \mathbb{F} , \mathbb{H} and \mathbb{G} as above and set $Z := (X, H)^\top$. We then have:*

(i) μ^Z is an $\mathbb{R}^d \times \mathbb{R}^\ell$ -valued marked point process, that is (Z, \mathbb{G}) is a step process.

(ii) \mathbb{G} is the smallest right-continuous filtration containing $\mathcal{R} := \mathcal{R}^X \vee \mathcal{R}^H$ and such that μ^Z is optional.

If furthermore $\mathcal{F} = \mathcal{G}_\infty$, then every $Y \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$ can be represented as

$$Y = Y_0 + W * \mu^Z - W * \nu^Z \quad (3.1)$$

where $(\omega, t, x_1, x_2) \mapsto W(\omega, t, x_1, x_2)$ is a $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^\ell)$ -measurable function such that $|W| * \mu^Z \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$ and ν^Z denotes the \mathbb{G} -dual predictable projection of the jump measure μ^Z of Z .

Proof: We start proving (i). Since Z is a sum of jumps, it is sufficient to show that μ^Z is a marked point process with respect to \mathbb{G} . To this aim, we observe that μ^X and μ^H are an \mathbb{R}^d -valued and an \mathbb{R}^ℓ -valued marked point process with respect to \mathbb{G} , respectively, (X, \mathbb{G}) and (H, \mathbb{G}) being an \mathbb{R}^d -valued and an \mathbb{R}^ℓ -valued step processes, respectively. Therefore, we have

$$\begin{aligned} \mu^Z((0, t] \times \mathbb{R}^d \times \mathbb{R}^\ell) &= \sum_{0 < s \leq t} 1_{\{\Delta Z_s \neq 0\}} = \sum_{0 < s \leq t} 1_{\{\Delta X_s \neq 0\} \cup \{\Delta H_s \neq 0\}} \\ &\leq \sum_{0 < s \leq t} (1_{\{\Delta X_s \neq 0\}} + 1_{\{\Delta H_s \neq 0\}}) \\ &= \mu^X((0, t] \times \mathbb{R}^d) + \mu^H((0, t] \times \mathbb{R}^\ell) < +\infty, \end{aligned}$$

meaning that μ^Z is an $\mathbb{R}^d \times \mathbb{R}^\ell$ -valued marked point process with respect to \mathbb{G} . This concludes the proof of (i).

We now come to (ii). Let us denote by \mathbb{G}' the smallest right continuous filtration such that μ^Z is optional. We first show the identity $\mathbb{G}' = \mathbb{F}^Z$. Since Z is an \mathbb{F}^Z -semimartingale, μ^Z is an \mathbb{F}^Z -optional

integer-valued random measure. So, $\mathbb{G}' \subseteq \mathbb{F}^Z$ holds. We now show the converse inclusion $\mathbb{G}' \supseteq \mathbb{F}^Z$. We denote $g_1(x_1, x_2) = x_1$ and $g_2(x_1, x_2) = x_2$. By definition of μ^Z we have

$$\begin{aligned} |g_1| * \mu_t^Z &= \sum_{0 < s \leq t} |\Delta X_s| 1_{\{\Delta Z \neq 0\}} \\ &= \sum_{0 < s \leq t} |\Delta X_s| (1_{\{\Delta X_s \neq 0, \Delta H_s = 0\}} + 1_{\{\Delta X_s = 0, \Delta H_s \neq 0\}} + 1_{\{\Delta X_s \neq 0, \Delta H_s \neq 0\}}) \\ &= \sum_{0 < s \leq t} |\Delta X_s| 1_{\{\Delta X_s \neq 0\}} \leq \text{Var}(X)_t < +\infty, \end{aligned}$$

where $\text{Var}(X)_t(\omega)$ denotes the total variation of $s \mapsto X_s(\omega)$ on $[0, t]$. Hence, the integral $g_1 * \mu^Z$ is well defined and satisfies $X = g_1 * \mu^Z$. Analogously, $H = g_2 * \mu^Z$ holds. This yields that X and H are \mathbb{G}' -optional processes. Since \mathbb{G}' is right-continuous, we get $\mathbb{G}' \supseteq \mathbb{F}^Z$. From Jacod (1979, Proposition 3.39 (a)) the filtration $\mathcal{R} \vee \mathbb{G}'$ is right-continuous. Therefore, $\mathcal{R} \vee \mathbb{G}'$ and \mathbb{G} coincide: They are both the smallest right continuous filtrations containing $\mathbb{F}^X, \mathbb{F}^H, \mathcal{R}^X$ and \mathcal{R}^H . The proof of (ii) is complete.

We now come to (3.1). If we assume $\mathcal{F} = \mathcal{G}_\infty$, this is an immediate consequence of (i), (ii) and Jacod and Shiryaev (2003, Theorem III.4.37). The proof is complete. \square

As an application of Theorem 3.1, we can easily show by induction the following result.

Corollary 3.2. *We consider the \mathbb{R}^{d_i} -valued step processes (X^i, \mathbb{F}^{X^i}) and the initial σ -fields \mathcal{R}^{X^i} , $i = 1, \dots, n$. We set $\mathbb{F}^i := \mathbb{F}^{X^i} \vee \mathcal{R}^{X^i}$ and denote by \mathbb{G} the smallest right-continuous filtration containing $\{\mathbb{F}^i, i = 1, \dots, n\}$. Then the $E := \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_n}$ -valued semimartingale $Z = (X^1, \dots, X^n)^\top$ satisfies:*

(i) μ^Z is an E -valued marked point process with respect to \mathbb{G} , that is Z is an E -valued step process with respect to \mathbb{G} .

(ii) \mathbb{G} is the smallest right continuous filtration containing $\mathcal{R} := \bigvee_{i=1}^n \mathcal{R}^{X^i}$ and such that μ^Z is an optional random measure.

If furthermore $\mathcal{F} = \mathcal{G}_\infty$, then every $Y \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$ can be represented as

$$Y = Y_0 + W * \mu^Z - W * \nu^Z \quad (3.2)$$

where $(\omega, t, x_1, \dots, x_n) \mapsto W(\omega, t, x_1, \dots, x_n)$ is a $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(E)$ -measurable function such that $|W| * \mu^Z \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$ and ν^Z denotes the \mathbb{G} -dual predictable projection of the jump measure μ^Z of Z .

4. The dual predictable projection in the enlarged filtration

Consider an \mathbb{R}^ℓ -valued step process H and introduce $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ by $\mathcal{H}_t := \mathcal{R}^H \vee \mathcal{F}_t^H$, $t \geq 0$, where \mathcal{R}^H denotes a σ -field. We denote by $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ the progressive enlargement of \mathbb{F} by \mathbb{H} .

The next result, which holds for general step processes, gives the structure of $\nu^{\mathbb{G}, Z}$ if H and X have no common jumps. Its proof, as well as all those one of other technical results from this section, is postponed to the Appendix A.

Theorem 4.1. *Let (X, \mathbb{F}) be an \mathbb{R}^d -valued step-process and let (H, \mathbb{H}) be an \mathbb{R}^ℓ -valued step-process. If $\Delta X \Delta H = 0$, then the following identities hold for the $\mathbb{R}^d \times \mathbb{R}^\ell$ -valued \mathbb{G} -step process $Z = (X, H)$.*

(i) $\mu^Z(\omega, dt, dx_1, dx_2) = \mu^X(\omega, dt, dx_1) \delta_0(dx_2) + \mu^H(\omega, dt, dx_2) \delta_0(dx_1)$.

(ii) $\nu^{\mathbb{G}, Z}(\omega, dt, dx_1, dx_2) = \nu^{\mathbb{G}, X}(\omega, dt, dx_1) \delta_0(dx_2) + \nu^{\mathbb{G}, H}(\omega, dt, dx_2) \delta_0(dx_1)$.

4.1. *Progressive enlargement by a random time.* We now denote by \mathbb{G} the progressive enlargement of \mathbb{F} by a random time $\tau : \Omega \rightarrow (0, +\infty]$: \mathbb{G} is the smallest right-continuous filtration containing \mathbb{F} and such that τ is a \mathbb{G} -stopping time.

For a given random time τ , we denote by $H = 1_{[\tau, +\infty)}$ the default process of τ and by $A = {}^o(1 - H) = {}^o 1_{[0, \tau]}$ the \mathbb{F} -optional projection of $(1 - H) = 1_{[0, \tau]}$ (see [Dellacherie \(1972, Theorem V.14 and V.15\)](#)). The process A is a càdlàg \mathbb{F} -supermartingale, called Azéma supermartingale, satisfying $A_t = \mathbb{P}[\tau > t | \mathcal{F}_t]$ a.s., for every $t \geq 0$. It is well-known that $\{A_- > 0\} \subseteq [0, \tau]$ (see, e.g., [Aksamit and Jeanblanc \(2017, Lemma 2.14\)](#)), so the process $\frac{1}{A_-} 1_{[0, \tau]}$ is well defined.

The \mathbb{G} -dual predictable projection $H^{p, \mathbb{G}}$ of H is denoted by $\Lambda^{\mathbb{G}}$ and, by [Aksamit and Jeanblanc \(2017, Proposition 2.15\)](#) it satisfies

$$\Lambda^{\mathbb{G}} = \int_0^{\tau \wedge \cdot} \frac{1}{A_{s-}} dH_s^{p, \mathbb{F}}, \quad (4.1)$$

$H^{p, \mathbb{F}}$ denoting the \mathbb{F} -dual predictable projection of H .

Because of the special structure of the enlarged filtration, the following result holds.

Lemma 4.2. *Let $(\omega, t, x) \mapsto W(\omega, t, x)$ be a \mathbb{G} -predictable function. Then, there exists an \mathbb{F} -predictable function $(\omega, t, x) \mapsto \bar{W}(\omega, t, x)$ such that $W(\omega, t, x) 1_{[0, \tau]}(\omega, t) = \bar{W}(\omega, t, x) 1_{[0, \tau]}(\omega, t)$. If furthermore W is bounded, then \bar{W} is bounded too.*

4.2. Quasi-left continuity in the enlarged filtration. From here on, X is an \mathbb{R}^d -valued step process and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ where $\mathcal{F}_t := \mathcal{F}_t^X$. In particular, \mathcal{F}_0 is trivial, since $X_0 = 0$, X being a step process. Moreover, $\tau : \Omega \rightarrow (0, +\infty]$ denotes a random time and $H := 1_{[\tau, +\infty)}$ the default process associated with τ . Then we set $Z := (X, H)$. We indicate by \mathbb{G} the progressive enlargement of \mathbb{F} by τ . We also use the notation $\nu^{\mathbb{F}, X}$ (resp. $\nu^{\mathbb{G}, X}$) for the \mathbb{F} -dual (resp. \mathbb{G} -dual) predictable projection of μ^X , while $\nu^{\mathbb{G}, Z}$ denotes the corresponding \mathbb{G} -dual predictable projection of jump measure μ^Z of Z .

Assume that $X \in \mathcal{A}_{\text{loc}}(\mathbb{F})$ is \mathbb{F} -quasi-left-continuous. We are interested in the following question: Is X \mathbb{G} -quasi-left-continuous? In general, this is not true: Intuitively, the larger filtration \mathbb{G} supports more predictable stopping times than \mathbb{F} . To see this we recall [Di Tella and Jeanblanc \(2021, Counterexample 4.8\)](#):

Counterexample 4.3. Let X be a homogeneous Poisson process with respect to \mathbb{F}^X and let $(\tau_n)_{n \geq 1}$ be the sequence of the jump-times of \mathbb{F}^X . The process X is not quasi-left-continuous in the filtration \mathbb{G} obtained enlarging \mathbb{F}^X progressively by the random time $\tau = \frac{1}{2}(\tau_1 + \tau_2)$. Indeed, the jump-time τ_2 of X is announced in \mathbb{G} by $(\vartheta_n)_{n \geq 1}$, $\vartheta_n := \frac{1}{n}\tau + (1 - \frac{1}{n})\tau_2$, and $\vartheta_n > \tau$ is a \mathbb{G} -stopping time for every $n \geq 1$ by [Dellacherie \(1972, Theorem III.16\)](#). Hence, τ_2 is a \mathbb{G} -predictable jump-time of X .

Notice that the quasi-left continuity of X can get lost only over $(\tau, +\infty]$, as the following result shows.

Proposition 4.4. *If X is \mathbb{F} -quasi-left continuous, then the \mathbb{G} -adapted stopped process X^τ (defined by $X_t^\tau := X_{t \wedge \tau}$, $t \geq 0$) is \mathbb{G} -quasi-left-continuous.*

We now state sufficient conditions for the \mathbb{G} -quasi-left continuity of Z .

Avoidance of \mathbb{F} -stopping times. The first property we are going to recall is the *avoidance of \mathbb{F} -stopping times*, from now on referred as assumption (\mathcal{A}) .

(\mathcal{A}) The random time τ is such that $\mathbb{P}[\tau = \sigma < +\infty] = 0$ for every \mathbb{F} -stopping time σ .

The interpretation of assumption (\mathcal{A}) is the following: The random time τ carries an information which is completely exogenous: Nothing about τ can be inferred from the information contained in the reference filtration \mathbb{F} .

Proposition 4.5. *Let τ satisfy (\mathcal{A}) . We then have:*

(i) H quasi-left continuous and τ is a \mathbb{G} -totally inaccessible stopping time.

(ii) $\Delta X \Delta H = 0$.

