CERME 13

13TH CONGRESS OF THE EUROPEAN SOCIETY FOR RESEARCH IN MATHEMATICS EDUCATION

10-14 July 2023
Budapest
Hungary

PROCEEDINGS OF THE THIRTEENTH CONGRESS OF THE EUROPEAN SOCIETY FOR RESEARCH IN MATHEMATICS EDUCATION

Editors: Paul Drijvers, Csaba Csapodi, Hanna Palmér, Katalin Gosztonyi and Eszter Kónya

Organised by: Alfréd Rényi Institute of Mathematics and Eötvös Loránd University
Budapest, Hungary

2023

erme european society for research in mathematics education

Alfréd Rényi Institute of Mathematics

Eötvös Loránd University Faculty of Science

Editors

Paul Drijvers[1], Csaba Csapodi[2,3], Hanna Palmér[4], Katalin Gosztonyi[2,5] and Eszter Kónya[6]

[1]Utrecht University, Utrecht, the Netherlands
[2]Eötvös Loránd University, Budapest, Hungary
[3]Alfréd Rényi Institute of Mathematics, Budapest, Hungary
[4]Linnaeus University, Växjö, Sweden
[5]University of Montpellier, Montpellier, France
[6]University of Debrecen, Debrecen, Hungary

Editorial Board

Publisher

Alfréd Rényi Institute of Mathematics, Budapest, Hungary and ERME
ISBN 978-963-7031-04-5

Figural component in geometrical reasoning: The case of a blind solver

Elisa Miragliotta[1], Carola Manolino[2,4] and Andrea Maffia[3]

[1]University of Pavia; [2]University of Valle d'Aosta; [3]University of Bologna;

[4]Laboratorio "S. Polin", University of Torino, Italy; c.manolino@univda.it

In this paper, we use the Theory of Figural Concepts to describe the geometrical reasoning when solvers cannot rely on visual information. We analyse a case study of a blind mathematician solving a geometrical task. Qualitative analyses allow observations of an effective figural and conceptual blending and problematizing the visual and spatial aspects that seem to be encapsulated into Fischbein's description of figural components.

Keywords: Geometrical reasoning, figural concepts, spatial reasoning, blindness, gestures.

Introduction

Researchers in mathematics education point out the tension between visuospatial and logical aspects of geometrical objects (e.g., Mariotti, 1995). Scholars agree that effective geometrical reasoning requires the interplay of theoretical and pictorial domains. From a developmental point of view, although gradually logical organisation becomes more important, visuospatial aspects maintain a role in geometrical reasoning, even at a more advanced stage (Sinclair & Gol Tabaghi, 2010). Solvers might differently elaborate representations of geometrical objects because of ineffective visuospatial abilities or visual impairment. Historically, as Sinclair and Gol Tabaghi state, "visual intelligence has been much more prominent than that of motor intelligence" (2010, p. 225). This raises questions about the exploratory power of the lens provided by traditional frameworks on geometrical reasoning. To explore these questions, the case of born blind solvers seems particularly fruitful for conducting this investigation. From the educational point of view, geometrical experience can be grounded on spatial experience but, in absence of visual perception, spatiality might be differently experienced and the hands become critical channels through which "the world is sensed, processed, and conceived" (Healy & Fernandes, 2014, p. 125). We assume that, while solving a geometrical task, a blind solver relies on figural components, as sighted solvers do, despite the fact that the figural component has not been developed through visual-perceptual experiences. Here, we intend to explore to what extent the theoretical lens provided by the *Theory of Figural Concepts* (Fischbein, 1993) can describe the geometrical reasoning of a blind solver. With this general aim, we report the analysis of a task-based interview with a born blind mathematician while she solves a geometrical task.

Theoretical framework and research aim

Visual and spatial aspects within Fischbein's *Theory of figural concepts*

Building on the research in cognitive psychology on imagery, Fischbein (1993) makes a distinction between two different mental entities: the *concept* that is ideal and general, and the *image* that is conceived as "a *sensorial* representation of an object or phenomenon" (1993, p. 139, italics in the original) and is "based on the perceptive-sensorial experience, like the image of a drawing" (1993, p. 148).

