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Young children's spontaneous representations of multiplicative semantic structures before formal introduction

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We present an exploratory study on young children's spontaneous representations of multiplicative problems. In contrast to prior research on this topic, we investigate children's responses to using different means of representation (manipulatives and drawings) and different multiplicative semantic structures including equal groups, allocation, comparison, and rectangular array. Results suggest that children aged 5- 6 years old can represent different multiplicative semantic structures, but some structures appear to be more difficult. We found that drawing provided an alternative tool to manipulatives when representing the problem, and sometimes it was easier for children than using manipulatives. Some of the difficulties seem to be related to their familiarity with the mathematical language. This observation opens possible paths for future research.

Keywords: Multiplication, kindergarten, representations, semantic structures.

Introduction

Research indicates that children can represent multiplicative problems prior to formal instruction. For instance, Carpenter and colleagues (1993) found that after specific instruction, kindergarten children (5- to 6-year-olds) could correctly solve multiplicative problems by modelling them with tally marks, counting, or recall of facts. Mellone et al. (2013) introduced 5-year-old children to rate problems through storytelling, dramatization, and through representations such as arrays, and these experiences enabled young children to demonstrate an understanding of the multiplicative structure. More recently, Vanluydt, Supply et al. (2020) and Vanluydt, Verschaffel and Van Dooren (2022) reported that some children at kindergarten level could identify a multiplicative relation (ratio) in problems about proportionality, and that young children may already have a preference for additive or multiplicative thinking, before a formal introduction to multiplication. Furthermore, Bakker and colleagues (2014) found that first graders (6- to 7-year-olds) could solve multiplicative word problems without knowing what multiplication is or how it is formally represented.

Representations, in their various forms, are important for the teaching and learning of mathematics (e.g., Anthony & Walshaw, 2007; NCTM, 2019). For example, NCTM (2019) highlighted the importance of using and making connections between mathematical representations for effective teaching, and classified representations as contextual, visual, verbal, physical, and symbolic. Also, children should make choices about the form of representation they use as tools for solving problems. Ultimately, good problem solvers are flexible in their use of representations and can switch between them to best emphasize the process towards a solution (Lesh et al., 1987).

Our exploratory study has a particular focus on two modes of representations: children's drawings and the use of concrete materials. The research questions underpinning our study are: How do young children represent different semantic structures (equal groups, rectangular array, multiplicative comparison, allocation/rate) before formal introduction of multiplication? What differences do we observe in children's use of manipulatives, drawings, and language?

Theoretical framework

Many authors acknowledge that multiplication is conceptually more complex than addition both in terms of the problem situations, otherwise referred to as semantic structures (e.g., Anghileri, 1989; Greer 1992; Kouba, 1989), and the conceptual understanding required (Clark & Kamii, 1996; Steffe, 1994). Greer (1992) proposed four broad classifications of semantic structures involving multiplication and division of positive integers (whole numbers). These were equal groups, multiplicative comparison, rectangular area, and Cartesian product. Within the equal groups category are sub-categories, such as part-whole relationships, rate, and problems involving measures that relate to real life contexts. With the exception of Cartesian product, the other categories can be generalized to situations involving rational numbers. Greer maintained that while the distinctions between models of situations are important pedagogically, and provide an analytical framework useful for guiding research, one must be aware of the way in which students may interpret a problem. Further, that students' lack of proportional reasoning was related to their limited experiences with the different semantic structures (Greer, 1992).

Prior to Greer (1992), Anghileri's (1989) outlined six different semantic structures for multiplication word problems: equal groups (repeated addition), allocation/rate, array, number line, scale factor/rate, and Cartesian product. The current study explores four of these semantic structures (excluding the number line, scale factor/rate, and Cartesian product) through story problems with young children. A detailed description of each of these semantic structures is provided in several contributions (e.g., Downton & Sullivan, 2017).

Equal Groups occurs when there are several groups with equal quantities. For example, 'There are four plates and three donuts on each plate. How many donuts are there altogether?' Within this conceptualisation of multiplication, the two numbers play different roles: one is the multiplier that operates on the multiplicand.

Allocation/Rate involves a many-to-one correspondence in which equal sets of objects are matched with a tally set. In problems such as these, the multiplier is a rate variable that stipulates the mapping relation between an individual target item and its multiple counterparts, whereas the multiplicand specifies the number of target items and hence the number of iterations of that mapping relation (Downton & Sullivan, 2017).

Multiplicative Comparison refers to making comparisons or an enlargement that may apply to either discrete objects or to continuous objects, rather than an iteration of equal groups (Greer, 1992). The multiplicative factor may be considered as the multiplier. For example, 'Kate has 3 times-as-many cherries as Jack. If Jack has 4 cherries, how many cherries does Kate have?' Although the language of times-as-many may be confusing for students, Greer argued that this conceptualisation is a preliminary stage to ratio and directly relates to the nature of multiplication.

