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Bayes Inference of Structural Safety under Extreme Wind Loads Based upon a Peak-Over-Threshold Process of Exceedances

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Abstract: In the present paper, the process of estimating the important statistical properties of extreme wind loads on structures is investigated by considering the effect of large variability. In fact, for the safety design and operating conditions of structures such as the ones characterizing tall buildings, wind towers, and offshore structures, it is of interest to obtain the best possible estimates of extreme wind loads on structures, the recurrence frequency, the return periods, and other stochastic properties, given the available statistical data. In this paper, a Bayes estimation of extreme load values is investigated in the framework of structural safety analysis. The evaluation of extreme values of the wind loads on the structures is performed via a combined employment of a Poisson process model for the peak-over-threshold characterization and an adequate characterization of the parent distribution which generates the base wind load values. In particular, the present investigation is based upon a key parameter for assessing the safety of structures, i.e., a proper safety index referred to a given extreme value of wind speed. The attention is focused upon the estimation process, for which the presented procedure proposes an adequate Bayesian approach based upon prior assumptions regarding (1) the Weibull probability that wind speed is higher than a prefixed threshold value, and (2) the frequency of the Poisson process of gusts. In the last part of the investigation, a large set of numerical simulations is analyzed to evaluate the feasibility and efficiency of the above estimation method and with the objective to analyze and compare the presented approach with the classical Maximum Likelihood method. Moreover, the robustness of the proposed Bayes estimation is also investigated with successful results, both with respect to the assumed parameter prior distributions and with respect to the Weibull distribution of the wind speed values.

Keywords: Bayes estimation; extreme value theory; peak-over-threshold; Poisson processes; wind loads on structures; structural safety analysis

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1. Introduction

In the present research work, a Bayes method is illustrated for the extreme wind speed characterization and estimation, with the purpose of its application in the framework of structural safety and wind engineering, by an adequate application of extreme value theory [1]. This topic has indeed brought about an increasing number of studies in recent years, both for the risk evaluation and reliability analysis of structures, see among others [2–10], and in wind energy production assessments [11,12]. This modeling procedure is challenging due to the various studies that exist in the literature regarding wind speed probability distributions. The recent advances in wind engineering motivated many studies in the literature to focus on wind speed (WS) probability distribution. For this purpose, significant interest has been focused on the extreme values (EV)'s characterization of WS, both for evaluating the risk, safety, and reliability of structures, and for assessing the maximum wind energy production [13].

It must be outlined that in most of the above studies devoted to wind energy characterization, the WS probability distribution, quantiles, and other relevant statistical attributes are generally evaluated by ignoring the temporal autocorrelation function, since they mostly depend on the probability distribution function, i.e., without explicitly taking time into account. This means that random variables are used instead of stochastic processes, as would be more realistic. In the following, instead, a dynamic approach is pursued in terms of stochastic processes.

The forecast of extreme wind speed values, or wind gusts, plays a crucial role both in structural safety, structural reliability, and risk evaluation analyses and in environmental studies. Indeed, this topic is of interest both in safety design, operating conditions of structures, and in environmental and energy studies. In fact, in case of intolerably high wind speeds, it is known that wind turbines are designed to be cut out as a means of safety and protection against possible damages [12,14–22]. With reference to the aspect of energy production and by keeping in mind the cubic rule relationship between the wind power and the wind speed, it is not trivial to put into evidence that the extreme upper quantiles of wind power are very sensitive to the corresponding quantiles of wind speed, so that an inaccurate quantiles estimation may involve relatively large errors in the evaluation of the expected wind energy production [23].

However, in the literature on wind energy studies and probability extremes, it has been emphasized that the EV estimation is a complex task, since large sample sizes are often required. In the case of insufficient sample sizes, many models can be employed, from the classical Weibull distribution to the more recent Log-logistic, Lomax, or Burr distribution, which generally perform quite similarly in the central part of the real WS distribution, i.e., with about the same values of central parameters such as the mean and the median values [24].

With reference to the structural safety design and the operating conditions of structures including, but not limited to, tall buildings, offshore structures, and wind towers, and starting from the available statistical data, it is critical to determine the best possible method for the estimation of extreme wind loads on structures, the return periods, and other stochastic properties. With the purpose of overcoming the above-mentioned concerns, the present paper proposes a Bayes approach for the estimation of the EV probability distribution, which may be suitable under various models based upon the characterization of extreme WS by means of a proper Poisson process of exceedances, by following a methodology introduced in the extreme value theory of stochastic processes and applied within the framework of structural safety analysis. This methodology can be regarded as strictly related to the so-called peaks-over-threshold (POT) method, which is based upon the stochastic process of the time instants in which the wind-speed exceeds a given threshold [25]. The results of the numerical simulations confirm the absolute and relative efficiency of the proposed model, as well as the robustness of the proposed estimation method.

2. Wind Speed Extreme Values Evaluation by Means of Stochastic Processes: The Peaks-Over-Threshold Approach

In the following, a dynamic approach is adopted, i.e., the wind speed is regarded as the realization of a stochastic process in time, as it is indeed from a rigorous point of view, see, e.g., [25–27]. By adopting a structural engineering framework, the focus is on the development of a methodology capable of guaranteeing adequate levels of structural safety margins. Since the operational lifetimes of such towers and structures are typically many years, the designers need to estimate the extreme values of wind speeds, i.e., the maximum wind gust amplitude over a predefined time. In this paper, we will denote as gust a wind gust amplitude over a given time interval [28].

First, the stochastic process of WS values over time is denoted by W = W(t), and let θ be a sufficiently high value, that is a threshold value, of WS such as that all WS values higher than θ can be considered as a gust. This threshold value depends on the structure under consideration and possibly on given guidelines and it is typically used for defining the "cut-off" value of the WS. In several cases such as for tower structures or structures which involve the use of machineries, the threshold value also depends on machine features, so it is left unspecified here.

Let NW(t) denote the stochastic counting process of the WS values which cross the barrier θ, i.e., the number of the peaks-over-threshold of WS, see, e.g., [29], and let us denote by T_k the time of the *k*-th gust occurrence. The literature on the extreme values of stochastic processes provides the necessary conditions, which are generally satisfied as discussed by [30,31], provided that the barrier level θ is large enough. Hereinafter, this process is defined as a gust counting process. In such conditions, the NW(t) process is characterized by the well-known Poisson probability law $p(k,t)$ expressing the probability that NW(t) attains a given integer value k. Such a probability law is given by:

$$P(j,t) \equiv P[NW(t) = j] = e^{-\alpha t} \cdot \frac{(\alpha t)^j}{j!} \quad j = 0, 1, \ldots, \infty \tag{1}$$

In (1), α is the mean number of up-crossings in the unit time. The mean, or expected value, and variance of the process NW(t) are numerically equal and given by:

$$E[NW(t)] = \ Var\ [NW(t)\ ] = \alpha t \tag{2}$$

The gust amplitude at time T_k is a random variable, herein indicated as WA_k, which represents the wind amplitude at time T_k. In the following, the times T_k will be denoted as the Poisson times. It is obvious that an adequate safety or reliability index for characterizing the extreme values of the stochastic process NW(t) is the maximum gust amplitude over the interest time interval, which is also an index of the damage that the gust process can cause to the system. This can be accomplished by associating to the stochastic process NW(t) and the random variables WA_k ($k = 1, 2, \ldots, NW(t)$), with the following stochastic process:

$$\begin{gathered} \mathrm{MW(t)} = max[\mathrm{WA}_1, \mathrm{WA}_2, \ldots, \mathrm{WA}_{NW}], \text{ if } NW(t) \geq 0 \\ \mathrm{MW(t)} = 0, \ otherwise \end{gathered} \tag{3}$$

MW(t) is a stochastic counting process, constituted by the maximum of a random number NW(t) of elements forming the succession of gust amplitudes. It is remarked that, for each time t, MW(t) is a (continuous) random variable RV which is given by the maximum of a countable random number N = N(t) of continuous RVs WA_k.