Although the Theory of Figural Concepts (TFC) seems to consider primarily visual perception – e.g., pointing out that a geometrical figure is not a mere concept, but a "visual image" too (p. 141) – the definition of image presented above is broad enough to include other sensorial experiences. Acting upon geometrical figures requires an image that is "not sensorially perceived but *thought*, the genuine object of our geometrical reasoning" (p. 143). Indeed, psychologically, geometry deals with objects intrinsically and simultaneously having a dual nature (Fischbein, 1993): as concepts they are controlled by a definition within a mathematical theory, but they maintain the figural aspects of images. These *conceptual* and *figural* components provide complementary contributions to a third mental entity, the *figural concept*, that is the actual object upon which geometrical reasoning is realized, a crucial achievement of geometry learning. In particular, the *conceptual components* refer to the ideal theorization of a geometrical object as part of an axiomatic system (as learned by the solver); on the other hand, the *figural components* refer to the "spatial properties (shape, position, magnitude)" (1993, p. 143) of a figural concept. Ideally, these two components are harmonically fused in the figural concept, but more frequently one of the two dominates the other. Effective geometrical reasoning needs to work on figural concepts, as it was constructed by discarding the visual aspects:

> I do not intend to affirm that the representation we have in mind […] is devoid of any sensorial quality (like color) except space properties. But I affirm that, while operating with a geometrical figure, we act *as if no other quality counts.* (Fischbein, 1993, p. 143, emphasis in the original)

In other words, a figural concept maintains only the *spatial* aspects of the perceptual experience. For instance, in the case of a rectangle we can refer to its global appearance as a whole, but what counts in geometry are the mutual relationships among its parts (e.g., sides, diagonals). As learners, we might have experienced "rectangularity" by perceiving rectangular objects, but for conceptualizing the rectangle as a geometrical figure we must discard many perceptual features. Nevertheless, researchers have shown that while sighted solvers are exploring the resolution of a geometrical task, they rely on spatial and visual aspects for locally and globally controlling the figure, respectively (Miragliotta & Soldano, 2022). Our hypothesis is that while solving a geometrical task, blind solvers rely on figural components, as sighted solvers do, even though the figural component is not visual in nature.

Figural and conceptual blending in geometric reasoning

Geometrical reasoning can be described as a dialectic between figural and conceptual components (Mariotti, 1995) and the effectiveness of geometrical reasoning reveals the harmonic blending of these two components. For instance, let us consider the following task proposed by Fischbein (1993).

In a circle of centre O, consider two perpendicular diameters AB and CD. Arbitrarily choose a point M on the circumference and consider the projections of M onto the two diameters: the segments MN and MP, respectively. What is the length of the segment PN? (a possible representation in Figure 1a)

The task seems to require a strong theoretical effort. Since the length of MP and MN depends on the position of M, by recalling the Pythagorean Theorem the solver might embark on a long process of calculation at the end of which the figural counterpart might be lost. On the other hand, focusing only on figural aspects and imagining moving M along the circumference, the conceptual counterpart might be lost and PN might be perceived as having different length depending on the location of M.

According to TFC a coherent solution is not reachable by considering the image and the theoretical constraints independently, but "by a unique process in which a distilled figure is considered, revealing logical relationships" (Fischbein, 1993, p. 142). The solution process is supported by an interplay between figural and conceptual aspects: looking at the given theoretical constraints (i.e., orthogonality between two diameters, projections of M on the diameters, M on a circle) the solver can deduce that there are three right angles, then MNOP is a rectangle. PN is the figural component of the figural concept 'diagonals of a rectangle'. Possibly, the solver might conceptualise MNOP as a prototypical rectangle; the drawing could suggest this interpretation. Now the solver can theoretically recall the rectangle's properties and figurally introduce the segment MO, which is congruent to PN being a diagonal of the rectangle. Then, conceptually controlling the figure, the solver can notice that MO is also the figural component of another figural concept: the radius of the circle. So, the conceptual and figural blending allows a simultaneous understanding of MO as the figural counterpart of different figural concepts. The solver can conclude that PN is equal to the radius

