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Calculus of Variations. – *A counterexample to the monotone increasing behavior of an Alt–Caffarelli–Friedman formula in the Heisenberg group*, by Fausto Ferrari and Nicolò Forcillo, communicated on 10 November 2022.

Abstract. – In this paper, we provide a counterexample about the existence of an increasing monotonicity behavior of a function introduced by Ferrari and Forcillo (2020), companion of the celebrated Alt–Caffarelli–Friedman monotonicity formula, in the noncommutative framework.

Keywords. – Alt–Caffarelli–Friedman monotonicity formula, Heisenberg group, two-phase free boundary problems.

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1. Introduction

In this paper, we continue the research about the existence of a monotonicity formula in the Heisenberg group started in [12]; see also [13, 14]. More precisely, we prove that there exists a function u such that, denoting $u^+ := \sup\{u, 0\}$ and $u^- := \sup\{-u, 0\}$, defined in a neighborhood of $0 \in \mathbb{H}^1$, if $\Delta_{\mathbb{H}^1} u^\pm \geq 0$ and $0 \in \mathcal{F}(u)$, then

$$J_u^{\mathbb{H}^1}(r) := \frac{1}{r^4} \int_{B_r^{\mathbb{H}^1}(0)} \frac{\left|\nabla_{\mathbb{H}^1} u^+(\xi)\right|^2}{|\xi|_{\mathbb{H}^1}^2}\, d\xi \int_{B_r^{\mathbb{H}^1}(0)} \frac{\left|\nabla_{\mathbb{H}^1} u^-(\xi)\right|^2}{|\xi|_{\mathbb{H}^1}^2}\, d\xi, \tag{1.1}$$

$\xi = (x, y, t) \in \mathbb{H}^1$, is not monotone increasing in a possibly small right neighborhood of 0, where $\mathbb{H}^1$ is the first Heisenberg group.

In order to better understand the profile of this result, we recall that in [1, 7] the celebrated monotonicity formula, in the Euclidean setting, was applied to prove regularity results about viscosity solutions of two-phase problems like

$$\begin{cases} \Delta u = 0 & \text{in } A^+(u) := \{x \in A : \ u(x) > 0\}, \\ \Delta u = 0 & \text{in } A^-(u) := \operatorname{Int}\left(\{x \in A : u(x) \leq 0\}\right), \\ |\nabla u^+|^2 - |\nabla u^-|^2 = 1 & \text{on } \mathcal{F}(u) := \partial A^+(u) \cap A, \end{cases} \tag{1.2}$$

where $A \subset \mathbb{R}^n$ is an open set and $u \in C(A)$ is a viscosity solution; see [6] or [5] for such a definition.

In particular, in [1] the authors proved that for every solution $u \in H^1(A)$ of (1.2) and for every $P_0 \in \mathcal{F}(u)$, the function

$$(1.3) \qquad J_u(r) := \frac{1}{r^4} \int_{B_r(P_0)} \frac{\left|\nabla u^+(P)\right|^2}{|P - P_0|^{n-2}}\, dP \int_{B_r(P_0)} \frac{\left|\nabla u^-(P)\right|^2}{|P - P_0|^{n-2}}\, dP$$

is monotone increasing in a right neighborhood of 0. Such a tool has been widely employed to prove regularity results for the solutions of (1.2). About this, we recall [11] for an overview concerning recent results on two-phase problems in the Euclidean framework; see also [5].

We worked on

$$(1.4) \qquad J_{\beta,u}^{\mathbb{H}^1}(r) := \frac{1}{r^\beta} \int_{B_r^{\mathbb{H}^1}(0)} \frac{\left|\nabla_{\mathbb{H}^1} u^+(\xi)\right|^2}{|\xi|_{\mathbb{H}^1}^2}\, d\xi \int_{B_r^{\mathbb{H}^1}(0)} \frac{\left|\nabla_{\mathbb{H}^1} u^-(\xi)\right|^2}{|\xi|_{\mathbb{H}^1}^2}\, d\xi$$

only.

