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ORIGINAL ARTICLE

Reference dependence and endogenous anchors

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Abstract

In a complete market, we find optimal portfolios for an investor whose satisfaction stems from both a payoff's intrinsic utility and its comparison with an endogenous reference as modeled by Kőszegi and Rabin. In the regular regime, arising when reference dependence is low, the marginal utility of the optimal payoff is proportional to a twist of the pricing kernel. High reference dependence leads to the anchors regime, whereby investors reduce disappointment by concentrating significant probability in one or few fixed outcomes, or "anchors." Multiple equilibria arise because anchors may not be unique. If stocks follow geometric Brownian motion, the model implies that investors with longer horizons choose larger stocks holdings.

KEYWORDS

complete market, endogenous anchors, loss aversion, personal equilibria, reference dependence

JEL CLASSIFICATION

G11, G12

1 | INTRODUCTION

Satisfaction depends on expectations as it does on results. This insight underlies the common advice to "under-promise and over-deliver" or "manage expectations," and is closely linked to loss aversion, the higher sensitivity to losses from a reference than to gains of similar size. The central question is how to rationally manage both one's expectations and actions as to strive for the best possible outcome, while mitigating future disappointment.

This paper answers this question for a rational investor who is free to choose any payoff in a complete market, and acknowledges that such choice determines both the investment outcome and the reference against which it will be compared. The dual role of payoffs as both outcomes and references is captured by the reference-dependent preferences of Kőszegi and Rabin (2006), which decompose utility into a classical component that reflects the intrinsic value of the outcome, and a reference-dependent component that represents the gains or losses from the comparison with the reference.

Our main result highlights the emergence of two qualitatively different regimes. In the *regular regime*, which attains when reference dependence is weak, the unique optimal payoff satisfies a modified first-order condition prescribing that marginal utility be proportional to the *twisted* state-price density. Such a *twist* function only depends on the reference-dependence parameters, but not on the utility function, as the twist is the same for all investors who have the same attitude to reference dependence, but otherwise arbitrarily different preferences. In particular, when the regular regime obtains, endogenous reference dependence is tantamount to utility maximization with a distorted state-price density, where the distortion depends on the loss aversion and reference dependence parameters. Thus, reference endogenization brings to light the connection among all three elements of prospect theory.

Strong reference dependence leads to the novel phenomenon of *endogenous anchors*: even if state prices have a diffuse distribution (as in typical models), the investor chooses a payoff that attains one or more values with positive probability—the anchors. These anchors do not comprise the entire payoff, which also includes a diffuse component spanning both very bad and very good outcomes: but for average states, the investor finds it more congenial to focus on specific targets, which are more effective at mitigating disappointment.

An interesting feature of endogenous anchors is their multiplicity: in contrast to the regular regime, where the optimal payoff is unique, in the anchors regime there are always infinitely many *personal equilibria*, [1] each of which represents an optimal choice for an investor already leaning towards such a payoff (as a reference). The *preferred personal equilibrium*, which represents the payoff chosen by an investor unencumbered with a reference (hence able to choose it optimally), tends to correspond to the most pessimistic personal equilibrium, thereby supporting the notion that keeping one's expectations relatively low is appropriate, even for a fully self-aware investor whose actions are exactly aligned with expectations.

This paper contributes to the literature on reference dependence, which finds its origins in the prospect theory of Kahneman and Tversky (1979) that emphasizes three intertwined deviations from the expected-utility framework: loss aversion, reference dependence, and probability distortion. Since then, research on reference dependence has developed in multiple strands of literature, with some models such as Sugden (2003) focusing on exogenous references, while others such as Kőszegi and Rabin (2006) seeking to endogenize references as the product of rational choice and self-awareness of loss aversion. In the context of portfolio choice, Berkelaar et al. (2004) incorporate the effect of exogenous references and find that loss aversion and risk aversion are empirically entangled, while Jin and Zhou (2008) also examine the effect of probability distortion and risk-seeking attitude to losses. For a thorough literature review on reference dependence, we refer to the general survey of O'Donoghue and Sprenger (2018), and to He and Strub (2022) for a sharper focus on portfolio choice.

The main modeling difference is between state-independent and state-dependent evaluation. In the state-independent evaluation of Kőszegi and Rabin (2006), an agent compares a payoff's value in any state with all the potential outcomes of the reference, including both the current state and others. Put differently, their model compares a payoff to an independent reference, therefore it

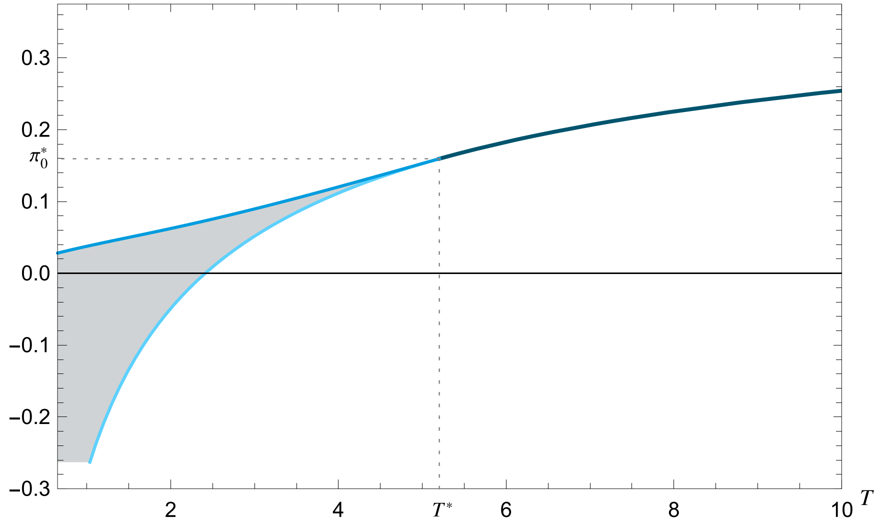

FIGURE 1 Initial optimal stock weights (vertical, as fraction) against investment horizon T (horizontal, in years) for an investor with logarithmic utility. For horizons longer than $T^* \approx 5.20$ years, the regular regime attains. For shorter horizons, the anchors regime leads to multiple personal equilibria that span an increasingly wide range of initial stock holdings. Parameters: initial capital $x_0 = 100$, Sharpe ratio $\kappa \approx 0.12$, stock volatility $\sigma = 24.5\%$, loss aversion $1/\lambda \approx 3.45$, and reference dependence $\eta = 0.71$. The initial optimal stock weight is $\pi_0^* \approx 16\%$ at $T^* \approx 5.20$. Without reference dependence, the Merton ratio is $\kappa/\sigma \approx 50\%$. [Color figure can be viewed at wileyonlinelibrary.com]

may not be appropriate for those problems in which it is natural for a payoff and its references to be dependent.

State-dependent evaluation addresses this issue by comparing a payoff's value to its reference in the same state. De Giorgi and Post (2011) argue that state-independent evaluation is ill-suited to some financial applications such as benchmarked portfolio management, in which dependence with the benchmark is a critical aspect of the problem. Thus, they propose a partially endogenous expectations model, whereby the endogenous reference is chosen within a neighborhood of an exogenous reference.

He and Strub (2022) study reference point endogenization under three models: the partially endogenous expectations of De Giorgi and Post (2011), optimal expectations, and mental updating with partial adaptation. Applying these models to portfolio selection, they establish that reference point endogenization reduces loss aversion across all models. Thus, agents with partially endogenized reference point behave as if they had an exogenous reference point with a lower degree of loss aversion. These findings underline difficulties in distinguishing between loss aversion and reference point from data, and confirm the robustness of models of exogenous references to mental adjustments, up to a reduction in loss aversion.

This paper focuses on the state-independent framework of Kőszegi and Rabin (2006) for three main reasons. First, this model, widely used in economics, describes the simplest form of endogenous reference dependence and offers an agnostic benchmark to investigate the conditions for uniqueness and multiplicity of personal equilibria. Second, characterizing optimality in complete markets in this setting is important to understand more complex models that include extra features. Third, the assumption of independence payoff and reference can be relaxed though a change of numeraire, as to accommodate more flexible dependence structures between payoff and references.[2]

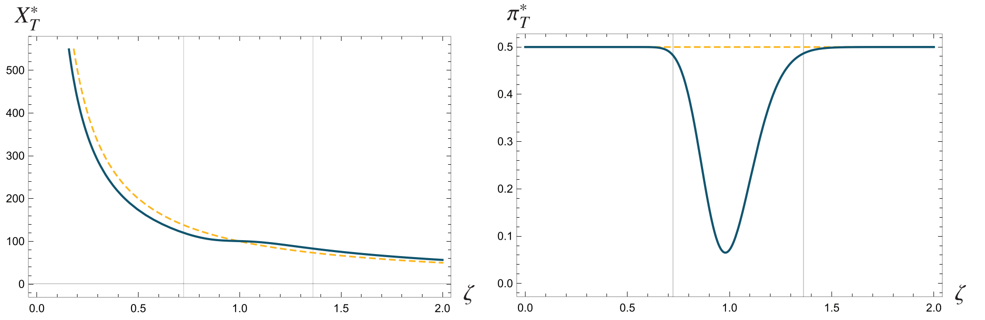

FIGURE 2 Optimal terminal wealth (vertical, left) and stock weight (vertical, right) against pricing kernel (horizontal) for an investor in the Black–Scholes model, with (solid) and without (dashed) reference dependence. Parameters: $x_0 = 100$, $\mu = 3\%$, $\sigma = 24.5\%$, $T = 1.0$, $\lambda = 0.29$, and $\eta = 0.33$ (hence $k \approx 1.31$, regular regime). The median of the pricing kernel is $\tilde{\zeta} \approx 0.99$, the Merton ratio is $\mu/\sigma^2 \approx 50\%$, and the minimum stock weight is approximately 7% (attained at $\zeta \approx 0.97$). The vertical solid lines give a 99% confidence band for ζ. [Color figure can be viewed at wileyonlinelibrary.com]

We illustrate these findings in the context of optimal investment with a finite horizon, focusing in particular on the trading policies that replicate personal equilibria. Figure 1 displays the initial stock allocation as a function of the investment horizon and clearly shows the transition between the *regular* and *anchor* regimes. Long horizons lead to the regular regime, in which the unique personal equilibrium satisfies the first-order condition with the twisted state-price density. For horizons shorter than critical horizon T^*, which is independent of the utility function, the anchor regime arises with its multiple equilibria. As a result, the initial stock allocation is no longer a single value but an entire range, corresponding to all the possible allocations spanned by personal equilibria.

In particular, the proportion of wealth allocated to stocks is higher for an investor with a longer horizon, who not only can achieve a higher average return, but also has the opportunity to devise a payoff that is less prone to disappointment. For this reason—and this reason alone—such an investor has a higher stock allocation. (Note that Figure 1 does not describe the change in portfolio weight for the same investor as time passes. In general, such weight is stochastic and depends on the relation between the portfolio and the reference. Figure 2 below displays the portfolio weight at the end of the horizon.)

Interestingly, for very short horizons, the range of stock allocations widens to include negative values, which correspond to suboptimal personal equilibria. This phenomenon arises because, when the horizon is short and hence the utility-maximizing payoff is relatively close to the initial capital, reference dependence becomes paramount in determining investors' choices. Their focus on minimizing disappointment can be so significant as to induce them to incur temporary expected losses to replicate payoffs that have the sole advantage of being comparatively insensitive to negative outcomes.

Finally, a comparison with our recent results in Guasoni and Meireles-Rodrigues (2020) is in order. Although both papers study optimal investment with reference-dependent preferences, their contexts are rather different: in Guasoni and Meireles-Rodrigues (2020), the focus is on a one-period setting, which only allows the investor to choose the scale of the payoff, but not its distribution. The implication is that, depending on the Gini coefficient and the gain–loss ratio of the asset return, the investor may participate in risky investments or forgo them completely.

By contrast, the present paper focuses on a complete market, where investors are free to choose any payoff that suits their budgets. (This framework includes, for example, continuous-time models based on Brownian motion, in which completeness is achieved through dynamic trading.) As a result, even if the pricing kernel has a density, the investor has the flexibility to combine payoffs with arbitrary distributions, including absolutely continuous, singular, and atomic components. Indeed, the emergence of the anchors regime shows that reference dependence can compel investors to prefer fixed outcomes with high probability to mitigate disappointment.

The outline of the paper is as follows: Section 2 specifies the model and portfolio choice problem under Kőszegi and Rabin's model of reference-dependent preferences; Section 3 presents the main results characterizing the personal equilibria, and discusses their implications; Section 4 studies in details the geometric Brownian motion model; Section 5 closes with concluding remarks. All proofs are in the Appendix.

2 | MODEL

Consider an investor with a finite investment horizon $T > 0$ and no intermediate consumption, trading in a frictionless and arbitrage-free financial market with underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The market contains a safe asset, used as numéraire, whose price is normalized to one. Assume further that the market is complete, and denote by ζ the unique pricing kernel determining all prices. The set of all non-negative payoffs at the horizon T that are affordable from initial capital $x_0 > 0$ is

$$\mathscr{C}(x_0) := \left\{X \in L^0_+ \,:\, \mathbb{E}[\zeta X] \le x_0\right\}.$$

Here, L^0 stands for the vector space of (equivalence classes of) $\mathscr{F}$-measurable real-valued random variables, and L^0_+ for the cone of $\mathbb{P}$-almost surely (a.s.) non-negative elements of L^0; moreover, we adopt the standard convention that the expectation of a random variable is $-\infty$ whenever its negative part has infinite expectation (i.e., $+\infty - \infty := -\infty$). Throughout the paper, we make the following:

Assumption 2.1.

(i) The *utility* function $u : (0, +\infty) \to \mathbb{R}$ is strictly increasing, twice-differentiable with $u''(\cdot) < 0$, and satisfies the Inada conditions:

$$u'(0+) := \lim_{x \to 0^+} u'(x) = +\infty \quad \text{and} \quad u'(+\infty) := \lim_{x \to +\infty} u'(x) = 0. \tag{1}$$

(ii) The pricing kernel ζ has a density function $f_\zeta(\cdot)$ (relative to the Lebesgue measure on $\mathbb{R}$) that is continuous and strictly positive on $(0, +\infty)$, with $\lim_{x \to 0^+} f_\zeta(x)x = 0$ and $\lim_{x \to +\infty} f_\zeta(x)x = 0$.

(iii) For all $y \in (0, +\infty)$,

$$\mathbb{E}[\zeta I(y\zeta)] < +\infty \quad \text{and} \quad \mathbb{E}[u(I(y\zeta))] < +\infty, \tag{2}$$

where $I(\cdot)$ denotes the (continuous and strictly decreasing) inverse function of $u'(\cdot)$.

Condition (i) states that investors' preferences are smooth and nonsatiable, with declining marginal utility spanning from zero to infinity. In view of monotonicity, we extend $u(\cdot)$ to $[0,+\infty)$ by setting $u(0) := \lim_{x\to 0^+} u(x) \in [-\infty,+\infty)$; likewise, $u(+\infty) := \lim_{x\to+\infty} u(x)$ exists, possibly infinite. Condition (ii) implies that the distribution function of ζ (under $\mathbb{P}$) and its generalized inverse, respectively, denoted by $F_\zeta(\cdot)$ and $q_\zeta(\cdot)$, are both continuous [3]; in particular, $q_\zeta(\cdot)$ is the proper inverse of $F_\zeta(\cdot)$ on $(0,+\infty)$. Condition (iii) ensures that the classical utility maximization problem from terminal wealth

$$\sup_{Z\in\mathscr{C}(x_0)} \mathbb{E}[u(Z)]$$

admits the solution $\tilde{X} := I(\tilde{y}\zeta)$, where $\tilde{y} \in (0,+\infty)$ is the unique Lagrange multiplier saturating the budget constraint (i.e., $\mathbb{E}[\zeta\tilde{X}] = x_0$).

We adopt the framework of Kőszegi and Rabin (2006), in which preferences depend on a (possibly stochastic) reference. Investors evaluate payoffs not only in absolute terms but also in comparison with the reference, resulting in an overall utility that combines classical expected utility with a reference-dependent component. The model also incorporates loss aversion (whereby losses are experienced more acutely than gains of similar size) by assuming that the function measuring the effect of gains and losses from the reference is nondifferentiable at the origin and steeper for losses. This paper focuses on piecewise linear gain–loss functions, so as to preserve risk-averse preferences.