Let Ω(ζ,t) be the cumulative distribution function (CDF) of MW(t) at time t, evaluated at a value ζ of WS:

$$\Omega\ (\zeta,t) = P[MW(t) < \zeta] \tag{4}$$

If, for safety reasons, a high threshold or a safety level *z* is assigned to the maximum of the EWS values ζ of (4) occurring in a given safety time interval (0, s), the following structural safety index (SSI), which is also a stochastic process with respect to the time index s, can be consequently defined:

$$\sigma(z,s) := \Omega\ (z,s) \tag{5}$$

Indeed, with reference to the time horizon (0, s) under study for safety, it is obvious that Ω(s,z) = P[MW(s) < z] is the probability that z is never exceeded and this can introduce a justification to the term safety index. The above SSI is expressible in an analytical form in terms of the probability distribution of the RVs WA_k. In fact, it is remarked that for every assigned integer value *n* of NW(t), the following relationship holds:

$$[\max[WA_1, WA_2, \ldots, WA_n] < z] \quad \textit{if and only if} \ [(WA < z) \cap \ldots \cap (WA_n < z)] \tag{6}$$

The RVs WA_k are assumed to be statistically independent and identically distributed with the common, time-independent, cumulative distribution function FW(x):

$$\mathrm{FW}(x) = P(WA_k \leq x), \forall\, \mathrm{k} = 1, 2, \ldots \mathrm{n} \tag{7}$$

After some manipulations, implying the power series expansion of a function like $[\Phi(x) = exp(g(x))]$, for any given continuous function g(x), the following compact expression can be obtained for the above function σ(s,z) under the Poisson hypothesis for NW(t), see Appendix A for details:

$$\sigma(\mathrm{z}, \mathrm{s}) = \exp[-\alpha\mathrm{s}(1 - \mathrm{FM}(\mathrm{z}))] \tag{8}$$

Such an equation can be further refined once the parent distribution FM(z) is known or estimated. A list of possible candidate parent distributions will be illustrated in Section 3.

As a function of time s, the SSI is an Exponential complementary CDF [32], as it may easily be noticed by expressing it as:

$$\sigma(\mathrm{z}, \mathrm{s}) = \exp(-\alpha \mathrm{s} \mathrm{q}(\mathrm{z})) \tag{9}$$

where it is recalled that α (denoted in the sequel as the Poisson frequency) is the expected gust frequency, i.e., the expected number of gust occurrences per unit of time, and q = q(z) = 1 − FM(z) = $P(WA_j > z)$ is the overcrossing probability (OP) of the safety level *z* by any single gust amplitude WA_j.

Equation (8) can be also expressed as

$$\sigma(\mathrm{z}, \mathrm{s}) = \exp(-\mathrm{s}/\mathrm{T}) \tag{10}$$

where T = 1/αq is the so-called return period [1] associated to the extreme values of the stochastic process MW(t). In this case, T = T(z) is to be interpreted as the mean recurrence time between two successive overcrossing occurrences of the safety level z, the dependence on z being related to the function q = q(z).

It is remarked that, under the assumed hypotheses, the OP q = q(z) neither depends on the index j, nor on the time, while of course the whole SSI depends on the time interval s. The function q = q(z), which is related to the gust CDF, may of course assume various expressions. According to the proposed approach, inference for the above model can be adequately accomplished by following a Bayesian approach as shown in the following Section 4, see also [33]. Such a Bayesian approach is applied both to the parameter α and to q = q(z)= 1 − FM(z) in Equation (8), once q has been made explicit by following some of the models, as discussed in Section 3.

3. An Overview on Parametric WS Extreme Value Distribution Models to Be Adopted as Parent Distributions in the POT Approach

This section provides an overview on the most adopted parametric WS extreme value distribution models which may be adopted as parent distributions in the POT approach, i.e., as analytical expressions of the function q = q(z) = 1 − FM(z) appearing in Equation (8). In particular, the CDF detailed in this section can be used for expressing the wind speed CDF FM(x), which should be evaluated at x = z, representing the safety level, in terms of WS values.

3.1. Weibull Distribution (Type III)

This distribution was first introduced by Weibull [34]. The two-parameter Weibull distribution with a scale parameter ϕ and a shape parameter ω can be written as follows:

PDF

$$f(x,\varphi,\omega) = \frac{\omega}{\varphi}\left(\frac{x}{\varphi}\right)^{\omega-1} exp\left(-\left(\frac{x}{\varphi}\right)^{\omega}\right) \tag{11}$$

CDF

$$F(x,\varphi,\omega) = 1 - exp\left(-\left(\frac{x}{\varphi}\right)^{\omega}\right) \tag{12}$$

An additional location parameter μ can be inserted as follows, with a new PDF and CDF denoted as $f_1(x)$ and $F_1(x)$ and expressed as functions of the above ones f(x) and F(x), respectively:

PDF

$$f_1(x) = f(x - \mu)\ (x > \mu) \tag{13}$$

CDF

$$F_1(x) = F(x - \mu)\ (x > \mu) \tag{14}$$

The two-parameter Weibull distribution has been widely used in the literature including the estimation of wind speeds. On the other hand, the three-parameter Weibull distribution expressed a high accuracy in the low wind speeds' estimation due to the shift caused by the location parameter [34,35]. Therefore, it has been used to estimate wind energy, characterize wind speed, and estimate the automatic wind power system [36]. Nevertheless, Weibull distribution is not suitable for the estimation of extreme values of wind speed which have little influence on the parameters of the Weibull distribution. Consequently, beyond a certain threshold, other extreme value distributions must be used [37].

3.2. Gumbel Distribution (Type I)

This distribution is widely used for modeling extreme events [38]. It fits extreme wind speeds according to many studies, see, e.g., [39]:

PDF:

$$f(x,\chi,\delta) = \frac{1}{\delta} exp\left(-\frac{(x-\chi)}{\delta} - exp\left(-\frac{(x-\chi)}{\delta}\right)\right) \tag{15}$$

where x is the random variable (RV), χ is the location parameter, and δ is the scale parameter ($\delta > 0$).

CDF:

$$F(x,\chi,\delta) = exp\left(-exp\left(-\frac{(x-\chi)}{\delta}\right)\right) \tag{16}$$

3.3. Inverse Weibull Distribution (Fréchet Distribution Type II)

The Inverse Weibull (IW) distribution was first introduced by Maurice Fréchet [40] to fit extreme events including extreme wind speeds, see also [41]. Two-parameter IW distribution, having a scale parameter φ and a shape parameter ω, can be expressed as:

PDF

$$f(x,\varphi,\beta) = \frac{\beta}{\varphi}\left(\frac{\varphi}{x}\right)^{\omega+1} exp\left(-\left(\frac{\varphi}{x}\right)^{\omega}\right) \tag{17}$$

CDF

$$F(x,\varphi,\omega) = 1 - exp\left(-\left(\frac{\varphi}{x}\right)^{\omega}\right) \tag{18}$$

Three-parameter IW distribution can be derived by adding a location parameter μ as follows:

PDF

$$f(x,\alpha,\beta,\mu) = \frac{\beta}{\alpha}\left(\frac{\alpha}{x-\mu}\right)^{\beta+1} exp\left(-\left(\frac{\alpha}{x-\mu}\right)^{\beta}\right) \tag{19}$$

CDF

$$F(x,\alpha,\beta,\mu) = 1 - exp\left(-\left(\frac{\alpha}{x-\mu}\right)^{\beta}\right) \tag{20}$$

3.4. The Generalized Extreme Value Distribution

The generalized extreme value (GEV) distribution is widely used to estimate extreme events and it can be deduced by combining the latter three distributions [42]. The CDF of the generalized value distribution is described as follows:

$$F(x,\alpha,\beta,\mu) = exp\left(-\left(1+\beta\left(\frac{x-\mu}{\alpha}\right)^{-1/\beta}\right)\right) \tag{21}$$

where $\alpha > 0$, the shape parameter β and the location parameter μ are real values. The GEV distribution is quite versatile and α has a substantial effect on its skewness [43]. The generalized extreme value distribution is a heavy-right-tail distribution compared to Weibull distribution. The latter allows for estimating extreme wind values above a certain threshold with high accuracy. However, the estimation of the threshold of the extreme wind speeds is still challenging [44].