On the other hand, since the function $u(x, y, t) = \alpha_1 x^+ - \alpha_2 x^-$, for some fixed numbers $\alpha_i \geq 0$, $i = 1, 2$, satisfies $\Delta_{\mathbb{H}^1} u = 0$ in $\{u > 0\}$, as well as $\Delta_{\mathbb{H}^1} u = 0$ in $\{u < 0\}$, we checked that if $\beta = 4$, then $J_u^{\mathbb{H}^1}(r) := J_{4,u}^{\mathbb{H}^1}(r)$ is constant. Hence, supposing that β is constant and assuming that $J_u^{\mathbb{H}^1}$ is monotone increasing as well, then necessarily $\beta \leq 4$.

We did not manage to conclude that $J_u^{\mathbb{H}^1}$ is monotone increasing for all the admissible functions, apparently since following the strategy described in [5], we would need some sharper results in geometric measure theory that, in the Heisenberg group, are not known yet; see [12] for the details.

As a consequence, in order to deepen that research, we decided to follow another strategy already available in the Euclidean case $\mathbb{R}^n$. More precisely, to select the right exponent β, and possibly to deduce the increasing monotone behavior of J_u in $\mathbb{R}^n$ as well, it is useful to study straightforwardly the behavior of the function

$$\mathcal{I}_u(r) := \frac{1}{r^2} \int_{B_r(0)} \frac{\left|\nabla u(P)\right|^2}{|P|^{n-2}}\, dP$$

when u is harmonic; see [18].

Nevertheless, on the contrary to what we supposed, we discover that there exists at least a function u such that $\Delta_{\mathbb{H}^1} u = 0$ and

$$(1.5) \qquad \mathcal{I}_u^{\mathbb{H}^1}(r) := \frac{1}{r^2} \int_{B_r^{\mathbb{H}^1}(0)} \frac{\left|\nabla_{\mathbb{H}^1} u(\xi)\right|^2}{|\xi|_{\mathbb{H}^1}^2}\, d\xi$$

is strictly monotone decreasing in a small right neighborhood of 0, differently from what happens in the Euclidean case.

Hence, starting from this result, we obtain that $J_u^{\mathbb{H}^1}$ is strictly monotone decreasing for a careful choice of u. More precisely, there exists a function u such that $J_u^{\mathbb{H}^1}$ is not monotone increasing. In particular, if $u = x$, we have that $J_u^{\mathbb{H}^1}$ is constant, while if

$$u = x - 3yt - 2x^3,$$

which satisfies $\Delta_{\mathbb{H}^1} u = 0$, then $J_u^{\mathbb{H}^1}$ is strictly monotone decreasing. This last fact depends on the lack of orthogonality of the intrinsic harmonic polynomials in the Heisenberg group; see e.g. [16].

Hence, our main result, whose proof is contained in Section 3, is the following one.

THEOREM 1.1. *Let* $u = x - 3yt - 2x^3$. *Then* $\Delta_{\mathbb{H}^1} u = 0$ *and* $J_u^{\mathbb{H}^1}(r)$ *is strictly monotone decreasing in a right neighborhood of* $r = 0$.

We point out that (1.4) is not the unique function that can be considered for proving a monotonicity formula. For instance,

$$J_{\beta,u}^{\mathbb{H}^1,C}(r) := \frac{1}{r^\beta} \int_{B_r^{\mathbb{H}^1,C}(0)} \frac{\left|\nabla_{\mathbb{H}^1} u^+(\xi)\right|^2}{d_{\mathbb{H}^1,C}(\xi,0)^2}\, d\xi \int_{B_r^{\mathbb{H}^1,C}(0)} \frac{\left|\nabla_{\mathbb{H}^1} u^-(\xi)\right|^2}{d_{\mathbb{H}^1,C}(\xi,0)^2}\, d\xi, \tag{1.6}$$

where $d_{\mathbb{H}^1,C} : \mathbb{H}^1 \times \mathbb{H}^1 \to [0,\infty)$ denotes the Carnot–Charathéodory distance in the Heisenberg group $\mathbb{H}^1$ and

$$B_r^{\mathbb{H}^1,C}(P) = \left\{Q \in \mathbb{H}^1 : d_{\mathbb{H}^1,C}(Q,P) < r\right\}$$