Definition 2.2 Piecewise linear gain–loss and reference-dependent utility; Kőszegi and Rabin, 2006. The piecewise linear *gain–loss* function $\nu : \mathbb{R} \to \mathbb{R}$ is of the form[4]

$$\nu(x) := \nu_+\left(x^+\right)\mathbb{1}_{[0,+\infty)}(x) - \nu_-(x^-)\mathbb{1}_{(-\infty,0)}(x), \quad \text{for all } x \in \mathbb{R}, \tag{3}$$

where, for some $\lambda, \eta \in (0,1)$,

$$\nu_+(x) := \frac{\lambda\eta}{1-\eta}x \quad \text{and} \quad \nu_-(x) := \frac{\eta}{1-\eta}x, \quad \text{for all } x \in [0,+\infty). \tag{4}$$

The *reference-dependent utility* of a payoff Z with respect to the reference B is defined as

$$U(Z|B) := \mathbb{E}[u(Z)] + \mathbb{E}\left[\int_{\mathbb{R}} \nu(u(Z) - u(b))d\mathbb{P}_B(b)\right] = \int_{\mathbb{R}}\int_{\mathbb{R}} [u(z) + \nu(u(z) - u(b))]d\mathbb{P}_B(b)d\mathbb{P}_Z(z), \tag{5}$$

where $\mathbb{P}_Z(\cdot)$ and $\mathbb{P}_B(\cdot)$ are the probability laws (under $\mathbb{P}$) of Z and B, respectively.

Here, the reference-dependence (or gain–loss sensitivity) parameter η is the relative weight attached to the gain–loss component in the overall utility, while $1/\lambda$ represents the investor's degree of loss aversion. In the limit cases of $\eta \to 0^+$ or $\lambda \to 1^-$, the overall utility (5) reduces to the classical expected utility; at the opposite extreme, $\eta \to 1^-$ combined with $\lambda \to 0^+$ results in exclusively reference-dependent preferences with complete disregard for gains.

To reflect the role of reference dependence in preferences, Kőszegi and Rabin (2006) also posit a rule for endogenously specifying references as the only expectations (about forthcoming payoffs) that are rational, in that they are indeed selected when adopted. Once these personal equilibria—that is, actual decisions that are consistent with prior expectations—have been determined, those with the highest overall utility (relative to themselves) are the preferred ones. We exclude from our study of personal equilibria the pathological case of a payoff B with $\mathbb{E}[u(B)^+] = +\infty$ and $\mathbb{E}[u(B)^-] = +\infty$, which leads to infinite grief, that is, $\mathbb{E}[u(B)] = -\infty$.

Definition 2.3 (Personal equilibria).

(i) A payoff $X \in \mathscr{C}(x_0)$ such that $\mathbb{E}[u(X)^+] < +\infty$ is a *personal equilibrium* if

$$\mathrm{v}(x_0, X) := \sup_{Z \in \mathscr{C}(x_0)} U(Z|X) = U(X|X). \tag{6}$$

PE(x_0) denotes the *set of personal equilibria*.

(ii) A payoff $X \in \mathrm{PE}(x_0)$ is a *preferred personal equilibrium* if

$$\mathrm{v}(x_0) := \sup_{Z \in \mathrm{PE}(x_0)} U(Z|Z) = U(X|X). \tag{7}$$

PPE(x_0) denotes the *set of preferred personal equilibria*.

Reference-dependent preferences give rise not to one, but a family of optimization problems, one for each potential reference. A key observation in determining the personal equilibria is that, under the assumption that $\nu(\cdot)$ is piecewise linear, problem (6) is equivalent to

$$\sup_{Z \in \mathscr{C}(x_0)} \mathbb{E}[\tilde{u}_B(Z)], \tag{8}$$

where

$$\tilde{u}_B(x) := \mathbb{E}\left[(u(x) - u(B))\mathbb{1}_{\{B \leq x\}}\right] + k\mathbb{E}\left[(u(x) - u(B))\mathbb{1}_{\{B > x\}}\right], \quad \text{for all } x \in (0, +\infty),$$

is a strictly increasing and globally concave function that depends on the reference payoff B (see Lemma A.1) and the parameter

$$k := \frac{1}{1 - \eta(1 - \lambda)} \in (1, +\infty) \tag{9}$$

is interpreted as the effective degree of loss aversion, that is, the impact on utility of losses, relative to gains, adjusted for the sensitivity η to reference dependence. In particular, k reduces to 1 in the absence of reference dependence ($\eta = 0$) while it coincides with the degree of loss aversion $1/\lambda$ with complete reference dependence ($\eta = 1$).[5]

Thus, every reference-dependent optimization problem boils down to solving a classical utility maximization problem, with the difference that the original smooth utility $u(\cdot)$ is replaced by the (possibly kinked) *reference-adjusted utility* function $\tilde{u}_B(\cdot)$.

Figure 3 plots $\tilde{u}_B(\cdot)$ for two reference payoffs: the safe payoff $X^f = x_0$ a.s. (i.e., from investing all wealth in the safe asset), and the classical utility maximizer $\tilde{X}$. Note that, while the reference-adjusted utility associated to the atomless reference $\tilde{X}$ is smooth, the one arising from taking the atomic safe payoff as reference is not differentiable at the atom of X^f.

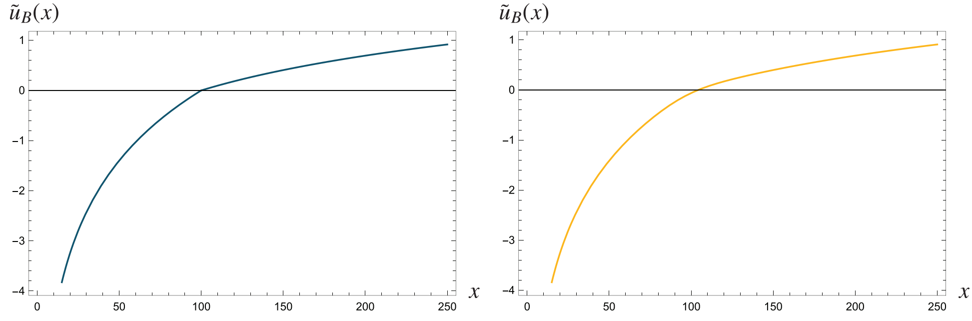

FIGURE 3 Reference-adjusted utility $\tilde{u}_B(\cdot)$ for a logarithmic investor (i.e., $u(x) := \log(x)$ for all $x > 0$) for different reference levels B: the safe payoff $X^f = x_0$ a.s. (left), and the classical utility maximizer $\tilde{X} = x_0/\zeta$ (right). ($\tilde{X}$ is identified by $U'(\tilde{X}) = 1/\tilde{X} = c\zeta$ for some $c > 0$, determined by $E[\tilde{X}\zeta] = x_0$, whence $\tilde{X} = x_0/\zeta$.) Parameters: $x_0 = 100$, $\lambda = 0.29$, and $\eta = 0.71$ (hence $k \approx 2.02$). [Color figure can be viewed at wileyonlinelibrary.com]

As references are endogenous, investors effectively choose their preferences in a way that reflects their expectations about payoffs. Note that, in view of loss aversion in the reference-dependent component, $\tilde{u}_B(\cdot)$ represents more risk-averse preferences than $u(\cdot)$ (refer to Remark A.2(i)).

As in classical portfolio choice, convex analysis arguments yield the optimal payoff for each reference (taking into account, however, that the first-order condition may fail for certain values due to the nondifferentiability of the utility). In this respect, Kőszegi and Rabin's framework does not depart from expected utility theory: the main challenge lies in establishing which of these references turn out to be optimal for their own maximization problem, that is, finding the "fixed points."

3 | MAIN RESULTS

The first result of the paper identifies the set of personal equilibria. It relies on the concept of *twist* function $w(\cdot)$ introduced in Equation (10) below, which is in turn motivated by the characterization of personal equilibria obtained in Lemma A.5.

The monotonicity of the twist entirely determines the number of personal equilibria, while their characterization requires the introduction of the family of its increasing envelopes, that is, the continuous increasing functions that lie between the upper and lower increasing envelopes of $w(\cdot)$ and that increase only when the twist increases. While the rigorous definition of such class of envelopes is somewhat convoluted, its visualization is relatively straightforward and is shown in Figure 4.

Definition 3.1 (Twist and increasing envelopes). Define

(i) the *twist* $w : [0, +\infty) \to [0, +\infty)$ as

$$w(x) := \frac{x}{1 - (1 - k^{-1})\Big(1 - F_\zeta(x)\Big)}, \quad \text{for all } x \in [0, +\infty); \tag{10}$$

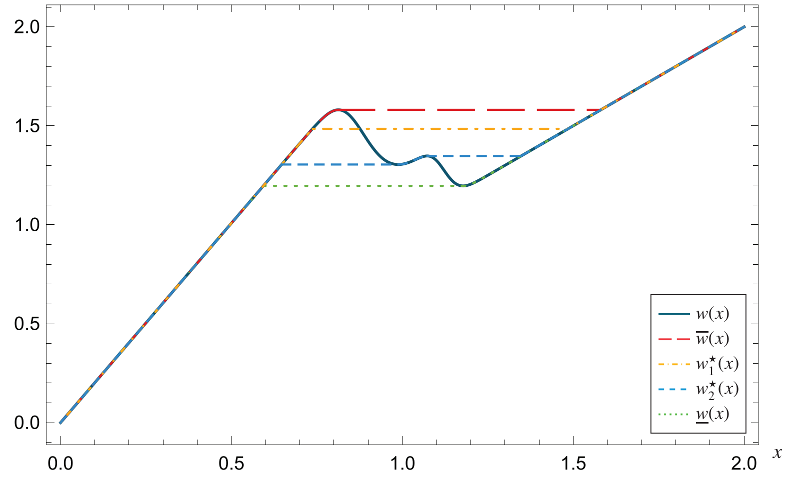

FIGURE 4 The twist (dark blue), its lower (green) and upper (red) envelopes, and two other monotonic envelopes $w_1^\star(\cdot)$ (ochre) and $w_2^\star(\cdot)$ (light blue). Here, ζ is a mixture (with mixing parameter $p \approx 0.42$) of two lognormal distributions (with parameters $\mu_1 \approx 0.12, \sigma_1 = 0.03$, and $\mu_2 = -0.10, \sigma_2 = 0.07$). Parameters: $\lambda = 0.29$ and $\eta = 0.71$ (hence $k \approx 2.02$). [Color figure can be viewed at wileyonlinelibrary.com]

(ii) the *upper increasing envelope* of $w(\cdot)$ as the smallest increasing function $\overline{w} : [0,+\infty) \to \mathbb{R}$ lying above $w(\cdot)$, that is,

$$\overline{w}(x) := \sup_{0\le y\le x} w(y), \quad \text{for all } x \in [0,+\infty);$$

(iii) the *lower increasing envelope* of $w(\cdot)$ as the largest increasing function $\underline{w} : [0,+\infty) \to \mathbb{R}$ lying below $w(\cdot)$, that is,

$$\underline{w}(x) := \inf_{y\ge x} w(y), \quad \text{for all } x \in [0,+\infty);$$

(iv) the *increasing envelopes* of $w(\cdot)$ as the set $\mathscr{W} \subseteq C([0,+\infty))$ of continuous functions such that
  (a) $w^\star(\cdot) \in C([0,+\infty))$ and is increasing;
  (b) $\underline{w}(\cdot) \le w^\star(\cdot) \le \overline{w}(\cdot)$;
  (c) for all $x_* > 0$, if $w^\star(x_*) \neq w(x_*)$, then $w^\star(\cdot)$ is constant in a neighborhood of x_*.

With these definitions, it is now possible to state the main result of this paper, which characterizes the optimal payoffs in the regular and anchors regimes.

Theorem 3.2. *Let Assumption 2.1 hold.*

(i) (REGULAR REGIME) *If $w(\cdot)$ is increasing,*[6] *then*

$$PE(x_0) = \{I(y^* w(\zeta))\}, \tag{11}$$

where $y^* \in (0,+\infty)$ is the unique solution of $\mathbb{E}[\zeta I(y^* w(\zeta))] = x_0$.

(ii) (ANCHORS REGIME) *If $w(\cdot)$ is not increasing, then*

$$PE(x_0) = \left\{ I\left(y^\star w^\star(\zeta)\right) \,:\, w^\star(\cdot) \in \mathscr{W}, y^\star \text{ such that } \mathbb{E}\left[\zeta I\left(y^\star w^\star(\zeta)\right)\right] = x_0 \right\}. \tag{12}$$

Proof. See Appendix A.2. □

In contrast to familiar expected utility, Theorem 3.2 shows that a personal equilibrium may not be unique. Indeed, reference dependence gives rise to two possible regimes: a regular regime, in which the investor has only one personal equilibrium, and an anchor regime, in which the investor is free to choose amongst infinitely many personal equilibria.

The twist inherits its continuity from that of the distribution function of the pricing kernel,[7] but the central question is whether it is increasing or not. When reference dependence or loss aversion is absent ($k = 1$), the twist is increasing because it coincides with the identity. Weak reference dependence or loss aversion (k near one) also preserve such monotonicity, which corresponds to the regular regime. But if they are sufficiently strong (k large) then the twist may cease to be increasing, and multiple equilibria with atomic outcomes arise, corresponding to the anchors' regime.

A striking feature of the twist is that it depends on the market (through the distribution of the pricing kernel) as well as on the investor's reference-dependence parameters, but not on the utility function $u(\cdot)$. (This property holds for piecewise-linear gain–loss functions, but not necessarily for others.)

In either regime, every personal equilibrium is of the form $X = I(y^\star \zeta^\star)$, where

$$\zeta^\star := w^\star(\zeta)$$

is understood as a *twisted pricing kernel*. In this sense, the personal equilibria resemble the classical optimizer, but with a different pricing kernel that reflects how expectations can twist the perception of the state of the economy. Nevertheless, the budget constraint still uses the original, real pricing kernel; moreover, even though the twisted kernel does not depend on the utility $u(\cdot)$, the multiplier $y^\star$ does.

If reference dependence is such that the twisted pricing kernel is increasing, then the regular regime holds: depending on whether or not monotonicity is strict, the unique personal equilibrium has either a continuous distribution (Proposition 3.3(i) below) or at least one constant interval.

When reference dependence and loss aversion are strong enough and the twist no longer increasing, we observe a remarkable change in optimal investment behavior: not only does the set of personal equilibria become infinite, but investors choose levels at which they keep their wealth in intermediate states of the economy and from which they are unwilling to deviate. In other words, agents endogenously create their own "anchors," that is, payoffs in which much of the probability is concentrated on one or few values. The novelty of this finding stems from the failure of the traditional first-order condition, which shapes the intuition of most results in this area.

When the twist function loses its monotonicity, heuristically applying the twisted first-order condition would lead to a payoff that is manifestly suboptimal, as it would be higher in worse states of nature, and thus is surpassed by another one, with the same distribution, but which preserves comonotonicity with the state of nature (Jin & Zhou, 2008, Lemma C.1). Yet, replacing the *twist* function with any of its increasing envelopes in Definition 3.1 yields a personal equilibrium, which explains the multiplicity of such equilibria.

Theorem 3.2 rules out personal equilibria with a discrete distribution, and in particular safe personal equilibria, so long as risk premia are nonzero, as implied by Assumption 2.1(ii). The intuition is that reference dependence and loss aversion induce preference for constant outcomes in intermediate states of the economy, but not in very good (high $\zeta \uparrow \infty$) or very bad ($\zeta \downarrow 0$) states: It is too costly to insure the same outcome in a sufficiently bad state and not profitable enough to accept it in a sufficiently good state. Put differently, very bad states are insured only in part, and very good states hedged only in part. (Lemma A.3 and Remark A.4 describe this phenomenon from a duality viewpoint, and its connection with the gain–loss ratios of Bernardo and Ledoit (2000).)

The fact that η and λ alone, but not $u(\cdot)$, govern the choice of regime is significant, as it draws a clear distinction between the roles of risk aversion and reference dependence. Anchors are an effect of reference dependence alone, and are independent of risk aversion. In particular, in a market where multiple investors share the same loss aversion and reference dependence, but have heterogeneous utility functions, identical regimes arise, in that all individuals adopt the same regime (i.e., either all opt to spread the atoms or all opt to stick closely to them, leading to no distinction between the types of investors). Furthermore, neither the occurrence nor the locations (i.e., the atoms of $w^\star(\zeta)$) of anchors depend on the utility function $u(\cdot)$, only the atoms' sizes $P(X^* = I(y^\star w^\star(\zeta)))$ do, as the personal equilibria of these heterogeneous investors may still differ, due to their different marginal preferences.

The next result examines the regularity and monotonicity properties of personal equilibria and their sensitivity to reference dependence.