The role of gestures in the mathematical practices of blind solvers

Healy and Fernandes (2011) have highlighted that blind solvers use gestures for communicating, reasoning upon, and haptically perceiving (representation of) mathematical objects. While we acknowledge that literature provides different definitions for what *gestures* are, for the sake of this research report, we will consider gestures as those "movements of the hands and arms that are produced when engaged in effortful cognitive activity" (Alibali, 2005, p. 309); indeed, our analysis is focused on the different ways in which an object can be explored using the hands (by touch and manipulation) during a problem solving activity, which is considered as an instance of effortful cognitive activity (Alibali, 2005). We conceive gestures as a "tool for seeing" since hands can provide access to the mathematical content as eyes do for sighted solvers, but they are also "tools for thinking", since haptic perception and conceptualization are intertwined: "Like the eyes of the sighted, the hands are moved in an intentional manner […] in order to perceive – and at the same time conceive – the object" (Healy & Fernandes, 2011, p. 159). The resulting conceptualization is potentially different from a visual one. Indeed, "whereas vision is synthetic and global, touch permits a gradual analysis, from parts to the whole" (Healy & Fernandes, 2011, p. 159) and therefore the two perceptions of a same object will not highlight the same properties. In tune with Healy and Fernandes (2011), we assume that the substitution of one tool (eye) to another (touch) would differently empower cognitive processes. Considering the twofold role of gesture as described above, in this study we look at blind solvers' gestures as windows onto their cognitive processes during the resolution of a geometric task, in order to observe the possible role played by haptic perception within the figural and conceptual blending of the blind solver's geometrical reasoning.

Research question

Starting from the assumption that, during the resolution of a geometrical task, a solver might recall the most suitable figural and conceptual components of the figural concepts in focus, we intend to explore to what extent the theoretical lens provided by the TFC intervenes in the interpretation of the geometrical reasoning of a blind solver. We address the following research question: *What is the role played by the figural components (if any) in a blind solver's geometrical reasoning?*

Methods

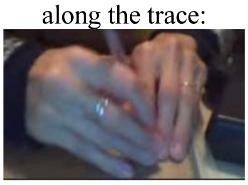
We design an instrumental case study aimed at describing a blind solver's effective geometrical reasoning. We conduct a task-based interview (Goldin, 2000) with Monica (fictious name), while she solves the geometrical task reported in the previous section. Monica is in her thirties, works as a business analyst, holds an honours degree in mathematics, and reads braille instead of using speech synthesis due to being blind since birth. At the beginning of the virtual meeting where the interview took place Monica was told that she could use whatever equipment or artifacts she wanted. She chose to use the rubber pad, in addition to her braille display. The rubber pad is used to make relief drawings; normally it is a 23x32 cm board. It consists of a wood and metal base, covered with a layer of rubber on which plastic sheets are clamped. The image is drawn with a pen (even without ink); when pressure is applied to the sheet, the plastic forms a relief perceptible to the touch.

The data consists of the video recording of the interview, from which frames and the verbatim transcription were retrieved. For the interview, a webcam was set up on Monica's hands. Therefore, the frames all depict her hands at work on the rubber pad. The data analysis followed a multimodal perspective, by focusing on all the solver's productions. Speech, gestures, and their blending were diachronically and synchronically analysed. Independent analyses of the transcripts were conducted by the three authors; the cases of disagreement were discussed until a consensus was reached.

Results of the analyses

In this section, we describe Monica's resolution path, by reporting on the most telling excerpts in terms of the contribution of conceptual and figural components. After reading the given construction, Monica manifests the need to have an "accurate drawing" which seems to be a personal need to exercise figural control upon the geometric situation by producing a perceptually accessible representation. This drawing needs to be isomorphic with the figural components of her personal figural concepts. She decides to perform a freehand drawing. Table 1 shows the corresponding excerpt.

Table 1: First excerpt from Monica's interview, time proceeds from left to right

Line	1	2	3
What is said	Ok, then let's do the diameters. One…	…and two. We said they must be perpendicular.	Yes, […] Well, it didn't come out so well, but theoretically I quite have it in mind.
What is done	With one hand she draws a line from a point on the circumference on her left and moves to the right and then the other hand moves along the trace:  She moves a whole hand on the drawing: .	She selects a point and draws a vertical trace from below to above. With one hand she draws and then the other hand moves along the trace:  Then, she moves the whole hand on the drawing.	With both hands, she touches the drawing, comprehending it between her hands. Then, one hand touches the intersection of the diameters while the other touches the rest of the drawing.  Finally, she touches the whole drawing with one hand.