is the metric ball centered at $P \in \mathbb{H}^1$ with radius r, keeps the same scaling properties of (1.4). In fact, $d_{\mathbb{H}^1,C}(\cdot,0)$ and $|\xi|_{\mathbb{H}^1}$ are equivalent; see e.g. [4, 9]. Nevertheless, Theorem 1.1 cannot be applied to the function

$$J_u^{\mathbb{H}^1,C}(r) := \frac{1}{r^4} \int_{B_r^{\mathbb{H}^1,C}(0)} \frac{\left|\nabla_{\mathbb{H}^1} u^+(\xi)\right|^2}{d_{\mathbb{H}^1,C}(\xi,0)^2}\, d\xi \int_{B_r^{\mathbb{H}^1,C}(0)} \frac{\left|\nabla_{\mathbb{H}^1} u^-(\xi)\right|^2}{d_{\mathbb{H}^1,C}(\xi,0)^2}\, d\xi,$$

obtained from (1.6) when $\beta = 4$. As a consequence, Theorem 1.1 is a counterexample to the increasing monotonicity behavior of (1.1) only.

We focus on (1.4) because $|\xi|_{\mathbb{H}^1}^{-2}$ is, up to a constant, the fundamental solution, computed at the pole 0, of the sub-Laplacian $\Delta_{\mathbb{H}^1}$; see e.g. [4, 9, 19]. Analogously, in the Euclidean case (1.3), when $n = 3$, the companion fundamental solution of the Laplacian in $\mathbb{R}^3$ with pole $P_0 = 0$, up to a multiplicative constant, appears in the formula. In particular, in the Euclidean proof of the Alt–Caffarelli–Friedman formula, the fundamental solution plays a role; see [1, 5] and also [12].

Hence, following the main stream of those results concerning the mean value properties of harmonic functions in the Heisenberg group and, more in general, the representation formulas of functions satisfying equations like $\Delta_{\mathbb{H}^1} u = f$, see [4, 10] as well, it appears useful to deal with (1.4).

On the other hand, it is well known, from [2, 3], that $d_{\mathbb{H}^1,C}(\cdot, 0)$ is not a $\Delta_{\mathbb{H}^1}$-gauge norm, so that $d_{\mathbb{H}^1,C}(\cdot, 0)^{-2}$ cannot be harmonic in $\mathbb{H}^1 \setminus \{0\}$.

The paper is organized as follows. In Section 2, we fix the notation. In Section 3, we show the explicit computation of the fact that $\mathcal{I}_u^{\mathbb{H}^1}$ is strictly monotone decreasing, obtaining the main tool useful to prove Theorem 1.1. In Section 4, we provide an extension of our argument and an application exhibiting a genuine nontrivial example of solution to a two-phase free boundary problem in the Heisenberg group.

2. The Heisenberg setting

In this section, we provide some basic notions about the Heisenberg group. For the sake of simplicity, we restrict ourselves to the $\mathbb{H}^1$ case, nevertheless the argument holds in $\mathbb{H}^n$ as well.

We recall here that $\mathbb{H}^n$ denotes the set $\mathbb{R}^{2n+1}$, $n \in \mathbb{N}$, $n \geq 1$, endowed with the noncommutative inner law in such a way that, for every $P \equiv (x_1, y_1, t_1) \in \mathbb{R}^{2n+1}$, $M \equiv (x_2, y_2, t_2) \in \mathbb{R}^{2n+1}$, $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}^n$, $i = 1, 2$, it holds that

$$P \circ M := \big(x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2\big(\langle x_2, y_1\rangle - \langle x_1, y_2\rangle\big)\big),$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$.

Let $X_i = (e_i, 0, 2y_i)$ and $Y_i = (0, e_i, -2x_i)$, $i = 1, \dots, n$, where $\{e_i\}_{1 \leq i \leq n}$ is the canonical basis for $\mathbb{R}^n$. The inverse of $P := (x, y, t) \neq 0$ is $(-x, -y, -t)$ and it is denoted by P^{-1}.