4. Bayes Inference for the Structural Safety Index under Extreme Wind Loads

4.1. Bayes Inference Methods

As it is well known [1,22,45–47], Bayesian inference methods consider the unknown parameters as RVs, with their own probability distributions, while for classical estimation methods, they are unknown constants. Under this framework, this feature of Bayesian inference methods allows us to use some a priori information to characterize an a priori distribution of the parameters. This paves the way, according to the Bayes theorem [44], to compute the distribution of the parameters conditioned to the observational data. Hence, such a procedure also displays the accuracy of the estimated parameters in terms of probability, i.e., it allows the computation of the probability in such a way that, for the given observed sample, the parameters lie in a specific interval. This interval constitutes the *Bayesian Credibility Interval*, i.e., the Bayesian counterpart of the *Confidence Interval* of classical estimation. The application of such methods in EV studies is discussed in Coles [1].

In this section the Bayesian estimation of the SSI is illustrated and developed, while in the next section, the results of a series of numerical simulations are presented, to show the feasibility and efficiency of the above estimation method. Furthermore, a brief account of the robustness analysis of the developed methodology is given in Section 6 to evaluate the efficiency of the estimation also in the presence of possible departures from the assumed hypotheses regarding the prior distributions. Indeed, assessing the prior distribution of the parameters under study is a crucial, and often difficult, point of the Bayesian inference methodology [45–47].

4.2. Bayesian Estimation of the SSI

In the expression of the SSI:

$$\sigma(\mathrm{z},\mathrm{s}) \;=\; \exp(-\alpha \mathrm{sq}) \tag{22}$$

it is recalled that α represents the expected gust frequency, i.e., the expected number of gust occurrence per unit time, and q = q(z) = 1 − FM(z) = P($WA_j > z$) is the probability that a single WS value exceeds the safety level z. According to most reported research, see, e.g., [11,12,29], in the following, it will be assumed that a two-parameter Weibull distribution is a well-fitting model for the WS distribution. Using a parametrization slightly different from the one in the previous section, and denoting the scale parameter by λ and a shape parameter by γ, the CDF of such a Weibull model can be written as follows for any given WS value x:

$$\mathrm{FM}(x) = 1 - \exp(-\lambda z^{\gamma}) \quad (23)$$

So, the complete expression of the SSI is:

$$\sigma(z,s) = \exp(-\alpha\, s\, \exp(-\lambda z^{\gamma})) \quad (24)$$

So, with respect to the variable z, the SSI assumes the form of an extreme value distribution, in particular, a Double Exponential distribution, belonging to the class of Gumbel distributions. It is remarked that in the present approach, a new demonstration of such extreme value distribution is proposed. Here, the Bayesian estimation process is introduced for the above SSI. More precisely, for numerical reasons, attention is focused on the minus logarithm of the SSI, i.e., the function:

$$\mathrm{H}(z,s) = -\log(\sigma(z,s)) = \alpha\, s\, \exp(-\lambda z^{\gamma}) \quad (25)$$

Apart from the numerical aspects, an interesting meaning can be assigned to the function H as the equivalent of the *cumulative hazard rate* (CHR) of reliability studies, as outlined in Appendix B. For simplicity, we here maintain the same name and acronym CHR for denoting such function, and it is also remarked that, differently from the SSI, which should be kept as large as possible, the CHR must be as small as possible. As obvious on intuitive grounds, it increases with time s and decreases with the safety level z.

Finally, the Bayesian estimation process is introduced for the above CHR function, for given values of s and z, to be considered as fixed values. It is noted that the variation of time s can be dealt with as briefly discussed in Appendix B. A large sample of estimation results based on numerical simulations [48] is illustrated and discussed in Section 5.

First, as in the majority of studies concerning Weibull distribution [11,12,34], it is assumed that the scale parameter λ is random, due to environment variations, while the shape parameter γ is assumed as known. Further studies will be devoted to allow for the case in which also the shape parameter is random. The input data for the estimation are a joint prior PDF, denoted as g(l,α), for interest parameters λ and α. As well known, such parameters to be estimated are regarded as RVs. Accordingly, they possess a PDF which can be integrated and updated with field data, denoted by D, by the Bayes' theorem reported below:

$$g(\lambda,\alpha|D) = Kg(\lambda,\alpha)L(D|\lambda,\alpha) \quad (26)$$

where:

- $L(D\,|\,\lambda,\alpha)$ is the Likelihood (PDF or, in the case of a discrete observation, as in the present case, *probability mass* functions) of the data D conditional to the parameters (λ,α);
- K is a constant (with respect to the parameter values), such that its reciprocal value is:

$$K^{-1} = \int_{o}^{+\infty}\int_{o}^{+\infty} g(\lambda,\alpha)L(D|\,\lambda,\alpha)d\lambda d\alpha \quad (27)$$

As well known, the best Bayes estimate, in the mean square error (MSE) sense of a given function x = x(λ, α), is provided by the posterior mean:

$$\xi^{\circ} = \mathrm{E}[\xi|D] = \int_{o}^{+\infty}\int_{o}^{+\infty} \xi(\lambda,\alpha)g(\lambda,\alpha|D)d\lambda d\alpha \quad (28)$$

where x° denotes an estimate of the generic parameter x.

In the case under study, the quantity to be estimated is the above CHR whose MSE estimate is expressed by:

$$H^{\circ} = \mathrm{E}[H|D] = \int_{o}^{+\infty} \int_{o}^{+\infty} \alpha s \exp(-\lambda\, z^{\gamma}) g(\lambda, \alpha|D) d\lambda d\alpha \tag{29}$$

The properties of such an estimator shall be compared with the ones of the traditional Maximum Likelihood (ML), H°_{L}, which can be easily expressed in terms of the ML estimates of (λ,α), denoted as (λ′,α′), which can be deduced as follows:

1. For what concerns the Weibull parameter λ, let a random sample $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ be available of RVs representing m WS values, assumed as statistically independent and identically distributed, following the above Weibull distribution. Therefore, as well known [32], if the shape parameter γ is known, the ML estimate of λ is given by:

$$\lambda' = \frac{m}{\sum_{k=1}^{m} X_k^{\gamma}} \tag{30}$$

2. As far as the Poisson parameter α is concerned, let a random sample $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ be available of RVs representing n gust inter-arrival times observed in the safety interval (0, s), i.e., Y_1 represents the time to the 1st gust, Y_2 represents the time between the 1st and 2nd gust, and so on. Accordingly, it is well known that such RVs are Exponential RVs with parameter α [32], and the ML estimate of α is given by:

$$\alpha' = \frac{n}{\sum_{k=1}^{n} Y_k} \tag{31}$$

The two random samples of WS values and Exponential times, $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_m)$, respectively, are assumed to be statistically independent from each other. This Bayes inference employs the well-known conjugate [45–47] priors for the RVs (λ,α), i.e., the Gamma prior PDF [32]. In Section 6, within the framework of robustness studies, also other prior models will be considered.