(iii) $\mu^Z(\omega, dt, dx_1, dx_2) = \mu^X(\omega, dt, dx_1) \delta_0(dx_2) + dH_t(\omega) \delta_1(dx_2) \delta_0(dx_1)$, and

$$\mathbf{v}^{\mathbb{G}, Z}(\omega, dt, dx_1, dx_2) = \mathbf{v}^{\mathbb{G}, X}(\omega, dt, dx_1) \delta_0(dx_2) + d\Lambda_t^{\mathbb{G}}(\omega) \delta_1(dx_2) \delta_0(dx_1). \quad (4.2)$$

Proof: Let τ satisfy (\mathcal{A}) . To see (i), we observe that (\mathcal{A}) is equivalent to the continuity of the \mathbb{F} -dual optional projection $H^{o, \mathbb{F}}$ of H (see, e.g., [Aksamit and Jeanblanc \(2017, Lemma 1.48\(a\)\)](#) or [Di Tella and Engelbert \(2022, Lemma 3.4\)](#)). Hence, the identity $H^{o, \mathbb{F}} = H^{p, \mathbb{F}}$ holds. Because of (4.1), this yields the continuity of $\Lambda^{\mathbb{G}}$. We now show (ii). By the definition of $[X, H]$, we get

$$[X, H]_t = \sum_{s \leq t} \Delta X_s \Delta H_s = \sum_{s \leq t} \Delta X_s \Delta H_s 1_{\{\Delta X_s \neq 0\} \cap \{\Delta H_s \neq 0\}}. \quad (4.3)$$

Let now $(\sigma_n)_{n \geq 1}$ denote a sequence of \mathbb{F} -stopping times exhausting the thin set $\{\Delta X \neq 0\}$. We obviously have $\{\Delta X \neq 0\} \cap \{\Delta H \neq 0\} = \bigcup_{n=1}^{\infty} [\sigma_n] \cap [\tau]$, where for a stopping time η we denote by $[\eta]$ the graph of η . By (\mathcal{A}) , the random set $[\sigma_n] \cap [\tau]$ is evanescent, for every $n \geq 1$. Hence, (4.3) yields $[X, H] = 0$ and therefore $\Delta X \Delta H = \Delta[X, H] = 0$. The statement (iii) follows by (ii) and by [Theorem 4.1](#), using the special form of H . The proof is complete \square

Immersion property. By [Proposition 4.5](#) (i), if τ satisfies assumption (\mathcal{A}) , then the process H is \mathbb{G} -quasi-left continuous. This however, does not imply that the joint process $Z = (X, H)$ is \mathbb{G} -quasi-left continuous. Indeed, the random time τ from [Counterexample 4.3](#) avoids \mathbb{F} -stopping times. However, X is not \mathbb{G} -quasi-left continuous. So, we need further assumptions to ensure the \mathbb{G} -quasi-left continuity of Z . We now therefore recall the *immersion property*, from now on referred as assumption (\mathcal{H}) .

(\mathcal{H}) The random time τ is such that \mathbb{F} -martingales remain \mathbb{G} -martingales.

The following proposition is an immediate consequence of [Jacod and Shiryaev \(2003, Theorem 2.21\)](#).

Proposition 4.6. *If τ satisfies assumption (\mathcal{H}) , then $\mathbf{v}^{\mathbb{G}, X} = \mathbf{v}^{\mathbb{F}, X}$ holds.*

As a consequence of the above discussion, we deduce the following result, whose proof is omitted.

Theorem 4.7. *Let τ be a random time satisfying both assumptions (\mathcal{A}) and (\mathcal{H}) . Then*

$$\mathbf{v}^{\mathbb{G}, Z}(\omega, dt, dx_1, dx_2) = \mathbf{v}^{\mathbb{F}, X}(\omega, dt, dx_1) \delta_0(dx_2) + \delta_1(dx_2) \delta_0(dx_1) d\Lambda_t^{\mathbb{G}}(\omega).$$

In particular, if X is \mathbb{F} -quasi-left continuous, then Z is \mathbb{G} -quasi-left continuous as well.

We stress that, although assumption (\mathcal{H}) is of technical nature, it is equivalent to require that the σ -fields \mathcal{F}_∞ and \mathcal{G}_t are conditionally independent given \mathcal{F}_t (see [Aksamit and Jeanblanc \(2017, Theorem 3.2\)](#)). Furthermore, because of the Cox construction (see [Aksamit and Jeanblanc \(2017, §2.3.1\)](#)), it is easy to construct random times τ satisfying the assumptions (\mathcal{A}) and (\mathcal{H}) , see [Di Tella and Engelbert \(2022, Remark 3.8\)](#) for details.

Jacod's absolute continuity condition. Let η denote the law of the random time τ , that is, $\eta(B) := \mathbb{P}(\tau^{-1}(B))$, for every $B \in \mathcal{B}(\mathbb{R})$. We denote by $P_t(\omega, B)$ a regular version of the conditional distribution $\mathbb{P}[\tau \in B | \mathcal{F}_t]$, $B \in \mathcal{B}(\mathbb{R})$. We assume that τ satisfies *Jacod's absolute continuity condition*, i.e.,

$$\eta \text{ is a diffused probability measure.} \quad (4.4)$$

$$P_t(du) \text{ is absolutely continuous with respect to } \eta(du). \quad (4.5)$$

We stress that random times satisfying Jacod's absolute continuity condition can be constructed following the approach presented in [Jeanblanc and Le Cam \(2009, §5\)](#).

Definition 4.8. We say that τ fulfils hypothesis (\mathcal{J}) if it satisfies Jacod's absolute continuity condition.

Remark 4.9. Let τ fulfill hypothesis (\mathcal{J}) . Then:

- (i) (\mathcal{A}) holds true, see [El Karoui et al. \(2010, Corollary 2.2\)](#).
- (ii) There exists a nonnegative and $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}([0, +\infty])$ -measurable function $(\omega, t, u) \mapsto p_t(\omega, u)$ such that $(p_t(u))_{t \geq 0}$ is an \mathbb{F} -martingale for every $u \in [0, +\infty]$ and

$$\mathbb{E}[f(\tau) | \mathcal{F}_t] = \int_{\mathbb{R}_+} f(u) p_t(u) \eta(du), \quad t \geq 0 \quad (4.6)$$

for every bounded Borel function f , see [Aksamit and Jeanblanc \(2017, Proposition 4.17\)](#).

Notice that hypothesis (\mathcal{J}) for τ does not imply, in general, that τ satisfies assumption (\mathcal{H}) . This is true if and only if $p_t(u) = p_u(u)$ η -a.s., if $u < t$, see [Aksamit and Jeanblanc \(2017, Proposition 5.28\)](#).

Remark 4.10. Let $(\omega, t, u) \mapsto p_t(\omega, u)$ be the optional function from [Remark 4.9](#) (ii). Since $p_t(u)$ is an \mathbb{F} -martingale for every u , we can represent it as $p_t(u) = W^u * \mu^X - W^u * \nu^{\mathbb{F}, X}$. As in the proof of [Jacod \(1985, Proposition 3.14 and Theorem 4.1\)](#), the function $(\omega, t, x, u) \mapsto W^u(\omega, t, x)$ can be chosen $\widetilde{\mathcal{P}}(\mathbb{F}) \otimes \mathcal{B}([0, +\infty])$ -measurable and such that the following two properties are satisfied:

- (1) $W^u(\omega, t, x) + p_{t-}(\omega, u) \geq 0$.
- (2) If $p_{t-}(u) = 0$, then $W^u(\omega, t, x) = 0$.

So, using the convention $\frac{0}{0} := 0$, one can define $V^u(\omega, t, x) := \frac{W^u(\omega, t, x)}{p_{t-}(\omega, u)}$ which satisfies $1 + V^u \geq 0$. Then

$$U(\omega, t, x) = V^{\tau(\omega)}(\omega, t, x) 1_{(\tau, +\infty)}(\omega, t) = \frac{W^{\tau(\omega)}(\omega, t, x)}{p_{t-}(\omega, \tau(\omega))} 1_{(\tau, +\infty)}(\omega, t)$$

is a \mathbb{G} -predictable function satisfying $1 + U \geq 0$.

In the next result we give the form of the \mathbb{G} -dual predictable projection of μ^X . To the best of our knowledge, this result is new and of independent interest.

Theorem 4.11. *Let X be an \mathbb{F} -quasi-left continuous step process and let τ be a random time satisfying hypothesis (\mathcal{J}) . Then*

$$\begin{aligned} \nu^{\mathbb{G}, X}(\omega, dt, dx) = \\ \left(1_{[0, \tau]}(\omega, t) \left(1 + \frac{W'(\omega, t, x)}{A_{t-}} \right) + 1_{(\tau, +\infty)}(\omega, t) (1 + U(\omega, t, x)) \right) \nu^{\mathbb{F}, X}(\omega, dt, dx), \end{aligned} \quad (4.7)$$

where W' is an \mathbb{F} -predictable function such that $A_- + W' \geq 0$ and U is the \mathbb{G} -predictable function given in [Remark 4.10](#). In particular, X is a \mathbb{G} -quasi-left continuous step process.

The following result is a direct application of [\(4.2\)](#), [Theorem 4.11](#), and [\(4.1\)](#).

Theorem 4.12. *Let X be an \mathbb{F} -quasi-left continuous step process and let τ satisfy hypothesis (\mathcal{J}) . Then the \mathbb{G} -predictable compensator $\mathbf{v}^{\mathbb{G},Z}$ of the jump measure μ^Z of $Z = (X, H)$ is given by*

$$\begin{aligned} & \mathbf{v}^{\mathbb{G},Z}(\omega, dt, dx_1, dx_2) \\ &= \left(1_{[0,\tau]}(\omega, t) \left(1 + \frac{W'(\omega, t, x_1)}{A_{t-}(\omega)} \right) + 1_{(\tau, +\infty)}(\omega, t) (1 + U(\omega, t, x_1)) \right) \mathbf{v}^{\mathbb{F},X}(\omega, dt, dx_1) \delta_0(dx_2) \\ &+ 1_{[0,\tau]}(\omega, t) d\Lambda_t^{\mathbb{G}}(\omega) \delta_0(dx_1) \delta_1(dx_2), \end{aligned}$$

where W' is an \mathbb{F} -predictable function such that $A_- + W' \geq 0$ and U is the \mathbb{G} -predictable function given in Remark 4.10.

We remark that, if τ fulfils condition (\mathcal{J}) , then, by Remark 4.9 (i) it satisfies (\mathcal{A}) , therefore $\Lambda^{\mathbb{G}}$ is continuous, see Proposition 4.5 (i). Therefore, under the assumptions of Theorem 4.12, Z is \mathbb{G} -quasi-left continuous.

5. Applications to stochastic control theory

Let $T > 0$ be a fixed maturity and let X be a marked point processes with respect to $\mathbb{F} = \mathbb{F}^X$. In this section we consider the problem of optimizing the value at T of an expected cost functional related to X under model uncertainty and in presence of an additional exogenous risk source that cannot be inferred from the information available in \mathbb{F} . The occurrence time of the additional risk source is modeled by an external random time τ . The uncertainty affects the model because, to each probability measure \mathbb{P}_u , under which the expectation of the cost functional is taken, corresponds a different \mathbb{G} -compensator $\mathbf{v}^{\mathbb{G},Z,u}$ of μ^Z . The compensator $\mathbf{v}^{\mathbb{G},Z,u}$ is absolutely continuous with respect to $\mathbf{v}^{\mathbb{G},Z}$ and its density is controlled.

We shall represent the value function associated to these control problems by means of suitable BSDEs and relying on the theory developed in Confortola and Fuhrman (2013). In order to apply it to the present context, the fundamental tools are our results about the \mathbb{G} -compensator of μ^X and the \mathbb{G} -WRP for Z , provided respectively in Theorem 4.7, Theorem 4.12 and Theorem 3.1.

Notice that, according to Jacod and Shiryaev (2003, Theorem II.1.8), one can always consider the decomposition

$$\mathbf{v}^{\mathbb{F},X}(dt, dx_1) = \phi_t^{\mathbb{F},X}(dx_1) dC_t^{\mathbb{F},X},$$

where $\phi^{\mathbb{F},X}$ is a transition probability from $(\Omega \times [0, T], \mathcal{P}(\mathbb{F}))$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and $C^{\mathbb{F},X} \in \mathcal{A}_{\text{loc}}^+(\mathbb{F})$ is a predictable process. Analogously,

$$\mathbf{v}^{\mathbb{G},X}(dt, dx_1) = \phi_t^{\mathbb{G},X}(dx_1) dC_t^{\mathbb{G},X},$$

where $\phi^{\mathbb{G},X}$ is a transition probability from $(\Omega \times [0, T], \mathcal{P}(\mathbb{G}))$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $C^{\mathbb{G},X} \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$ is \mathbb{G} -predictable.

We state the following assumptions.

Assumption 5.1. The process X is \mathbb{F} -quasi-left continuous.

Assumption 5.2. τ satisfies (\mathcal{A}) and (\mathcal{H}) .

Assumption 5.3. τ satisfies (\mathcal{J}) (hence (\mathcal{A}) , see Remark 4.9 (i)).

In the present section we always assume Assumption 5.1 together with Assumption 5.2 or alternatively together with Assumption 5.3. This, in particular, implies that (\mathcal{A}) is always assumed.