Proposition 3.3. *Let Assumption 2.1 hold. Then,*

(i) For all $X \in PE(x_0)$, the mapping $\zeta \mapsto X$ is continuous and decreasing, with

$$\lim_{\zeta\to 0} X = +\infty \quad \text{and} \quad \lim_{\zeta\to+\infty} X = 0. \tag{13}$$

(ii) If $w(\cdot)$ is strictly increasing, then the unique personal equilibrium

$$X^* := I(y^* w(\zeta)), \tag{14}$$

where $y^* \in (0, +\infty)$ is the unique solution of $\mathbb{E}[\zeta X^*] = x_0$, has an absolutely continuous distribution. Moreover, recalling that $\tilde{X}$ denotes the classical expected utility optimizer, $X^* < \tilde{X}$ a.s. (respectively, $X^* > \tilde{X}$ a.s.) on the event $\{\zeta < \tilde{\zeta}\}$ (respectively, $\{\zeta > \tilde{\zeta}\}$), where

$$\tilde{\zeta} := q_\zeta\left(\frac{y^* - k\tilde{y}}{(1 - k^{-1})\tilde{y}}\right). \tag{15}$$

In addition, $y^* \in (0, +\infty)$ is strictly increasing in λ, η and strictly decreasing in x_0.

(iii) If η is sufficiently small or λ is sufficiently close to 1, then $PE(x_0) = \{X^*\}$ with X^* as in Equation (14). Moreover,

$$\lim_{\lambda\to 1} X^* = \tilde{X} \quad \text{and} \quad \lim_{\eta\to 0} X^* = \tilde{X}. \tag{16}$$

Proof. See Appendix A.2. □

Remark 3.4. The limits in Equation (16) are equivalent to $\lim_{k\to 1} X^* = \tilde{X}$.

Property (i) establishes the continuous dependence of personal equilibria on the pricing kernel, in that sufficiently small changes in the state of nature produce arbitrarily small changes in the optimal payoffs. Moreover, each personal equilibrium is antimonotonic with the pricing kernel, steadily declining as the state of nature worsens: in bad states of nature (as ζ increases to $+\infty$) wealth gradually tends to 0, whereas in good states (as ζ decreases to 0) wealth grows without bound. Therefore, in this sense, all personal equilibria follow the market, thus exhibiting a similar behavior to that of the classical utility maximizer.

Property (ii) states that, when the twist is strictly increasing, the unique personal equilibrium has no atoms, and we call it the *diffuse equilibrium*. Even though the diffuse equilibrium qualitatively resembles the utility maximizer $\tilde{X}$, note that reference-dependent investors are willing to give up some gains in good states of the world (when the pricing kernel is low), compared to reference-indifferent investors, so as to reduce their losses in bad states of the world, due to loss aversion. The effect of changes in the parameters on the diffuse equilibrium is clear: an increase in the initial capital leads to an increase in the diffuse equilibrium in all states of the world; however, the monotonicity of X^* with respect to either η or λ is ambiguous, as it depends on the state of nature (see Section 4).

Finally, property (iii) shows that the twist monotonicity is strict if either reference dependence or loss aversion is sufficiently weak. Then, similar to classical portfolio theory, spreading atoms makes investors better off. According to Equation (13) and Remark 3.4, as $k \to 1$ (i.e., reference dependence fades away), the twist approaches the identity function, meaning that there is virtually no deformation of the pricing kernel and the unique personal equilibrium reduces to the utility-maximizing payoff.

In the case where reference dependence causes the strict monotonicity of the twist to fail, the result below identifies ranges of values of the pricing kernel for which all personal equilibria are constant. It also provides an upper bound on the number of atoms of personal equilibria.

Proposition 3.5. *Let Assumption 2.1 hold. Define the* concentration function $H : (0, +\infty) \to \mathbb{R}$ *as*

$$H(x) := 1 - \left(1 - k^{-1}\right)\Big(1 - F_\zeta(x) + f_\zeta(x)x\Big), \quad \text{for all } x \in (0, +\infty), \tag{17}$$

and the underwater set

$$N := \{x \in (0, +\infty) : H(x) \le 0\}. \tag{18}$$

(i) The twist $w(\cdot)$ is everywhere strictly increasing if and only if N has empty interior (we write $\mathrm{int}(N) = \emptyset$*).*
(ii) If N has nonempty interior, then

$$\mathrm{int}(N) = \bigcup_{i\in I} (a_i, b_i), \quad \text{for some } \emptyset \neq I \subseteq \mathbb{N},\ 0 < a_i < b_i, \text{ and } (a_i, b_i) \cap (a_j, b_j) = \emptyset \text{ for } i \neq j. \tag{19}$$

Hence, any personal equilibrium is constant on each of the events $\{a_i \leq \zeta \leq b_i\}$*. Moreover, if the index set* I *is finite, then any personal equilibrium has at most* $|I|$ *atoms. (*$|S|$ *denotes the cardinality of the set* S*.)*

Proof. See Appendix A.2. □

The concentration function, which is associated with a reduced first-order condition, measures the trade-off between spreading an anchor around its value and remaining committed to it: dispersion is more attractive if $H(\cdot)$ is positive in all states of the world; on the contrary, if it falls into negative territory, then anchors are preferable for such states of the world.

The sign of the concentration function and consequently the monotonicity of the twist are closely linked to the elasticity of the pricing kernel, defined as

$$E(x) := \frac{x f_\zeta(x)}{F_\zeta(x)}, \quad \text{for all } x \in (0, +\infty).$$

Indeed, if $\bar{E} := \sup_{x\in(0,+\infty)} E(x) \leq 1$, meaning that the pricing kernel has an inelastic distribution (in other words, a small relative change in the state of nature results in a small relative change in the accumulation of probability), then $H(x) \geq 1 - (1 - k^{-1})(1 + 0) > 0$ for all $x \in (0, +\infty)$, whence $w(\cdot)$ is always strictly increasing; in this case, all investors choose the regular regime over the anchors, irrespective of their degrees of reference dependence and loss aversion. However, in a market with an elastic pricing kernel (for which $\bar{E} > 1$), there exist combinations of sufficiently strong loss aversion and reference dependence that drive investors towards the anchors, as in this case, we can find $x_*, \varepsilon > 0$ such that $H(x_*) \leq 1 - (1 - k^{-1})(1 + \varepsilon F_\zeta(x_*))$.

Mixtures of lognormal distributions are a common tool to estimate empirical pricing kernels implied by option prices (cf. Liu et al., 2009, for an options' application). As Figure 4 demonstrates, even in the lognormal-mixture case, the twist can have several local extrema, making it difficult to identify personal equilibria in general, and in particular how many atoms each anchor has. By way of illustration, the personal equilibrium associated with the envelope $w_1^\star(\cdot)$ has a single atom, whereas the personal equilibrium associated with $w_2^\star(\cdot)$ has two.

We conclude this section with two results focusing on more familiar settings: The first one considers a pricing kernel with a decreasing density, which corresponds to an inelastic case, where reference dependence is not meaningful enough relative to the terminal distribution of market states to cause a shift in investment behaviors from the regular regime to the anchors regime.

Corollary 3.6. *Let Assumption 2.1 hold. Assume further that* $f_\zeta(\cdot)$ *is decreasing. Then,* $PE(x_0) = \{X^*\}$ *for all* $\lambda, \eta \in (0, 1)$*, where* X^* *is as in Equation (14). Moreover,* $\lim_{k\to+\infty} X^* = \bar{X}$*, where*

$$\bar{X} := I\left(\frac{\bar{y}\zeta}{F_\zeta(\zeta)}\right) \tag{20}$$

and $\bar{y} \in (0, +\infty)$ *is the unique solution of* $\mathbb{E}[\zeta \bar{X}] = x_0$*.*

Proof. See Appendix A.2. □

Interestingly, a somewhat unexpected phenomenon arises for investors who focus entirely on reference dependence and care solely about losses: if the (relative) likelihood that the value of the pricing kernel is close to zero cannot become arbitrarily large, then $\lim_{\zeta\to 0}\bar{X} < +\infty$; put differently, the optimal limit payoff (20) remains bounded even in very good states of the world. Intuitively, losses—which are infinitely grievous, however small—are simply too likely for such investors if they take too ambitious a reference payoff when the market is extremely good, unless those states have overwhelmingly greater odds of happening compared to others.

Our last result, which encompasses the Black–Scholes model as a particular case, provides a "1-2-1 rule" whereby reference dependence combined with a pricing kernel having exactly one mode generates two regimes, and every anchor has one single endogenous atom.

Corollary 3.7. *Let Assumption 2.1 hold. Assume further that $f_\zeta(\cdot)$ is strictly unimodal with mode $\theta > 0$.* [8]

(i) (REGULAR REGIME) *If*

$$k \leq 1 + \frac{1}{f_\zeta(\theta)\theta - F_\zeta(\theta)}, \tag{21}$$

then $PE(x_0) = \{X^*\}$ as in Equation (14).

(ii) (ANCHORS REGIME) *If Equation (21) fails, then $PE(x_0) = \{X^\star_\alpha \,:\, \alpha \in [x_1, \bar{x}_2]\}$, where*

$$X^\star_\alpha := I\big(y^\star_\alpha w(\alpha)\big)\mathbb{1}_{\{\zeta\in[\alpha,\bar{\alpha}]\}} + I\big(y^\star_\alpha w(\zeta)\big)\mathbb{1}_{\{\zeta\notin[\alpha,\bar{\alpha}]\}} \tag{22}$$

and $y^\star_\alpha \in (0,+\infty)$ is uniquely determined by $\mathbb{E}[\zeta X^\star_\alpha] = x_0$. Here, $\bar{x}_2 < \theta < \bar{x}_1$ denote the only two zeros of the concentration function, that is, the solutions to

$$1 - \left(1 - k^{-1}\right)\Big(1 - F_\zeta(x) + f_\zeta(x)x\Big) = 0. \tag{23}$$

Also, x_i is the unique solution of $w(x) = w(\bar{x}_i)$ for each $i \in \{1,2\}$, and $\bar{\alpha}$ is the unique number in $[\bar{x}_1, x_2]$ satisfying $w(\alpha) = w(\bar{\alpha})$. Moreover, for all $\alpha_1, \alpha_2 \in [x_1, \bar{x}_2]$ such that $\alpha_1 < \alpha_2$, there exist $\underline{\zeta} \in (\alpha_1, \alpha_2)$ and $\overline{\zeta} \in (\bar{\alpha}_1, \bar{\alpha}_2)$ such that $X^\star_{\alpha_1} > X^\star_{\alpha_2}$ on $\{\zeta \in (\underline{\zeta}, \overline{\zeta})\}$ and $X^\star_{\alpha_1} < X^\star_{\alpha_2}$ on $\{\zeta \notin (\underline{\zeta}, \overline{\zeta})\}$.

Proof. See Appendix A.2. □

Note that $F_\zeta(x) < f_\zeta(x)x$ for all $x \in (0, \theta]$, therefore an infinitesimal relative change in the value of ζ about reasonably good market states causes a significant relative change in the accumulation of probability; consequently, a strictly unimodal distribution is an example of an elastic distribution.

In line with the findings of Proposition 3.3(iii) and Proposition 3.5, we see that the regular regime emerges when at least one of the elements of reference dependence (whether it is loss aversion or reference dependence) is relatively weak, as measured by Equation (21); by contrast, the anchors stand out as the most appealing choice for investors whose combination of the two preference parameters is sufficiently strong. Put differently, the personal equilibria set consists of a single diffuse payoff so long as reference dependence remains below some threshold near the mode of the pricing kernel; when it rises above this threshold, the set of personal equilibria explodes from one in the regular regime to an infinity of equilibria with anchors.

Furthermore, in this case, we can characterize explicitly the endogenous anchors: each of them entails following the market when it is either very good or very bad (i.e., when the pricing kernel is below α or above $\bar{\alpha}$), because these extreme outcomes are too far from any plausible reference. Instead, intermediate market states ($\alpha \leq \zeta \leq \bar{\alpha}$) elicit a unique endogenous atom—the anchor—to which it is optimal to compare the outcome.

Note that the outcome on this middle event concentrating most of the probability—the optimal threshold on which investors wish their wealth to remain—is chosen endogenously. To determine which personal equilibrium is the preferred one, it suffices to numerically maximize the overall utility over the finite interval $[x_1, \bar{x}_2]$.

4 | REFERENCE DEPENDENCE WITH GEOMETRIC BROWNIAN MOTION

This section solves in detail the Merton model with reference dependence. Because the conditions under which the regular and anchors regimes attain are independent of the utility function, the discussion below focuses on the logarithmic utility function to simplify calculations.

Assumption 4.1. Investors have constant relative risk aversion coefficient $p = 1$, that is,

$$u(x) := \log(x), \quad \text{for all } x \in (0, +\infty). \tag{24}$$

As it is often the case with logarithmic preferences, it is possible to derive closed-form expressions for both the optimal wealth processes and the optimal investment strategies.

Consider a standard Brownian motion $W = \{W_t\}_{t\in[0,T]}$ under $\mathbb{P}$ representing the source of risk in the market, and let (the $\mathbb{P}$-augmentation of) its natural filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ describe the flow of information available to investors over time. Without loss of generality, set $\mathcal{F} = \mathcal{F}_T$. Two assets are traded continuously: one money-market account with null interest rate, and one stock whose price process $S = \{S_t\}_{t\in[0,T]}$ follows a geometric Brownian motion with constant expected return $\mu \in \mathbb{R} \setminus \{0\}$ and volatility $\sigma > 0$, that is,

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t), \quad t \in [0, T], \quad S_0 = s_0 > 0.$$

Denoting the market price of risk by $\kappa := \mu/\sigma$, recall that the unique martingale density process $\{\zeta_t\}_{t\in[0,T]}$ is

$$\zeta_t = \exp\left\{-\kappa W_t - \frac{\kappa^2 t}{2}\right\} = \left(\frac{S_t}{s_0}\right)^{-\frac{\kappa}{\sigma}} e^{\frac{t}{2}(\kappa^2-\mu)}, \quad \text{for all } t \in [0, T].$$

Consequently, $\zeta := \zeta_T$ is lognormally distributed under $\mathbb{P}$, with density function

$$f_\zeta(x) = \frac{1}{x\sqrt{\kappa^2 T}}\phi\left(\frac{\log(x) + \kappa^2 T/2}{\sqrt{\kappa^2 T}}\right)\mathbb{1}_{(0,+\infty)}(x), \quad \text{for all } x \in \mathbb{R},$$

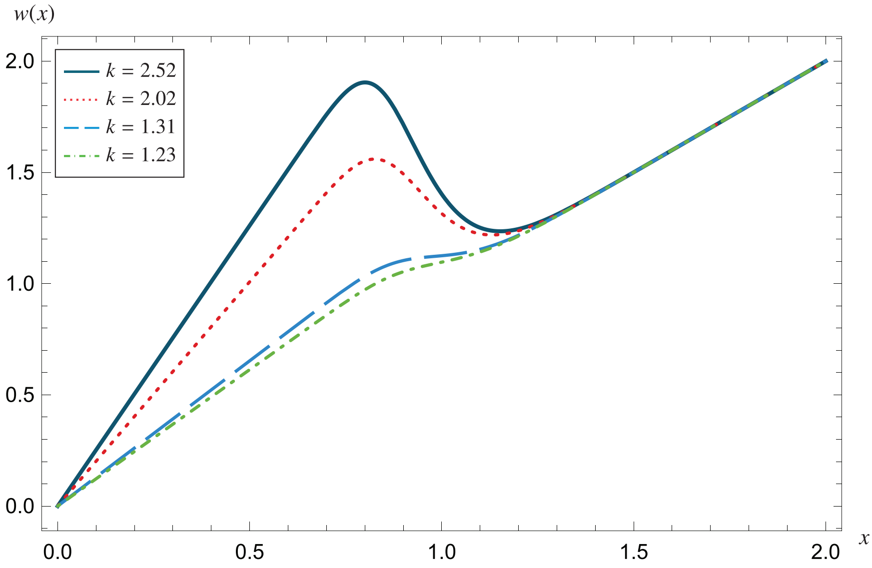

FIGURE 5 Twist in the Black–Scholes model for different values of k. Parameters: $x_0 = 100$, $\mu = 3\%$, $\sigma = 24.5\%$, and $T = 1.0$. [Color figure can be viewed at wileyonlinelibrary.com]

and mode $\theta := \exp\{-3\kappa^2 T/2\}$. Here and henceforth, $\phi(\cdot)$ and $\Phi(\cdot)$, respectively, denote the density and distribution functions of a standard univariate normal random variable.

A self-financing portfolio starting from initial wealth $x_0 > 0$ is an $\mathbb{F}$-progressively measurable process $\vartheta = \{\vartheta_t\}_{t\in[0,T]}$ such that $\int_0^T |\vartheta_t|^2\, dt < +\infty$ a.s., and the associated wealth process $X^\vartheta = \{X_t^\vartheta\}_{t\in[0,T]}$ satisfies

$$dX_t^\vartheta = X_t^\vartheta(\mu\vartheta_t\, dt + \sigma\vartheta_t\, dW_t), \quad t \in [0,T], \quad X_0^\vartheta = x_0.$$

We regard ϑ_t as the proportion of wealth held in the stock at any time t. Because the logarithmic utility is finite only for positive values, an admissible self-financing portfolio has necessarily an a.s. non-negative wealth process.