The Gamma PDF is a PDF with positive parameters n and δ (the shape and scale parameter, respectively) expressed, for a given parameter ρ, by:

$$gampdf(\rho, n, \delta) = (1/\ \delta^{n} \Gamma(n)) \rho^{n-1} exp\left(-\frac{\rho}{\delta}\right) \tag{32}$$

where Γ(ν) is the Euler–Gamma Function evaluated at ν (ν >0):

$$\Gamma(v) = \int_{0}^{\infty} t^{v-1} exp(-t) dt \tag{33}$$

The mean value and variance of the distribution are:

$$E[\alpha] = n\delta, \qquad Var[\alpha] = n\delta^2 \tag{34}$$

Moreover, the above two RVs (λ,α) are assumed to be statistically independent. Accordingly, by denoting with the suffix "0" the prior PDF parameters, in the Gamma prior case, the joint prior PDF is expressed as:

$$g(\lambda, \alpha) = gampdf(\lambda; m_0, x_0) gampdf(\alpha; n_0, \delta_0) \tag{35}$$

The prior hyper-parameters (m_0, χ_0, n_0, and δ_0) are chosen based on prior knowledge or expert's opinion. The fact that the above PDF are conjugate priors lies in the property that the posterior PDF is still a Gamma PDF with updated parameters. Once the above two random samples $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_m)$ are observed, by denoting by suffix

"1" the posterior PDF parameters, it is straightforward to deduce that the prior joint PDF g(λ,α) is given by:

$$g(\lambda, \alpha|D) = gampdf(\lambda; m_1, x_1)gampdf(\alpha; n_1, \delta_1) \tag{36}$$

where:

$$m_1 = m_0 + m; x_1 = \frac{x_0}{1 + U(\mathbf{X})x_0} \tag{37}$$

$$n_1 = n_0 + m; \delta_1 = \frac{\delta_0}{1 + V(\mathbf{Y})\delta_0} \tag{38}$$

$$\mathrm{U}(\mathbf{X}) = \sum_{k=1}^{m} X_k^{\gamma};\ \mathrm{V}(\mathbf{Y}) = \sum_{k=1}^{n} Y_k \tag{39}$$

Moreover, once the data D are assigned, the posterior conditional independence of λ and α also holds. Finally, by using the properties of Gamma distribution and of the Euler–Gamma function, the Bayes estimate of the SSI are evaluated as follows, according to (14):

$$\begin{aligned} H^{\circ} = \mathrm{E}[H|D] = \int_o^{+\infty} \int_o^{+\infty} \alpha s \exp(-\lambda\, z^{\gamma}) g(\lambda, \alpha|D) d\lambda d\alpha \\ = \int_o^{+\infty} \int_o^{+\infty} \alpha s \exp(-\lambda\, z^{\gamma}) gampdf(\lambda; m_1, x_1) gampdf(\alpha; n_1, \delta_1) d\lambda d\alpha \end{aligned} \tag{40}$$

Then, let us consider the following integrals:

$$\int_o^{+\infty} \alpha gampdf(\alpha; n_1, \delta_1) d\alpha = n_1 \delta_1 \tag{41}$$

which is indeed the expected value of a Gamma RV with parameters (n_1,δ_1), see also Equation (34), and

$$\int_0^{\infty} exp(-x\, z^{\gamma})\ gampdf(x, m_1, x_1)dx = \left(\frac{\beta}{\beta + z^{\gamma}}\right)^{m_1}; \quad \beta = \frac{1}{x_1} \tag{42}$$

(see Appendix C). The above equations imply that the Bayes estimator of H is expressed as follows:

$$\mathrm{E}[H|D] = n_1 \delta_1 s \left(\frac{\beta}{\beta + z^{\gamma}}\right)^{m_1} \tag{43}$$

Thus, it has been shown that using the conjugate priors for the RVs (λ,α), i.e., the Gamma prior PDF for both λ and α, implies a compact analytical expression for the Bayes estimator of index H. The same may not hold, as it will be briefly discussed in the framework of robustness studies, if other prior models are considered. However, such an issue poses no particular problems since nowadays numerical methods for the evaluation of posterior means are well established [45–47].

5. Numerical Applications for the Evaluation of the Performances of the Proposed Bayesian Estimators

In the present section, and in the following one, some large sets of numerical applications are illustrated in order to measure the efficiency and accuracy of the Bayesian estimator. Since analytical results about estimate errors (to be defined below) are not available, such numerical applications consist of the Monte Carlo simulation [47], and aim to:

1. Evaluate various indexes of performance (such as the bias and the Mean Square Error) of the Bayes estimator;
2. Compare the proposed Bayesian estimators with the classical ones, in particular with the more largely adopted Maximum Likelihood (ML) estimates.

The numerical applications were conducted for various sample sizes and various input data values. For the sake of brevity, only a significant subset of the results is reported.

In this section, the simulated data (i.e., the parameter λ governing the PDF of wind speed) and the Poisson frequency α, together with their relevant samples, allowing the SSI estimation as outlined in previous section, were generated from the assumed prior PDFs on (λ,α), while in the next section, the *robustness* of the estimates is analyzed with respect to different prior PDFs.

In every simulation:

- First, the Poisson frequency α, and the scale Weibull parameter λ are generated according to their assumed Gamma prior PDFs.
- Data are generated on the succession of times between gusts over the given safety time interval *s* (here taken as s = 1 year) by a Poisson Process of mean frequency α;
- Data are generated on the observed WS values by means of a Weibull RV with scale parameter λ generated according to its Gamma prior PDF introduced above, while the shape parameter γ is fixed according to the assumed values of the OP q = P(X > z), X being the WS RV and z an extremal WS value, here assumed equal to 35 m/s as a typical gust value [12,21–23].

Among the many sets of possible parameter values, the following twelve sets of values (A1, A2, B1, B2, C1, C2, D1, D2, F1, and F2) are illustrated herein. The first six sets (A1 through C2) are relevant to a return period of 20 years, the following six sets (D1 through C2) are relevant to a return period of 50 years. The sets are characterized by the mean or expected values of α and λ, E[α] and E[λ], respectively, and by their coefficient of variation (CV), CV[α] and CV[λ].

The CV Values were chosen equal to 0.20 (for the cases with suffix 1, such as A1, B1, . . .) and 0.40 (for the cases with suffix 2, such as, A2, B2, . . .) to represent different degrees of prior information. The values E[λ] and γ were chosen in order to correspond to expected q values typical for an extremal WS value, i.e., q = 0.020 (cases Ak), q = 0.010 (cases Bk), and q = 0.005 (cases Ck). The 12 cases are summarized in Table 1.

Table 1. Combinations of prior values with respect to mean value and coefficient of variation (CV) of the input random variable (RV) α and λ.

Case	E[α]	E[λ]	CV[α]	CV[λ]
A1	2.5	0.0125	0.20	0.20
A2	2.5	0.0125	0.40	0.40
B1	5.0	0.0085	0.20	0.20
B2	5.0	0.0085	0.40	0.40
C1	10	0.0062	0.20	0.20
C2	10	0.0062	0.40	0.40
D1	1.0	0.0125	0.20	0.20
D2	1.0	0.0125	0.40	0.40
E1	2.0	0.0085	0.20	0.20
E2	2.0	0.0085	0.40	0.40
F1	4.0	0.0062	0.20	0.20
F2	4.0	0.0062	0.40	0.40

In each simulation, two samples are generated, the n-sized sample of the Poisson times (T_1, T_2, . . ., T_n) and the m-sized sample of the WS values (X_1, X_2, . . ., X_m). The value m has been chosen equal to 30. Subsequently, for each sample size n a number of N = 10^4 replications have been performed, and the Bayes estimate of H was deduced. In particular, the results for various sample sizes (n = 5, n = 15, n = 30, n = 50, indicated in the 1st column) are reported in Tables 2–15, in terms of the following indices (indicated in columns 2 through 7):

I1 = Average Bias of the Bayes estimator;
I2 = Average Bias of the ML estimator;
I3 = Mean Square Error of the Bayes estimator (MSEB);
I4 = Mean Square Error of the ML estimator (MSEL);

R1 = ratio of I2 to I1;
R2 = ratio of I4 to I3, i.e., relative efficiency of the Bayes estimator.