Proposition 5.4. *Under Assumption 5.1 together with Assumption 5.2 or Assumption 5.3, Z is a \mathbb{G} -quasi-left continuous step process with*

$$v^{\mathbb{G},Z}(\omega, dt, dx_1, dx_2) = \delta_0(dx_2) \phi_t^{\mathbb{G},X}(\omega, dx_1) dC_t^{\mathbb{G},X}(\omega) + \delta_0(dx_1) \delta_1(dx_2) d\Lambda_t^{\mathbb{G}}(\omega). \quad (5.1)$$

In particular, under Assumptions 5.1-5.2,

$$\phi_t^{\mathbb{G},X}(\omega, dx_1) = \phi_t^{\mathbb{F},X}(\omega, dx_1), \quad dC_t^{\mathbb{G},X}(\omega) = dC_t^{\mathbb{F},X}(\omega),$$

while under Assumptions 5.1-5.3,

$$\begin{aligned} \phi_t^{\mathbb{G},X}(\omega, dx_1) &= \frac{D(\omega, t, x_1)}{\int_{\mathbb{R}^d} D(\omega, t, x_1) \phi_t^{\mathbb{F},X}(\omega, dx_1)} \phi_t^{\mathbb{F},X}(\omega, dx_1), \\ dC_t^{\mathbb{G},X}(\omega) &= \int_{\mathbb{R}^d} D(\omega, t, x_1) \phi_t^{\mathbb{F},X}(\omega, dx_1) dC_t^{\mathbb{F},X}(\omega), \end{aligned}$$

where

$$D(\omega, t, x_1) := 1_{[0, \tau]}(\omega, t) \left(1 + \frac{W'(\omega, t, x_1)}{A_{t-}(\omega)} \right) + 1_{(\tau, +\infty)}(\omega, t) (1 + U(\omega, t, x_1))$$

is the density function appearing in Theorem 4.11.

Proof: By Assumption 5.1, $C^{\mathbb{F},X}$ is a continuous process. On the other hand, thanks to condition (\mathcal{A}) , by Proposition 4.5-2 we have (5.1), where $\Lambda_t^{\mathbb{G}}$ is continuous by Proposition 4.5-1. Finally, thanks to Assumption 5.1 together with Assumption 5.2 or Assumption 5.3, $C^{\mathbb{G},X}$ is a continuous process by Proposition 4.6 or Theorem 4.11, that also specify the forms of $\phi^{\mathbb{G},X}$ and $C^{\mathbb{G},X}$. \square

We notice that, according to Jacod and Shiryaev (2003, Theorem II.1.8), (5.1) can be rewritten as

$$v^{\mathbb{G},Z}(\omega, dt, dx_1, dx_2) = \phi_t^{\mathbb{G},Z}(\omega, dx_1, dx_2) dC_t^{\mathbb{G},Z}(\omega) \quad (5.2)$$

with

$$C_t^{\mathbb{G},Z}(\omega) := C_t^{\mathbb{G},X}(\omega) + \Lambda_t^{\mathbb{G}}(\omega), \quad (5.3)$$

$$\phi_t^{\mathbb{G},Z}(\omega, dx_1, dx_2) := d_1(\omega, t) \delta_0(dx_2) \phi_t^{\mathbb{G},X}(\omega, dx_1) + d_2(\omega, t) \delta_0(dx_1) \delta_1(dx_2), \quad (5.4)$$

where d_1, d_2 are non negative processes such that $d_1 + d_2 = 1$, and

$$d_1(\omega, t) dC_t^{\mathbb{G},Z}(\omega) = dC_t^{\mathbb{G},X}(\omega), \quad (5.5)$$

$$d_2(\omega, t) dC_t^{\mathbb{G},Z}(\omega) = d\Lambda_t^{\mathbb{G}}. \quad (5.6)$$

5.1. The control problem up to maturity. The data specifying the optimal control problem are an action space U , a running cost function l , a terminal cost function g , and another function r specifying the effect of the control process. They are assumed to satisfy the following conditions.

Assumption 5.5. (U, \mathcal{U}) is a measurable space.

Assumption 5.6. The functions $r, l : \Omega \times [0, T] \times \mathbb{R}^d \times U \rightarrow \mathbb{R}$ are $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$ -measurable and there exist constants $M_r > 1, M_l > 0$ such that, \mathbb{P} -a.s.,

$$0 \leq r_t(x_1, u) \leq M_r, \quad |l_t(x_1, u)| \leq M_l, \quad t \in [0, T], x_1 \in \mathbb{R}^d, u \in U. \quad (5.7)$$

Assumption 5.7. The function $g : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ is $\mathcal{G}_T \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable. Furthermore, there exists a constant β such that $\beta > \sup |r - 1|^2$ and the following estimates hold:

$$\mathbb{E}[e^{\beta C_T^{\mathbb{G},Z}}] < +\infty, \quad (5.8)$$

$$\mathbb{E}[|g(X_T)|^2 e^{\beta C_T^{\mathbb{G},Z}}] < +\infty. \quad (5.9)$$

With every admissible control process $u \in \mathcal{C}$ we associate the cost functional

$$J(u) = \mathbb{E}_u \left[\int_0^T l_t(X_t, u_t) dC_t^{\mathbb{G},X} + g(X_T) \right], \quad (5.10)$$

where \mathbb{E}_u denotes the expectation under a probability measure \mathbb{P}_u that is absolutely continuous with respect to \mathbb{P} and will be specified below.

The control problem consists in minimizing J over all admissible controls. Because of the structure of the control problem, it is evident that in general it cannot be directly solved in the filtration \mathbb{F} . Therefore, we have to allow \mathbb{G} -predictable strategies: The set of admissible control processes, denoted \mathcal{C} , consists of all U -valued and \mathbb{G} -predictable processes $u(\cdot) = (u_t)_{t \in [0, T]}$.

Remark 5.8. In (5.10) one can consider for example a defaultable terminal cost $g : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ of the form $g(\omega, x_1) = g_1(x_1)1_{\{T < \tau(\omega)\}} + g_2(x_1)1_{\{T \geq \tau(\omega)\}}$. In this case, g_1 is the terminal cost if the default does not occur before the maturity T while g_2 is the cost to pay in case of default up to maturity. Similarly, one can allow a defaultable running cost l . We can also regard the minimization problem associated to (5.10) as the problem of an insider who disposes of private information about τ and whose strategies are U -valued and \mathbb{G} -predictable.

With every control $u \in \mathcal{C}$, we associate the predictable random measure

$$\begin{aligned} v^{\mathbb{G},Z,u}(\omega, dt, dx_1, dx_2) &= r_t(\omega, x_1, u_t(\omega)) \delta_0(dx_2) \phi_t^{\mathbb{G},X}(\omega, dx_1) dC_t^{\mathbb{G},X}(\omega) + \delta_1(dx_2) \delta_0(dx_1) d\Lambda_t^{\mathbb{G}}(\omega) \\ &= R_t(\omega, x_1, x_2, u_t(\omega)) v^{\mathbb{G},Z}(\omega, dt, dx_1, dx_2), \end{aligned} \quad (5.11)$$

where

$$R_t(\omega, x_1, x_2, u) := r_t(\omega, x_1, u) 1_{\{x_2=0\}} + 1_{\{x_2 \neq 0\}}, \quad \omega \in \Omega, x_1 \in \mathbb{R}^d, x_2 \in \{0, 1\}, u \in U. \quad (5.12)$$

We denote by $(T_n)_{n \geq 1}$ the sequence of jump times of Z and, for any $u \in \mathcal{C}$, we consider the process

$$\kappa_t^u = \exp \left(\int_0^t \int_{\mathbb{R}^{d+1}} (1 - R_s(x_1, x_2, u_s)) v^{\mathbb{G},Z}(ds, dx_1, dx_2) \right) \prod_{n \geq 1: T_n \leq t} R_{T_n}(X_{T_n}, H_{T_n}, u_{T_n}), \quad (5.13)$$

with the convention that the last product equals 1 if there are no indices $n \geq 1$ satisfying $T_n \leq t$. We notice that κ^u is a Doléans-Dade stochastic exponential and it is the solution to

$$\kappa_t^u = 1 + \int_0^t \int_{\mathbb{R}^{d+1}} \kappa_{s-}^u (R_s(x_1, x_2, u_s) - 1) (\mu^Z - \nu^Z)(ds, dx_1, dx_2).$$

Hence, κ^u is a \mathbb{G} -local martingale, for every $u \in \mathcal{C}$. Furthermore, κ^u is nonnegative, see Jacod (1975, Proposition 4.3) for details. Thus κ^u is a nonnegative \mathbb{G} -supermartingale. Taking into account (5.12), we remark that

$$\int_0^t \int_{\mathbb{R}^{d+1}} (1 - R_s(x_1, x_2, u_s)) v^{\mathbb{G},Z}(ds, dx_1, dx_2) = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} (1 - (r_s(x_1, u_s)) \delta_0(dx_2) \phi_s^{\mathbb{G},X}(dx_1) dC_s^{\mathbb{G},X}$$

so that (5.13) reads

$$\kappa_t^u = \exp \left(\int_0^t \int_{\mathbb{R}^d} (1 - r_s(x_1, u_s)) \phi_s^{\mathbb{G},X}(dx_1) dC_s^{\mathbb{G},X} \right) \prod_{n \geq 1: T_n \leq t} (r_{T_n}(X_{T_n}, u_{T_n}) 1_{\{H_{T_n}=0\}} + 1_{\{H_{T_n} \neq 0\}}). \quad (5.14)$$

The result below follows from [Confortola and Fuhrman \(2013, Lemma 4.2\)](#) with $\gamma = 2$.

Lemma 5.9. *Assume that*

$$\mathbb{E}[e^{(3+M^4)C_T}] < +\infty. \quad (5.15)$$

Then, for every $u \in \mathcal{C}$, $\sup_{t \in [0, T]} \mathbb{E}[|\kappa_t^u|^2] < \infty$ and $\mathbb{E}[\kappa_T^u] = 1$. In particular, κ^u is a square integrable \mathbb{G} -martingale for every $u \in \mathcal{C}$.

By Lemma 5.9 we can define an absolutely continuous probability measure \mathbb{P}_u by setting

$$d\mathbb{P}_u(\omega) = \kappa_T^u(\omega) d\mathbb{P}(\omega).$$

It can then be proven (see e.g. [Jacod \(1975, Theorem 4.5\)](#)) that the \mathbb{G} -compensator $v^{Z,u}$ of μ^Z under \mathbb{P}_u is given by (5.11). We consider the control problem

$$\inf_{u \in \mathcal{C}} J(u) = \inf_{u \in \mathcal{C}} \mathbb{E}_u \left[\int_0^T l_t(X_t, u_t) dC_t^{\mathbb{G}, X} + g(X_T) \right]. \quad (5.16)$$

Notice that in the optimal control problem (5.16) the action under \mathbb{P}_u consists in multiplying by $r_t(x_1, u_t)$ the density with respect to $C^{\mathbb{G}, X}$ of the \mathbb{G} -predictable compensator of μ^X . At the same time, the \mathbb{G} -predictable compensator of H under \mathbb{P}_u remains unchanged, see (5.11).

Remark 5.10. The cost functional J in (5.10) is finite for every admissible control. Indeed, we have

$$\mathbb{E}_u[|g(X_T)|] = \mathbb{E}[|\kappa_T^u g(X_T)|] \leq (\mathbb{E}[|\kappa_T^u|^2])^{1/2} (\mathbb{E}[|g(X_T)|^2])^{1/2} < \infty, \quad (5.17)$$

where the latter estimate follows from (5.9) in Assumptions 5.7. This shows that $g(X_T)$ is \mathbb{P}_u -integrable. Moreover, under Assumption 5.6, by (5.5) and (5.7), we get

$$\mathbb{E}_u \left[\int_0^T l_t(X_t, u_t) dC_t^{\mathbb{G}, X} \right] = \mathbb{E}_u \left[\int_0^T l_t(X_t, u_t) d_1(t) dC_t^{\mathbb{G}, Z} \right] \leq M_l \mathbb{E}_u[C_T] < \infty.$$

The associated BSDE. We next proceed to the solution of the optimal control problem formulated above. A fundamental role is played by the following BSDE

$$Y_t + \int_t^T \int_{\mathbb{R}^{d+1}} \Theta_s(x_1, x_2) (\mu^Z - v^{\mathbb{G}, Z})(ds, dx_1, dx_2) = g(X_T) + \int_t^T f(s, X_s, \Theta_s(\cdot)) dC_s^{\mathbb{G}, X} \quad (5.18)$$

whose generator is the Hamiltonian function

$$f(\omega, t, y_1, \theta(\cdot)) := \inf_{u \in U} \left\{ l_t(\omega, y_1, u) + \int_{\mathbb{R}^d} \theta(x_1, 0) (r_t(\omega, x_1, u) - 1) \phi_t^{\mathbb{G}, X}(\omega, dx_1) \right\}, \quad (5.19)$$

for every $\omega \in \Omega$, $t \in [0, T]$, $y_1 \in \mathbb{R}^d$, $y_2 \in \mathbb{R}$ and $\theta \in \mathcal{L}^1(\mathbb{R}^{d+1}, \mathcal{B}(\mathbb{R}^{d+1}), \phi_t^{\mathbb{G}, Z}(\omega, dx_1, dx_2))$, where \mathbb{L}^1 denotes the usual space of integrable functions.

For $\beta > 0$, we look for a solution $(Y, \Theta(\cdot))$ to (5.18) in the space $L_{\text{Prog}}^{2, \beta}(\Omega \times [0, T], \mathbb{G}) \times L^{2, \beta}(\mu^Z, \mathbb{G})$, where $L_{\text{Prog}}^{2, \beta}(\Omega \times [0, T], \mathbb{G})$ denotes the set of real-valued \mathbb{G} -progressively measurable processes Y such that $\mathbb{E}[\int_0^T e^{\beta C_t^{\mathbb{G}, Z}} |Y_t|^2 dC_t^{\mathbb{G}, Z}] < \infty$, and $L^{2, \beta}(\mu^Z, \mathbb{G})$ the set of $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^{d+1})$ -measurable functions Θ such that $\mathbb{E}[\int_0^T \int_{\mathbb{R}^d} e^{\beta C_t^{\mathbb{G}, Z}} |\Theta_t(x_1, 0)|^2 \phi_t^{\mathbb{G}, X}(dx_1) dC_t^{\mathbb{G}, X}] + \mathbb{E}[\int_0^T e^{\beta C_t^{\mathbb{G}, Z}} |\Theta_t(0, 1)|^2 d\Lambda_t^{\mathbb{G}}] < \infty$.