For a classical utility-maximizing investor, the best strategy is to keep a constant proportion—equal to the Merton ratio μ/σ^2—of the total wealth in the stock, and the optimal wealth process $\tilde{X} = \{\tilde{X}_t\}_{t\in[0,T]}$ is $\tilde{X}_t = x_0/\zeta_t$. In particular, the optimal stock weight depends neither on the investment horizon nor on the state of nature.

To find the personal equilibria for an investor with reference-dependent preferences *à la* Kőszegi and Rabin, we need to determine the twist and concentration functions, which in the Black–Scholes model are

$$w(x) = \frac{x}{1-(1-k^{-1})\Phi(-d_1(x,T,T))},$$

$$H(x) = 1-\left(1-k^{-1}\right)\left(\Phi(-d_1(x,T,T)) + \frac{1}{\sqrt{\kappa^2 T}}\phi(d_1(x,T,T))\right),$$

respectively, (see Figures 5 and 6), with

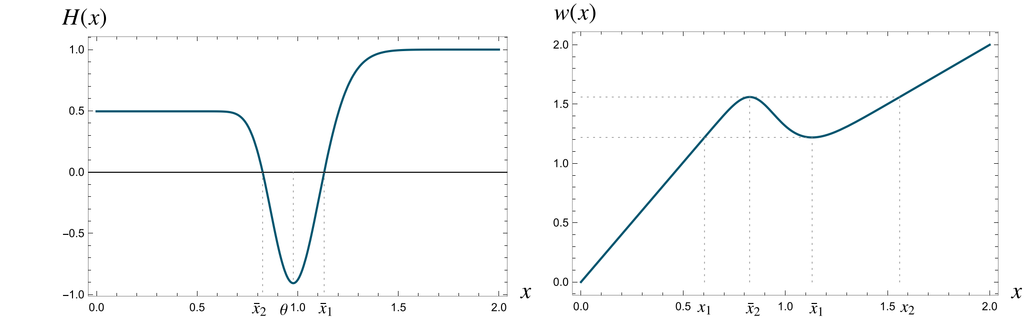

FIGURE 6 Concentration function (left) and twist (right) in the Black–Scholes model (Section 4). Parameters: Sharpe ratio $\kappa \approx 0.12$, stock volatility $\sigma = 24.5\%$, $T = 1.0$, $\lambda = 0.29$, and $\eta = 0.71$ (hence $k \approx 2.02$, anchors regime). The pricing kernel is lognormal with mode $\theta \approx 0.98$, and the concentration function has roots $\bar{x}_1 \approx 1.13$ and $\bar{x}_2 \approx 0.83$. [Color figure can be viewed at wileyonlinelibrary.com]

$$d_1(x,t,T) := \frac{\log(x) + \kappa^2 t/2}{\sqrt{\kappa^2(2T-t)}}.$$

Note that $\lim_{T\to+\infty} w(x) = x$ for all $x \in [0,+\infty)$, therefore the behavior of the twist in the long run approaches that of the identity.

For the remainder of this example, following Tversky and Kahneman (1991, 1992), we use a loss aversion parameter $1/\lambda = 1/0.29 \approx 3.45$. Also, we choose $\mu = 3\%$ and $\sigma = 24.5\%$, therefore $\kappa \approx 0.12$ and the optimal proportion of wealth invested in the stock is $\mu/\sigma^2 \approx 50\%$. In the first part, we keep the investment horizon fixed at 1 year and vary the value of the reference-dependence parameter η, so as to highlight both the regular and the anchors regimes.

4.1 | Regular regime

Let $\lambda, \eta \in (0,1)$ such that

$$k \leq 1 + \frac{\sqrt{\kappa^2 T}}{\phi\left(\sqrt{\kappa^2 T}\right) - \sqrt{\kappa^2 T}\,\Phi\left(-\sqrt{\kappa^2 T}\right)}. \tag{25}$$

By Corollary 3.7, $\mathrm{PE}(x_0) = \{X_T^*\}$, where the investor's optimal wealth at time $t \in [0,T]$ is

$$X_t^* = \tilde{X}_t\left(1 - \frac{k-1}{k+1}\left(1 - 2\Phi(d_1(\zeta_t,t,T))\right)\right).$$

Therefore, the unique personal equilibrium of an investor with relatively weak reference-dependent preferences differs from the classical utility maximizer by the zero-price payoff

$$-\frac{x_0}{\zeta}\left(\frac{k-1}{k+1}\right)\left(1 - 2\Phi(d_1(\zeta,T,T))\right).$$

As observed in Proposition 3.3(ii) and visible in Figure 2, whether this payoff is negative or positive depends on the state of the market: it is negative in good states to mitigate losses in bad states, where it becomes positive.

Also consistent with the same proposition, the personal equilibrium always increases with the initial capital, whereas its sensitivity to preference parameters depends on the state of the market: if it is good (respectively, bad), then the more loss averse or reference-dependent investors are, the smaller (respectively, larger) their wealth level is.

As for the time-t optimal proportion of wealth invested in the stock, it equals

$$\pi_t^* = \frac{\mu}{\sigma^2}\left(1-\left(1-k^{-1}\right)\frac{\phi(-d_1(\zeta_t,t,T))}{\sqrt{\kappa^2(2T-t)}\Big(1-(1-k^{-1})\Phi(-d_1(\zeta_t,t,T))\Big)}\right).$$

Clearly, the optimal stock proportion consists of two terms: the first one is the Merton ratio, while the second one is a correction term that accounts for the investor's reference-dependent preferences and loss aversion. As a consequence, the personal equilibrium still entails taking a long position in the stock, but reference dependence depresses demand for stocks, more significantly so in intermediate states of nature (see Figure 2). In good states of the world, investors keep the stock weight close to the Merton proportion. In intermediate states, they decrease the proportion of stock in their portfolios, which attains the minimum value at the median market state. In bad states, they start increasing their stock investments, eventually bringing them close again to the Merton proportion.

Another important difference is that, while usual logarithmic investors invest myopically, now the optimal stock weight not just depends on the preference parameters λ and η (via k only), but it also changes with the investment horizon and the state of the world.

Finally, note that a greater degree of loss aversion or reference dependence translates into a reduction in the stock investment, which on the other hand approaches the Merton ratio as reference dependence disappears ($k \to 1$).

4.2 | Anchors regime

Let $\lambda, \eta \in (0,1)$ such that Equation (25) fails. By Corollary 3.7, $\mathrm{PE}(x_0) = \{X_\alpha^\star : \alpha \in [x_1, \bar{x}_2]\}$, where

$$X_\alpha^\star = \frac{x_0}{\ell_\alpha}\left(\frac{1-\left(1-k^{-1}\right)\Phi(-d_1(\alpha,T,T))}{\alpha}\mathbb{1}_{\{\zeta\in[\alpha,\bar{\alpha}]\}} + \frac{1-\left(1-k^{-1}\right)\Phi(-d_1(\zeta,T,T))}{\zeta}\mathbb{1}_{\{\zeta\notin[\alpha,\bar{\alpha}]\}}\right).$$

Here,

$$\begin{aligned}\ell_\alpha &:= \frac{1-\left(1-k^{-1}\right)\Phi(-d_1(\alpha,T,T))}{\alpha}\Big(\Phi(d_2(\bar{\alpha},0,T)) - \Phi(d_2(\alpha,0,T))\Big)\\ &\quad + \frac{1+k^{-1}}{2} + \Big(\Phi(d_1(\alpha,T,T)) - \Phi(d_1(\bar{\alpha},T,T))\Big)\left(1-\left(1-k^{-1}\right)\frac{\Phi(-d_1(\alpha,T,T))+\Phi(-d_1(\bar{\alpha},T,T))}{2}\right)\end{aligned}$$

and

$$d_2(x,t,T) := \frac{\log(x) - \kappa^2(T-t)/2}{\sqrt{\kappa^2(T-t)}}.$$

Also, $\bar{x}_2 < \theta < \bar{x}_1$ denote the two roots of the concentration function, and x_i is the unique solution of

$$\frac{1-\left(1-k^{-1}\right)\Phi(-d_1(x,T,T))}{x} = \frac{1-\left(1-k^{-1}\right)\Phi(-d_1(\bar{x}_i,T,T))}{\bar{x}_i}$$

for each $i \in \{1,2\}$ (see Figure 6), while $\bar{\alpha}$ is the unique number in $[\bar{x}_1, x_2]$ such that

$$\frac{1-\left(1-k^{-1}\right)\Phi(-d_1(\bar{\alpha},T,T))}{\bar{\alpha}} = \frac{1-\left(1-k^{-1}\right)\Phi(-d_1(\alpha,T,T))}{\alpha}.$$

Furthermore, for each $\alpha \in [x_1, \bar{x}_2]$, the wealth process $\{X^{\star}_{\alpha,t}\}_{t\in[0,T]}$ that replicates the corresponding endogenous anchor is

$$\begin{aligned} X^{\star}_{\alpha,t} = {} & \frac{x_0}{\ell_\alpha}\frac{1-\left(1-k^{-1}\right)\Phi(-d_1(\alpha,T,T))}{\alpha}\left[\Phi\left(d_2\left(\frac{\bar{\alpha}}{\zeta_t},t,T\right)\right) - \Phi\left(d_2\left(\frac{\alpha}{\zeta_t},t,T\right)\right)\right] \\ & + \frac{x_0}{\ell_\alpha \zeta_t}\left[\Phi\left(d_2\left(\frac{\zeta_t}{\bar{\alpha}},t,T\right)\right) - \left(1-k^{-1}\right)\mathrm{M}\left(d_2\left(\frac{\zeta_t}{\bar{\alpha}},t,T\right), -d_1(\zeta_t,t,T); -\sqrt{\frac{T-t}{2T-t}}\right)\right] \\ & + \frac{x_0}{\ell_\alpha \zeta_t}\left[\Phi\left(-d_2\left(\frac{\zeta_t}{\alpha},t,T\right)\right) - \left(1-k^{-1}\right)\mathrm{M}\left(-d_2\left(\frac{\zeta_t}{\alpha},t,T\right), -d_1(\zeta_t,t,T); \sqrt{\frac{T-t}{2T-t}}\right)\right], \end{aligned}$$

where

$$\mathrm{M}(h,k;\rho) := \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{h}\int_{-\infty}^{k}\exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\}dx\,dy, \quad \text{for all } h,k \in \mathbb{R},$$

is the joint distribution function of the standard bivariate normal distribution with covariance $\rho \in (-1,1)$.

Remarkably, once reference dependence and loss aversion become large enough, investors completely abandon diffuse payoffs; instead, they are better off keeping their wealth constant over a range of intermediate market states concentrating most of the probability. In fact, investors can choose infinitely many wealth levels between their most pessimistic and most optimistic rational expectations. Figure 7 displays three such anchors, arising, respectively, from the lowest increasing envelope, the highest increasing envelope, and an intermediate one.

In the example considered, it can be determined numerically that the preferred personal equilibrium is the one corresponding to the upper envelope; in other words, such an investor achieves the highest welfare by adopting the most pessimistic endogenous anchor (i.e., having the lowest wealth threshold; see Figure 8).

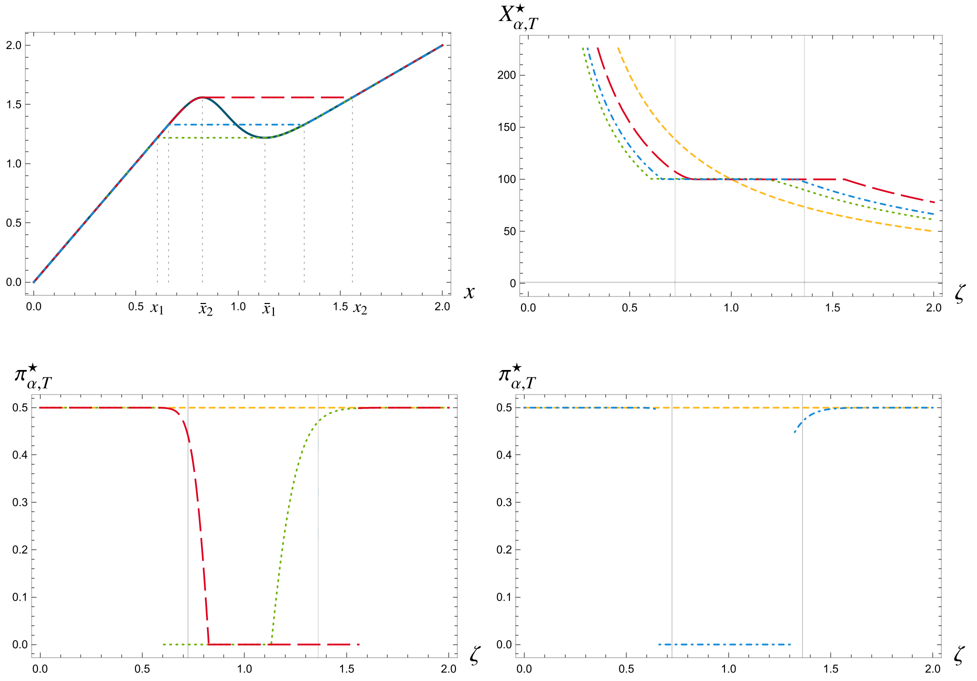

FIGURE 7 Three endogenous anchors (vertical, top right; classical utility maximizer in ochre) and corresponding terminal stock weights (vertical, bottom; Merton ratio $\mu/\sigma^2 \approx 50\%$ in ochre) against pricing kernel (horizontal) for an investor with reference-dependent preferences in the Black–Scholes model. At the top left, the twist (dark blue) and monotonic envelope associated with each anchor (lower envelope in green, an intermediate monotonic envelope with $\alpha = 0.66$ and $\bar{\alpha} \approx 1.32$ in light blue, and upper envelope in red). Parameters: $x_0 = 100$, $\mu = 3\%$, $\sigma = 24.5\%$, $T = 1.0$, $\lambda = 0.29$, and $\eta = 0.71$ (hence $k \approx 2.02$, anchors regime). The vertical solid lines give a 99% confidence band for ζ. [Color figure can be viewed at wileyonlinelibrary.com]

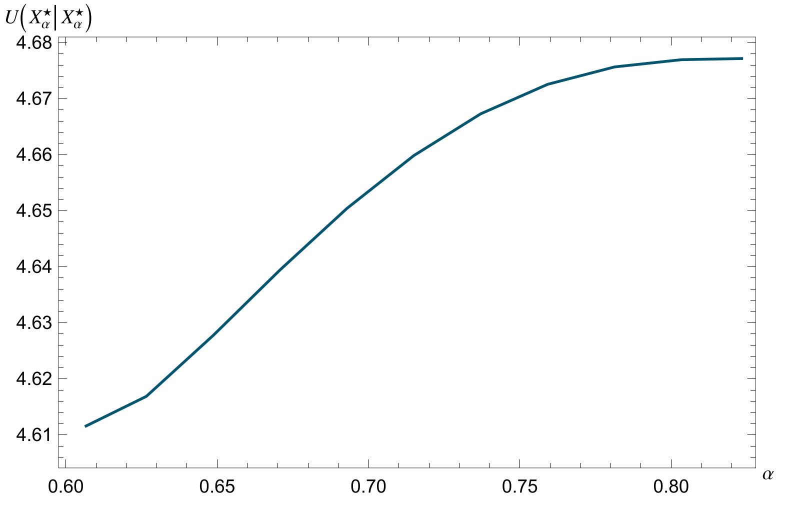

FIGURE 8 Reference-dependent utility of the endogenous anchors (relative to themselves) for a logarithmic investor with reference-dependent preferences in the Black–Scholes model. Parameters: $x_0 = 100$, $\mu = 3\%$, $\sigma = 24.5\%$, $\lambda = 0.29$, and $\eta = 0.71$ (hence $k = 2.02$, anchors regime). [Color figure can be viewed at wileyonlinelibrary.com]

We also obtain an analytic expression for the optimal stock weight process associated with each endogenous anchor (refer to Appendix A.2); in particular, the terminal stock weight is

$$\pi^{\star}_{\alpha,T} = \frac{\mu}{\sigma^2}\left(1 - \left(1 - k^{-1}\right)\frac{\phi(-d_1(\zeta,T,T))}{\sqrt{\kappa^2 T}(1-(1-k^{-1})\Phi(-d_1(\zeta,T,T)))}\right)\mathbb{1}_{\{\zeta\notin[\alpha,\bar{\alpha}]\}}.$$

As in the regular regime, reference dependence entails a lower investment in stocks also for the anchors. However, while the terminal payoff of each endogenous anchor has a smooth pasting property at the endpoints α and $\bar{\alpha}$ (meaning that a small change in the market about these two states does not cause a jump in wealth relative to the atom), this is no longer the case for the corresponding terminal stock weight. In fact, if the market moves from a state that is better than α to another one that is worse, no matter how slightly, the investor immediately pulls out of the market by suddenly dropping the stock investment to zero (as for market changes at the less advantageous state $\bar{\alpha}$). Note that this discontinuity of the mapping $\zeta_t \mapsto \pi^{\star}_{\alpha,t}$ occurs only at the terminal time, at which the investor has no time left to recover losses. As the discontinuity is inherited from the slope of the envelope $w^{\star}(\cdot)$, it occurs only at the points where $w^{\star}(\cdot)$ merges into $w(\cdot)$ with a different slope.