In order to explicit the above indexes, let ω be a parameter to be estimated and let ω° be its estimator (be it the Bayes or the ML estimator), and let M[ω°] and M[ω] be, respectively, the average values of the $\omega_j{}^\circ$ and ω_j values over the N performed simulations, i.e.,

$$M[\omega^\circ] = (1/N)\sum_{k=1}^{N} \omega^\circ{}_k \; ; \;\; \mathrm{M}[\omega] = (1/N)\sum_{k=1}^{N} \omega_k \tag{44}$$

Then, the Average Bias of the estimator is expressed by:

$$\mathrm{M}[\omega^\circ] - \mathrm{M}[\omega] \tag{45}$$

In the Tables, both for I1 and I2, and also for the ratio R1, the absolute value of the Average Bias is reported.

The Mean Square Error (MSE) is evaluated on the estimated ($\omega^\circ{}_j$) and true (ω_j) values of the parameter ω of the N simulated samples as:

$$MSE = \frac{1}{N}\sum_{j=1}^{N} \left(\omega^\circ{}_j - \omega j\right)^2 \tag{46}$$

The accuracy and the efficiency of Bayesian estimates are respectively measured by the ratios R1 and R2: the more such indexes exceed 1, the better the Bayesian estimates perform compared to the ML ones.

The above MSEs have been obtained at the end of each simulation as the averages over the N sampled estimator's square errors.

By observing the MSEB and MSEL indexes and their ratio, it is remarked that the efficiency of the Bayesian estimation increases, as always occurs when the number of data are exiguous. Moreover, in the considered framework, it is useful to highlight that the proposed Bayes estimation is performs very well, and it is more effective with respect to the classical ML estimation as clearly shown by the R2 values, also when many data are available, whereas, typically, it is well known that the ML estimation becomes more efficient as the sample size n increases, see, e.g., [45–47]. As expected, the results are more favorable to the Bayes estimation in the case of a smaller CV, i.e., when the CV is 0.20 with respect to the case in which the CV is 0.40, since a smaller CV implies a smaller degree of uncertainty in the prior hypotheses.

Table 2. Simulation results showing the efficiency of the Bayes estimator, n = sample size (CASE A1).

n	I1	I2	I3	I4	R1	R2
3	0.0003	0.0402	0.0013	0.0328	134.00	**24.459**
10	0.0001	0.0128	0.0013	0.0043	1280.0	**3.3843**
30	0.0004	0.0072	0.0012	0.0026	18.000	**2.1199**
50	0.0001	0.0067	0.0012	0.0024	67.000	**2.0881**

Table 3. Simulation results showing the efficiency of the Bayes estimator, n = sample size (CASE A2).

n	I1	I2	I3	I4	R1	R2
3	0.0015	0.0604	0.0081	0.0937	40.267	**11.510**
10	0.0009	0.0147	0.2632	0.0064	16.333	**1.8751**
30	0.0007	0.0074	0.2275	0.0054	10.5714	**1.3242**
50	0.0007	0.0065	0.0050	0.0064	9.2857	**1.2727**

Table 4. Simulation results showing the efficiency of the Bayes estimator, n = sample size (CASE B1).

n	I1	I2	I3	I4	R1	R2
3	0.0001	0.0393	0.0012	0.0356	393.00	**30.391**
10	0.0001	0.0121	0.0011	0.0039	121.00	**3.4329**
30	0.0001	0.0076	0.0011	0.0023	76.00	**2.0557**
50	0.0007	0.0070	0.0010	0.0022	10.000	**2.1293**

Table 5. Simulation results showing the efficiency of the Bayes estimator, n = sample size (CASE B2).

n	I1	I2	I3	I4	R1	R2
3	0.0023	0.0797	0.0184	0.2008	34.652	**10.896**
10	0.0010	0.0204	0.0144	0.0255	20.400	**1.7716**
30	0.0011	0.0135	0.0086	0.0121	12.273	**1.4104**
50	0.0013	0.0092	0.0096	0.0120	7.0769	**1.2497**

Table 6. Simulation results showing the efficiency of the Bayes estimator, n = sample size (CASE C1).

n	I1	I2	I3	I4	R1	R2
3	0.0001	0.0612	0.0042	0.0895	689.26	**21.378**
10	0.0005	0.0228	0.0039	0.0133	48.134	**3.4005**
30	0.0001	0.0188	0.0042	0.0105	212.51	**2.5315**
50	0.0008	0.0167	0.0036	0.0095	21.429	**2.6396**

Table 7. Simulation results showing the efficiency of the Bayes estimator, n = sample size (CASE C2).

n	I1	I2	I3	I4	R1	R2
3	0.0003	0.1441	0.0534	0.8182	480.33	**15.323**
10	0.0022	0.0295	0.0374	0.0629	13.409	**1.6813**
30	0.0002	0.0224	0.0333	0.0471	112.00	**1.4151**
50	0.0009	0.0182	0.0326	0.0426	20.222	**1.3097**

Table 8. Simulation results showing the efficiency of the Bayes estimator, n = sample size (CASE D1).

n	I1	I2	I3	I4	R1	R2
3	0.0001	0.0182	0.0002	0.0064	303.00	**30.874**
10	0.0001	0.0058	0.0002	0.0008	57.856	**4.0475**
30	0.0001	0.0033	0.0002	0.0005	40.680	**2.6513**
50	0.0001	0.0030	0.0002	0.0005	30.284	**2.4474**

Table 9. Simulation results showing the efficiency of the Bayes estimator, n = sample size (CASE D2).

n	I1	I2	I3	I4	R1	R2
3	0.0003	0.0273	0.0013	0.0184	86.989	**13.819**
10	0.0002	0.0058	0.0010	0.0019	37.953	**1.8903**
30	0.0003	0.0035	0.0008	0.0011	13.351	**1.3916**
50	0.0004	0.0021	0.0008	0.0010	5.9105	**1.2317**

Table 10. Simulation results showing the efficiency of the Bayes estimator, n = sample size (CASE E1).

n	I1	I2	I3	I4	R1	R2
3	0.0001	0.0030	0.0002	0.0049	30.284	**2.4471**
10	0.0001	0.0055	0.0002	0.0008	54.692	**4.3385**
30	0.0001	0.0034	0.0002	0.0005	34.352	**2.5000**
50	0.0007	0.0070	0.0010	0.0020	10.000	**2.0293**

Table 11. Simulation results showing the efficiency of the Bayes estimator, n = sample size (CASE E2).

n	I1	I2	I3	I4	R1	R2
3	0.0023	0.0360	0.0030	0.0394	15.663	**13.354**
10	0.0010	0.0092	0.0023	0.0050	9.2208	**2.1669**
30	0.0011	0.0061	0.0096	0.0120	5.5473	**1.7217**
50	0.0013	0.0092	0.0014	0.0024	7.0769	**1.4104**

Table 12. Simulation results showing the efficiency of the Bayes estimator, n = sample size (CASE F1).

n	I1	I2	I3	I4	R1	R2
3	0.0001	0.0277	0.0007	0.0176	276.62	**26.076**
10	0.0005	0.0103	0.0006	0.0026	20.611	**4.1730**
30	0.0001	0.0085	0.0007	0.0021	84.976	**3.0592**
50	0.0008	0.0075	0.0006	0.0019	9.4355	**3.2291**

Table 13. Simulation results showing the efficiency of the Bayes estimator, n = sample size (CASE F2).

n	I1	I2	I3	I4	R1	R2
3	0.0003	0.0651	0.0086	0.1607	217.11	**18.749**
10	0.0022	0.0133	0.0006	0.0124	6.0609	**2.0580**
30	0.0002	0.0101	0.0053	0.0093	50.624	**1.7308**
50	0.0009	0.0082	0.0052	0.0084	9.1404	**1.5990**