We use the same symbols to denote the vector fields associated with the previous vectors so that, for $i = 1, \dots, n$, we have

$$X_i := \partial_{x_i} + 2y_i \partial_t, \quad Y_i := \partial_{y_i} - 2x_i \partial_t.$$

The commutator between the vector fields is

$$[X_i, Y_i] := X_i Y_i - Y_i X_1 = -4\partial_t, \quad i = 1, \dots, n,$$

otherwise is 0. The intrinsic gradient of a real valued smooth function u in a point P is

$$\nabla_{\mathbb{H}^n} u(P) := \sum_{i=1}^{n} \big(X_i u(P) X_i(P) + Y_i u(P) Y_i(P)\big).$$

Now, there exists a unique metric on

$$H\mathbb{H}^n_P := \operatorname{span}\big\{X_1(P), \dots, X_n(P), Y_1(P), \dots, Y_n(P)\big\}$$

which makes the set of vectors $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ orthonormal. Thus, for every $P \in \mathbb{H}^n$ and for every $U, W \in H\mathbb{H}^n_P$,

$$U = \sum_{j=1}^{n} \big(\alpha_{1,j} X_j(P) + \beta_{1,j} Y_j(P)\big),$$
$$V = \sum_{j=1}^{n} \big(\alpha_{2,j} X_j(P) + \beta_{2,j} Y_j(P)\big),$$

it holds that

$$\langle U, V\rangle = \sum_{j=1}^{n} (\alpha_{1,j}\alpha_{2,j} + \beta_{1,j}\beta_{2,j}).$$

Since we mainly work on $\mathbb{H}^1$, that is the case in which $n = 1$, we simply introduce the remaining notation in $\mathbb{H}^1$. In particular, we define a norm associated with the metric on the space span$\{X, Y\}$ as follows:

$$|U| := \sqrt{\sum_{j=1}^{1} (\alpha_{1,j}^2 + \beta_{1,j}^2)} = \sqrt{\alpha_{1,1}^2 + \beta_{1,1}^2}.$$

For example, the norm of the intrinsic gradient of a smooth function u in P is

$$\big|\nabla_{\mathbb{H}^1} u(P)\big| = \sqrt{\big(Xu(P)\big)^2 + \big(Yu(P)\big)^2}.$$

Moreover, if $\nabla_{\mathbb{H}^1} u(P) \neq 0$, then

$$\left| \frac{\nabla_{\mathbb{H}^1} u(P)}{\big|\nabla_{\mathbb{H}^1} u(P)\big|} \right| = 1.$$

If $\nabla_{\mathbb{H}^1} u(P) = 0$, instead, we say that the point P is characteristic for the smooth surface $\{u = u(P)\}$. In particular, for every point $M \in \{u = u(P)\}$, which is not characteristic, the intrinsic normal to the surface $\{u = u(P)\}$ is well defined, given by

$$\nu(M) := \frac{\nabla_{\mathbb{H}^1} u(M)}{\big|\nabla_{\mathbb{H}^1} u(M)\big|}.$$

The Kohn–Laplace operator is

$$\Delta_{\mathbb{H}^1} := X^2 + Y^2,$$

where

$$\Delta_{\mathbb{H}^1} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4y \frac{\partial^2}{\partial x \partial t} - 4x \frac{\partial^2}{\partial y \partial t} + 4(x^2 + y^2) \frac{\partial^2}{\partial t^2},$$

so that $\Delta_{\mathbb{H}^1}$ is a degenerate elliptic operator, because the smallest eigenvalue associated with the matrix

$$\begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \\ 2y & -2x & 4(x^2+y^2) \end{pmatrix}$$

is always 0.

At this point, we introduce in the Heisenberg group $\mathbb{H}^1$ the Koranyi norm of $P \equiv (x, y, t) \in \mathbb{H}^1$ as

$$\big|(x,y,t)\big|_{\mathbb{H}^1} := \sqrt[4]{(x^2+y^2)^2+t^2}.$$

In particular, for every positive number r, the gauge ball of radius r centered at 0 is

$$B_r^{\mathbb{H}^1}(0) := \big\{P \in \mathbb{H}^1 : |P|_{\mathbb{H}^1} < r\big\}.$$

It is worth to say that this structure is endowed by suitable properties, like the left invariance with respect to the inner law. More precisely, for every point $P \in \mathbb{H}^1$,

$$P \circ B_r^{\mathbb{H}^1}(0) = B_r^{\mathbb{H}^1}(P) = \big\{S \in \mathbb{H}^1 : |P^{-1} \circ S|_{\mathbb{H}^1} < r\big\},$$

which implies that

$$\mathrm{meas}_3\big(B_r^{\mathbb{H}^1}(0)\big) = \mathrm{meas}_3\big(B_r^{\mathbb{H}^1}(P)\big),$$

where meas_3 denotes the usual Lebesgue measure in $\mathbb{R}^3$.