Finally, the second part of this example focuses on how the investment horizon influences investment decisions of individuals with reference-dependent preferences, so this time, we fix $\lambda, \eta \in (0,1)$ and consider different values for T. Elementary calculus shows that condition (21) holds if and only if $T \geq T^* := \varrho^2/\kappa^2$, where $\varrho \in (0,+\infty)$ is the unique solution of the transcendental equation

$$x\left(1 - \left(1 - k^{-1}\right)\Phi(x)\right) - \left(1 - k^{-1}\right)\phi(x) = 0.$$

Put differently, there is a critical investment horizon T^* at which individuals switch from one regime to the other: when the horizon is long enough so that they can still recover from eventual losses, investors follow the regular regime, otherwise it is simply too risky to deviate from the atom and they remain attached to the endogenous anchors.

Clearly, T^* is strictly decreasing with the absolute value of κ: when the Sharpe ratio is low, there is little to gain and anchors are relevant for longer horizons; when it is high, expected utility takes over and investors give up on reference dependence.

Moreover, $\lim_{k\to 1} T^* = 0$, meaning that investors abandon endogenous anchors almost instantly, as reference dependence vanishes (either through λ or η). On the other hand, $\lim_{k\to+\infty} T^* = +\infty$, so an investor focusing solely on reference dependence and having an arbitrarily high level of loss aversion commits to anchors indefinitely (see Figure 9).

Observe also that $\lim_{T\to+\infty} X^*_t = \tilde{X}_t$ for all $t \in [0,T)$; intuitively, for an investor with an arbitrarily large investment horizon, the diffuse personal equilibrium becomes virtually indistinguishable from the classical optimizer at intermediate times. However,

$$\lim_{T\to+\infty}\left(\frac{X^*_T - \tilde{X}_T}{\tilde{X}_T}\right) = \frac{k-1}{k+1} > 0,$$

which justifies why in Figure 10 the personal equilibrium for a long horizon (viz., $T = 100$) still does not coincide with $\tilde{X}$; in fact, the relative change between X^*_T and $\tilde{X}$ approaches the ratio $(k-1)/(k+1)$ as T goes to infinity. It is also worth noting that, as the investment horizon becomes

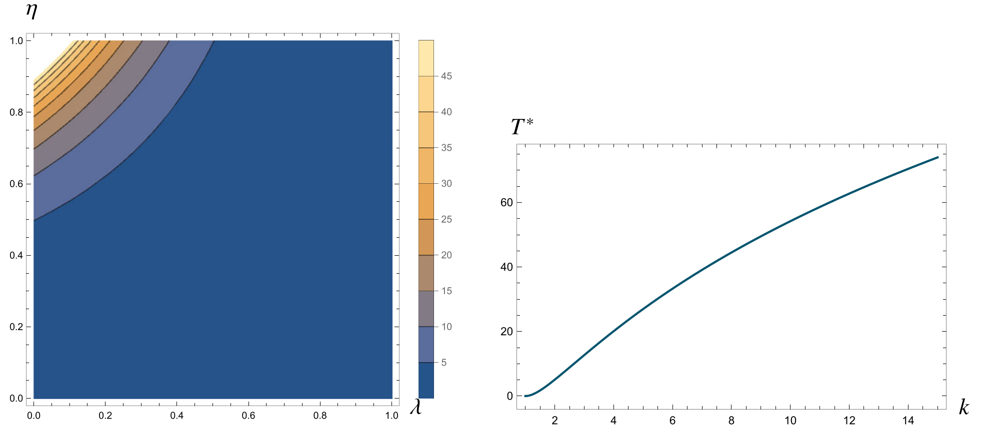

FIGURE 9 Critical horizon for an investor with reference-dependent preferences in the Black–Scholes model. On the left, contour plot of the critical horizon T^* in the (λ, η)-plane. On the right, the critical horizon T^* (vertical, in years) as a function of k (horizontal). Parameters: $x_0 = 100$, $\mu = 3\%$, and $\sigma = 24.5\%$. [Color figure can be viewed at wileyonlinelibrary.com]

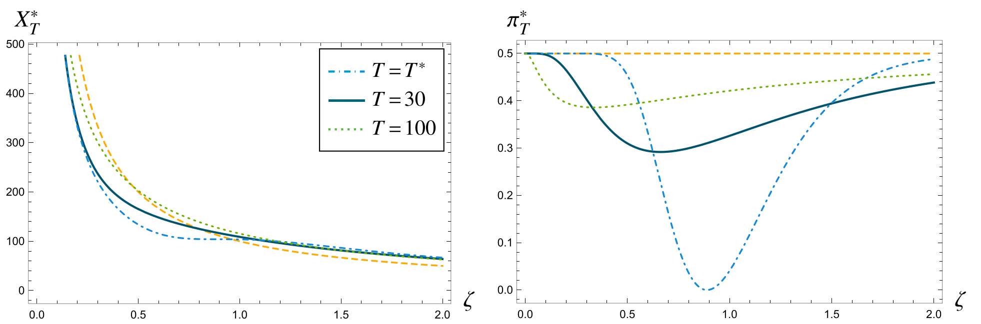

FIGURE 10 Optimal terminal wealth (vertical, left; classical utility maximizer w_0/ζ in ochre) and stock weight (vertical, right; Merton ratio $\mu/\sigma^2 \approx 50\%$ in ochre) against pricing kernel (horizontal) for an investor with reference-dependent preferences and different (longer) horizons in the Black–Scholes model (regular regime). Parameters: $x_0 = 100$, $\mu = 3\%$, $\sigma = 24.5\%$, $\lambda = 0.29$, and $\eta = 0.71$ (hence $k \approx 2.02$); the critical horizon is $T^* \approx 5.20$ years. [Color figure can be viewed at wileyonlinelibrary.com]

very large, π_t^* converges to the Merton proportion at all times $t \in [0, T]$, so the behavior of the optimal stock weight gradually moves closer to the Merton line as T increases to infinity.

Turning to the case of investors with short investment horizons, we see in Figure 11 that, as T approaches the critical horizon, the interval of market states on which agents wish to keep their wealth constant and their terminal stock investment equal to zero begins to shrink, until they eventually abandon the anchors in favor of the diffuse personal equilibrium.

We conclude by emphasizing that T^* does not depend on the utility $u(\cdot)$, hence the critical horizon is the same in the Black–Scholes model whether we consider an investor with logarithmic preferences or an investor with, for example, another CRRA utility, provided that they all have in common the same reference-dependent preferences.

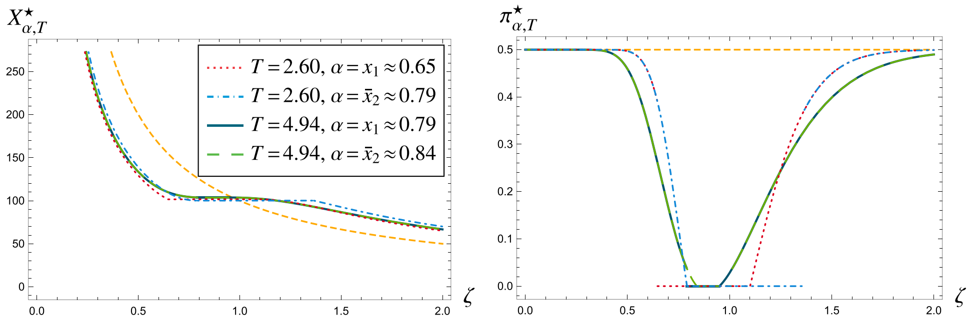

FIGURE 11 Lower and upper endogenous anchors (vertical, left) and corresponding terminal stock weights (vertical, right) against pricing kernel (horizontal) for an investor with reference-dependent preferences and two different (shorter) horizons in the Black–Scholes model (anchors regime). In ochre, the classical utility maximizer (left) and Merton ratio $\mu/\sigma^2 \approx 50\%$ (right). Parameters: $x_0 = 100$, $\mu = 3\%$, $\sigma = 24.5\%$, $\lambda = 0.29$, and $\eta = 0.71$ (hence $k \approx 2.02$); the critical horizon is $T^* \approx 5.20$ years. [Color figure can be viewed at wileyonlinelibrary.com]

5 | CONCLUSION

This paper solves the portfolio choice problem in a complete market for an investor with the reference-dependent preferences of Kőszegi and Rabin (2006). Unlike in the one-period setting of Guasoni and Meireles-Rodrigues (2020), in a complete market investors are free to choose payoffs with arbitrary distributions, regardless of asset price dynamics.

With a fixed reference payoff, the reference-dependent utility maximization problem reduces to a familiar expected utility problem, in which the utility function is replaced by a reference-adjusted utility. Each personal equilibrium—a payoff that is optimal when taken as reference—is identified as a fixed point for the reference-optimizer map.

We characterize the set of personal equilibria, highlighting two regimes. The regular regime arises when reference dependence is low and entails a unique personal equilibrium satisfying a first-order condition with respect to a twisted pricing kernel. By contrast, investors with a strong combination of loss aversion and reference dependence endogenously create their own "anchors"—payoffs that concentrate significant probability on one or few values. Importantly, such anchors are not unique.

In the usual geometric Brownian motion model, which is dynamically complete, we show that the two regimes arise for different investment horizons: long-term investors naturally gravitate in the regular regime, choosing diffuse payoffs. Shorter horizons lead to lower stock holdings, and each short-term investor stays committed to the respective endogenous anchor.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as it did not generate or analyze new datasets.

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ENDNOTES

[1] A personal equilibrium is a payoff that, when used as reference, is optimal among all alternative payoffs.

[2] Formulating the utility maximization problem for discounted quantities with respect to some auxiliary numeraire, the independence of the discounted payoff and discounted reference under such numeraire implies that the payoff and reference are in fact dependent in the original units.

[3] Recall that, given a random variable X with distribution function $F_X(\cdot)$, the *(right-continuous) generalized inverse* $q_X : (0,1) \to \mathbb{R}$ is defined by

$$q_X(p) := \inf\left\{x \in \mathbb{R} : F_X(x) > p\right\}, \quad \text{for all } p \in (0,1).$$

[4] Here, $x^{\pm} := \max\{\pm x, 0\}$ for all $x \in \mathbb{R}$. In addition, $\mathbf{1}_A : X \to \{0,1\}$ is the *indicator function* of the set $A \subseteq X$, defined by

$$\mathbf{1}_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

[5] We are very grateful to an anonymous referee for pointing out this simplification.

[6] Note that $w(\cdot)$ is increasing if and only if $\underline{w}(\cdot) = \overline{w}(\cdot)$, in which case these three functions coincide.

[7] An examination of the proof shows that Assumption 2.1(ii) is not needed in its full strength; in fact, to obtain the results of Theorem 3.2 describing the set of personal equilibria, it suffices that the pricing kernel ζ has a continuous and invertible distribution function on $[0, +\infty)$.

[8] A density function is *strictly unimodal* if it has a unique maximum attained at θ (called the *mode* of the distribution), and is strictly monotonic on both sides of θ.

[9] The *essential supremum* and the *essential supremum* of a random variable X are defined, respectively, as

$$\operatorname{ess\,sup} \zeta := \inf\{a \in \mathbb{R} : \mathbb{P}\{\zeta > a\} = 0\} \quad \text{and} \quad \operatorname{ess\,inf} \zeta := \sup\{a \in \mathbb{R} : \mathbb{P}\{\zeta < a\} = 0\},$$

with the usual conventions $\inf \varnothing := +\infty$ and $\sup \varnothing := -\infty$.

[10] Recall that a random variable X is *degenerate* if $\mathbb{P}\{X = c\} = 1$ for some constant $c \in \mathbb{R}$.

[11] The *signum function* $\operatorname{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ is defined by

$$\operatorname{sgn}(x) := \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

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APPENDIX

This appendix contains auxiliary results as well as the proofs of the results stated in the main body. Appendix A presents the technical lemmata that are used in Appendix B, which in turn provides the proofs of the results in Sections 3 and 4.

In what follows, we assume $u(1) = 0$; such a normalization does not restrict generality, as adding a constant to $u(\cdot)$ simply causes a vertical shift of the reference-dependent utility, leaving the optimization problem (6) unchanged.

A | Auxiliary results

The first lemma is instrumental in that it enables us to rewrite the reference-dependent optimization problem for each reference B as a classical utility maximization problem for the new, suitably modified utility $\bar{u}_B(\cdot)$ defined in Equation (A.1). Put differently, adopting a different reference level corresponds to choosing different preferences. Note that, since $\bar{u}_B(\cdot)$ is "kinked" at the atoms of

B, its right derivative $\partial_+\bar{u}_B(\cdot)$ has a generalized inverse $\bar{I}_B(\cdot)$ that may be flat on some regions of the domain, so the optimal payoff for the reference B may have atoms.

Lemma A.1. *Let $B \in L^0_+$ such that $|\mathbb{E}[u(B)]| < +\infty$, and let Assumptions 2.1(i) and 2.1(iii) hold. Define the* reference-adjusted utility $\bar{u}_B : (0,+\infty) \to [-\infty,+\infty]$ *as*

$$\bar{u}_B(x) := u(x) + \int_{\mathbb{R}} \nu_+(u(x)-u(b))\mathbb{1}_{[0,x]}(b)\, d\mathbb{P}_B(b) - \int_{\mathbb{R}} \nu_-(u(b)-u(x))\mathbb{1}_{(x,+\infty)}(b)\, d\mathbb{P}_B(b), \quad \text{for all } x \in (0,+\infty). \tag{A.1}$$

(i) The function $\bar{u}_B(\cdot)$ is finite everywhere on $(0,+\infty)$, continuous, strictly increasing, strictly concave, with right $(+)$ and left $(-)$ derivatives at every point $x \in (0,+\infty)$ equal to

$$\partial_\pm\bar{u}_B(x) = u'(x) + u'(x)\int_{\mathbb{R}} \nu'_+(u(x)-u(b))\mathbb{1}_{[0,x)}(b)\, d\mathbb{P}_B(b) + u'(x)\int_{\mathbb{R}} \nu'_-(u(b)-u(x))\mathbb{1}_{(x,+\infty)}(b)\, d\mathbb{P}_B(b) + u'(x)\nu'_\pm(0)\mathbb{P}\{B=x\}. \tag{A.2}$$

In particular, $\bar{u}_B(\cdot)$ is differentiable outside the countable set $\Delta_B := \{x \in \mathbb{R} : \mathbb{P}\{B=x\} > 0\}$.

(ii) The right derivative $\partial_+\bar{u}_B : (0,+\infty) \to (0,+\infty)$ in Equation (A.2) is strictly decreasing and càdlàg, with downward jumps of size $u'(x)(\nu'_-(0)-\nu'_+(0))\mathbb{P}\{B=x\}$ at the points $x \in \Delta_B$. Moreover,

$$\partial_+\bar{u}_B(0+) := \lim_{x\to 0^+} \partial_+\bar{u}_B(x) = +\infty \quad \text{and} \quad \partial_+\bar{u}_B(+\infty) := \lim_{x\to+\infty} \partial_+\bar{u}_B(x) = 0. \tag{A.3}$$

(iii) The generalized inverse of $\partial_+\bar{u}_B(\cdot)$, defined by

$$\bar{I}_B(y) := \inf\{x \in (0,+\infty) : \partial_+\bar{u}_B(x) \le y\}, \quad \text{for all } y \in (0,+\infty), \tag{A.4}$$

is finite everywhere on $(0,+\infty)$, continuous, strictly positive, decreasing, with

$$\bar{I}_B(y) \le I\left(\frac{y}{1+\nu'_-(0)}\right) \quad \text{for all } y \in (0,+\infty), \tag{A.5}$$

$\bar{I}_B(0+) := \lim_{y\to 0^+}\bar{I}_B(y) = +\infty$ and $\bar{I}_B(+\infty) := \lim_{y\to+\infty}\bar{I}_B(y) = 0$. Moreover,

$$\bar{I}_B(y) \le x \Leftrightarrow y \ge \partial_+\bar{u}_B(x) \quad \text{for all } x, y \in (0,+\infty), \tag{A.6}$$

and

$$\{y \in (0,+\infty) : \bar{I}_B(y) = x\} = [\partial_+\bar{u}_B(x), \partial_+\bar{u}_B(x-)] \quad \text{for all } x \in (0,+\infty), \tag{A.7}$$

where $\partial_+\bar{u}_B(x-) := \lim_{z\to x^-}\partial_+\bar{u}_B(z)$.