All of Monica's drawing processes follow a common path: one hand holds the pen and two fingers of the other hand follow the graphic mark as it is generated (e.g., first frame in Table 1, line 1); then Monica moves the whole hand over the newly obtained drawing (e.g., second frame in Table 1, line 1). The first gesture seems to be used to produce the "accurate drawing", while the second gesture acts as a sort of figural control for the accuracy being sought. The latter hand movement echoes the moment when a sighted solver decides to glance at the global drawing as it is so far. Therefore, during the drawing process, the haptic interaction is *local* when Monica follows the trace-mark she is drawing, while it becomes *global* when she is checking the final and whole drawing.

After the drawing is performed, Monica places two fingers at specific positions to check the local consistency of the drawing. This is evident in line 3, when she focuses on the diameters with one hand and on their intersection point with the other. This kind of static gesture reveals the figural components in focus at a certain moment. The claim at line 3 supports our interpretation of Monica's need to have a drawing that incorporates and "*well*" mirrors the figural component of the figural concepts she has recalled so far. Indeed, the drawing Monica has produced contains a mistake (see Figure 1b): the two lines are perpendicular but one is not a diameter; hence, the circle is not split into four equal sectors. Here we see the interplay between the conceptual and figural components: Monica seems to be quite aware of the conceptual aspects required of her drawing by the problem, but she does not find a trustworthy figural counterpart in the drawing.

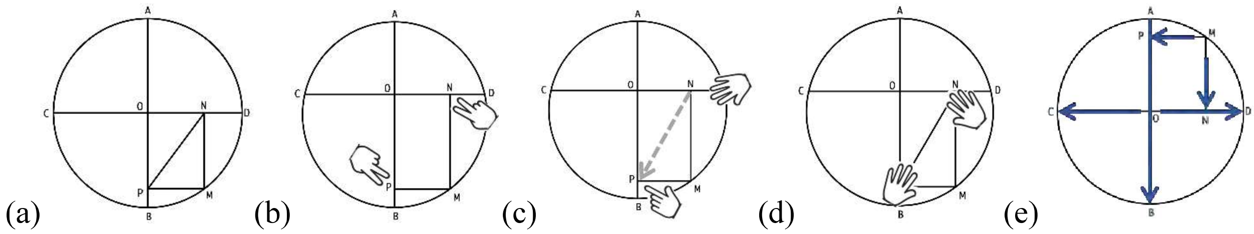


Figure 1: (a) An instance of task's representation; (b-e) Monica's drawing and exploration process

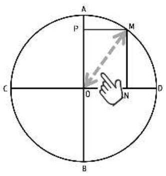
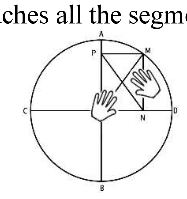
Then, Monica starts exploring the configuration. She holds with two fingers N and P (Figure 1b), while she claims: "*I must calculate the length of the line segment NP*". We interpret this gesture coupled with the utterance as a focus on the figural component that is requested in the task. She adds "*of this one*" following with one finger the possible trajectory of the unrealized segment (see Figure 1c), while the other hand holds the circle that marks the boundaries of the configuration. This gesture externalises the inner process of imagining an object that is not present in the drawing. After this brief exploration, she traces the segment NP using the two hands as she did before (same process as in Table 1, line 1) and she starts exploring the new configuration using two hands (Figure 1d). This global gesture corresponds to a broader focus that is gradually narrowing on the graphical traces within the sector DOB when the gestures move there. This interaction with the drawing suggests to us that Monica is looking for figural regularities that suggest a solution. The haptic perception shifts from local to global and then from global to local, revealing a continuous change of focus.

Monica continues the exploration frenetically moving her fingers on all the segments (NO, OP, MP, MN, NP) in the sector DOB. We interpret the speed of these gestures as a sign that she is looking for a sub-configuration to be further explored. During this exploration, the gestures slow down when Monica seems to find an interesting configuration. Indeed, we observe a new two-handed gesture:

with one hand she touches the angle at N, and she uses two fingers of the other hand to hold M and the segment PN at the same time. Now the focus is local and global at the same time. This moment turns out to be crucial for the following development of the resolution path.