Moreover, if u is a C^1 function in $\mathbb{H}^1$ and for every $P \in \mathbb{H}^1$ we define $v(S) := u(P \circ S)$, then

$$\begin{aligned} Xv(S) &= Xu(P \circ S), \\ Yv(S) &= Yu(P \circ S), \\ \Delta_{\mathbb{H}^1} v(S) &= \Delta_{\mathbb{H}^1} u(P \circ S). \end{aligned}$$

In addition, a dilation semigroup is defined as follows: for every $r > 0$ and for every $P \equiv (x, y, t) \in \mathbb{H}^1$, let

$$\delta_r(P) := (rx, ry, r^2 t).$$

As a consequence, denoting $u_r(S) := u(\delta_r(S))$, it holds that

$$\begin{aligned} Xu_r(S) &= r(Xu)\big(\delta_r(S)\big), \\ Yu_r(S) &= r(Yu)\big(\delta_r(S)\big), \\ \Delta_{\mathbb{H}^1} u_r(S) &= r^2(\Delta_{\mathbb{H}^1} u)\big(\delta_r(S)\big). \end{aligned}$$

The details of all previous properties can be found in [19] or in other handbooks like [9] or [4]. See also [8, 15, 17] for further developments.

3. Nonexistence of an Alt–Caffarelli–Friedman-type monotonicity formula in $\mathbb{H}^1$

In this section, we exhibit a function u such that (1.1) is strictly monotone decreasing.

Indeed, we explicitly check that fixing the polynomial

$$u = x - 3yt - 2x^3,$$

$\mathcal{I}_u^{\mathbb{H}^1}$ is monotone decreasing in a right neighborhood of $r = 0$.

Lemma 3.1. *Let $u = x - 3yt - 2x^3$. Then $\Delta_{\mathbb{H}^1} u = 0$ and $\mathcal{I}_u^{\mathbb{H}^1}(r)$ is strictly monotone decreasing in a right neighborhood of $r = 0$.*

Proof. We immediately check that if $u = x - 3yt - 2x^3$, then $\Delta_{\mathbb{H}^1} u = 0$. By straightforward computation, we get

$$Xu = 1 - 6x^2 - 6y^2, \quad Yu = 3(-t + 2xy), \tag{3.1}$$

which implies that

$$\Delta_{\mathbb{H}^1} u = -12x + 12x = 0.$$

Now, we focus on the behavior of $\mathcal{I}_u^{\mathbb{H}^1}$. Substituting (3.1) in (1.5), it holds that

$$\mathcal{I}_u^{\mathbb{H}^1}(r) = \frac{1}{r^2} \int_{B_r^{\mathbb{H}^1}(0)} \frac{\big(1 - 6(x^2 + y^2)\big)^2 + 9(-t + 2xy)^2}{|\xi|_{\mathbb{H}^1}^2} \, d\xi.$$

Next, we have

$$\begin{aligned}
\mathcal{I}_u^{\mathbb{H}^1}(r) &= \frac{1}{r^2}\Bigg(\int_0^r \rho \Bigg(\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{\big(1 - 6\rho^2(\sigma_x^2 + \sigma_y^2)\big)^2}{\sqrt{\sigma_x^2 + \sigma_y^2}} \, d\sigma_{\mathbb{H}^1} \Bigg) d\rho \\
&\quad + \int_0^r \rho \Bigg(\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{9\rho^4(-\sigma_t + 2\sigma_x\sigma_y)^2}{\sqrt{\sigma_x^2 + \sigma_y^2}} \, d\sigma_{\mathbb{H}^1} \Bigg) d\rho \Bigg) \\
&= \frac{1}{r^2}\Bigg(\int_0^r \rho \Bigg(\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{1}{\sqrt{\sigma_x^2 + \sigma_y^2}} \, d\sigma_{\mathbb{H}^1} \Bigg) d\rho \\
&\quad - \int_0^r 12\rho^3 \Bigg(\int_{\partial B_1^{\mathbb{H}^1}(0)} \sqrt{\sigma_x^2 + \sigma_y^2} \, d\sigma_{\mathbb{H}^1} \Bigg) d\rho \\
&\quad + \int_0^r \rho^5 \Bigg(\int_{\partial B_1^{\mathbb{H}^1}(0)} \Bigg(36(\sigma_x^2 + \sigma_y^2)^{3/2} + \frac{9(-\sigma_t + 2\sigma_x\sigma_y)^2}{\sqrt{\sigma_x^2 + \sigma_y^2}} \Bigg) d\sigma_{\mathbb{H}^1} \Bigg) d\rho \Bigg)
\end{aligned}$$