Table 14. Simulation results showing the robustness of the Bayes estimator with respect to the prior PDF, n = sample size (CASE G—Lognormal prior PDF).

n	I1	I2	I3	I4	R1	R2
3	0.0014	0.0238	0.0010	0.0126	17.345	**12.755**
10	0.0008	0.0067	0.0009	0.0019	8.2474	**2.1920**
30	0.0008	0.0031	0.0007	0.0011	3.7931	**1.4668**
50	0.0012	0.0030	0.0007	0.0009	2.4435	**1.3598**

Table 15. Simulation results showing the robustness of the Bayes estimator with respect to the prior PDF, n = sample size (CASE H—Uniform prior PDF).

n	I1	I2	I3	I4	R1	R2
3	0.0046	0.0637	0.0096	0.1206	13.848	**12.565**
10	0.0040	0.0196	0.0078	0.0188	4.9000	**2.4005**
30	0.0036	0.0089	0.0066	0.0094	2.4722	**1.4163**
50	0.0054	0.0047	0.0060	0.0071	0.8704	**1.1849**

6. A Robustness Analysis with Respect to the Assumed Prior and Wind Speed PDFs

In this last section, a small subset of many other simulations referring to different prior PDFs of the relevant RVs and parameters to be estimated is reported and illustrated. The purpose is to perform an adequate robustness analysis of the estimation process herein proposed in two cases:

(1) with respect to departures from the assumptions of the Gamma prior PDFs of the previous section;
(2) with respect to departures from the assumptions of the Weibull PDF of WS samples of the previous section.

First, a robustness analysis with respect to prior PDFs is investigated. Indeed, satisfactory results are obtained, as those illustrated in the previous section, as consistent with Bayesian statistical theory, as long as the Bayes estimates are evaluated assuming the *consistent* a priori distribution of Z, i.e., the one (herein the Gamma distribution with the given parameters) actually used in performing the simulation of the random samples.

However, choosing the prior distribution is not trivial, especially in view of its subjectivity, since it represents the information available on the possible values of the parameters to be estimated, together with the degree of uncertainty about them. Accordingly, it is useful to assess the robustness of the proposed methodology when the prior hypotheses about such distribution are not valid. In the sequel, simulations have been performed and robustness analyses have been successfully carried out by assuming different models, such as the Lognormal and the Uniform ones [32], as prior PDFs for both α and λ, instead of the previously assumed Gamma PDF.

For the sake of brevity, in this section, only the results relevant to the Lognormal model (case G) and the Uniform model (case H) as a prior PDF for both α and λ are reported. Moreover, only the most unfavorable cases are reported in Tables 14 and 15, i.e., the ones with a higher CV, while maintaining the same mean values of the prior PDF, as in previous section.

In summary, the characteristic features of the two cases selected for the robustness analysis are reported here below:

(1) Case G: Lognormal prior PDFs for both α and λ with the same mean value and CV of the Gamma PDF assumed for the computations;
(2) Case H: Uniform prior PDFs for both α and λ with same mean value and CV of the Gamma PDF assumed for the computations.

In Tables 14 and 15, for the sake of brevity, only the results relevant to case D2 (see previous section) are shown, since this is generally the most unfavorable case, i.e., with ratios R1 and R2 smaller than in other cases.

The results of the robustness analysis, as reported in Tables 14 and 15, still confirm the adequacy of the presented estimation procedure. Indeed, the Bayes estimate errors for the various assumed prior PDFs are not excessive even in the case of a very limited sample size and a remarkably unfavorable assumption, such as the Uniform PDF. In particular, it is worth noticing that in this unfavorable case, the results are still much better with respect to the ML ones.

Indeed, both indexes, R_1 and R_2, are remarkably greater than one for every sample size of every case. This is a noteworthy result, especially in the case of a Uniform prior PDF, which is obviously very different from the Gamma prior PDF assumed in the computations, whereas the Lognormal model is not very different from the Gamma model with the same mean value and CV.

Subsequently, a robustness analysis with respect to a sample PDF of the WS values is investigated. This investigation implies that, instead of the assumptions of the Weibull PDF of the WS samples of the previous section, different sample PDFs are assumed, with the same mean and CV. In the following, by making reference to the case D2, the results of two more cases are illustrated in Tables 16 and 17, and denoted as I and L, respectively. They are referred to as:

A Lognormal PDF for the WS (Table 16);
An Uniform PDF for the WS (Table 17).

In both cases, the prior PDFs are assumed as the Gamma PDFs of the previous section. Also in these cases, it is noticed that while the results for the Lognormal cases are similar to those for the Weibull case, as expected, a Uniform PDF is very different from the assumed Weibull model, and yet the simulation results are comparable with those previously illustrated in the Weibull case. Only in one case of Table 15 does the ratio R1 assume a value of less than one.

Table 16. Simulation results showing the robustness of the Bayes estimator with respect to the WS PDF, n = sample size (CASE I—Lognormal WS PDF).

n	*I*1	*I*2	*I*3	*I*4	R_1	R_2
3	0.0013	0.0058	0.0003	0.0030	4.6027	**8.8804**
10	0.0010	0.0019	0.0003	0.0006	1.8689	**1.8726**
30	0.0014	0.0039	0.0003	0.0004	2.7857	**1.3333**
50	0.0014	0.0042	0.0003	0.0004	3.0976	**1.4242**

Table 17. Simulation results showing the robustness of the Bayes estimator with respect to the WS PDF, n = sample size (CASE L—Uniform WS PDF).

n	*I*1	*I*2	*I*3	*I*4	R_1	R_2
3	0.0548	0.1002	0.0037	0.0316	1.8283	**8.4545**
10	0.0548	0.0693	0.0039	0.0080	1.2653	**2.0430**
30	0.0545	0.0631	0.0040	0.0059	1.1566	**1.4916**
50	0.0542	0.0615	0.0040	0.0055	1.1347	**1.3924**

Finally, it is noted that the estimation results are always scarcely sensitive with respect to the assumed Gamma prior distributions, and with respect to the assumed Weibull distribution of the WS values. This observation is confirmed by many other performed simulation analyses.

7. Conclusions

In structural safety design and in the analyses of the operating conditions of structures such as the ones related to tall buildings, offshore structures, and wind towers, it is useful to determine the best possible estimates of extreme wind loads on structures, the return periods, and other stochastic properties, given the available statistical data.

The present paper proposes a novel approach for the estimation of the probability that wind speed is lower than a prefixed extreme value, which might be dangerous in terms of the operating conditions of structures, the safety of structures and wind towers, and structural reliability. From a probabilistic point of view, the proposed method is based on the POT method for describing the stochastic processes of WS extremes in time and on a PD for the parent distribution by exploiting the Bayes estimation method for inference on the above probability, which allows one to define a proper safety index for the structure with respect to some pre-established operating conditions. A large set of numerical simulations have been performed and described in the last part of the paper. The performed numerical simulations show the absolute and relative efficiency of the model and the effectiveness of the proposed method of estimation.

In addition, the robustness of the proposed estimation method has also been investigated and discussed in detail. At this regard, it has been remarked that the performed investigations show that the present approach provides satisfactory estimates also when the true prior models are different from the ones assumed in the present work, namely the conjugate Gamma PDFs, and the same happens with respect to the Weibull PDF of wind speed.

Further investigations will also be conducted in future studies to estimate the autocorrelation function of the wind speed time series [49]. We also highlight that the Weibull distribution has been adopted in this analysis as a typical and widely adopted model for the sole purpose of evaluating how the estimation process works. The same efficiency might be achieved when dealing with other wind speed distributions, such as those examined in Section 3 or others available in the relevant literature [35,50–52]. The adoption of a double-period probability model is also worth investigating for future studies, to address the typical double periodicity of wind [53]. This kind of analysis will be developed in forthcoming research works.