The *Rectangular Array* structure gives a visual pattern to both multiplication and division. Items are arranged in a row-by-column structure with the same number of items in each row and the same number of items in each column. It is argued that such a representation encourages students to develop their thinking about multiplication as a binary operation (Greer, 1992), and about its properties (Maffia & Mariotti, 2018).

These semantic structures and their representations (i.e., physical models) are different, but interconnected, which highlights the complexity associated with learning multiplication and division. As students construct an appropriate representation of a problem, different kinds of knowledge play an important role, in particular: schemata of problem situations, and linguistic knowledge. Nesher (1988) found that linguistic variables such as *each, altogether, times-as-many as, row,* or *column* have a considerable effect on children's performance. Further, that having an understanding of the textural structure of word problems is critical to understanding the underlying mathematical structure of the problem. Vanluydt and colleagues (2020) found that the familiarity with words such as *three more* and *double,* before beginning primary school, could predict results in rate problems.

Knowledge of the different semantic structures and of their possible representations (verbal, physical, and graphical) may be available to children before formal schooling, but there is little evidence of this in the literature – especially for drawings. In this paper we present findings of a study that investigated whether young children are able to interpret and represent multiplicative problems relating to each of the semantic structures.

Methods

In this paper we report on a small aspect of a larger study that we are conducting. We are interested in understanding how children represent multiplicative problems and if these representations lead them to a correct solution. Considering the young age of the children and the different modes of representations (manipulatives and drawings) we decided to adopt a multimodal semiotic approach by referring to the construct of 'semiotic bundle' as presented by Arzarello et al. (2009), which is "a system of signs […] produced by one or more interacting subjects and that evolves in time. [It] is made of the signs that are produced by a student or by a group of students while solving a problem" (Arzarello et al., 2009, p. 100).

We conducted our research in two countries with different languages (Australia and Italy, so English and Italian languages). These languages differ in the way in which multiplication is verbally represented. In English (as in many other languages) the word 'times' is used to read multiplications (like expressions such as 3×4). This is not the case in Italian, where the symbol $\times$ is read 'per' which is unrelated to the word 'volte' ('times' in Italian) but refers only to the name of the symbol itself. Also, in Italian there is no single word to say 'twice', but it must be translated in a phrase like 'two times' or referring to the 'double'. Therefore, we acknowledge that even with translating the proposed problems (Table 1) from English to Italian, the relative difficulty that children have experienced may be influenced by the nature of the language. Indeed, the direct translation may have led to the use of words that were not as familiar to children speaking one language as to the others. This issue will be further discussed in the Discussion section.

The two countries differ also in terms of the transition from kindergarten to primary school. In Australia, children start preschool (kindergarten) when they are 4-year-olds; from there they transition to their first year of primary school (Foundation) at the age of 5. In Italy, children start kindergarten at the age of 3 and then move directly to primary school when they are 6 years old. All children were interviewed in their first year of formal schooling. Australian children were in their Foundation year, while Italian children were interviewed at the very beginning of first grade (6-year-olds).

Prior to commencing the main study, the interview tasks (Table 1) and recording sheet were trialled, following which refinements were made to the language and the number range. Each child was asked to represent up to six problems, meaning that each of them did not represent all those listed in Table 1 (this is why not all problems have been asked the same number of times, as shown in Table 2). In half of the cases the child was asked to represent the problem using a drawing and in the other situations manipulatives were provided. Problems that appeared easier in the pilot study were asked more frequently to reduce any frustration experienced due to the complexity of the problems.

Table 1: Multiplicative problems used in the interviews

Equal groups	Three children have two cookies each. How many cookies do they have altogether?	I have three plates with four cherries on each. How many cherries altogether?
Rectangular Array	There are 3 teddies in each of 4 rows. How many teddies are there altogether?	I planted some carrots in my veggie garden. There were three carrots in each row and there were two rows. How many carrots did I plant?
Multiplicative Comparison	John has 3 times-as-many apples as Mary. If Mary has 4 apples, how many apples does John have?	Mary made a tower using three blocks. Max made a tower that was twice as tall. How many blocks did Max use?
Allocation/Rate	If there are four cookies per child, how many cookies do three children have?	One person has two shoes. How many shoes would three people have?

Participants in our study consisted of 19 children with an average age of 6 years (6 males and 4 females were Australian; 5 females and 4 males were Italian). The researcher interviewed the children individually using an interview script to ensure consistency. The task could be repeated as many times as needed. Each interview took for approximately 15 minutes and was video recorded with parent consent. Video recordings were transcribed verbatim, and the transcript was enriched with images of gestures and drawings to describe the semiotic bundle (Radford & Sabena, 2015).

Results

As it was described in the Methods section, each child was asked to represent up to six problems. Not all type of problems were experienced by the same number of children. Table 2 reports how many children, among those that were asked to do so, successfully represented the problems classified according to the theoretical framework presented above.