$$= \frac{1}{r^2}\left(\frac{r^2}{2}\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{1}{\sqrt{\sigma_x^2+\sigma_y^2}}\, d\sigma_{\mathbb{H}^1} - 3r^4 \int_{\partial B_1^{\mathbb{H}^1}(0)} \sqrt{\sigma_x^2+\sigma_y^2}\, d\sigma_{\mathbb{H}^1}\right.$$
$$\left. + \frac{r^6}{6}\int_{\partial B_1^{\mathbb{H}^1}(0)} \left(36(\sigma_x^2+\sigma_y^2)^{3/2} + \frac{9(-\sigma_t+2\sigma_x\sigma_y)^2}{\sqrt{\sigma_x^2+\sigma_y^2}}\right) d\sigma_{\mathbb{H}^1}\right),$$

which yields

$$\mathcal{I}_u^{\mathbb{H}^1}(r) = a_1^{\mathbb{H}^1} - 2a_{3,1}^{\mathbb{H}^1} r^2 + a_3^{\mathbb{H}^1} r^4, \tag{3.2}$$

with

$$a_1^{\mathbb{H}^1} = \frac{1}{2}\int_{\partial B_1^{\mathbb{H}^1}(0)} \frac{1}{\sqrt{\sigma_x^2+\sigma_y^2}}\, d\sigma_{\mathbb{H}^1},$$
$$2a_{3,1}^{\mathbb{H}^1} = 3\int_{\partial B_1^{\mathbb{H}^1}(0)} \sqrt{\sigma_x^2+\sigma_y^2}\, d\sigma_{\mathbb{H}^1},$$
$$a_3^{\mathbb{H}^1} = \int_{\partial B_1^{\mathbb{H}^1}(0)} \left(36(\sigma_x^2+\sigma_y^2)^{3/2} + \frac{9(-\sigma_t+2\sigma_x\sigma_y)^2}{\sqrt{\sigma_x^2+\sigma_y^2}}\right) d\sigma_{\mathbb{H}^1}.$$

Explicitly calculating the derivative of $\mathcal{I}_u^{\mathbb{H}^1}$ as in (3.2) and letting $r \to 0$, we reach the thesis since $a_{3,1}^{\mathbb{H}^1}$ is positive by definition. ■

Proof of Theorem 1.1. We first note that

$$J_u^{\mathbb{H}^1}(r) = \mathcal{I}_{u^+}^{\mathbb{H}^1}(r)\mathcal{I}_{u^-}^{\mathbb{H}^1}(r). \tag{3.3}$$

So, since $\mathcal{I}_{u^+}^{\mathbb{H}^1}$ and $\mathcal{I}_{u^-}^{\mathbb{H}^1}$ are nonnegative, we reach the desired result if we prove that they are both monotone decreasing.

We claim that $\mathcal{I}_{u^+}^{\mathbb{H}^1}(r) = \mathcal{I}_{u^-}^{\mathbb{H}^1}(r)$. Before proving it, we show that it immediately implies the monotone decreasing behavior of $\mathcal{I}_{u^+}^{\mathbb{H}^1}(r)$ and $\mathcal{I}_{u^-}^{\mathbb{H}^1}(r)$.