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Appendix A. Analytical Details on the Expression of the SSI

Herein, some details of the series expansion leading to Equation (8) are reported:

$$\begin{gathered}\sum_{k=0}^{\infty} \exp(-\alpha t)\frac{(\alpha t)^k}{k!}(\mathrm{FM}(z))^k = \\ \sum_{k=0}^{\infty} \exp(-\alpha t)\frac{(\alpha \mathrm{FM}(z) t)^k}{k!} = \\ \exp(-\alpha t)\sum_{k=0}^{\infty} \frac{(\alpha \mathrm{FM}(z) t)^k}{k!} = \\ \exp(-\alpha t)\exp(\alpha \mathrm{FM}(z) t) = \\ \exp(-\alpha t(1-\mathrm{FM}(z)))\end{gathered}$$

Appendix B. A Reliability Interpretation of the SSI: The CHR Function

Let us introduce some reliability-related quantities, assuming that the threshold value z of WS with respect to the safety index is the maximum gust amplitude for some operating conditions of the structure, see also [22]. In other words, let us assume that the value z is such that the structure fails to satisfy some operating conditions as soon as z is reached. Then, the Reliability R(t) of the structure in the interval (0,t) is given by σ(t,z) of Equation (8), where t is a generic time instant.

By introducing the quantity MW(t) of (3), the following identity holds for the Reliability function with respect to the pre-established operating conditions of the structure:

$$R(t) = P(MW(t) < z) = \sigma(t,z)$$

Formally, at any time $t \geq 0$ for which $R(t) \neq 0$, it is possible to define a *hazard rate* function

$$h(t) = -\frac{R'(t)}{R(t)} = -\frac{d}{dt}[\log(R(t))] = -\frac{d}{dt}[\log(\sigma(t,z))]$$

As it is well known, the meaning of the hazard rate function *h(t),* as well as the origin of its name, lies in the following property, which is easily seen to be equivalent to the above definition: as $\Delta x \to 0^+$, the product $h(x)\Delta x$ equals the conditional probability, where the deficiency in satisfying the operating conditions occurs in the interval $(x, x+\Delta x)$, given that the device has been satisfactory until age x, i.e., such a product may be interpreted as the *instantaneous* conditional failure probability for a device of age x.

Then, introducing the *cumulative hazard rate* (CHR) function:

$$H(t) = \int_0^t h(x)dx$$

one simply obtains the following relationship between H and σ:

$$H(t) = -[\log(\sigma(t, z))]$$

On the basis of such an analogy, the CHR has been introduced in this paper, although, in the present framework, the term deficiency in satisfying some serviceability conditions used herein could also be interpreted as safety reduction. However, here, as in reliability studies, the CHR H(t) must be kept as small as possible, while the SSI σ(t,z), on the contrary, must be kept as large as possible.

Appendix C. On the Analytical Evaluation of the Bayes Estimate of H

$$E[H(s)] = \int_0^\infty \int_0^\infty \alpha s \exp(-xz^\gamma) g(\alpha, \lambda) d\alpha d\lambda$$

$$\frac{1}{\Gamma(m_1)} \int_0^\infty \exp(-\lambda z^\gamma) \beta_1^{m_1} \lambda^{m_1-1} \exp(-\lambda \beta_1) d\lambda$$

$$= \frac{1}{\Gamma(m_1)} \int_0^\infty \beta_1^{m_1} \lambda^{m_1-1} \exp(-\lambda(\beta_1 + z^\gamma)) d\lambda$$

$$= \frac{1}{\Gamma(m_1)} \int_0^\infty \frac{\beta_1^{m_1}}{(\beta_1+z^\gamma)^{m_1}} t^{m_1-1} \exp(-t) dt$$

$$= \frac{\Gamma(m_1)}{\Gamma(m_1)} \left(\frac{\beta_1}{(\beta_1+z^\gamma)}\right)^{m_1} = \left(\frac{\beta_1}{(\beta_1+z^\gamma)}\right)^{m_1}$$

References

1. Coles, S.G. *An Introduction to Statistical Modeling of Extreme Values*; Springer: London, UK, 2001.
2. Roncallo, L.; Solari, G.; Muscolino, G.; Tubino, F. Maximum dynamic response of linear elastic SDOF systems based on an evolutionary spectral model for thunderstorm outflows. *J. Wind. Eng. Ind. Aerodyn.* **2022**, *224*, 104978. [CrossRef]
3. Pagnini, L.C.; Solari, G. Joint modeling of the parent population and extreme value distributions of the mean wind velocity. *J. Struct. Eng.* **2016**, *142*, 04015138. [CrossRef]
4. Grigoriu, M.; Samorodnitsky, G. Reliability of dynamic systems in random environment by extreme value theory. *Probabilistic Eng. Mech.* **2014**, *38*, 54–69. [CrossRef]
5. Gioffrè, M.; Cluni, F.; Gusella, V. Characterization of an Equivalent Coupled Flexural-Torsional Beam Model for the Analysis of Tall Buildings under Stochastic Actions. *J. Struct. Eng.* **2020**, *146*, 04020239. [CrossRef]
6. Ciano, M.; Gioffrè, M.; Gusella, V.; Grigoriu, M.D. Non-stationary dynamic structural response to thunderstorm outflows. *Probabilistic Eng. Mech.* **2020**, *62*, 103103. [CrossRef]
7. Cluni, F.; Gioffrè, M.; Gusella, V. Dynamic response of tall buildings to wind loads by reduced order equivalent shear-beam models. *J. Wind. Eng. Ind. Aerodyn.* **2013**, *123*, 339–348. [CrossRef]
8. Gioffrè, M.; Gusella, V.; Grigoriu, M.D. Simulation of non-Gaussian field applied to wind pressure fluctuations. *Probabilistic Eng. Mech.* **2000**, *15*, 339–345. [CrossRef]
9. Gioffrè, M.; Gusella, V.; Grigoriu, M.D. Non-Gaussian wind pressure on prismatic buildings. I: Stochastic field. *J. Struct. Eng.* **2001**, *127*, 981–989. [CrossRef]
10. Gioffrè, M.; Gusella, V. Numerical analysis of structural systems subjected to non-Gaussian random fields. *Meccanica* **2002**, *37*, 115–128. [CrossRef]
11. Shi, H.; Dong, Z.; Xiao, N.; Huang, Q. Wind Speed Distributions Used in Wind Energy Assessment: A Review. *Front. Energy Res.* **2021**, *9*, 769920. [CrossRef]
12. An, Y.; Pandey, M.D. A comparison of methods of extreme wind speed estimation. *J. Wind. Eng. Ind. Aerodyn.* **2005**, *93*, 535–545. [CrossRef]