The set of $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^{d+1})$ -measurable functions Θ satisfying

$$\mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} |\Theta_t(x_1, 0)| \phi_t^{\mathbb{G}, X}(dx_1) dC_t^{\mathbb{G}, X} \right] + \mathbb{E} \left[\int_0^T |\Theta_t(0, 1)| d\Lambda_t^{\mathbb{G}} \right] < \infty$$

is denoted by $L^{1, 0}(\mu^Z, \mathbb{G})$. Notice that the inclusion $L^{2, \beta}(\mu^Z) \subseteq L^{1, 0}(\mu^Z)$ holds, for all $\beta > 0$, see [Confortola and Fuhrman \(2013, Remark 3.2-2\)](#).

To ensure the existence of the optimal control, we are going to work under the following additional assumption:

Assumption 5.11. For every $\Theta \in L^{1,0}(\mu^Z, \mathbb{G})$ there exists a \mathbb{G} -predictable U -valued process (i.e., an admissible control) $\underline{u}^\Theta : \Omega \times [0, T] \rightarrow U$, such that, for $d_1(\omega, t) dC_t^{\mathbb{G}, Z}(\omega) \mathbb{P}(d\omega)$ -almost all (ω, t) , we have

$$\begin{aligned} f(\omega, t, X_{t-}(\omega), \Theta_t(\omega, \cdot)) &= l_t(\omega, X_{t-}(\omega), \underline{u}^\Theta(\omega, t)) \\ &+ \int_{\mathbb{R}^d} \Theta_t(\omega, x_1, 0) (r_t(\omega, x_1, \underline{u}^\Theta(\omega, t)) - 1) \phi_t^{\mathbb{G}, X}(\omega, dx_1). \end{aligned} \quad (5.20)$$

Remark 5.12. Using appropriate measurable selection theorems, it is possible to state general sufficient conditions ensuring Assumption 5.11. For example, this is the case if U is a compact metric space and $l_t(\omega, x, \cdot), r_t(\omega, x, \cdot) : U \rightarrow \mathbb{R}$ are continuous functions, see Confortola and Fuhrman (2013, Proposition 4.8)).

From here on we will denote

$$L := \operatorname{ess\,sup}_\omega \left(\sup \{ |r_t(\omega, x_1, u) - 1| : t \in [0, T], x_1 \in \mathbb{R}^d, u \in U \} \right) \quad (5.21)$$

Thanks to the WRP for Z with respect to \mathbb{G} provided in Theorem 3.1, one can show the existence and the uniqueness of the solution of BSDE (5.18). The proof of the proposition below is postponed to Appendix B.

Proposition 5.13. *Let Assumptions 5.1, 5.5, 5.6 and 5.11 hold true. Assume that Assumption 5.2 or 5.3 holds true. Let Assumption 5.7 hold true with $\beta > L^2$. Then BSDE (5.18) admits a unique solution $(Y, \Theta(\cdot)) \in L_{Prog}^{2,\beta}(\Omega \times [0, T], \mathbb{G}) \times L^{2,\beta}(\mu^Z, \mathbb{G})$.*

Solution to the optimal control problem. At this point we can give the main result of the section.

Theorem 5.14. *Let Assumptions 5.1, 5.5, 5.6 and 5.11 hold true. Assume also that Assumption 5.7 holds true with $\beta > L^2$, and that condition (5.15) holds true. Let Assumption 5.2 or 5.3 holds true, and let $(Y, \Theta(\cdot)) \in L_{Prog}^{2,\beta}(\Omega \times [0, T], \mathbb{G}) \times L^{2,\beta}(\mu^Z, \mathbb{G})$ denote the unique solution to BSDE (5.18), with corresponding admissible control $\underline{u}^\Theta \in \mathcal{C}$ satisfying (5.20). Then \underline{u}^Θ is optimal and Y_0 is the optimal cost, i.e.*

$$Y_0 = J(\underline{u}^\Theta) = \inf_{u \in \mathcal{C}} J(u).$$

Proof: The proof consists in proving the so-called fundamental relation. We first recall that, by Lemma 5.9, for every $u \in \mathcal{C}$ we have $\sup_{t \in [0, T]} \mathbb{E}[|\kappa_t^u|^2] < \infty$. Moreover, by (5.17), $\mathbb{E}_u[|g(X_T)|] < +\infty$.

Let $u \in \mathcal{C}$ be fixed. Then, Hölder's inequality and Assumption 5.6 yield $\Theta(\cdot) \in L^{1,0}(\mu^Z, \mathbb{G})$ under \mathbb{P}_u . Setting $t = 0$ and taking the expectation $\mathbb{E}_u[\cdot]$ in BSDE (5.18), we get

$$\begin{aligned} Y_0 + \mathbb{E}_u \left[\int_0^T \int_{\mathbb{R}^d} \Theta(x_1, 0) (r_s(x_1, u) - 1) \phi_s^{\mathbb{G}, X}(dx_1) dC_s^{\mathbb{G}, X} \right] = \\ \mathbb{E}_u[g(X_T)] + \mathbb{E}_u \left[\int_0^T f(s, X_s, \Theta_s(\cdot)) dC_s^{\mathbb{G}, X} \right]. \end{aligned}$$

Then, adding and subtracting $\mathbb{E}_u \left[\int_0^T l_s(X_s, u) dC_s^{X, \mathbb{G}} \right]$, we obtain

$$\begin{aligned} Y_0 = J(u) \\ + \mathbb{E}_u \left[\int_0^T \left[f(s, X_s, \Theta_s(\cdot)) - l_s(X_s, u) - \int_{\mathbb{R}^d} \Theta_s(x_1, 0) (r_s(x_1, u) - 1) \phi_s^{\mathbb{G}, X}(dx_1) \right] dC_s^{\mathbb{G}, X} \right] \end{aligned}$$

where we have also used the continuity of C . The conclusion follows from the definition of f in (5.19), noticing that the term in the square brackets is non positive, and it equals 0 if $u(\cdot) = \underline{u}^\ominus(\cdot)$. \square

5.2. The control problem up to default. We now consider the problem of an agent for whom the available information is exclusively given by \mathbb{F} (that is, she pursues \mathbb{F} -predictable strategies) but, for some reasons, she has only access to the market up to the occurrence of the exogenous shock event, whose occurrence time is modeled by τ . For example, the problem over $[0, T \wedge \tau]$ can be regarded as the optimization problem of an agent who minimizes running costs not up to the maturity $T > 0$, but only up to $T \wedge \tau$.

We still consider a measurable space (U, \mathcal{U}) satisfying Assumption 5.5. The other data specifying the optimal control problem are a running cost function \bar{l} and function \bar{r} that are $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$ -measurable, and a terminal cost \bar{g} that is $\mathcal{G}_{T \wedge \tau} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable. We assume that \bar{l} and \bar{r} verify (5.7) and that \bar{r} and \bar{g} verify (5.8)-(5.9) with $\beta > L^2$, with L the constant in (5.21) where $r = \bar{r}$.

Let \mathcal{C} be the set of admissible strategies for the optimization problem introduced in Section 5.1. For any $u \in \mathcal{C}$, we define $\hat{u} := 1_{[0, T \wedge \tau]}u$. Clearly $\hat{u} \in \mathcal{C}$. We then introduce the new set of admissible strategies as

$$\hat{\mathcal{C}} := \{u \in \mathcal{C} : 1_{[T \wedge \tau, T]}u = 0\} \subseteq \mathcal{C}. \quad (5.22)$$

Since \mathbb{F} -predictable and \mathbb{G} -predictable processes coincide on $[0, \tau]$ (see Jeulin (1980, Lemma 4.4. b))), the set $\hat{\mathcal{C}}$ given in (5.22) consists of strategies which are morally \mathbb{F} -predictable.

For any $\hat{u} \in \hat{\mathcal{C}}$, we consider then the Doléans-Dade exponential martingale $\kappa^{\hat{u}}$ defined as in (5.14) with r replaced by \bar{r} , and we introduce the absolutely continuous probability measure $\mathbb{P}_{\hat{u}}$ defined as $d\mathbb{P}_{\hat{u}}(\omega) = \kappa_T^{\hat{u}}(\omega)d\mathbb{P}(\omega)$. We then take a cost functional of the form

$$\bar{J}(\hat{u}) = \mathbb{E}_{\hat{u}} \left[\int_0^{T \wedge \tau} \bar{l}_t(X_t, \hat{u}_t) dC_t^{\mathbb{F}, X} + \bar{g}(X_{T \wedge \tau}) \right], \quad \hat{u} \in \hat{\mathcal{C}}, \quad (5.23)$$

where $\mathbb{E}_{\hat{u}}$ denotes the expectation under $\mathbb{P}_{\hat{u}}$. The control problem is now

$$\inf_{\hat{u} \in \hat{\mathcal{C}}} \bar{J}(\hat{u}) = \inf_{\hat{u} \in \hat{\mathcal{C}}} \mathbb{E}_{\hat{u}} \left[\int_0^{T \wedge \tau} \bar{l}_t(X_t, \hat{u}_t) dC_t^{\mathbb{F}, X} + \bar{g}(X_{T \wedge \tau}) \right]. \quad (5.24)$$

Remark 5.15. For simplicity, in the present section we consider only the case where Assumptions 5.1-5.2 are satisfied, so that, according to Proposition 5.4, $dC^{\mathbb{F}, X} = dC^{\mathbb{G}, X}$ and $\phi^{\mathbb{F}, X}(dx_1) = \phi^{\mathbb{G}, X}(dx_1)$. This seems to be a natural assumption here. Indeed, let Jacod's assumption hold for τ . Then, by Jeanblanc and Le Cam (2009, Corollary 3.1), τ satisfies (\mathcal{H}) if and only if $p_\cdot(u)$ is constant after u , that is, $p_t(u) = p_t(t)$, $t \geq u$, a.s. As observed in El Karoui et al. (2010, p.1016), this is substantially equivalent to say that the ‘‘information contained in the reference filtration after the default time gives no new information on the conditional distribution of the default’’. But, since we restrict our attention to $[0, T \wedge \tau]$, that is, before the default, we are neglecting all information after default.

Remark 5.16. The control problem in (5.24) can be interpreted as the one of an agent who only controls X using \mathbb{F} -predictable strategies but only up to the occurrence τ of an external risky event. Hence, because of the exogenous risk source, this control problem cannot be solved in \mathbb{F} .

The functional cost in optimal control problem in (5.24) can be equivalently rewritten in the form

$$\mathbb{E}_{\hat{u}} \left[\int_0^T l_t(X_t, \hat{u}_t) 1_{[0, T \wedge \tau]}(t) dC_t^{\mathbb{F}, X} + g \right], \quad \hat{u} \in \hat{\mathcal{C}},$$

with

$$l_t(\omega, x_1, u) := \bar{l}_t(\omega, x_1, u) 1_{[0, T \wedge \tau(\omega)]}(t), \quad (5.25)$$

$$g(\omega) = g(\omega, x_1) := \bar{g}(\omega, X_{T \wedge \tau}(\omega)). \quad (5.26)$$

Clearly l in (5.25) is $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$ -measurable and g in (5.26) is \mathcal{G}_T -measurable, so that the control problem in (5.24) can be seen as one of the type studied in Section 5.1 (where the subclass \mathcal{C} of admissible controls is considered).

Let us now consider the *enlarged* optimal control problem obtained from (5.24) by taking the infimum over all the \mathbb{G} -predictable processes $u(\cdot)$:

$$\inf_{u \in \mathcal{C}} \bar{J}(u) = \mathbb{E}_u \left[\int_0^T l_t(X_t, u_t) dC_t^{\mathbb{F}, X} + g \right]. \quad (5.27)$$

According to Section 5.1, one can solve optimal control problem (5.27) by considering the following BSDE: \mathbb{P} -a.s., for all $t \in [0, T]$,

$$Y_t + \int_t^T \int_{\mathbb{R}^{d+1}} \Theta_s(x_1, x_2) (\mu^Z - \nu^{\mathbb{G}, Z})(ds, dx_1, dx_2) = g + \int_t^T f(s, X_s, \Theta_s(\cdot)) dC_t^{\mathbb{F}, X}, \quad (5.28)$$

with f in (5.19) with $r = \bar{r}$ and $\phi^{\mathbb{G}, X} = \phi^{\mathbb{F}, X}$. Notice that l in (5.25) and g in (5.26) satisfy respectively Assumptions 5.6 and 5.7 with $\beta > L$. Then, under Assumption 5.11, there exists by Proposition 5.13 a unique solution $(Y, \Theta(\cdot)) \in L_{\text{Prog}}^{2, \beta}(\Omega \times [0, T], \mathbb{G}) \times L^{2, \beta}(\mu^Z, \mathbb{G})$ of BSDE (5.28) with corresponding admissible control $\underline{u}^\Theta \in \mathcal{C}$. Moreover, under (5.15), by Theorem 5.14

$$Y_0 = \bar{J}(\underline{u}^\Theta) = \inf_{u \in \mathcal{C}} \bar{J}(u).$$

Let us now go back to optimal control problem (5.24): we aim at finding an admissible process $\hat{u} \in \mathcal{C}$ such that

$$\bar{J}(\hat{u}) = \inf_{\hat{u} \in \mathcal{C}} \bar{J}(\hat{u}).$$

We show below that such an optimal control process exists and is provided by

$$\hat{u}^\Theta := \underline{u}^\Theta 1_{[0, T \wedge \tau]} \in \mathcal{C}, \quad (5.29)$$

and that the value functions of the optimal control problems (5.24) and (5.27) coincide (the corresponding proof is provided in Appendix B).