Indeed, since $\mathcal{I}_u^{\mathbb{H}^1}(r) = \mathcal{I}_{u^+}^{\mathbb{H}^1}(r) + \mathcal{I}_{u^-}^{\mathbb{H}^1}(r)$, we deduce that $\mathcal{I}_u^{\mathbb{H}^1}(r) = 2\mathcal{I}_{u^+}^{\mathbb{H}^1}(r) = 2\mathcal{I}_{u^-}^{\mathbb{H}^1}(r)$, which immediately gives the decreasing monotonicity of $\mathcal{I}_{u^+}^{\mathbb{H}^1}$ and $\mathcal{I}_{u^-}^{\mathbb{H}^1}$ from Lemma 3.1.

As a byproduct of this remark, we prove that $J_u^{\mathbb{H}^1}(r)$ is monotone decreasing, because it is the product of two positive monotone decreasing functions; see (3.3).

It remains to show that the claim holds. To this end, we write

$$\mathcal{I}_{u^+}^{\mathbb{H}^1}(r) = \frac{1}{r^2}\int_{B_r^{\mathbb{H}^1}(0)\cap\{u>0\}} \frac{\left(1-6(x^2+y^2)\right)^2 + 9(-t+2xy)^2}{|\xi|_{\mathbb{H}^1}^2}\, d\xi$$
$$= \frac{1}{r^2}\int_{B_r^{\mathbb{H}^1}(0)\cap\{x-3yt-2x^3>0\}} \frac{\left(1-6(x^2+y^2)\right)^2 + 9(-t+2xy)^2}{|\xi|_{\mathbb{H}^1}^2}\, d\xi,$$

and we apply the change of variables

$$\xi = (x, y, t) = T(\eta) = T(w, z, s) := (-w, -z, s), \tag{3.4}$$

which yields

$$\begin{aligned}
&\mathcal{I}_{u^+}^{\mathbb{H}^1}(r)\\
&= \frac{1}{r^2}\int_{B_r^{\mathbb{H}^1}(0)\cap\{-w-3(-z)s-2(-w)^3>0\}} \frac{\big(1-6(w^2+z^2)\big)^2+9(-s+2wz)^2}{|\eta|^2_{\mathbb{H}^1}}\,d\eta\\
&= \frac{1}{r^2}\int_{B_r^{\mathbb{H}^1}(0)\cap\{w-3zs-2w^3<0\}} \frac{\big(1-6(w^2+z^2)\big)^2+9(-s+2wz)^2}{|\eta|^2_{\mathbb{H}^1}}\,d\eta = J_{u^-}^{\mathbb{H}^1}(r),
\end{aligned}$$

and thus the claim follows. ■

4. A further generalization with application

Let us consider now the following two-phase continuous function:

$$u_{\alpha_1,\alpha_2} = \alpha_1 u^+ - \alpha_2 u^-$$

in $\mathbb{H}^1$, where, as usual, $u(x, y, t) = x - 3yt - 2x^3$.

We conclude the paper by proving the following result, where a nontrivial solution of a two-phase free boundary problem in the Heisenberg group is given.

Corollary 4.1. *Let $\alpha_1, \alpha_2 > 0$ be given. Then $\mathcal{I}_{u_{\alpha_1,\alpha_2}}^{\mathbb{H}^1}$, $\mathcal{I}_{\alpha_2 u^-}^{\mathbb{H}^1}$, $\mathcal{I}_{\alpha_1 u^+}^{\mathbb{H}^1}$ are monotone decreasing functions in a right neighborhood of $r = 0$.*

Moreover, if $\alpha_1^2 - \alpha_2^2 > 0$, then u_{α_1,α_2} is a solution of the following two-phase problem in the Heisenberg group:

$$\begin{cases}
\Delta_{\mathbb{H}^1} u = 0 & \textit{in } \Omega^+(u) := \big\{x \in \Omega : u(x) > 0\big\},\\
\Delta_{\mathbb{H}^1} u = 0 & \textit{in } \Omega^-(u) := \operatorname{Int}\big(\big\{x \in \Omega : u(x) \le 0\big\}\big),\\
|\nabla_{\mathbb{H}^1} u^+|^2 - |\nabla_{\mathbb{H}^1} u^-|^2 = g_{\alpha_1,\alpha_2} & \textit{on } \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega,
\end{cases}$$

where

$$g_{\alpha_1,\alpha_2}(x, y, t) = (\alpha_1^2 - \alpha_2^2)\big(\big(1-6(x^2+y^2)\big)^2 + 9(-t+2xy)^2\big).$$