Table 2: Number of children who correctly represented the proposed multiplicative problems over the number of total children who were asked to represent that semantic structures

	Equal Groups	Rectangular Array	Multiplicative Comparison	Allocation/Rate
Manipulatives	2/5	5/8	5/15	4/6
Drawing	8/9	4/6	1/4	8/9

All the problems were represented correctly by at least one child. However, we can see that the equal groups problems were represented correctly more often by means of drawing. Also, in the case of allocation/rate problems, using manipulatives does not appear easier than representing the same problem using drawing. These results suggest that, for the children in our sample, representing multiplicative problems through manipulatives is not necessarily easier or more intuitive than representing them through drawings; however, this varies according to the type of problem.

Although all the semantic structures were accessible to at least one child in this sample, this does not mean that they all were easy to access for each child interviewed. Comparing the data in Table 2, suggests that equal groups and rate problems were more frequently represented correctly in drawings. This may depend on the familiarity of children with the language that is used to describe a certain type of problem. We consider the role of language to be particularly important since we observed some differences in the responses by the Italian and Australian children. For instance, in the case of Comparison problems, almost all the correct representations belong to Australian children. For the "Apples" Comparison problem, all the Italian children who responded to this question represented a character with three apples and a character with four apples (Fig. 1). Doing so, suggests these children could not distinguish the roles of multiplicand and multiplier in this story and interpreted it as an addition problem.

Figure 1: Two Italian children's representation of the Apples problem by means of drawings

The only situation in which an Italian child was able to correctly represent a Comparison semantic structure was by using manipulatives to represent the "Tower" problem. After the interviewer read the problem, the child built a tower made of three blocks (Fig. 2a). Then, the child created two other towers of three blocks each. When asked how many blocks were used in the second case she correctly answered "six". However, the representation provided with the blocks leaves open the possibility that the child interpreted the original problem as, “Max made a tower that was twice as tall” as if it was “Max made the tower twice”. In the case of some Australian children we have a representation that resembles more clearly the words ‘twice as tall’ (Fig. 2b).

(a)  (b) 

Figure 2: Two children’s representation of the Tower problem by means of manipulatives.

Many of the children who had difficulties in correctly representing the “Tower” problem made or drew a tower of five blocks. When asked to justify their choice, they provided arguments like “twice means two more” which suggests that the difficulty found in representing this problem is given by the meaning of the word ‘twice’. Those that were able to correctly represent the “Tower” problem were also able to explain their interpretation of the word ‘twice’, for example by saying “twice is another way of saying two of the same number, so it is six”. The fact that the difficulty may reside in the language may be one of the reasons for the differences of results between the Italian and the Australian children. The language difficulty will be further discussed in the final section of this paper.

Discussion

We wondered how children would represent the different multiplicative semantic structures before their formal introduction in school. Our video analysis allowed us to gain insights into whether the children in our study could interpret and represent all the problems. Fewer children were successful in solving and representing the Comparison problems compared to the other semantic structures. We noticed that, in the context of our sample, the use of manipulatives was more effective than drawing for the Comparison problems. It is possible that this result depends on the particular context of building towers with blocks. When doing so, children could construct composite units (of three blocks in our case), which become the unit of comparison. Representing the problem with apples was more difficult, possibly because it is more difficult to compare discrete sets rather than a continuous set. Indeed, Bakker et al. (2014) suggest that the characteristics of the problem might influence student performance.

The specific difficulty of Comparison problems compared to other problems could also relate to the language that is used to describe those problems. In particular, the phrase ‘time-as-many’ can be difficult to interpret (Nesher, 1988). The fact that the Italian children we interviewed found this kind of problem more difficult may suggest that the familiarity with such phrase (which is rarely used in Italian everyday language) could be crucial for representing Comparison problems. This observation may have an educational implication that familiarity with the words used to describe different semantic structures could be a pre-requisite for accessing multiplicative problem solving

(Vanluydt et al., 2020). While we recognize that our sample is too small for any generalization, we agree with Vanluydt et al. (2020) in that topic-specific linguistic interventions may provide a base for future arithmetical learning. However, this assumption requires additional research before it can be substantiated.

Until recently, most studies about young children's representation of multiplicative problems involved mainly manipulatives (e.g. Carpenter et al., 1993; Vanluydt et al., 2022). However, the children we observed often (but not always) found it easier to represent problems by means of drawing, which substantiates what other researchers have reported: drawing can become an important tool for problem solving in early years mathematics (e.g. Soundy & Drucker, 2009).

The aim of our work was to begin exploring a research topic that has been under-researched. The small number of subjects involved allowed us to analyse qualitatively the interviews considering different representations (drawings, movements, words). We acknowledge a larger data set would be needed to generalize these results. Our intention is to develop the study on a larger scale in the future to address this limitation. Furthermore, we intend to investigate if and how linguistic interventions can enable young children to represent different multiplicative semantic structures.

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