Theorem 5.17. *Let Assumption 5.11 hold true and assume the validity of condition (5.15). Then*

$$Y_0 = \bar{J}(\hat{u}^\Theta) = \inf_{\hat{u} \in \mathcal{C}} \bar{J}(\hat{u}) = \inf_{u \in \mathcal{C}} \bar{J}(u). \quad (5.30)$$

6. An Example

In this section we give an example on the optimization of the expected exponential utility of the terminal wealth under the worst-case scenario. To this aim, we assume that X is a compound Poisson process, $\mathbb{F} = \mathbb{F}^X$, and the random time τ satisfies assumptions (\mathcal{A}) - (\mathcal{H}) . The \mathbb{G} -predictable compensator $\Lambda^{\mathbb{G}}$ is of the form

$$\Lambda_t^{\mathbb{G}}(\omega) = \int_0^{t \wedge \tau(\omega)} \lambda_s(\omega) ds,$$

where λ is a bounded and \mathbb{F} -predictable process. Denoting by ρ the Lévy measure of X , we have $\phi^{\mathbb{F}, X}(dx_1) = \rho(dx_1)\rho^{-1}(\mathbb{R})$ and $dC_t^{\mathbb{F}, X} = \rho(\mathbb{R})dt$. So, by Theorem 4.7, the \mathbb{G} -dual predictable projection of μ^Z is

$$v^{\mathbb{G},Z}(\omega, dt, dx_1, dx_2) = \rho(dx_1)\delta_0(dx_2)dt + \delta_1(dx_2)\delta_0(dx_1)d\Lambda_t^{\mathbb{G}}(\omega).$$

Finally, by Proposition 4.6, we have $C_t^{\mathbb{G},X} = C_t^{\mathbb{F},X}$. Therefore,

$$C_t^{\mathbb{G},Z}(\omega) = \rho(\mathbb{R})t + \int_0^{t \wedge \tau(\omega)} \lambda_s(\omega) ds.$$

Notice that, because of the boundedness of λ , $C_T^{\mathbb{G}}$ obviously satisfies (5.8) for every β .

We now follow Becherer et al. (2019, §5.1.2) and consider a stock price of the form $S = S_0 \mathcal{E}(L)$ ($\mathcal{E}(\cdot)$ denoting the stochastic exponential), where, for an \mathbb{F} -predictable bounded process b and an \mathbb{F} -predictable bounded function G , we set

$$L_t := \int_0^t b_s ds + G * (\mu^X - v^{\mathbb{F},X})_t$$

We also assume $G > -1$, so that $S > 0$ holds. Let us consider an agent who maximizes the expected exponential utility of the terminal wealth under the worst-case scenario in presence of a bounded and defaultable (that is \mathcal{G}_T -measurable) claim ξ which may represent a liability $\xi > 0$ or an asset $\xi < 0$.

An admissible trading strategy π is a \mathbb{G} -predictable process taking values in a compact set $D \subseteq \mathbb{R}$. We denote by \mathcal{D} the set of admissible strategies. The wealth process associated to $\pi \in \mathcal{D}$, and with the initial endowment $w > 0$, is defined by $W^{w,\pi} := w + \int_0^\cdot \pi_s dL_s$. The optimization problem becomes

$$\sup_{\pi \in \mathcal{D}} \inf_{u \in \mathcal{C}} \mathbb{E}_u \left[-\exp(-\alpha(W_T^{w,\pi} - \xi)) \right], \quad \alpha > 0, \quad (6.1)$$

meaning that the agent first selects the worst-case model and then she optimizes over the trading strategies. The inner problem

$$\inf_{u \in \mathcal{C}} \mathbb{E}_u \left[-\exp(-\alpha(W_T^{w,\pi} - \xi)) \right], \quad \alpha > 0, \quad (6.2)$$

corresponds to (5.10) with $l \equiv 0$ and $g(\omega, x) = g(\omega) = -\exp(-\alpha(W_T^{w,\pi}(\omega) - \xi(\omega)))$.

Observe that g satisfies (5.9). Indeed, for every $\beta > 0$, since $0 \leq \lambda_t(\omega) \leq c$, for a certain $c > 0$, we have

$$\mathbb{E}[g^2 e^{\beta C_T^{\mathbb{G},Z}}] \leq \mathbb{E} \left[\exp(-2\alpha(W_T^{w,\pi} - \xi)) \right] e^{\beta(\rho(\mathbb{R})+c)T} < +\infty,$$

where, in the last estimate, we have used Morlais (2010, Lemma 2) because D is compact, ξ is bounded and $\mu^X - v^{\mathbb{F},X}$ is a compensated Poisson random measure under the reference probability \mathbb{P} .

To ensure the existence of an optimal control \underline{u}^\ominus satisfying (5.20), we require that U is a compact subset of \mathbb{R} (see Assumption 5.5), and hence we assume $\mathcal{U} = \mathcal{B}(U)$. Furthermore, the \mathbb{G} -predictable function $r_t(x_1, u)$ that rules the model uncertainty on the compensator of μ^X under \mathbb{P}_u (see (5.11)), is assumed to be bounded (see (5.7)), and continuous in u , see Remark 5.12.

According to Theorem 5.14, we can then represent the value function of (6.2) as

$$Y_0^\pi = \mathbb{E}_{\underline{u}^\ominus} \left[-\exp(-\alpha(W_T^{w,\pi} - \xi)) \right],$$

where Y^π is the first component of the solution of BSDE (5.18) with the generator given in (5.20) with $l \equiv 0$ and $\phi^{\mathbb{G},X} = \phi^{\mathbb{F},X}$. Therefore, to solve (6.1), it is now sufficient to solve

$$\sup_{\pi \in \mathcal{D}} \mathbb{E}_{\underline{u}^\ominus} \left[-\exp(-\alpha(W_T^{w,\pi} - \xi)) \right]. \quad (6.3)$$

To this aim, we observe that, since $\mathbb{P}_{\underline{u}^\ominus} \ll \mathbb{P}$, the step process $Z = (X, H)$ has the WRP with respect to \mathbb{G} also under $\mathbb{P}_{\underline{u}^\ominus}$, see [Jacod and Shiryaev \(2003, Theorem III.5.24\)](#). Furthermore, under $\mathbb{P}_{\underline{u}^\ominus}$ the \mathbb{G} -dual predictable projection $\mathbf{v}^{\mathbb{G}, Z, \underline{u}^\ominus}$ of μ^Z is given by (5.11) and, from [Theorem 4.7](#), we get

$$\mathbf{v}^{\mathbb{G}, Z, \underline{u}^\ominus}(\omega, dt, dx_1, dx_2) = \zeta_t^{\mathbb{G}, \underline{u}^\ominus}(\omega, x_1, x_2) \eta(dx_1, dx_2) dt,$$

where

$$\begin{aligned} \zeta_t^{\mathbb{G}, \underline{u}^\ominus}(\omega, x_1, x_2) &= r_t(\omega, x_1, \underline{u}^\ominus) 1_{\{x_1 \neq 0, x_2 = 0\}} + \lambda_t(\omega) 1_{[0, \tau]}(\omega, t) 1_{\{x_1 = 0, x_2 \neq 0\}}, \\ \eta(dx_1, dx_2) &= \rho(dx_1) \delta_0(dx_2) + \delta_0(dx_1) \delta_1(dx_2). \end{aligned}$$

To represent the value function of (6.3), we aim at constructing a family $\{R^\pi, \pi \in \mathcal{D}\}$ of processes satisfying the martingale optimality principle, that is, such that: (1) R^π is a supermartingale whose initial value R_0^π does not depend on π and (2) there exists $\pi^* \in \mathcal{D}$ for which R^{π^*} is a true martingale. In this way, one can show that π^* is optimal and the value function in (6.3) equals $R_0^{\pi^*}$.

To this end, we set

$$R_t^\pi := -\exp(-\alpha(W_t^{w, \pi} - Y_t)),$$

where Y is the first component of the solution of the BSDE

$$Y_t = \xi + \int_t^T f(s, V_s(\cdot)) ds + \int_t^T \int_{\mathbb{R} \times \mathbb{R}} V_s(x_1, x_2) (\mu^Z - \mathbf{v}^{\mathbb{G}, Z, \underline{u}^\ominus})(ds, dx_1, dx_2) \quad (6.4)$$

with generator

$$f(\omega, t, v(\cdot)) := \inf_{\pi \in D} \left(-\pi b_t(\omega) + \int_{\mathbb{R} \times \mathbb{R}} g_\alpha(v(x_1, x_2) - \pi \tilde{G}_t(\omega, x_1, x_2)) \zeta_t^{\mathbb{G}, \underline{u}^\ominus}(\omega, x_1, x_2) \eta(dx_1, dx_2) \right), \quad (6.5)$$

where

$$g_\alpha(x) := \frac{(e^{\alpha x} - \alpha x - 1)}{\alpha}, \quad \tilde{G}_t(\omega, x_1, x_2) := G_t(\omega, x_1) 1_{\{x_2 = 0\}}.$$

Thanks to [Becherer \(2006, Theorem 3.5\)](#), BSDE (6.4) admits a unique bounded solution $(Y, V(\cdot))$. By a standard computation relying on [Jacod and Shiryaev \(2003, Theorem II.8.10\)](#), we obtain the representation

$$R_t^\pi = M_t^\pi A_t^\pi \exp(-\alpha(w - Y_0)),$$

with

$$\begin{aligned} M_t^\pi &:= \mathcal{E} \left(\int_0^t \int_{\mathbb{R} \times \mathbb{R}} \left(\exp(\alpha(V_s(x_1, x_2) - \pi_s \tilde{G}_s(x_1, x_2))) - 1 \right) (\mu^Z - \mathbf{v}^{\mathbb{G}, Z, \underline{u}^\ominus})(dx_1, dx_2) \right), \\ A_t^\pi &:= -\exp \left[\alpha \int_0^t \left(-f(s, V_s(\cdot)) - \pi_s b_s + \int_{\mathbb{R} \times \mathbb{R}} g_\alpha(V_s(x_1, x_2) - \pi_s \tilde{G}_s(x_1, x_2)) \zeta_s^{\mathbb{G}, \underline{u}^\ominus}(x_1, x_2) \eta(dx_1, dx_2) \right) ds \right]. \end{aligned}$$

So, M^π is a uniformly integrable martingale by [Izumisawa et al. \(1979, Theorem 2\)](#)¹ and A^π is decreasing. Therefore, R^π is a supermartingale for every $\pi \in \mathcal{D}$. By dominated convergence the mapping

$$\pi \mapsto -\pi b_t(\omega) + \int_{\mathbb{R}} g_\alpha(V_t(\omega, x_1, x_2) - \pi \tilde{G}_t(\omega, x_1, x_2)) \zeta_t^{\mathbb{G}, \underline{u}^\ominus}(\omega, x_1, x_2) \eta(dx_1, dx_2)$$

¹Indeed, by the boundedness of $\alpha(V - \pi \tilde{G})$, η being a finite measure, we get that $\exp(\alpha(V - \pi \tilde{G})) - 1 * (\mu^Z - \mathbf{v}^{\mathbb{G}, Z, \underline{u}^\ominus})$ is a true martingale in $BMO(\mathbb{G})$ over $[0, T]$ under $\mathbb{P}_{\underline{u}^\ominus}$, and its jumps are bounded away from -1 .

is continuous, for every (t, ω) . Hence, by optimal selection (see e.g. Bertsekas and Shreve (1996, Proposition 7.33)), we find a \mathbb{G} -predictable process $\pi^* \in \mathcal{D}$ such that $R_t^{\pi^*} = -M_t^{\pi^*} \exp(-\alpha(w - Y_0))$. Therefore, $\{R^\pi, \pi \in \mathcal{D}\}$ satisfies the martingale optimality principle under $\mathbb{P}_{\hat{\mu}^\theta}$, and we finally get

$$\sup_{\pi \in \mathcal{D}} \inf_{u \in \mathcal{C}} \mathbb{E}_u \left[-\exp(-\alpha(W_T^{w,\pi} - \xi)) \right] = \mathbb{E}_{\hat{\mu}^\theta} \left[-\exp(-\alpha(W_T^{w,\pi^*} - \xi)) \right] = -\exp(-\alpha(w - Y_0)).$$

We stress that, if we introduce $\hat{\mathcal{D}} = \{1_{[0, T \wedge \tau]} \pi, \pi \in \mathcal{D}\}$, we can solve the problem

$$\sup_{\hat{\pi} \in \hat{\mathcal{D}}} \inf_{\hat{u} \in \hat{\mathcal{C}}} \mathbb{E}_{\hat{u}} \left[-\exp(-\alpha(W_{T \wedge \tau}^{w,\hat{\pi}} - \xi)) \right], \quad \alpha > 0, \quad \xi \text{ bounded and } \mathcal{G}_{T \wedge \tau} \text{ measurable}$$

exactly in the same way. The main difference is that now (6.4) has to be considered over the random time interval $[0, T \wedge \tau]$ and the generator f in (6.5) must be replaced by $1_{[0, T \wedge \tau]} f$.

Appendix.