(iv) *The Fenchel–Legendre transform of $\bar{u}_B$, defined by*

$$\bar{u}_B^*(y) := \sup_{x\in(0,+\infty)} \{\bar{u}_B(x) - xy\}, \quad \text{for all } y \in (0,+\infty), \tag{A.8}$$

is finite everywhere on $(0,+\infty)$*, differentiable, decreasing, convex, with*

$$\bar{u}_B^*(y) = \bar{u}_B(\bar{I}_B(y)) - y\bar{I}_B(y) \quad \text{for all } y \in (0,+\infty). \tag{A.9}$$

(v) *The function* $\Xi_B : (0,+\infty) \to [-\infty,+\infty]$ *defined by*

$$\Xi_B(y) := \mathbb{E}[\zeta \bar{I}_B(y\zeta)], \quad \text{for all } y \in (0,+\infty), \tag{A.10}$$

maps $(0,+\infty)$ *onto* $(0,+\infty)$ *and is both continuous and decreasing.*

(vi) *For all* $x_0 > 0$*, the optimization problem*

$$\mathrm{v}(x_0, B) = \sup_{Z\in\mathscr{C}(x_0)} \mathbb{E}[\bar{u}_B(Z)] \tag{A.11}$$

has unique (up to a $\mathbb{P}$*-null set) solution* $\hat{Z}_B := \bar{I}_B(\hat{y}\zeta)$*, where* $\hat{y} \in (0,+\infty)$ *solves* $\mathbb{E}[\zeta\hat{Z}_B] = x_0$*. If, in addition,* $F_\zeta(\cdot)$ *is continuous and strictly increasing on* $[0,+\infty)$*, then* $\hat{y}$ *is unique and the distribution of* $\hat{Z}_B$ *under* $\mathbb{P}$ *is*

$$F_{\hat{Z}_B}(x) = 1 - F_\zeta\left(\frac{\partial_+ \bar{u}_B(x)}{\hat{y}}\right) \quad \text{for all } x \in (0,+\infty). \tag{A.12}$$

Proof. First, observe that $\mathbb{E}[u(B)]$ is finite if and only if

$$\int_{\mathbb{R}} u(b)\mathbb{1}_{[0,1]}(b)\, d\mathbb{P}_B(b) > -\infty \quad \text{and} \quad \int_{\mathbb{R}} u(b)\mathbb{1}_{(1,+\infty)}(b)\, d\mathbb{P}_B(b) < +\infty$$

(recall our convention $+\infty - \infty := -\infty$). Note also that, if $u(0) = -\infty$, then $B > 0$ must hold a.s. For readability, we split the proof into several steps.

(i) To see that $\bar{u}_B(\cdot)$ takes on a finite value at every point of its domain, let $x \in (0,+\infty)$.

(a) Since $\nu_-(\cdot)$ is non-negative and $\nu_+(y) \le \nu_+'(0)y$ for all $y \ge 0$,

$$\bar{u}_B(x) \le u(x)\big(1 + \nu_+'(0)\mathbb{P}\{0 \le B \le x\}\big) - \nu_+'(0) \int_{\mathbb{R}} u(b)\mathbb{1}_{[0,1]}(b)\, d\mathbb{P}_B(b) < +\infty. \tag{A.13}$$

An identical argument yields $\bar{u}_B(x) > -\infty$. It is useful to notice that combining Equation (A.13) with Equation (2) yields $\mathbb{E}[\bar{u}_B(Z)] \le (1+\nu_+'(0))\mathbb{E}[u(Z)] - \nu_+'(0)\mathbb{E}[u(B)\mathbb{1}_{\{B\le 1\}}] < +\infty$ for all $Z \in \mathscr{C}(x_0)$.

(b) Fix an arbitrary $x > 0$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with limit x. Assume without loss of generality that $x_n < 3x/2$ for all $n \in \mathbb{N}$. Because $u(\cdot)$ and $\nu_+(\cdot)$ are both continuous, the sequence

$$\big\{\nu_+(u(x_n) - u(\cdot))\mathbb{1}_{[0,x_n]}(\cdot)\big\}_{n\in\mathbb{N}}$$

converges pointwise to $\nu_+(u(x)-u(\cdot))\mathbb{1}_{[0,x]}(\cdot)$. Furthermore, it is dominated by the integrable function $\nu_+(u(3x/2)-u(\cdot))\mathbb{1}_{[0,3x/2]}(\cdot)$. It follows from Lebesgue's dominated convergence theorem that

$$\lim_{n\to+\infty}\int_{\mathbb{R}}\nu_+(u(x_n)-u(b))\mathbb{1}_{[0,x_n]}(b)\,d\mathbb{P}_B(b)=\int_{\mathbb{R}}\nu_+(u(x)-u(b))\mathbb{1}_{[0,x]}(b)\,d\mathbb{P}_B(b).$$

In a similar way, we obtain the convergence of the integral $\int_{\mathbb{R}}\nu_-(u(b)-u(x_n))\mathbb{1}_{(x_n,+\infty)}(b)\,d\mathbb{P}_B(b)$ to $\int_{\mathbb{R}}\nu_-(u(b)-u(x))\mathbb{1}_{(x,+\infty)}(b)\,d\mathbb{P}_B(b)$. Hence, $\bar{u}_B(x_n)$ tends to $\bar{u}_B(x)$ as $n\to+\infty$.

(c) The strict monotonicity of $\bar{u}_B(\cdot)$ is inherited from that of $u(\cdot)$ and $\nu_\pm(\cdot)$.

(d) Let $x>0$. Consider a sequence $\{h_n\}_{n\in\mathbb{N}}$ of strictly positive real numbers converging to zero and bounded above by $x/2$. For all $n\in\mathbb{N}$, the difference quotient of $\bar{u}_B(\cdot)$ at x with increment h_n equal to

$$\begin{aligned}\frac{\bar{u}_B(x)-\bar{u}_B(x-h_n)}{h_n}&=\frac{u(x)-u(x-h_n)}{h_n}\\&+\int_{\mathbb{R}}\frac{\nu_+(u(x)-u(b))-\nu_+(u(x-h_n)-u(b))}{h_n}\mathbb{1}_{[0,x-h_n]}(b)d\mathbb{P}_B(b)\\&+\int_{\mathbb{R}}\frac{\nu_+(u(x)-u(b))}{h_n}\mathbb{1}_{(x-h_n,x]}(b)d\mathbb{P}_B(b)\\&-\int_{\mathbb{R}}\frac{\nu_-(u(b)-u(x))-\nu_-(u(b)-u(x-h_n))}{h_n}\mathbb{1}_{(x,+\infty)}(b)d\mathbb{P}_B(b)\\&+\int_{\mathbb{R}}\frac{\nu_-(u(b)-u(x-h_n))}{h_n}\mathbb{1}_{(x-h_n,x]}(b)d\mathbb{P}_B(b).\end{aligned}\tag{A.14}$$

Using the dominated convergence theorem twice, both with the upper bound $\nu_+(0)u'(x/2)$, we obtain that the first and second integrals of Equation (A.14) have the respective limits (as $n\to+\infty$)

$$\int_{\mathbb{R}}\nu_+'(u(x)-u(b))u'(x)\mathbb{1}_{[0,x)}(b)\,d\mathbb{P}_B(b)\quad\text{and}\quad\int_{\mathbb{R}}0\,d\mathbb{P}_B(b)=0.$$

Two further applications of the dominated convergence theorem, this time with upper bound $\nu_-(0)u'(x/2)$, yield that the third and fourth integrals in Equation (A.14) also converge as $n\to+\infty$, respectively, to

$$\int_{\mathbb{R}}\left(-\nu_-'(u(x)-u(b))u'(x)\right)\mathbb{1}_{(x,+\infty)}(b)\,d\mathbb{P}_B(b)\quad\text{and}\quad\int_{\mathbb{R}}\nu_-'(0)u'(x)\mathbb{1}_{\{x\}}(b)\,d\mathbb{P}_B(b).$$

The proof for the right derivative is analogous. Note that, for all $x\in(0,+\infty)$,

$$0<u'(x)\le\partial_\pm\bar{u}_B(x)\le u'(x)\left(1+\nu_+'(0)\right)<+\infty.\tag{A.15}$$

(e) For all $x_1, x_2 \in (0, +\infty)$ such that $x_1 < x_2$,

$$\begin{aligned}
&\partial_- \bar{u}_B(x_2) - \partial_+ \bar{u}_B(x_1) \\
&\quad \leq \int_{\mathbb{R}} \Big[u'(x_2)\Big(1 + \nu'_+(u(x_2) - u(b))\Big) - u'(x_1)\Big(1 + \nu'_+(u(x_1) - u(b))\Big)\Big] \mathbb{1}_{[0,x_1]}(b)\, d\mathbb{P}_B(b) \\
&\quad + \int_{\mathbb{R}} \Big[u'(x_2)\Big(1 + \nu'_-(u(b) - u(x_2))\Big) - u'(x_1)\Big(1 + \nu'_-(u(b) - u(x_1))\Big)\Big] \mathbb{1}_{[\,x_2,+\infty)}(b)\, d\mathbb{P}_B(b) \\
&\quad + \Big[u'(x_2)\Big(1 + \nu'_-(0)\Big) - u'(x_1)\Big(1 + \nu'_-(u(x_2) - u(x_1))\Big)\Big] \mathbb{P}\{B \in (x_1, x_2)\}.
\end{aligned} \tag{A.16}$$

The first integral in Equation (A.16) is strictly negative, because the mapping $u'(\cdot)\Big(1 + \nu'_+(u(\cdot) - u(y))\Big)$ is strictly decreasing on $(y, +\infty)$ for every fixed $y > 0$; indeed, it follows from the concavity of both $u(\cdot)$ and $\nu_+(\cdot)$ that

$$u''(\cdot)\Big(1 + \nu'_+(u(\cdot) - u(y))\Big) + u'(\cdot)^2 \nu''_+(u(\cdot) - u(y)) < 0.$$

The second integral and the last term of Equation (A.16) are both non-negative because $u'(\cdot)\Big(1 + \nu'_-(u(y) - u(\cdot))\Big)$ is decreasing on $(0, y)$ for every fixed $y > 0$. Combining these inequalities with the strict concavity of $u(\cdot)$ yields $\partial_+ \bar{u}_B(x_1) > \partial_- \bar{u}_B(x_2)$.

On the other hand,

$$\partial_- \bar{u}_B(x_2) - \partial_+ \bar{u}_B(x_2) = \big(\nu'_-(0) - \nu'_+(0)\big) u'(x_2) \mathbb{P}\{B = x_2\} \geq 0.$$

Hence, the one-sided derivatives $\partial_\pm \bar{u}_B(\cdot)$ are both strictly decreasing on $(0, +\infty)$, and $\bar{u}_B(\cdot)$ is strictly concave on $(0, +\infty)$ (refer to Fischer, 2012, Theorem 2.5).

(ii) In view of the previous step, $\partial_+ \bar{u}_B(\cdot)$ is well-defined, strictly positive, and strictly decreasing. We prove its other properties separately.

(a) To establish the existence of left limits, fix $x \in (0, +\infty)$ and let $\{x_n\}_{n\in\mathbb{N}} \subseteq (0, +\infty)$ be a sequence converging to x in a strictly increasing way. By the dominated convergence theorem with upper bound $\nu'_\pm(0)$,

$$\begin{aligned}
\lim_{n\to+\infty} \partial_+ \bar{u}_B(x_n) &= u'(x)\Big(1 + \int_{\mathbb{R}} \nu'_+(u(x) - u(b)) \mathbf{1}_{[\,0,x)}(b)\, \mathbb{P}_B(b) + \int_{\mathbb{R}} \nu'_-(u(b) - u(x)) \mathbf{1}_{[\,x,+\infty)}(b)\, \mathbb{P}_B(b)\Big) \\
&= \partial_- \bar{u}_B(x).
\end{aligned}$$

Right-continuity follows analogously.

(b) Consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq (0, +\infty)$ such that $\lim_{n\to+\infty} x_n = 0$. Then, $\liminf_{n\to+\infty} \partial_+ \bar{u}_B(x_n) \geq \liminf_{n\to+\infty} u'(x_n) = +\infty$.

(c) Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers diverging to $+\infty$. It is an immediate consequence of the compound inequality in Equation (A.15) together with the squeeze principle that $\{\partial_+ \bar{u}_B(x_n)\}_{n\in\mathbb{N}}$ has limit zero.

(iii) That $\bar{I}_B(y)$ is a strictly positive real number for all $y \in (0, +\infty)$ is trivial by Equation (A.3). Furthermore, $\{x \in (0, +\infty) : \partial_+ \bar{u}_B(x) \leq y_1\} \subseteq \{x \in (0, +\infty) : \partial_+ \bar{u}_B(x) \leq y_2\}$ for all $0 < y_1 < y_2 < +\infty$, which in turn implies the monotonicity of $\bar{I}_B(\cdot)$. The proof of the remaining properties relies on similar arguments to those of Embrechts and Hofert (2013, Proposition 1).

(a) We claim that, for all $y \in (0,+\infty)$,

$$S_y := \{x \in (0,+\infty) : \partial_+ \bar{u}_B(x) \le y\} = [\bar{I}_B(y), +\infty). \tag{A.17}$$

Indeed, the inclusions $(\bar{I}_B(y), +\infty) \subseteq S_y \subseteq [\bar{I}_B(y), +\infty)$ are straightforward, because $\bar{I}_B(y)$ is the greatest lower bound for S_y. In addition, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq S_y$ converging to $\bar{I}_B(y)$ in a strictly decreasing way, which, combined with the right-continuity of $\partial_+\bar{u}_B(\cdot)$ yields $\partial_+\bar{u}_B(\bar{I}_B(y)) = \lim_{n\to+\infty} \partial_+\bar{u}_B(x_n) \le y$.

(b) Fix $y \in (0,+\infty)$, and consider a sequence $\{y_n\}_{n\in\mathbb{N}} \subseteq (0,+\infty)$ converging to y in a strictly increasing way. Then $\lim_{n\to+\infty} \bar{I}_B(y_n) = \inf_{n\in\mathbb{N}} \bar{I}_B(y_n) =: x_* \in [\bar{I}_B(y), +\infty)$. Suppose that $x_* > \bar{I}_B(y)$. Then,

$$y_n < \partial_+\bar{u}_B\left(\bar{I}_B(y_n) - \frac{x_* - \bar{I}_B(y)}{2}\right) \le \partial_+\bar{u}_B\left(\bar{I}_B(y) + \frac{x_* - \bar{I}_B(y)}{2}\right),$$

where the first inequality follows by Equation (A.17), while the second one is due to the monotonicity of $\partial_+\bar{u}_B(\cdot)$. Hence $y = \lim_{n\to+\infty} y_n \le \partial_+\bar{u}_B(\bar{I}_B(y) + \frac{x_*-\bar{I}_B(y)}{2}) < \partial_+\bar{u}_B(\bar{I}_B(y)) \le y$, a contradiction. An identical reasoning shows that $\bar{I}_B(\cdot)$ is right-continuous as well.

(c) The inequality $x \ge \inf S_{\partial_+\bar{u}_B(x)}$ is trivial for all $x \in (0,+\infty)$. On the other hand, $x > \bar{I}_B(\partial_+\bar{u}_B(x))$ would imply $\partial_+\bar{u}_B(x) < \partial_+\bar{u}_B(\bar{I}_B(\partial_+\bar{u}_B(x)))$, as $\partial_+\bar{u}_B(\cdot)$ is strictly decreasing. But this would contradict $\partial_+\bar{u}_B(\bar{I}_B(\partial_+\bar{u}_B(x))) \le \partial_+\bar{u}_B(x)$ (recall Equation A.17), hence

$$\bar{I}_B(\partial_+\bar{u}_B(x)) = x. \tag{A.18}$$

(d) For all $y \in (0,+\infty)$,

$$\bar{I}_B(y) = \bar{I}_B\left((1+\nu'_-(0))u'\left(I\left(\frac{y}{1+\nu'_-(0)}\right)\right)\right) \le \bar{I}_B\left(\partial_+\bar{u}_B\left(I\left(\frac{y}{1+\nu'_-(0)}\right)\right)\right) = I\left(\frac{y}{1+\nu'_-(0)}\right),$$

where the inequality is a consequence of Equation (A.15) and of the decreasing property of $\bar{I}_B(\cdot)$, while the last identity follows from Equation (A.18).