In the following, the frenetic movements on the triangle PMN reveal her need to explore further due to a lack of conceptual components ("*I don't know anything*"). Monica needs "*a more accurate thing*", that is a new drawing (Figure 1e) where all the figural components, that she considers essential for obtaining an accurate rectangle, are carefully traced. Then a new exploration starts.

Table 2: Second excerpt from Monica's interview

Line	4	5	6	7	8
What is said	[...] I don't have accurate information.	Ah! But, but… Then there's this thing: this diagonal in the rectangle	is like if it were a radius.	And then also this other diagonal, which is the segment I wanted to calculate, will be necessarily equal to the radius,	because I am talking about a rectangle and the rectangle has to equal diagonals.
What is done	She fixes a right-hand finger on the point of the circumference and moves repeatedly two fingers of the left hand on pairs of segments MN, MP, ON, OP.	She moves twice a finger between O and M. 	She draws MO using two hands, as usual.	She draws PN: she puts two fingers on P and N; then she moves the P-finger to the N-finger following the pen.	She moves the two hands on the rectangle and touches all the segments. 

Monica frantically explores a specific portion of the drawing, focusing on the sides of the rectangle MNOP. The haptic exploration is driven by the need to gather "accurate" conceptual components through the figural ones that the drawing embeds. Differently from the previous explorations, gestures are focused on the mutual spatial relationship between the figural components (see line 4).

Suddenly, she reaches a new insight into the solution (Table 2, line 5) as if during the exploration she has developed a different conceptualization of the figural component OM. Indeed, we infer that, so far, OM was narrowly conceptualised as a diagonal of the rectangle (see "*this diagonal*") and now it assumes another conceptual status (see "*like if it were a radius*") while the figural component remains the same. Therefore, OM is seen as the figural component of two different figural concepts: the circle radius and the diagonal of a rectangle. In this process, figural and conceptual components are deeply intertwined: the control of conceptual components of the figural concept at stake has suggested focusing on the mutual relationship between the radii and the sides of the rectangle; through the haptic exploration coupled with the absence of tracemarks inside the rectangle Monica develops a twofold conceptualization of OM as part of different figural concepts. Indeed, this absence allows Monica to focus more deeply on the rectangle and the radii, and only after on the diagonals. Moreover, the figural component of the recalled figural concept plays an important role precisely because she talks about OM in absence of a physical support but using an ostensive lexicon (see "*this diagonal*") as if the figural component is embedded into the drawing. Finally, the gesture-speech coordination reveals how the figural concepts are recalled in their figural components (see the gesture at line 5) and only then in their conceptual components (see the utterance at line 5). The last two utterances (line 7-8) show the intervention of conceptual components for concluding that PN is always equal to the radius.

Discussion and conclusions

We focused on geometrical reasoning of a blind solver in order to shed light on the role played by the figural component. The reported case study allows us to describe an effective interplay between figural and conceptual components and, in retrospect, reconsider the multifaceted aspects that are encapsulated into TFC's description of figural components. The qualitative analysis provides an insight into Monica's geometrical reasoning and confirms our hypothesis on the role played by figural components, upon which Monica frequently relies during the task resolution. Much evidence supports this claim: from the start Monica creates a drawing that mainly supports figural elaboration; then the investigation is driven by conceptual aspects, but it is devoted to exploring the figure to gather additional properties. The variety of Monica's gestures manifests these different dimensions of haptic perception. Finally, the decision to redraw the configuration manifests the need to better control the mutual relationship between figural components. Monica seems to conceive the conceptual elaboration as reliable enough for solving the task, but also misaligned with the figural components embedded into the drawing. The new drawing allows her to recompose the break between the figural component, haptically perceived, and the conceptual component. Overall, the figural component seems to be essentially supported and exploited using drawing and gestures.