Proof. We first note that

$$J_{u_{\alpha_1,\alpha_2}}^{\mathbb{H}^1}(r) = \mathcal{I}_{\alpha_1 u^+}^{\mathbb{H}^1}(r)\,\mathcal{I}_{\alpha_2 u^-}^{\mathbb{H}^1}(r). \tag{4.1}$$

Since the zero-level set of u_{α_1,α_2} coincides with the zero-level set of u, we remark that

$$\mathcal{I}^{\mathbb{H}^1}_{\alpha_1 u^+}(r) = \frac{\alpha_1^2}{r^2}\int_{B_r^{\mathbb{H}^1}(0)\cap\{x-3yt-2x^3>0\}} \frac{\big(1-6(x^2+y^2)\big)^2+9(-t+2xy)^2}{|\xi|^2_{\mathbb{H}^1}}\,d\xi,$$

as well as

$$\mathcal{I}^{\mathbb{H}^1}_{\alpha_2 u^-}(r) = \frac{\alpha_2^2}{r^2}\int_{B_r^{\mathbb{H}^1}(0)\cap\{x-3yt-2x^3<0\}} \frac{\big(1-6(x^2+y^2)\big)^2+9(-t+2xy)^2}{|\xi|^2_{\mathbb{H}^1}}\,d\xi.$$

Hence, performing the same change of variables introduced in (3.4), we obtain

$$\mathcal{I}^{\mathbb{H}^1}_{\alpha_1 u^+}(r) = \frac{\alpha_1^2}{\alpha_2^2}\mathcal{I}^{\mathbb{H}^1}_{\alpha_2 u^-}(r). \tag{4.2}$$

On the other hand, in this case, keeping in mind (4.2) it follows that

$$\begin{aligned}\mathcal{I}^{\mathbb{H}^1}_{u_{\alpha_1,\alpha_2}}(r) &= \mathcal{I}^{\mathbb{H}^1}_{\alpha_1 u^+}(r) + \mathcal{I}^{\mathbb{H}^1}_{\alpha_2 u^-}(r) = \left(\frac{\alpha_1^2}{\alpha_2^2}+1\right)\mathcal{I}^{\mathbb{H}^1}_{\alpha_2 u^-}(r)\\ &= (\alpha_1^2+\alpha_2^2)\mathcal{I}^{\mathbb{H}^1}_{u^-}(r) = \frac{\alpha_1^2+\alpha_2^2}{2}\mathcal{I}^{\mathbb{H}^1}_{u}(r).\end{aligned} \tag{4.3}$$

As a consequence of Lemma 3.1, (4.3) implies that

$$\mathcal{I}^{\mathbb{H}^1}_{u_{\alpha_1,\alpha_2}},\quad \mathcal{I}^{\mathbb{H}^1}_{\alpha_2 u^-},\quad \mathcal{I}^{\mathbb{H}^1}_{\alpha_1 u^+}$$

are monotone decreasing in a right neighborhood of $r = 0$. Thus, we conclude from (4.1) that $J^{\mathbb{H}^1}_{u_{\alpha_1,\alpha_2}}$ is monotone decreasing, because it is the product of the two positive monotone decreasing functions $\mathcal{I}^{\mathbb{H}^1}_{\alpha_2 u^-}$ and $\mathcal{I}^{\mathbb{H}^1}_{\alpha_1 u^+}$.

In addition, under our hypotheses, u_{α_1,α_2} is a solution of the following two-phase free boundary problem in the Heisenberg group $\mathbb{H}^1$:

$$\begin{cases}\Delta_{\mathbb{H}^1} u = 0 & \text{in } \Omega^+(u) := \big\{x\in\Omega : u(x)>0\big\},\\ \Delta_{\mathbb{H}^1} u = 0 & \text{in } \Omega^-(u) := \operatorname{Int}\big(\big\{x\in\Omega : u(x)\le 0\big\}\big),\\ |\nabla_{\mathbb{H}^1} u^+|^2 - |\nabla_{\mathbb{H}^1} u^-|^2 = g_{\alpha_1,\alpha_2} & \text{on } \mathcal{F}(u) := \partial\Omega^+(u)\cap\Omega.\end{cases} \qquad \blacksquare$$

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