A. Proofs of technical results of Section 4.

Proof of Theorem 4.1: Let us start by proving (i). We have

$$\begin{aligned} \mu^Z(\omega, dt, dx_1, dx_2) &= \sum_{s>0} 1_{\{\Delta Z_s(\omega) \neq 0\}} \delta_{(s, \Delta Z_s(\omega))}(dt, dx_1, dx_2) \\ &= \sum_{s>0} 1_{\{\Delta X_s(\omega) \neq 0, \Delta H_s(\omega) \neq 0\}} \delta_{(s, (\Delta X_s(\omega), \Delta H_s(\omega)))}(dt, dx_1, dx_2) \\ &\quad + \sum_{s>0} 1_{\{\Delta X_s(\omega) \neq 0, \Delta H_s(\omega) = 0\}} \delta_{(s, (\Delta X_s(\omega), 0))}(dt, dx_1, dx_2) \\ &\quad + \sum_{s>0} 1_{\{\Delta X_s(\omega) = 0, \Delta H_s(\omega) \neq 0\}} \delta_{(s, (0, \Delta H_s(\omega)))}(dt, dx_1, dx_2). \end{aligned}$$

Now we notice that, since by assumption $\Delta X \Delta H = 0$, we have $\{\Delta X \neq 0\} \cap \{\Delta H \neq 0\} = \emptyset$. Therefore, noticing that moreover $\{\Delta X \neq 0\} \subseteq \{\Delta H = 0\}$ and $\{\Delta H \neq 0\} \subseteq \{\Delta X = 0\}$, previous expression reads

$$\begin{aligned} \mu^Z(\omega, dt, dx_1, dx_2) &= \sum_{s>0} 1_{\{\Delta X_s(\omega) \neq 0\}} \delta_{(s, \Delta X_s(\omega))}(dt, dx_1) \delta_0(dx_2) \\ &\quad + \sum_{s>0} 1_{\{\Delta H_s(\omega) \neq 0\}} \delta_{(s, \Delta H_s(\omega))}(dt, dx_2) \delta_0(dx_1) \\ &= \mu^X(\omega, dt, dx_1) \delta_0(dx_2) + \mu^H(\omega, dt, dx_2) \delta_0(dx_1). \end{aligned}$$

Let us now prove (ii). Set

$$v^{\mathbb{G}, Z}(\omega, dt, dx_1, dx_2) := v^{\mathbb{G}, X}(\omega, dt, dx_1) \delta_0(dx_2) + v^{\mathbb{G}, H}(\omega, dt, dx_2) \delta_0(dx_1).$$

We have to prove that $v^{\mathbb{G}, Z}$ is the \mathbb{G} -dual predictable projection of μ^Z . To this end it is sufficient to show that, for every \mathbb{G} -predictable function W satisfying $W \geq 0$ and $W * \mu^Z \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$, the process $W * \mu^Z - W * v^{\mathbb{G}, Z} \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$. So, let us consider such a \mathbb{G} -predictable function W . We then have

$$W * \mu_t^Z = \int_0^t \int_{\mathbb{R}^d} W(\omega, s, x_1, 0) \mu^X(\omega, ds, dx_1) + \int_0^t \int_{\mathbb{R}^\ell} W(\omega, s, 0, x_2) \mu^H(\omega, ds, dx_2)$$

and

$$W * v_t^{\mathbb{G}, Z} = \int_0^t \int_{\mathbb{R}^d} W(\omega, s, x_1, 0) v^{\mathbb{G}, X}(\omega, ds, dx_1) + \int_0^t \int_{\mathbb{R}^\ell} W(\omega, s, 0, x_2) v^{\mathbb{G}, H}(\omega, ds, dx_2).$$

Since $W * \mu^Z \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$ and $W \geq 0$, we get

$$\int_0^\cdot \int_{\mathbb{R}^d} W(s, x_1, 0) \mu^X(ds, dx_1), \int_0^\cdot \int_{\mathbb{R}^\ell} W(s, 0, x_2) \mu^H(ds, dx_2) \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$$

and therefore

$$\int_0^\cdot \int_{\mathbb{R}^d} W(s, x_1, 0) \mathbf{v}^{\mathbb{G}, X}(ds, dx_1), \int_0^\cdot \int_{\mathbb{R}^\ell} W(s, 0, x_2) \mathbf{v}^{\mathbb{G}, H}(ds, dx_2) \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$$

that yields $W * \mathbf{v}^{\mathbb{G}, Z} \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$. It remains to show that $W * \mu^Z - W * \mathbf{v}^{\mathbb{G}, Z} \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$. By definition of $\mathbf{v}^{\mathbb{G}, Z}$ we have

$$\begin{aligned} W * \mu^Z - W * \mathbf{v}^{\mathbb{G}, Z} &= \int_0^\cdot \int_{\mathbb{R}^d} W(\omega, s, x_1, 0) \mu^X(\omega, ds, dx_1) - \int_0^\cdot \int_{\mathbb{R}^d} W(\omega, s, x_1, 0) \mathbf{v}^{\mathbb{G}, X}(\omega, ds, dx_1) \\ &\quad + \int_0^\cdot \int_{\mathbb{R}^\ell} W(\omega, s, 0, x_2) \mu^H(\omega, ds, dx_2) - \int_0^\cdot \int_{\mathbb{R}^\ell} W(\omega, s, 0, x_2) \mathbf{v}^{\mathbb{G}, H}(\omega, ds, dx_2). \end{aligned}$$

By linearity, it follows that $W * \mu^Z - W * \mathbf{v}^{\mathbb{G}, Z} \in \mathcal{H}_{\text{loc}}^1(\mathbb{G})$, $\mathbf{v}^{\mathbb{G}, X}$ and $\mathbf{v}^{\mathbb{G}, H}$ being the \mathbb{G} -compensator of μ^X and μ^H , respectively. Let now W be an arbitrary \mathbb{G} -predictable function such that $|W| * \mu^Z \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$. Applying the previous step to W^+ and W^- we get that $|W| * \mathbf{v}^{\mathbb{G}} \in \mathcal{A}_{\text{loc}}^+(\mathbb{G})$ and $W * \mu^Z - W * \mathbf{v}^{\mathbb{G}, Z} \in \mathcal{M}_{\text{loc}}^+(\mathbb{G})$. \square

Proof of Lemma 4.2: Let W be a \mathbb{G} -predictable bounded function of the form $W(\omega, t, x) = f(x)J_t(\omega)$, where f is a bounded $\mathcal{B}(\mathbb{R}^d)$ -measurable function and J is a bounded \mathbb{G} -predictable process. Then, by [Jeulin \(1980, Lemma 4.4 \(b\)\)](#) we have $W(\omega, t, x)1_{[0, \tau]}(\omega, t) = f(x)\bar{J}_t(\omega)1_{[0, \tau]}(\omega, t)$, where \bar{J} is an \mathbb{F} -predictable bounded process. By a monotone class argument we get the statement for arbitrary bounded \mathbb{G} -predictable functions. Then, by approximation, we get the statement for arbitrary \mathbb{G} -predictable functions. \square

Let m be the martingale defined by

$$m_t = \mathbb{E}[H_\infty^{\mathbb{F}} + 1_{\{\tau = +\infty\}} | \mathcal{F}_t] \quad a.s., \quad t \geq 0, \quad (\text{A.1})$$

where $H^{\mathbb{F}}$ denotes the \mathbb{F} -dual optional projection of H . For every \mathbb{F} -local martingale Y , the process $[Y, m]$ belongs to $\mathcal{A}_{\text{loc}}(\mathbb{F})$. Therefore, the \mathbb{F} -dual predictable projection $[Y, m]^{\mathbb{F}}$ of $[Y, m]$ is well defined and we set $\langle Y, m \rangle^{\mathbb{F}} := [Y, m]^{\mathbb{F}}$.

Proof of Proposition 4.4: Because of the \mathbb{F} -quasi-left continuity of X , $X^{p, \mathbb{F}}$ is an \mathbb{F} -adapted continuous process, see [He et al. \(1992, Corollary 5.28 \(3\)\)](#). Furthermore, by [Aksamit and Jeanblanc \(2017, Theorem 5.1\)](#), the process

$$X^\tau - (X^{p, \mathbb{F}})^\tau - \int_0^{\tau \wedge \cdot} \frac{1}{A_{s-}} d\langle X - X^{p, \mathbb{F}}, m \rangle_s^{\mathbb{F}}$$

is a \mathbb{G} -local martingale, where m is the martingale defined in (A.1). This means that $(X^{p, \mathbb{F}})^\tau + \int_0^{\tau \wedge \cdot} \frac{1}{A_{s-}} d\langle X - X^{p, \mathbb{F}}, m \rangle_s^{\mathbb{F}}$ is the \mathbb{G} -dual predictable projection of X . We denote this process by $(X^{p, \mathbb{G}})^\tau$. Since X is \mathbb{F} -quasi-left continuous, $\langle X - X^{p, \mathbb{F}}, m \rangle^{\mathbb{F}}$ is a continuous process. Indeed, be the property of the dual predictable projection, for every \mathbb{F} -predictable finite valued stopping time σ we have

$$\Delta \langle X - X^{p, \mathbb{F}}, m \rangle_\sigma^{\mathbb{F}} = \mathbb{E}[\Delta \langle X - X^{p, \mathbb{F}}, m \rangle_\sigma | \mathcal{F}_{\sigma-}] = \mathbb{E}[\Delta X_\sigma \Delta m_\sigma | \mathcal{F}_{\sigma-}] = 0.$$

Hence, by the predictable section theorem, $\langle X - X^{p, \mathbb{F}}, m \rangle^{\mathbb{F}}$ being \mathbb{F} -predictable, $\Delta \langle X - X^{p, \mathbb{F}}, m \rangle^{\mathbb{F}} = 0$ up to an evanescent set. Therefore, $(X^{p, \mathbb{F}})^\tau$ being continuous, we deduce that $(X^{p, \mathbb{G}})^\tau$ is continuous as well. Hence, by [He et al. \(1992, Corollary 5.28 \(3\)\)](#), X^τ is \mathbb{G} -quasi-left continuous. \square

Proof of Theorem 4.11: Let $(\omega, t, x) \mapsto W(\omega, t, x)$ be a \mathbb{G} -predictable bounded and nonnegative function. We then have that $W * \mu^X$ is locally bounded, and hence belongs to $\mathcal{A}_{\text{loc}}^+(\mathbb{G})$, because of the estimate $W * \mu^X \leq cN^X$, where $c > 0$ is such that $W(\omega, t, x) \leq c$ and N^X is the point process associated to X . Because of Lemma 4.2 there exists a bounded \mathbb{F} -predictable function \bar{W} such that

$$W = \bar{W}1_{[0, \tau]} + W1_{(\tau, +\infty)}.$$

We now analyse separately the two integrals $\bar{W}1_{[0, \tau]} * \mu^X$ and $W1_{(\tau, +\infty)} * \mu^X$.

We start with $\bar{W}1_{[0, \tau]} * \mu^X$. We observe that $\bar{W} * \mu^X$ is locally bounded and hence it belongs to $\mathcal{A}_{\text{loc}}^+(\mathbb{F})$, \bar{W} being an \mathbb{F} -predictable bounded function. Hence, $(\bar{W} * \mu^X)^{p, \mathbb{F}}$ exists and is equal to $\bar{W} * \nu^{\mathbb{F}, X}$. So, the process $\bar{W} * \mu^X - \bar{W} * \nu^{\mathbb{F}, X}$ is an \mathbb{F} -local martingale and, by [Aksamit and Jeanblanc \(2017, Theorem 5.1\)](#),

$$\bar{W}1_{[0, \tau]} * \mu^X - \bar{W}1_{[0, \tau]} * \nu^{\mathbb{F}, X} - \int_0^{\tau \wedge \cdot} \frac{1}{A_s^-} d\langle \bar{W} * \mu^X - \bar{W} * \nu^{\mathbb{F}, X}, m \rangle^{\mathbb{F}}$$

is a \mathbb{G} -local martingale. Let m be the martingale defined in (A.1). Since m is an \mathbb{F} -martingale, we find an \mathbb{F} -predictable function W^m such that $m = m_0 + W^m * \mu^X - W^m * \nu^{\mathbb{F}, X}$. Furthermore, m is in the class *BMO*. So, the \mathbb{F} -predictable covariation $\langle \bar{W} * \mu^X - \bar{W} * \nu^{\mathbb{F}, X}, m \rangle^{\mathbb{F}}$ is well defined and satisfies $\langle \bar{W} * \mu^X - \bar{W} * \nu^{\mathbb{F}, X}, m \rangle^{\mathbb{F}} = \bar{W}W^m * \nu^{\mathbb{F}, X}$, where we used [Jacod and Shiryaev \(2003, Theorem II.1.13\)](#) and that $\nu^{\mathbb{F}, X}$ is non-atomic in t , X being \mathbb{F} -quasi-left continuous, see [Jacod and Shiryaev \(2003, Proposition II.2.9\)](#). So, by linearity we deduce that

$$\bar{W}1_{[0, \tau]} * \mu^X - \bar{W} \left(1 + \frac{W^m}{A_-} \right) 1_{[0, \tau]} * \nu^{\mathbb{F}, X}$$

is a \mathbb{G} -local martingale. We also have that $\Delta m = W^m(\cdot, \cdot, \Delta X)1_{\{\Delta X \neq 0\}} = \Delta A$. Therefore, we obtain

$$A_- + W^m(\cdot, \cdot, \Delta X)1_{\{\Delta X \neq 0\}} = A_- + \Delta A = A \geq 0.$$

We now introduce the $\tilde{\mathcal{F}}(\mathbb{F})$ -measurable set $D := \{(\omega, t, x) \in \tilde{\Omega} : A_{t-}(\omega) + W^m(\omega, t, x) < 0\}$. We denote by M_{μ^X} the Doléans measure induced by μ^X , that is, $M_{\mu^X}(B) = \mathbb{E}[1_B * \mu_{\infty}^X]$, for every $B \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d)$. We then have $M_{\mu^X}(D) = 0$. Therefore, we can define the \mathbb{F} -predictable function $W'(\omega, t, x) := W^m(\omega, t, x)1_{D^c}(\omega, t, x)$ which again satisfies $m = W' * \mu^X - W' * \nu^{\mathbb{F}, X}$ and moreover $A_{t-}(\omega) + W'(\omega, t, x) \geq 0$ *identically*. We now define the \mathbb{G} -predictable measure

$$\nu^{\mathbb{G}, \leq \tau}(\omega, dt, dx) = 1_{[0, \tau]} \left(1 + \frac{W'(\omega, t, x)}{A_{t-}(\omega)} \right) \nu^{\mathbb{F}, X}(\omega, dt, dx).$$

We then clearly have that $W1_{[0, \tau]} * \mu^X - W * \nu^{\mathbb{G}, \leq \tau}$ is a \mathbb{G} -local martingale.