(e) Let $\{y_n\}_{n\in\mathbb{N}} \subseteq (0,+\infty)$ be a strictly decreasing sequence such that $\lim_{n\to+\infty} y_n = 0$. Since $\bar{I}_B(\cdot)$ is decreasing, $\{\bar{I}_B(y_n)\}_{n\in\mathbb{N}}$ converges to $x_* := \sup_{n\in\mathbb{N}} \bar{I}_B(y_n) \in (0,+\infty]$. If $x_* < M$ for some $M > 0$, then

$$y_n \ge \partial_+\bar{u}_B\left(\bar{I}_B(y_n) + \frac{M - x_*}{2}\right) \ge \partial_+\bar{u}_B\left(M - \frac{M - x_*}{2}\right),$$

leading to $0 = \lim_{n\to+\infty} y_n \ge \partial_+\bar{u}_B(M - \frac{M-x_*}{2}) > 0$, which is absurd. Hence, $x_* = +\infty$. Analogously, $\bar{I}_B(+\infty) = 0$.

(f) Let $x, y \in (0,+\infty)$. Using the decreasing property of $\bar{I}_B(\cdot)$ and Equation (A.18), it is immediate that $\partial_+\bar{u}_B(x) \le y$ implies $\bar{I}_B(y) \le \bar{I}_B(\partial_+\bar{u}_B(x)) = x$. Conversely, if $\bar{I}_B(y) \le x$, then

$$\partial_+ \bar{u}_B(x) \leq \partial_+ \bar{u}_B(\bar{I}_B(y)) \leq y$$

where the monotonicity of $\partial_+ \bar{u}_B(\cdot)$ and Equation (A.17) yields the first and second inequalities, respectively.

(g) Fix an arbitrary $x \in (0, +\infty)$, and set $E_x := \{y \in (0, +\infty) : \bar{I}_B(y) = x\}$. If $y < \partial_+ \bar{u}_B(x)$, then x does not belong to the set S_y, so by Equation (A.17), we have $\bar{I}_B(y) > x$. Next, suppose that $y > \partial_- \bar{u}_B(x)$. Since $\partial_- \bar{u}_B(x)$ is the left-hand limit of $\partial_+ \bar{u}_B(\cdot)$ at the point x, there exists $x_* < x$ such that $\partial_+ \bar{u}_B(x_*) < y$, which in turn implies $\bar{I}_B(y) \leq \bar{I}_B(\partial_+ \bar{u}_B(x_*)) = x_* < x$ (recall Equation A.18). Combining these two observations gives $E_x \subseteq [\partial_+ \bar{u}_B(x), \partial_- \bar{u}_B(x)]$. To show the reverse inclusion, let $y \in [\partial_+ \bar{u}_B(x), \partial_- \bar{u}_B(x)]$. As $\bar{I}_B(\cdot)$ is decreasing,

$$x = \bar{I}_B(\partial_+ \bar{u}_B(x)) \geq \bar{I}_B(y) \geq \bar{I}_B(\partial_- \bar{u}_B(x)).$$

Noticing that $\bar{I}_B(\partial_- \bar{u}_B(x)) = x$ (otherwise $\partial_+ \bar{u}_B(x-) \geq \partial_+ \bar{u}_B(x_*)$ for some $x_* < x$, a contradiction) concludes.

(iv) It follows from the definition of supremum that $\bar{u}_B^*(y) \geq \bar{u}_B(\bar{I}_B(y)) - y\bar{I}_B(y)$ for all $y \in (0, +\infty)$. To show that the reverse inequality also holds, let $x, y \in (0, +\infty)$. Then (Föllmer & Schied, 2004, Proposition A.4),

$$\bar{u}_B(\bar{I}_B(y)) = \bar{u}_B(x) + \int_x^{\bar{I}_B(y)} \partial_+ \bar{u}_B(z)\, dz \geq \bar{u}_B(x) + \int_x^{\bar{I}_B(y)} y\, dz = \bar{u}_B(x) + y(\bar{I}_B(y) - x),$$

where the inequality is due to Equation (A.17). The differentiability, monotonicity, and convexity of $\bar{u}_B^*(\cdot)$ are known facts from convex analysis (see Pham, 2009, Appendix B.2).

(v) By Equation (A.5) and Assumption 2.1(iii), $\Xi_B(y) \leq \mathbb{E}[\zeta I(\frac{y\zeta}{1+\nu'_-(0)})] < +\infty$ for all $y \in (0, +\infty)$. On the other hand, $\Xi_B(y) > 0$, because $\bar{I}_B(\cdot)$ is strictly positive and $0 < \zeta < +\infty$ a.s. The monotonicity of $\Xi_B(\cdot)$ is an immediate consequence of $\bar{I}_B(\cdot)$ being decreasing. Continuity is justified by the dominated convergence theorem, with dominating random variable $\zeta I(\frac{y\zeta/2}{1+\nu'_-(0)})$. Similarly, $\Xi_B(+\infty) := \lim_{y\to+\infty} \Xi_B(y) = 0$. Finally, Fatou's lemma yields $\Xi_B(0+) := \lim_{y\to 0^+} \Xi_B(y) = +\infty$.

(vi) Let $x_0 > 0$. First observe that (v) ensures the existence of some $\hat{y} \equiv \hat{y}(x_0) \in (0, +\infty)$, not necessarily unique, such that $\Xi_B(\hat{y}) = x_0$. Next, define the random variable $\hat{Z}_B := \bar{I}_B(\hat{y}\zeta)$. By construction, $\hat{Z}_B$ is strictly positive a.s. and satisfies the budget constraint $\mathbb{E}[\zeta \hat{Z}_B] = x_0$, thus it belongs to the set $\mathscr{C}(x_0)$ of affordable payoffs. In addition, (iv) implies that, for all $Z \in \mathscr{C}(x_0)$,

$$\begin{aligned}
\mathbb{E}[\bar{u}_B(Z)] &\leq \mathbb{E}\left[\bar{u}_B^*(\hat{y}\zeta)\right] + \hat{y}\mathbb{E}[\zeta Z] \\
&= \mathbb{E}[\bar{u}_B(\bar{I}_B(\hat{y}\zeta)) - \hat{y}\zeta\bar{I}_B(\hat{y}\zeta)] + \hat{y}\mathbb{E}[\zeta Z] \\
&= \mathbb{E}\left[\bar{u}_B\left(\hat{Z}_B\right)\right] - \hat{y}(x_0 - \mathbb{E}[\zeta Z]) \leq \mathbb{E}\left[\bar{u}_B\left(\hat{Z}_B\right)\right].
\end{aligned}$$

Hence, $\hat{Z}_B$ is optimal for Equation (A.11). The (almost sure) uniqueness of the optimizer follows from the strict concavity of $\bar{u}_B(\cdot)$ combined with the convexity of the set $\mathscr{C}(x_0)$. We conclude with the remark that, for all $x \in (0, +\infty)$,

$$F_{\hat{Z}_B}(x) = \mathbb{P}\{\bar{I}_B(\hat{y}\zeta) \leq x\} = \mathbb{P}\{\hat{y}\zeta \geq \partial_+ \bar{u}_B(x)\} = 1 - F_\zeta\left(\frac{\partial_+ \bar{u}_B(x)}{\hat{y}}\right) \quad \text{for all } x \in (0, +\infty). \tag{A.19}$$

Here, we use Equation (A.6) in the second equality, and the nonatomicity of ζ in the last one. In particular, Equation (A.19) entails the uniqueness of the Lagrange multiplier $\hat{y}$.

□

Remark A.2.

(i) For almost every (a.e.) $x \in (0, +\infty)$,

$$\bar{u}'_B(x) = \frac{u'(x)}{1-\eta}\big(1 - \eta(1-\lambda)F_B(x)\big),$$

$$\bar{u}''_B(x) = \frac{u''(x)}{1-\eta}\big(1 - \eta(1-\lambda)F_B(x)\big) - \frac{u'(x)}{1-\eta}\eta(1-\lambda)f_B(x),$$

whence the *Arrow–Pratt coefficient of absolute risk aversion* (Arrow, 1963; Pratt, 1964) of $\bar{u}_B(\cdot)$ at x satisfies

$$\mathrm{ARA}_{\bar{u}_B}(x) := -\frac{\bar{u}''_B(x)}{\bar{u}'_B(x)} = \mathrm{ARA}_u(x) + \frac{\eta(1-\lambda)f_B(x)}{1 - \eta(1-\lambda)F_B(x)} \geq \mathrm{ARA}_u(x).$$

For almost every fixed wealth level, the absolute risk aversion is strictly increasing in η and strictly decreasing in λ.

Another, more direct way of establishing that $\bar{u}_B(\cdot)$ displays more risk aversion than $u(\cdot)$ is to notice that the former utility is "more concave" than the latter in the sense that $\bar{u}_B(\cdot) = \varphi(u(\cdot))$, where

$$\varphi(x) := x + \int_{\mathbb{R}} \nu(x - u(b))\, d\mathbb{P}_B(b)$$

is strictly increasing and concave (by the concavity of the piecewise linear gain–loss $\nu(\cdot)$).

(ii) Gain–loss functions admit the following more general definition: $\nu(\cdot)$ is of the form given by Equation (3), where $\nu_\pm : [0, +\infty) \to \mathbb{R}$ satisfy the conditions below.

(A1) $\nu_\pm(\cdot)$ are continuous and strictly increasing on $[0, +\infty)$, twice-differentiable on $(0, +\infty)$, with $\nu_\pm(0) = 0$.

(A2) (Risk aversion on gains and risk propensity on losses) $\nu''_\pm(x) \leq 0$ for all $x > 0$.

(A3) (Loss aversion) $\nu_+(y) - \nu_+(x) < \nu_-(y) - \nu_-(x)$ for all $x, y \in (0, +\infty)$ with $x < y$, and

$$\lambda := \frac{\nu'_+(0)}{\nu'_-(0)} \in (0, 1),$$

where $\nu'_\pm(0)$ denote the right (+) and left (−) derivatives of ν at 0.

It is easy to check that Lemma A.1 as well as Lemma A.3 below still hold if we replace the piecewise linear gain–loss function with a more general $\nu(\cdot)$ such that

$$\text{for all } y > 0,\ x \mapsto u'(x)\Big(1 + \nu'_-(u(y) - u(x))\Big) \text{ is decreasing on } (0, y). \tag{A.20}$$

This condition ensures that the risk aversion level of the investors' utility $u(\cdot)$ outweighs their risk propensity on losses, so that for every fixed, deterministic reference $b > 0$ the utility

$$\bar{u}_b(x) = u(x) + \nu(u(x) - u(b))$$

remains globally concave—and consequently so does each overall reference-dependent optimization problem $\mathrm{v}(x_0, B) = \sup_{Z \in \mathscr{C}(x_0)} \mathbb{E}[\bar{u}_B(Z)]$. Notice that Equation (A.20) not only covers the particular case of linear $\nu_-(\cdot)$, but also allows for S-shaped gain–loss functions (e.g., it holds for $u(x) := \log(x)$ together with $\nu_-(x) := 1 - e^{-\gamma x}$ for some $\gamma \in (0, 1)$).

Finally, since

$$\begin{aligned}\tilde{u}_B(x) :=\ & \frac{1}{1 + \nu_+(0)} \mathbb{E}\Big[\Big((u(x) - u(B)) + \nu_+(u(x) - u(B))\Big)\mathbb{1}_{\{B \le x\}}\Big] \\ & - \frac{1}{1 + \nu_+(0)} \mathbb{E}\Big[\Big((u(B) - u(x)) + \nu_-(u(B) - u(x))\Big)\mathbb{1}_{\{B > x\}}\Big]\end{aligned}$$

is an affine transformation of $\bar{u}_B(\cdot)$, it immediately follows that solving problem (6) is equivalent to solving problem (8).

We start by investigating the existence of the safe personal equilibrium. The lemma below provides a necessary and sufficient condition for the optimality of the safe payoff $X^f = x_0$ a.s.

Lemma A.3. *Let Assumptions 2.1(i) and 2.1(iii) hold. Then, $X^f \in PE(x_0)$ if and only if*[9]

$$\frac{\operatorname{ess\,sup} \zeta}{\operatorname{ess\,inf} \zeta} \le \frac{1 + \nu'_-(0)}{1 + \nu'_+(0)}. \tag{A.21}$$

Proof. Clearly $\mathbb{E}[u(X^f)] = u(x_0) \in (-\infty, +\infty)$, therefore $\mathrm{v}(x_0, X^f) = U(\hat{Z}^f | X^f)$ by Lemma A.1. Here, $\hat{Z}^f = \bar{I}_{X^f}(\hat{y}\zeta)$ a.s., where $\hat{y} \in (0, +\infty)$ solves $\mathbb{E}[\zeta \hat{Z}^f] = x_0$, and $\bar{I}_{X^f}(\cdot)$ is the generalized inverse of

$$\partial_+ \bar{u}_{X^f}(x) = \begin{cases} u'(x)\big(1 + \nu'_-(u(x_0) - u(x))\big), & \text{if } 0 < x < x_0, \\ u'(x)\big(1 + \nu'_+(u(x) - u(x_0))\big), & \text{if } x \ge x_0. \end{cases}$$

To prove necessity, suppose that $\mathrm{v}(x_0, X^f) = U(X^f | X^f)$. By almost sure uniqueness of the optimizer, $x_0 = X^f = \hat{Z}^f = \bar{I}_{X^f}(\hat{y}\zeta)$ a.s. Using Equation (A.7), this identity is equivalent to

$$\partial_+ \bar{u}_{X^f}(x_0) \le \hat{y}\zeta \le \partial_+ \bar{u}_{X^f}(x_0 -) \text{ a.s.,}$$

which in turn implies both ess inf $\zeta \geq \partial_+\bar{u}_{X^f}(x_0)/\hat{y}$ and ess sup $\zeta \leq \partial_+\bar{u}_{X^f}(x_0-)/\hat{y}$. Hence,

$$\frac{\operatorname{ess\,sup}\zeta}{\operatorname{ess\,inf}\zeta} \leq \frac{\partial_+\bar{u}_{X^f}(x_0-)}{\partial_+\bar{u}_{X^f}(x_0)} = \frac{u'(x_0)\big(1+\nu'_-(0)\big)}{u'(x_0)\big(1+\nu'_+(0)\big)}.$$

Turning to sufficiency, let Equation (A.21) hold. In particular, ess inf $\zeta > 0$ and ess sup $\zeta < +\infty$. Setting $\hat{y} := u'(x_0)(1+\nu'_+(0))/\operatorname{ess\,inf}\zeta \in (0,+\infty)$, it holds almost surely that

$$\partial_+\bar{u}_{X^f}(x_0) = u'(x_0)\big(1+\nu'_+(0)\big) \leq \hat{y}\zeta \leq u'(x_0)\big(1+\nu'_-(0)\big) = \partial_+\bar{u}_{X^f}(x_0-),$$

thus $\mathbb{E}[\zeta \bar{I}_{X^f}(\hat{y}\zeta)] = \mathbb{E}[\zeta x_0] = x_0$. Consequently, $\hat{Z}^f = \bar{I}_{X^f}(\hat{y}\zeta) = x_0$ a.s., which shows that X^f is a personal equilibrium. □

Remark A.4.

(i) Intuitively, in a market where either very good states of the world (with low pricing kernel ζ) or very bad states (corresponding to a high pricing kernel ζ) can occur, good states and bad ones are so markedly different that loss aversion and reference dependence are overridden, whence an investor considering to invest exclusively in the safe asset (i.e., taking such reference) actually has an incentive to reconsider and bet in the market. In other words, the safe payoff cannot be a personal equilibrium unless ζ is bounded both from above and away from zero—that is, the market remains in intermediate “lukewarm” states.

(ii) It is worth comparing the necessary and sufficient condition (A.21) for the existence of the safe personal equilibrium in a complete market with the one obtained in a single-period, generically incomplete model (see condition (7) of Guasoni & Meireles-Rodrigues, 2020). In both settings, neither the initial capital nor the investor's marginal utility play a role in whether the safe personal equilibrium exists; the only relevant parameters are loss aversion, reference dependence, and how markedly different bad market states are from good ones—as measured by the ratio of largest to smallest values of the pricing kernel. In view of the duality relationship between the latter and the market best gain–loss ratio (see Section III of Bernardo & Ledoit, 2000, in a finite sample space), the existence of the safe personal equilibrium implies that the model must have finite best gain–loss ratio.