In this study, gestures have a significant multifaceted and intertwined role both for the solver and for the researchers. As for the latter, the analysis of gestures allows us to identify a shift of the solver's focus and infer ongoing processes. This provides evidence of the importance of the *spatial* dimension which is embedded into the figural component within the TFC. Indeed, the analysis of the haptic interaction with the drawing reveals the crucial role played by a narrow focus on the mutual relationship between graphical traces (e.g., the simultaneous focus on the right angle and the hypotenuse in the right triangle or the exploration of the rectangle's sides), more than the analysis of sighted solvers' processes, where researchers have less access to the specific elements in focus. Broadly speaking, blind solvers' interviews offer a rich situation for learning more about geometrical reasoning precisely because the perception is haptic and directly observable by the researchers. As for the solver, Monica combines the use of different gestures – with one/two hands and one/two fingers – which correspond to different aims: following the drawing process; checking the figural consistency of the drawing; anticipating the position of unrealised figural components; maintaining the focus on a specific object; investigating the mutual relationships between figural components.

We have observed other dimensions of Monica's gestures that can be summarised in three dichotomies: *global* and *local*, *static* and *dynamic*, *fast* and *slow* gestures. It seems that Monica uses a skilful combination of these dimensions to manipulate figural components while harmonising the conceptual components she has developed and, thereby, navigate the solution process. Although further investigations are needed, our hypothesis is that Monica strategically combines different gestures in line with previously developed geometrical problem-solving heuristics. In this sense, gestures are not just "tools for seeing", but crucial "tools for thinking". Despite starting from different theoretical assumptions on cognitive processes, our results are consistent with Healy and Fernandes' findings on gestures as "part of thinking" (2014, p. 142). We had assumed that the haptic perception would differently empower cognitive processes; this point is well demonstrated by the *spatial* nature of Monica's exploration and conceptualization.

From the theoretical point of view, TFC turns out to provide a suitable interpretative lens to analyse effective geometrical reasoning also when solvers do not have access to visual information. Monica provides a telling example of what it could mean to work on a "purified version of the image" (Fischbein, 1993, p. 148): to be effective the solver's figural component has to be spatially determined and as little as possible visually biased. Therefore, Monica's interview allows us to stress that the visual aspects are less crucial for geometrical reasoning than spatial ones. However, other visually impaired solvers could still rely on their visual past experience; so, other research is needed to unfold personal differences in figural and conceptual blending in the case of visual impairments. From the educational point of view, since Monica provides an example of an effective figural-conceptual blending not rooted in a visual experience, we would like to stress the importance of designing geometrical activities to promote the development of the spatial aspects of geometrical objects. We conjecture that diversifying the channels of access to the geometrical information is a useful design principle for strengthening the teaching-learning processes for each and everyone.

Acknowledgement

The study was partially funded by INdAM (GNSAGA project CUP_E55F22000270001).

References

Alibali, M. W. (2005). Gesture in spatial cognition: Expressing, communicating, and thinking about spatial information. *Spatial Cognition and Computation*, *5*(4), 307–31. https://doi.org/10.1207/s15427633scc0504_2

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics*, *24*(2), 139–162. http://dx.doi.org/10.1007/BF01273689

Goldin, G. (2000). A scientific perspective on structures, task-based interviews in mathematics education research. In R. Lesh & A.E. Kelly (Eds.), *Research design in mathematics and science education*, (pp. 517–545). Erlbaum.

Healy, L., & Fernandes, S. H. A. A. (2011). The role of gestures in the mathematical practices of those who do not see with their eyes. *Educational Studies in Mathematics*, *77*(2-3), 157–174.

Healy, L., & Fernandes, S. H. A. A. (2014). The gestures of blind mathematics learners. In L. D. Edwards, F. Ferrara & D. Moore-Russo (Eds.), *Emerging perspectives on gesture and embodiment in mathematics* (pp. 125–150). Information Age Publishing.

Mariotti, M. A. (1995). Images and concepts in geometrical reasoning. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education* (pp. 97–116). Springer. https://doi.org/10.1007/978-3-642-57771-0_7

Miragliotta, E. & Soldano, C. (2022). From figural to theoretical control (and back): a first proposal for framing the interplay between different controls. In J. Hodgen, E. Geraniou, G. Bolondi & F. Ferretti. (Eds.), *Proceedings of the Twelfth Congress of European Research Society in Mathematics Education* (CERME12) (pp. 4288–4295). ERME/Free University of Bozen-Bolzano.

Sinclair, N., & Tabaghi, S. G. (2010). Drawing space: Mathematicians' kinetic conceptions of eigenvectors. *Educational Studies in Mathematics, 74*(3), 223–240.