13. Castillo, E.; Hadi, A.S.; Balakrishnan, N.; Sarabia, J.M. *Extreme Value and Related Models with Applications in Engineering and Science*; John Wiley & Sons: Chichester, UK, 2004.
14. Villanueva, D.; Feijoo, A. Wind Power Distributions: A review of their Applications. *Renew. Sustain. Energy Rev.* **2010**, *14*, 1490–1495. [CrossRef]
15. Tabbuso, P.; Spence, S.M.J.; Palizzolo, L.; Pirrotta, A.; Kareem, A. An efficient framework for the elasto-plastic reliability assessment of uncertain wind excited systems. *Struct. Saf.* **2016**, *58*, 69–78. [CrossRef]
16. De Angelis, F.; Cancellara, D. Dynamic analysis and vulnerability reduction of asymmetric structures: Fixed base vs base isolated system. *Compos. Struct.* **2019**, *219*, 203–220. [CrossRef]
17. De Angelis, F.; Cancellara, D. Multifield variational principles and computational aspects in rate plasticity. *Comput. Struct.* **2017**, *180*, 27–39. [CrossRef]
18. De Angelis, F. A multifield variational formulation of viscoplasticity suitable to deal with singularities and non-smooth functions. *Int. J. Eng. Sci.* **2022**, *172*, 103616. [CrossRef]
19. De Angelis, F. An internal variable treatment of evolutive problems in hardening plasticity and viscoplasticity with singularities. *Contin. Mech. Thermodyn.* **2023**, *35*, 1807–1819. [CrossRef]
20. Chiodo, E.; De Falco, P. Inverse Burr Distribution for Extreme Wind Speed Prediction: Genesis, Identification and Estimation. *Electr. Power Syst. Res.* **2016**, *141*, 549–561. [CrossRef]
21. Volpi, E. On return period and probability of failure in hydrology. *WIREs Water* **2019**, *6*, e1340. [CrossRef]
22. Chiodo, E.; Lauria, D. Bayes prediction of wind gusts for Wind Power Plants Reliability Estimation. In Proceedings of the Clean Electrical Power (ICCEP) 2011, International Conference on Clean Electrical Power, Ischia, Italy, 14–16 June 2011; pp. 498–506.
23. Rajabi, M.R.; Modarres, R. Extreme value frequency analysis of wind data from Isfahan, Iran. *J. Wind. Eng. Ind. Aerodyn.* **2008**, *96*, 78–82. [CrossRef]
24. Wang, J.; Qin, S.; Jin, S.; Wu, J. Estimation methods review and analysis of offshore extreme wind speeds and wind energy resources. *Renew. Sustain. Energy Rev.* **2015**, *42*, 26–42. [CrossRef]
25. Palutikof, J.P.; Brabson, B.B.; Lister, D.H.; Adcock, S.T. A review of methods to calculate extreme wind speeds. *Meteorological* **1999**, *6*, 119–132. [CrossRef]
26. Leadbetter, M.R.; Lindgren, G.; Rootzén, H. *Extremes and Related Properties of Random Sequences and Processes*; Springer: New York, NY, USA, 1983.
27. Lechner, J.A.; Simiu, E.; Heckert, N.A. Assessment of ‘peaks over threshold methods for estimating extreme value distribution tails. *Struct. Saf.* **1993**, *12*, 305–314. [CrossRef]
28. Beirlant, J.; Goegebeur, Y.; Segers, J.J.; Teugels, J. *Statistics of Extremes: Theory and Applications*; Wiley: Chichester, UK, 2004.
29. Chiodo, E.; Diban, B.; Mazzanti, G.; De Angelis, F. A review on Wind Speed Extreme Values Modelling and Estimation for wind power plant design and construction. *Energies* **2023**, *16*, 5456. [CrossRef]
30. Coles, S.G.; Pericchi, L.R. Anticipating catastrophes through extreme value modeling. *Appl. Stat.* **2003**, *52*, 405–416.
31. Simiu, E.; Heckert, N. Extreme wind distribution tails: A “peaks over threshold” approach. *J. Struct. Eng.* **1996**, *122*, 539–547. [CrossRef]
32. Johnson, N.L.; Kotz, S.; Balakrishnan, N. *Continuous Univariate Distributions*; John Wiley & Sons: Hoboken, NJ, USA, 1995; Volume 1–2.
33. Chiodo, E.; Lauria, D.; Pisani, C. Bayes estimation of wind speed extreme values. In Proceedings of the 3rd Renewable Power Generation Conference (RPG 2014), Naples, Italy, 24–26 September 2014.
34. Wais, P. Two and Three-Parameter Weibull Distribution in Available Wind Power Analysis. *Renew. Energy* **2017**, *103*, 15–29. [CrossRef]
35. Kiss, P.; Jánosi, I.M. Comprehensive Empirical Analysis of ERA-39 Surface Wind Speed Distribution over Europe. *Energy Convers. Manag.* **2008**, *48*, 2142–2151. [CrossRef]
36. Bilir, L.; İmir, M.; Devrim, Y.; Albostan, A. An Investigation on Wind Energy Potential and Small Scale Wind Turbine Performance at İncek Region–Ankara. *Turkey. Energy Convers. Manag.* **2015**, *103*, 910–923. [CrossRef]
37. Perrin, O.; Rootzén, H.; Taesler, R. A Discussion of Statistical Methods Used to Estimate Extreme Wind Speeds. *Theor. Appl. Climatol.* **2006**, *85*, 203–215. [CrossRef]
38. Gumbel, E.J. *Statistics of Extremes*; Columbia University Press: New York, NY, USA, 1958.
39. Cheng, E.; Yeung, C. Generalized extreme gust wind speeds distributions. *J. Wind Eng. Ind. Aerodyn.* **2002**, *90*, 1657–1669. [CrossRef]
40. Fréchet, M. Sur la loi de probabilité de l’écart maximum. *Ann. Soc. Polon. Math.* **1927**, *6*, 93–116.
41. Sarkar, A.; Deep, S.; Datta, D.; Vijaywargiya, A.; Roy, R.; Phanikanth, V.S. Weibull and Generalized Extreme Value Distributions for Wind Speed Data Analysis of Some Locations in India. *KSCE J. Civ. Eng.* **2019**, *23*, 3476–3492. [CrossRef]
42. Alves, M.F.; Neves, C. *Extreme Value Distributions*; Springer: Berlin, Germany, 2011.
43. Pinheiro, E.C.; Ferrari, S.L.P. A comparative review of generalizations of the Gumbel extreme value distribution with an application to wind speed data. *J. Stat. Comput. Simul.* **2016**, *86*, 2241–2261. [CrossRef]
44. Kang, D.; Ko, K.; Huh, J. Determination of Extreme Wind Values Using the Gumbel Distribution. *Energy* **2015**, *86*, 51–58. [CrossRef]
45. Bernardo, J.M.; Smith, A.F.M. *Bayesian Theory*; John Wiley & Sons: Hoboken, NJ, USA, 2000.

46. Cowles, M.K. *Applied Bayesian Statistics*; Springer: Berlin/Heidelberg, Germany, 2013.
47. Press, S.J. *Subjective and Objective Bayesian Statistics: Principles, Models, and Applications*, 2nd ed.; John Wiley & Sons: Hoboken, NJ, USA, 2002.
48. Casella, G.; Robert, C.P. *Monte Carlo Statistical Methods*; Springer: Berlin/Heidelberg, Germany, 2000.
49. Dimitriadis, P.; Koutsoyiannis, D.; Iliopoulou, T.; Papanicolaou, P. A global-scale investigation of stochastic similarities in marginal distribution and dependence structure of key hydrological-cycle processes. *Hydrology* **2021**, *8*, 59. [CrossRef]
50. Morgan, E.C.; Lackner, M.; Vogel, R.M.; Baise, L.G. Probability distributions for offshore wind speeds. *Energy Convers. Manag.* **2011**, *52*, 15–26. [CrossRef]
51. Koutsoyiannis, D.; Dimitriadis, P.; Lombardo, F.; Stevens, S. *From Fractals to Stochastics: Seeking Theoretical Consistency in Analysis of Geophysical Data, Advances in Nonlinear Geosciences*; Tsonis, A.A., Ed.; Springer: Berlin/Heidelberg, Germany, 2018; pp. 237–278. [CrossRef]
52. Knoop, H.; Ament, F.; Maronga, B. A generic gust definition and detection method based on wavelet-analysis. *Adv. Sci. Res.* **2019**, *16*, 143–148. [CrossRef]
53. Katikas, L.; Dimitriadis, P.; Koutsoyiannis, D.; Kontos, T.; Kyriakidis, P. A stochastic simulation scheme for the long-term persistence, heavy-tailed and double periodic behavior of observational and reanalysis wind time-series. *Appl. Energy* **2021**, *295*, 116873. [CrossRef]