We now come to the integral $W1_{(\tau, +\infty)} * \mu^X$. To begin with, we introduce the filtration $\mathbb{G}^\tau = (\mathcal{G}_t^\tau)_{t \geq 0}$ by $\mathcal{G}_t^\tau := \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon} \vee \sigma(\tau)$, that is, \mathbb{G}^τ is the initial enlargement of \mathbb{F} by τ . It is clear that $\mathbb{G} \subseteq \mathbb{G}^\tau$ and that $\mathbb{G} = \mathbb{G}^\tau$ over the stochastic interval $(\tau, +\infty]$. Following the proof of [Jacod \(1985, Proposition 3.14 and Theorem 4.1\)](#) we can show that there exists a \mathbb{G}^τ -predictable function U such that $1 + U \geq 0$ and

$$\tilde{\nu}(\omega, dt, dx) = (1 + U(\omega, t, x)) \nu^{\mathbb{F}, X}(\omega, dt, dx)$$

is the \mathbb{G}^τ -dual predictable projection of μ^X . In particular, $W1_{(\tau, +\infty)} * \mu^X$ being \mathbb{G}^τ -adapted and locally bounded, we deduce that $W1_{(\tau, +\infty)} * \mu^X - W1_{(\tau, +\infty)} * \tilde{\nu}$ is a \mathbb{G}^τ local martingale. We now observe that the function $1_{(\tau, +\infty)}(1 + U)$ is indeed \mathbb{G} -predictable, since \mathbb{G} and \mathbb{G}^τ coincides over $(\tau, +\infty]$. This implies that the \mathbb{G}^τ -local martingale $W1_{(\tau, +\infty)} * \mu^X - W1_{(\tau, +\infty)} * \tilde{\nu}$ is actually \mathbb{G} -adapted. Furthermore, this is a martingale with bounded jumps, the process $W1_{(\tau, +\infty)} * \tilde{\nu}$ being continuous and $W1_{(\tau, +\infty)}$

being bounded. We can therefore apply [Jacod \(1979, Proposition 9.18 \(iii\) and the subsequent comment\)](#) to obtain that $W1_{(\tau,+\infty)} * \mu^X - W1_{(\tau,+\infty)} * \tilde{\nu}$ is indeed a \mathbb{G} -local martingale. We now define the \mathbb{G} -predictable random measures $\nu^{\mathbb{G},>\tau}(\omega, dt, dx) := 1_{(\tau,+\infty)}(\omega, t)(1 + U(\omega, t, x))\nu^{\mathbb{F},X}(\omega, dt, dx)$ and

$$\begin{aligned} \nu^{\mathbb{G}}(\omega, dt, dx) &= \nu^{\mathbb{G},\leq\tau}(\omega, dt, dx) + \nu^{\mathbb{G},>\tau}(\omega, dt, dx) \\ &= \left(1_{[0,\tau]}(\omega, t) \left(1 + \frac{W'(\omega, t, x)}{A_{t-}(\omega)} \right) + 1_{(\tau,+\infty)}(\omega, t)(1 + U(\omega, t, x)) \right) \nu^{\mathbb{F},X}(\omega, dt, dx). \end{aligned}$$

Putting together the two previous steps, we get that the process $W * \mu^X - W * \nu^{\mathbb{G}}$ is a \mathbb{G} -local martingale, for every bounded nonnegative \mathbb{G} -predictable function W .

Let now W be a nonnegative \mathbb{G} -predictable function and define $W^n(\omega, t, x) := W(\omega, t, x) \wedge n$. Because of the previous step, the process $W^n * \mu^X - W^n * \nu^{\mathbb{G}}$ is a \mathbb{G} -local martingale. Let $(\sigma_n)_n$ be a localizing sequence. For every $n \geq 0$ we get $\mathbb{E}[W^n 1_{[0,\sigma_n]} * \mu_\infty^X] = \mathbb{E}[W^n 1_{[0,\sigma_n]} * \nu_\infty^{\mathbb{G}}]$. Since $W^n 1_{[0,\sigma_n]}$ converges monotonically to W , by monotone convergence we obtain the identity $\mathbb{E}[W * \mu_\infty^X] = \mathbb{E}[W * \nu_\infty^{\mathbb{G}}]$, for every nonnegative \mathbb{G} -predictable function W . By [Jacod and Shiryaev \(2003, Theorem II.1.18 \(i\)\)](#) we deduce the identity $\nu^{\mathbb{G}} = \nu^{\mathbb{G},X}$. In particular, since $\nu^{\mathbb{F},X}(\{t\} \times \mathbb{R}^d) = 0$ for every t , X being \mathbb{F} -quasi-left continuous, we deduce that $\nu^{\mathbb{G},X}(\{t\} \times \mathbb{R}^d) = 0$ for every t , meaning that X is also \mathbb{G} -quasi-left continuous.

B. Proofs of technical results of Section 5.

Proof of Proposition 5.13: To show the result we apply [Confortola and Fuhrman \(2013, Theorem 3.4\)](#) to the present framework. Let us then check that all the hypotheses of the above-mentioned theorem are satisfied. More precisely, setting

$$F(\omega, t, X_t(\omega), \Theta_t(\omega)) = d_1(\omega, t)f(\omega, t, X_t(\omega), \Theta_t(\omega)),$$

with d_1 the function appearing in (5.5), we have to verify that:

- (1) The terminal cost $g(X_T)$ is \mathcal{G}_T -measurable and there exists $\beta > 0$ such that

$$\mathbb{E}[e^{\beta C_T^{\mathbb{G},Z}} |g(X_T)|^2] < \infty \text{ and } \mathbb{E}\left[\int_0^T e^{\beta C_t^{\mathbb{G},Z}} |F(t, X_t, 0)|^2 dC_t^{\mathbb{G},Z}\right] < \infty.$$

- (2) For every $\omega \in \Omega$, $t \in [0, T]$, $\Theta(\cdot) \in L^{2,\beta}(\mu^Z, \mathbb{G})$, the mapping $(t, \omega) \mapsto F(\omega, t, X_t(\omega), \Theta_t(\omega, \cdot))$ is \mathbb{G} -progressively measurable.
(3) For every $\omega \in \Omega$, $t \in [0, T]$, and $\zeta, \zeta' \in \mathcal{L}^2(\mathbb{R}^{d+1}, \mathcal{B}(\mathbb{R}^{d+1}), \phi_t(\omega, dx_1, dx_2))$, there exists a constant $L_F > 0$ such that

$$|F(t, \omega, X_t(\omega), \zeta) - F(t, \omega, X_t(\omega), \zeta')| \leq L_F \left(\int_{\mathbb{R}^{d+1}} |\zeta(x_1, x_2) - \zeta'(x_1, x_2)| \phi_t^{\mathbb{G},Z}(\omega, dx_1, dx_2) \right)^{1/2}. \tag{B.1}$$

Point (1) follows from Assumption 5.7 with $\beta > L^2$, being

$$\mathbb{E}\left[\int_0^T e^{\beta C_t^{\mathbb{G},Z}} |F(t, X_t, 0)|^2 dC_t^{\mathbb{G},Z}\right] \leq \mathbb{E}\left[\int_0^T e^{\beta C_t^{\mathbb{G},Z}} \inf_{u \in U} l_t(X_t, u)^2 dC_t^{\mathbb{G},Z}\right] \leq M_t^2 \beta^{-1} \mathbb{E}[e^{\beta C_T^{\mathbb{G},Z}}] < +\infty.$$

Concerning (3), by the boundedness conditions (5.7) in Assumption 5.6, it is easy to check that estimate (B.1) holds with $L_F = L$. Finally, the measurability requirements in (2) for the Hamiltonian f hold thanks to Assumption 5.11. By (5.20), for every $\Theta(\cdot) \in L^{2,\beta}(\mu^Z, \mathbb{G})$ (recalling the inclusion $L^{2,\beta}(\mu^Z) \subseteq L^{1,0}(\mu^Z)$ for all $\beta > 0$), the map $(t, \omega) \mapsto f(\omega, t, X_{t-}(\omega), \Theta_t(\omega, \cdot))$ is \mathbb{G} -progressively

measurable; since by Proposition 5.4 the process $C^{\mathbb{G},Z}$ is continuous and X has piecewise constant paths, the same holds (after modification on a set of measure zero) for $(\omega, t) \mapsto F(\omega, t, X_t(\omega), \Theta_t(\omega, \cdot))$.

Proof of Theorem 5.17: We divide the proof into two steps.

Step 1. The BSDE

$$\bar{Y}_t + \int_t^T \int_{\mathbb{R}^{d+1}} \bar{\Theta}_s(x_1, x_2) (\mu^Z - \nu^{\mathbb{G},Z})(ds, dx_1, dx_2) = \bar{g}(X_{T \wedge \tau}) + \int_t^T \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) 1_{[0, T \wedge \tau]}(s) dC_s^{\mathbb{F},X} \quad (\text{B.2})$$

with

$$\bar{f}(\omega, t, y_1, \theta(\cdot)) = \inf_{u \in U} \left\{ \bar{l}_t(\omega, y_1, u) + \int_{\mathbb{R}^d} \theta(x_1, 0) (\bar{r}_t(\omega, x_1, u) - 1) \phi_t^{\mathbb{F},X}(\omega, dx_1) \right\}$$

admits a unique solution $(\bar{Y}, \bar{\Theta}(\cdot)) \in L_{\text{Prog}}^{2,\beta}(\Omega \times [0, T], \mathbb{G}) \times L^{2,\beta}(\mu^Z, \mathbb{G})$ for $\beta > L^2$. Moreover, $\bar{Y} = \bar{Y}_{\cdot \wedge \tau}$, $\mathbb{P}(d\omega)$ -a.e. and $\bar{\Theta} = \bar{\Theta} 1_{[0, T \wedge \tau]}$, $\phi_t^{\mathbb{G},Z}(\omega, dx_1, dx_2) dC_t^{\mathbb{G},Z}(\omega) \mathbb{P}(d\omega)$ -a.e.

Step 2. $(Y, \Theta(\cdot))$ coincides with $(\bar{Y}, \bar{\Theta}(\cdot))$.

Step 3. Identity (5.30) holds true.

Proof of Step 1. The well-posedness of BSDE (B.2) follows from Confortola and Fuhrman (2013, Theorem 3.4), recalling the assumptions on \bar{l} , \bar{r} and \bar{g} , and noting that the map

$$\bar{F}(\omega, t, X_t(\omega), \Theta_t(\omega)) = d_1(\omega, t) \bar{f}(\omega, t, X_t(\omega), \Theta_t(\omega)) 1_{[0, T \wedge \tau(\omega)]}(t)$$

is \mathbb{G} -progressively measurable, $\mathbb{E} \left[\int_0^T e^{\beta C_t^{\mathbb{G},Z}} |\bar{F}(t, X_t, 0)|^2 dC_t^{\mathbb{G},Z} \right]$ is finite for every $\beta > L^2$, and

$$|\bar{F}(t, \omega, X_t(\omega), \zeta) - \bar{F}(t, \omega, X_t(\omega), \zeta')| \leq L \left(\int_{\mathbb{R}^{d+1}} |\zeta(x_1, x_2) - \zeta'(x_1, x_2)| \phi_t^{\mathbb{G},Z}(\omega, dx_1, dx_2) \right)^{1/2}.$$

Let us now prove that $\bar{Y} = \bar{Y}_{\cdot \wedge \tau}$, and $\bar{\Theta} = \bar{\Theta} 1_{[0, T \wedge \tau]}$. The process \bar{Y} is defined as

$$\bar{Y}_t = \mathbb{E} \left[\bar{g}(X_{T \wedge \tau}) + \int_0^T \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) 1_{[0, T \wedge \tau]}(s) dC_s^{\mathbb{F},X} \middle| \mathcal{G}_t \right] - \int_0^t \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) 1_{[0, T \wedge \tau]}(s) dC_s^{\mathbb{F},X} \quad (\text{B.3})$$

and moreover

$$\begin{aligned} & \bar{g}(X_{T \wedge \tau}) + \int_0^T \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) 1_{[0, T \wedge \tau]}(s) dC_s^{\mathbb{F},X} \\ &= \mathbb{E} \left[\bar{g}(X_{T \wedge \tau}) + \int_0^T \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) 1_{[0, T \wedge \tau]}(s) dC_s^{\mathbb{F},X} \right] + \int_0^T \int_{\mathbb{R}^{d+1}} \bar{\Theta}_s(x_1, x_2) (\mu^Z - \nu^{\mathbb{G},Z})(ds, dx_1, dx_2), \end{aligned}$$

where the last integral is a martingale. Consequently, for all $t \in [0, T]$,

$$\begin{aligned} \bar{Y}_t &= \mathbb{E} \left[\bar{g}(X_{T \wedge \tau}) + \int_0^T \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) 1_{[0, T \wedge \tau]}(s) dC_s^{\mathbb{F},X} \right] \\ &+ \int_0^t \int_{\mathbb{R}^{d+1}} \bar{\Theta}_s(x_1, x_2) (\mu^Z - \nu^{\mathbb{G},Z})(ds, dx_1, dx_2) - \int_0^t \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) 1_{[0, T \wedge \tau]}(s) dC_s^{\mathbb{F},X}. \end{aligned}$$

By Doob's stopping theorem and (B.2), previous expression gives

$$\begin{aligned} \bar{Y}_{t \wedge \tau} &= \mathbb{E} \left[\bar{g}(X_{T \wedge \tau}) + \int_0^{T \wedge \tau} \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) dC_s^{\mathbb{F},X} \right] \\ &+ \int_0^{t \wedge \tau} \int_{\mathbb{R}^{d+1}} \bar{\Theta}_s(x_1, x_2) (\mu^Z - \nu^{\mathbb{G},Z})(ds, dx_1, dx_2) - \int_0^{t \wedge \tau} \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) dC_s^{\mathbb{F},X}. \end{aligned} \quad (\text{B.4})$$

Now we notice that for $t = 0$ and $t = T \wedge \tau$ in (B.3) we obtain respectively

$$\bar{Y}_0 = \mathbb{E} \left[\bar{g}(X_{T \wedge \tau}) + \int_0^{T \wedge \tau} \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) dC_s^{\mathbb{F}, X} \right], \quad \bar{Y}_{T \wedge \tau} = \bar{g}(X_{T \wedge \tau}).$$

Then,

$$\begin{aligned} \bar{Y}_{t \wedge \tau} &+ \int_{t \wedge \tau}^{T \wedge \tau} \int_{\mathbb{R}^{d+1}} \bar{\Theta}_s(x_1, x_2) 1_{[0, T \wedge \tau]}(s) (\mu^Z - \nu^{\mathbb{G}, Z})(ds, dx_1, dx_2) \\ &= \bar{g}(X_{T \wedge \tau}) + \int_{t \wedge \tau}^{T \wedge \tau} \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) 1_{[0, T \wedge \tau]}(s) dC_s^{\mathbb{F}, X}, \quad t \in [0, T], \end{aligned}$$

or, equivalently,

$$\begin{aligned} \bar{Y}_{t \wedge \tau} &+ \int_t^T \int_{\mathbb{R}^{d+1}} \bar{\Theta}_s(x_1, x_2) 1_{[0, T \wedge \tau]}(s) (\mu^Z - \nu^{\mathbb{G}, Z})(ds, dx_1, dx_2) \\ &= \bar{g}(X_{T \wedge \tau}) + \int_t^T \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) 1_{[0, T \wedge \tau]}(s) dC_s^{\mathbb{F}, X}, \quad t \in [0, T]. \end{aligned}$$

Proof of Step 2. It is enough to show the identity

$$\bar{f}(\omega, s, X_s(\omega), \bar{\Theta}_s(\omega, \cdot)) 1_{[0, T \wedge \tau(\omega)]}(s) = f(\omega, s, X_s(\omega), \bar{\Theta}_s(\omega, \cdot)), d_1(\omega, s) dC_s^{\mathbb{G}, Z}(\omega) \mathbb{P}(d\omega)\text{-a.e.} \quad (\text{B.5})$$

As a matter of fact, plugging (B.5) in BSDE (B.2) we would get BSDE (5.28). Then, by the uniqueness of the solution, $(Y, \Theta(\cdot))$ would coincide with $(\bar{Y}, \bar{\Theta}(\cdot))$ and, by Step 1., $Y = \bar{Y}$, $\mathbb{P}(d\omega)$ -a.e. and $\Theta = \bar{\Theta} 1_{[0, T \wedge \tau]}$, $\phi_t^{\mathbb{G}, Z}(\omega, dx_1, dx_2) dC_t^{\mathbb{G}, Z}(\omega) \mathbb{P}(d\omega)$ -a.e.

Let us thus prove (B.5). By Step 1,

$$f(\omega, s, X_s(\omega), \bar{\Theta}_s(\omega, \cdot) 1_{[0, T \wedge \tau(\omega)]}(s)) = f(\omega, s, X_s(\omega), \bar{\Theta}_s(\omega, \cdot)). \quad (\text{B.6})$$

On the other hand, recalling (5.19), we have that, $d_1(\omega, s) dC_s^{\mathbb{G}, Z}(\omega) \mathbb{P}(d\omega)$ -almost surely on $\Omega \times [0, T]$,

$$\begin{aligned} &f(\omega, s, X_s(\omega), \bar{\Theta}_s(\omega, \cdot) 1_{[0, T \wedge \tau(\omega)]}(s)) \\ &= \inf_{u \in U} \left\{ \bar{l}_s(\omega, X_s(\omega), u) 1_{[0, T \wedge \tau(\omega)]}(s) + \int_{\mathbb{R}^d} \bar{\Theta}(x_1, 0) 1_{[0, T \wedge \tau(\omega)]}(s) (\bar{r}_s(\omega, x_1, u) - 1) \phi_s^{\mathbb{F}, X}(\omega, dx_1) \right\} \\ &= 1_{[0, T \wedge \tau(\omega)]}(t) \inf_{u \in U} \left\{ \bar{l}_s(\omega, X_s(\omega), u) + \int_{\mathbb{R}^d} \bar{\Theta}(x_1, 0) (\bar{r}_s(\omega, x_1, u) - 1) \phi_s^{\mathbb{F}, X}(\omega, dx_1) \right\} \\ &= \bar{f}(\omega, s, X_s(\omega), \bar{\Theta}_s(\omega, \cdot)) 1_{[0, T \wedge \tau(\omega)]}(s). \end{aligned} \quad (\text{B.7})$$

Collecting (B.6) and (B.7), we get (B.5).

Proof of Step 3. Recalling the proof of Theorem 5.14, it is enough to show that, for almost all (ω, t) such that $t \leq T \wedge \tau(\omega)$ with respect to the measure $d_1(\omega, t) dC_t^{\mathbb{G}, Z}(\omega) \mathbb{P}(d\omega)$,

$$\begin{aligned} \bar{f}(\omega, t, X_{t-}(\omega), \Theta_t(\omega, \cdot)) &= \bar{l}_t(\omega, X_{t-}(\omega), \underline{u}^\Theta(\omega, t)) \\ &+ \int_{\mathbb{R}^d} \Theta_t(\omega, x_1, 0) (\bar{r}_t(\omega, x_1, \underline{u}^\Theta(\omega, t)) - 1) \phi_t^{\mathbb{F}, X}(\omega, dx_1). \end{aligned} \quad (\text{B.8})$$

Identities (B.5) and (5.20), together with Step 1, yield

$$\begin{aligned} \bar{f}(\omega, t, X_t(\omega), \Theta_t(\omega, \cdot)) 1_{[0, T \wedge \tau(\omega)]}(t) &= f(\omega, t, X_t(\omega), \Theta_t(\omega, \cdot)) \\ &= 1_{[0, T \wedge \tau(\omega)]}(t) \bar{l}_t(\omega, X_{t-}(\omega), \underline{u}^\Theta(\omega, t)) \\ &+ \int_{\mathbb{R}^d} \Theta_t(\omega, x_1, 0) 1_{[0, T \wedge \tau(\omega)]}(t) (\bar{r}_t(\omega, x_1, \underline{u}^\Theta(\omega, t)) - 1) \phi_t^{\mathbb{F}, X}(\omega, dx_1) \\ &= 1_{[0, T \wedge \tau(\omega)]}(t) \left\{ \bar{l}_t(\omega, X_{t-}(\omega), \underline{u}^\Theta(\omega, t)) + \int_{\mathbb{R}^d} \Theta_t(\omega, x_1, 0) (\bar{r}_t(\omega, x_1, \underline{u}^\Theta(\omega, t)) - 1) \phi_t^{\mathbb{F}, X}(\omega, dx_1) \right\}. \end{aligned}$$

□

References

- Aksamit, A. and Jeanblanc, M. *Enlargement of filtration with finance in view*. SpringerBriefs in Quantitative Finance. Springer, Cham (2017). ISBN 978-3-319-41254-2; 978-3-319-41255-9. [MR3729407](#).
- Aksamit, A., Li, L., and Rutkowski, M. Generalized BSDE and reflected BSDE with random time horizon. *Electron. J. Probab.*, **28**, Paper No. 40, 41 (2023). [MR4557762](#).
- Ankirchner, S., Blanchet-Scalliet, C., and Eyraud-Loisel, A. Credit risk premia and quadratic BSDEs with a single jump. *Int. J. Theor. Appl. Finance*, **13** (7), 1103–1129 (2010). [MR2738764](#).
- Becherer, D. Bounded solutions to backward SDE's with jumps for utility optimization and indifference hedging. *Ann. Appl. Probab.*, **16** (4), 2027–2054 (2006). [MR2288712](#).
- Becherer, D., Büttner, M., and Kentia, K. On the monotone stability approach to BSDEs with jumps: extensions, concrete criteria and examples. In *Frontiers in stochastic analysis—BSDEs, SPDEs and their applications*, volume 289 of *Springer Proc. Math. Stat.*, pp. 1–41. Springer, Cham (2019). [MR4008339](#).
- Bertsekas, D. P. and Shreve, S. E. *Stochastic optimal control. The discrete time case*. Athena Scientific, Belmont, Massachusetts (1996). ISBN 1-886529-03-5. Available at https://web.mit.edu/dimitrib/www/SOC_1978.pdf.
- Calzolari, A. and Torti, B. Martingale representations in progressive enlargement by multivariate point processes. *Int. J. Theor. Appl. Finance*, **25** (3), Paper No. 2250015, 21 (2022). [MR4446298](#).
- Confortola, F. and Fuhrman, M. Backward stochastic differential equations and optimal control of marked point processes. *SIAM J. Control Optim.*, **51** (5), 3592–3623 (2013). [MR3105784](#).
- Confortola, F., Fuhrman, M., and Jacod, J. Backward stochastic differential equation driven by a marked point process: an elementary approach with an application to optimal control. *Ann. Appl. Probab.*, **26** (3), 1743–1773 (2016). [MR3513605](#).
- Dellacherie, C. *Capacités et processus stochastiques*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 67. Springer-Verlag, Berlin-New York (1972). [MR448504](#).
- Di Tella, P. On the weak representation property in progressively enlarged filtrations with an application in exponential utility maximization. *Stochastic Process. Appl.*, **130** (2), 760–784 (2020). [MR4046519](#).
- Di Tella, P. and Engelbert, H.-J. Martingale representation in progressively enlarged Lévy filtrations. *Stochastics*, **94** (2), 311–333 (2022). [MR4372659](#).
- Di Tella, P. and Jeanblanc, M. Martingale representation in the enlargement of the filtration generated by a point process. *Stochastic Process. Appl.*, **131**, 103–121 (2021). [MR4151216](#).
- El Karoui, N., Jeanblanc, M., and Jiao, Y. What happens after a default: the conditional density approach. *Stochastic Process. Appl.*, **120** (7), 1011–1032 (2010). [MR2639736](#).

- Gapeev, P., Jeanblanc, M., and Wu, D. Projections in enlargements of filtrations under Jacod's equivalence hypothesis for marked point processes. *HAL preprint* (2021). [HAL: 03207058](#).
- He, S. W., Wang, J. G., and Yan, J. A. *Semimartingale theory and stochastic calculus*. Kexue Chubanshe (Science Press), Beijing; CRC Press, Boca Raton, FL (1992). ISBN 7-03-003066-4. [MR1219534](#).
- Izumisawa, M., Sekiguchi, T., and Shiota, Y. Remark on a characterization of BMO-martingales. *Tohoku Math. J. (2)*, **31** (3), 281–284 (1979). [MR547642](#).
- Jacod, J. Multivariate point processes: predictable projection, Radon-Nikodým derivatives, representation of martingales. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **31**, 235–253 (1975). [MR380978](#).
- Jacod, J. *Calcul stochastique et problèmes de martingales*, volume 714 of *Lecture Notes in Mathematics*. Springer, Berlin (1979). ISBN 3-540-09253-6. [MR542115](#).
- Jacod, J. Grossissement initial, hypothèse (H') et théorème de Girsanov. In *Grossissements de filtrations: exemples et applications*, pp. 15–35. Springer (1985). [DOI: 10.1007/BFb0075768](#).
- Jacod, J. and Shiryaev, A. N. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition (2003). ISBN 3-540-43932-3. [MR1943877](#).
- Jeanblanc, M. and Le Cam, Y. Progressive enlargement of filtrations with initial times. *Stochastic Process. Appl.*, **119** (8), 2523–2543 (2009). [MR2532211](#).
- Jeanblanc, M., Mastrolia, T., Possamaï, D., and Réveillac, A. Utility maximization with random horizon: a BSDE approach. *Int. J. Theor. Appl. Finance*, **18** (7), 1550045, 43 (2015). [MR3423182](#).
- Jeulin, T. *Semi-martingales et grossissement d'une filtration*, volume 833 of *Lecture Notes in Mathematics*. Springer, Berlin (1980). ISBN 3-540-10265-5. [MR604176](#).
- Jiao, Y. and Pham, H. Optimal investment with counterparty risk: a default-density model approach. *Finance Stoch.*, **15** (4), 725–753 (2011). [MR2863640](#).
- Kharroubi, I. and Lim, T. Progressive enlargement of filtrations and backward stochastic differential equations with jumps. *J. Theoret. Probab.*, **27** (3), 683–724 (2014). [MR3245982](#).
- Kharroubi, I., Lim, T., and Ngoupeyou, A. Mean-variance hedging on uncertain time horizon in a market with a jump. *Appl. Math. Optim.*, **68** (3), 413–444 (2013). [MR3131502](#).
- Lim, T. and Quenez, M.-C. Exponential utility maximization in an incomplete market with defaults. *Electron. J. Probab.*, **16**, no. 53, 1434–1464 (2011). [MR2827466](#).
- Morlais, M.-A. A new existence result for quadratic BSDEs with jumps with application to the utility maximization problem. *Stochastic Process. Appl.*, **120** (10), 1966–1995 (2010). [MR2673984](#).
- Pham, H. Stochastic control under progressive enlargement of filtrations and applications to multiple defaults risk management. *Stochastic Process. Appl.*, **120** (9), 1795–1820 (2010). [MR2673975](#).