(iii) With the piecewise linear gain–loss function given by Equation (4), the safe payoff is a personal equilibrium if and only if

$$\frac{\operatorname{ess\,sup}\zeta}{\operatorname{ess\,inf}\zeta} \leq \frac{1}{1-\eta(1-\lambda)}. \tag{A.22}$$

Clearly, the right-hand side of Equation (A.22) is increasing in η (respectively, decreasing in λ), therefore a rising reference dependence (respectively, falling loss aversion) results in an expansion (respectively, a shrinking) of the safe personal equilibrium region.

The next result presents an alternative characterization of nondegenerate personal equilibria in terms of the generalized inverse of the respective distribution function.[10] Condition (i) excludes (nonconstant) discrete distributions from being personal equilibria. The crux of the lemma is identity (A.23), which is a consequence of two conditions having to hold simultaneously: not only does a personal equilibrium need to satisfy a first-order optimality condition: it should also “close the loop"—in that it must be optimal for its own maximization problem. The last condition (iii) gives the link between the personal equilibria and the pricing kernel, while ensuring that we choose the best possible payoff (i.e., the cheapest) among the ones with a fixed distribution. (Bernard et al., 2014, building on the work of Dybvig, 1988b, 1988a, call such payoffs cost-efficient.)

Lemma A.5. *Let $B \in L^0_+$ be nondegenerate with $|\mathbb{E}[u(B)]| < +\infty$, and let Assumptions 2.1(i) and 2.1(iii) hold. Assume further that $F_\zeta(\cdot)$ is continuous and strictly increasing on $[0, +\infty)$. Then, $B \in PE(x_0)$ if and only if all three conditions below hold.*

(i) $F_B(\cdot)$ is strictly increasing on $(0, +\infty)$.
(ii) For all $x \in (0, +\infty)$,

$$q_B\big(F_B(x)\big) = I\left(\hat{y}\frac{q_\zeta\big(1 - F_B(x)\big)}{1 - \eta(1-\lambda)F_B(x)}\right), \tag{A.23}$$

where $\hat{y} \in (0, +\infty)$ solves $\mathbb{E}[\zeta q_B(1 - F_\zeta(\zeta))] = x_0$.
(iii) $B = q_B(1 - F_\zeta(\zeta))$ a.s.

Proof. We prove necessity and sufficiency separately.

(i) Suppose that B is a personal equilibrium. By Lemma A.1, we must have $B = \hat{Z}_B = \bar{I}_B(\check{y}\zeta)$ a.s., where $\check{y} \in (0, +\infty)$ is uniquely determined by $\mathbb{E}[\zeta \bar{I}_B(\check{y}\zeta)] = x_0$. In particular, $\mathbb{E}[\zeta B] = x_0$. Next, note that because the distribution function of ζ is strictly increasing on $(0, +\infty)$, so is $F_{\hat{Z}_B}(\cdot)$ (recall Equation A.12). Since almost sure equality implies equality in distribution, (i) follows.

As a consequence, $q_B(F_B(x)) = x$ holds for all $x \in (0, +\infty)$. Combining this identity with the equality in law of B and $\hat{Z}_B$, it follows that

$$F_B(x) = F_{\hat{Z}_B}(x) = 1 - F_\zeta\left(\frac{u'(x)}{(1-\eta)\check{y}}\big(1 - \eta(1-\lambda)F_B(x)\big)\right) = 1 - F_\zeta\left(\frac{u'\big(q_B\big(F_B(x)\big)\big)}{(1-\eta)\check{y}}\big(1 - \eta(1-\lambda)F_B(x)\big)\right)$$

for all $x \in (0, +\infty)$. Solving for $q_B(F_B(x))$ yields Equation (A.23) with $\hat{y} := (1-\eta)\check{y} \in (0, +\infty)$.

Finally, set $Y := q_B(1 - F_\zeta(\zeta))$. Because Y has the same distribution as B (see Lemma A.19 in Föllmer & Schied, 2004) and $U(\cdot\,|B)$ is law invariant, $\mathrm{v}(x_0, B) = U(B|B) = U(Y|B)$. In addition, Y is feasible for Equation (6), because

$$\mathbb{E}[\zeta Y] = \int_0^1 q_\zeta(s) q_Y(1-s)\,ds = \int_0^1 q_\zeta(s) q_B(1-s)\,ds \leq \mathbb{E}[\zeta B] = x_0. \tag{A.24}$$

Here, the first and penultimate steps are due to the Hardy–Littlewood inequality (Theorem A.24 in Föllmer & Schied, 2004). Hence, the uniqueness of the optimizer implies (iii), and a fortiori the equality in Equation (A.24) follows.

(ii) Conversely, let conditions (i)–(iii) hold. By (i) and (ii), we have for all $x \in (0, +\infty)$ that

$$x = q_B\big(F_B(x)\big) = I\left(\hat{y}\frac{q_\zeta\big(1 - F_B(x)\big)}{1 - \eta(1-\lambda)F_B(x)}\right) = I\left(\check{y}\frac{(1-\eta)q_\zeta\big(1 - F_B(x)\big)}{1 - \eta(1-\lambda)F_B(x)}\right)$$

with $\check{y} := \hat{y}/(1-\eta) \in (0, +\infty)$, or equivalently

$$F_B(x) = 1 - F_\zeta\left(\frac{u'(x)}{(1-\eta)\check{y}}\big(1 - \eta(1-\lambda)F_B(x)\big)\right) = F_{\bar{I}_B(\check{y}\zeta)}(x).$$

In addition,

$$\begin{aligned}\mathbb{E}[\zeta \bar{I}_B(\check{y}\zeta)] &= \int_0^1 q_\zeta(s) q_{\bar{I}_B(\check{y}\zeta)}(1-s)\,ds = \int_0^1 q_\zeta(s) q_B(1-s)\,ds \\ &= \mathbb{E}\Big[q_\zeta\Big(F_\zeta(\zeta)\Big)q_B\Big(1 - F_\zeta(\zeta)\Big)\Big] = \mathbb{E}\Big[\zeta q_B\Big(1 - F_\zeta(\zeta)\Big)\Big] = x_0,\end{aligned}$$

where the first step is again due to the Hardy–Littlewood inequality, the second step is a consequence of the equality in distribution between B and $\bar{I}_B(\check{y}\zeta)$, and the third step uses that $F_\zeta(\zeta)$ has a standard uniform distribution. Therefore, it follows from Lemma A.1 that $\bar{I}_B(\check{y}\zeta)$ is the (a.s.) unique optimizer of Equation (A.11). But then, by law invariance, $\mathrm{v}(x_0, B) = U(\bar{I}_B(\check{y}\zeta)|B) = U(B|B)$. Furthermore, $B \in \mathscr{C}(x_0)$ because (iii) implies $\mathbb{E}[\zeta B] = \mathbb{E}[\zeta q_B(1 - F_\zeta(\zeta))] = x_0$. Hence, B is a personal equilibrium.

□

B | Proofs of Sections 3 and 4

Proof of Theorem 3.2. Let $\lambda, \eta \in (0,1)$. It is easy to check that both $\underline{w}(\cdot)$ and $\overline{w}(\cdot)$ inherit the continuity of $w(\cdot)$, along with $\underline{w}(0) = \overline{w}(0) = w(0) = 0$, $\underline{w}(+\infty) := \lim_{x\to+\infty} \underline{w}(x) = +\infty$, and $\overline{w}(+\infty) := \lim_{x\to+\infty} \overline{w}(x) = +\infty$. Moreover, $\underline{w}(x) < w(x)$ (respectively, $\underline{w}(x) > w(x)$) implies that $\underline{w}(\cdot)$ (respectively, $\overline{w}(\cdot)$) must be constant on some neighborhood of x. Observe also that the compound inequality $x \le w(x) \le x/(1 - \eta(1-\lambda))$ holds for all $x \in [0, +\infty)$, therefore

$$x \le \underline{w}(x) \le \overline{w}(x) \le \frac{x}{1 - \eta(1-\lambda)} \quad \text{for all } x \in [0, +\infty).$$

We carry out the proof in four parts.

(i) We show that a payoff $B \in \mathscr{C}(x_0)$ with $\mathbb{E}[u(B)\mathbb{1}_{\{B>1\}}] < +\infty$ is a personal equilibrium only if $\mathbb{E}[u(B)] > -\infty$. Suppose that $\mathbb{E}[u(B)] = -\infty$, or equivalently $\mathbb{E}[u(B)\mathbb{1}_{\{B\le 1\}}] = -\infty$. On one hand, $U(B|B) \le \mathbb{E}[u(B)] = -\infty$ (see Lemma 1 in De Giorgi & Post, 2011). On the other hand, since $\nu_+(\cdot)$ is non-negative and $\nu_-(x) \le \nu_-'(0)x$ for all $x \ge 0$,

$$U\Big(X^f \Big| B\Big) \ge u(x_0)\big(1 + \nu_-'(0)\big) - \nu_-'(0)\mathbb{E}\big[u(B)\mathbb{1}_{\{B>1\}}\big] > -\infty.$$

Hence, B is not a personal equilibrium, because $U(X^f|B) > U(B|B)$ (i.e., an investor expecting the payoff B leading to infinite grief is actually better off investing in the safe asset).

(ii) Next, we claim that X^f is the only candidate for a degenerate personal equilibrium. Indeed, let $B = b$ a.s. for some $b \geq 0$ be a personal equilibrium. Clearly, $\mathbb{E}[u(B)] = u(b)$ and $\mathbb{E}[\zeta B] = \mathbb{E}[\zeta b] = b$. It follows that either $b > 0$ or $u(0) > -\infty$. Since $|\mathbb{E}[u(B)]| < +\infty$ and the optimal solution to Equation (A.11) is almost surely unique, it follows that $B = \bar{I}_B(\hat{y}\zeta)$ a.s., where $\hat{y} \in (0, +\infty)$ solves $\mathbb{E}[\zeta \bar{I}_B(\hat{y}\zeta)] = x_0$. Hence, $b = \mathbb{E}[\zeta B] = x_0$.

(iii) This step establishes the inclusion $\mathrm{PE}(x_0) \supseteq \{I(y^\star w^\star(\zeta)) : w^\star(\cdot) \in \mathscr{W}\}$. Let $w^\star(\cdot) \in \mathscr{W} \neq \emptyset$. Because

$$0 < I\big(y w^\star(\zeta)\big) \leq I\big(y \underline{w}(\zeta)\big) \leq I(y\zeta) < +\infty \text{ a.s.} \quad \text{for all } y \in (0, +\infty)$$

and Assumption 2.1(iii) holds, the function defined by

$$\Xi^\star(y) := \mathbb{E}\big[\zeta I\big(y w^\star(\zeta)\big)\big], \quad \text{for all } y \in (0, +\infty),$$

is finite everywhere, continuous, strictly decreasing, with $\Xi^\star(0+) := \lim_{y \to 0^+} \Xi^\star(y) = +\infty$ and $\Xi^\star(+\infty) := \lim_{y \to +\infty} \Xi^\star(y) = 0$ (the proof is similar to the one of Lemma A.1(v)). Therefore, the equation $\Xi^\star(y) = x_0$ admits a unique solution in $(0, +\infty)$, denoted by $y^\star$. Moreover, recall that under Assumption 2.1(i) and (iii), the function

$$\tilde{\Xi}(y) := \mathbb{E}[\zeta I(y\zeta)], \quad \text{for all } y \in (0, +\infty),$$

maps $(0, +\infty)$ onto $(0, +\infty)$, and is both continuous and strictly decreasing, so $\tilde{y}$ is the unique solution to $\tilde{\Xi}(\tilde{y}) = x_0$. It is a consequence of $\tilde{\Xi}(y^\star) = \mathbb{E}[\zeta I(y^\star \zeta)] \geq \Xi^\star(y^\star) = x_0 = \tilde{\Xi}(\tilde{y})$ that

$$y^\star \leq \tilde{y}. \tag{A.25}$$

A similar reasoning yields

$$y^\star \geq (1 - \eta(1 - \lambda))\tilde{y}. \tag{A.26}$$

Next, we see that the function $q^\star : (0,1) \to \mathbb{R}$ defined by

$$q^\star(p) := I\Big(y^\star w^\star\Big(q_\zeta(1-p)\Big)\Big), \quad \text{for all } p \in (0,1),$$

is increasing, continuous, with $q^\star(0+) := \lim_{p \to 0^+} q^\star(p) = 0$ and $q^\star(1-) := \lim_{p \to 1^-} q^\star(p) = +\infty$. Consequently, $q^\star(\cdot)$ is the generalized inverse of some strictly increasing distribution function $F^\star(\cdot)$ on $(0, +\infty)$. Furthermore, $\mathbb{E}[\zeta q^\star(1 - F_\zeta(\zeta))] = \Xi^\star(y^\star) = x_0$.

We also claim that $p_0 = F^\star(x_*)$ for some $x_* > 0$ implies that

$$q^\star(p_0) = I\Big(y^\star w\Big(q_\zeta(1-p_0)\Big)\Big) = I\left(y^\star \frac{q_\zeta(1-p_0)}{1 - \eta(1-\lambda)p_0}\right).$$

In fact, suppose otherwise, or equivalently that $w^\star(\check{x}) \neq w(\check{x})$ with $\check{x} := q_\zeta(1-p_0)$. Then there exists $\delta > 0$ such that $w^\star(x) = w^\star(\check{x})$ for all $x \in (\check{x}-\delta, \check{x}+\delta)$. By continuity of $q_\zeta(\cdot)$, we can find some $\rho > 0$ such that $\check{x} - \delta < q_\zeta(1-p) < \check{x}+\delta$ for all $p \in (p_0 - \rho, p_0 + \rho)$, whence $w^\star(q_\zeta(1-p)) = w^\star(\check{x})$. But this in turn leads to $q^\star(\cdot)$ being constant on $(p_0 - \rho, p_0+\rho)$, so p_0 cannot be in the range of $F^\star(\cdot)$.

Finally, using Assumption 2.1(iii) again,

$$\mathbb{E}\left[u\left(I\left(y^\star \frac{\zeta}{1-\eta(1-\lambda)}\right)\right)\right] \leq \mathbb{E}\left[u\left(I\left(y^\star \overline{w}(\zeta)\right)\right)\right] \leq \mathbb{E}\left[u\left(I\left(y^\star w^\star(\zeta)\right)\right)\right] \leq \mathbb{E}[u(\tilde{X})] < +\infty.$$

On the other hand, it follows from (i) that there exists some $C \geq 0$ such that $u(x) \geq -u'(x)(1+x) - C$. for all $x \in (0, +\infty)$, which, combined with Equation (2) yields

$$\mathbb{E}[u(I(y\zeta))] \geq -y(1 + \mathbb{E}[\zeta I(y\zeta)]) - C > -\infty$$

for all $y \in (0, +\infty)$, so $\mathbb{E}[u(I(y^\star \frac{\zeta}{1-\eta(1-\lambda)}))] > -\infty$ as well.

Hence, the random variable

$$X^\star := q^\star\left(1 - F_\zeta(\zeta)\right) = I\left(y^\star w^\star(\zeta)\right) \text{ a.s.}$$

is a personal equilibrium by Lemma A.5.

(iv) We conclude by proving the reverse inclusion $\mathrm{PE}(x_0) \subseteq \{I(y^\star w^\star(\zeta)) : w^\star(\cdot) \in \mathscr{W}\}$, so let $B \in \mathrm{PE}(x_0)$. By Lemma A.5, $B = q_B(1 - F_\zeta(\zeta))$ a.s., where $\hat{y} \in (0, +\infty)$ solves $\mathbb{E}[\zeta q_B(1 - F_\zeta(\zeta))] = x_0$. It suffices to show that the function $w_B^\star : [0, +\infty) \to [0, +\infty)$ defined by

$$w_B^\star(x) := \begin{cases} 0, & \text{if } x = 0, \\ u'\left(q_B\left(1 - F_\zeta(x)\right)\right)/\hat{y}, & \text{if } x > 0, \end{cases}$$

belongs to $\mathscr{W}$. The continuity and monotonicity of $w_B^\star(\cdot)$ are a consequence of the continuity and monotonicity of $F_\zeta(\cdot)$, $q_B(\cdot)$, and $u'(\cdot)$, combined with $\operatorname{ess\,sup} B = +\infty$. On the other hand, $w_B^\star(+\infty) := \lim_{x\to+\infty} w_B^\star(x) = +\infty$ follows from $\operatorname{ess\,inf} B = 0$.

Next, let $x_* \in (0, +\infty)$ such that $w_B^\star(x_*) \neq w(x_*)$. We claim that $w_B^\star(x) = w_B^\star(x_*)$ for all x in some neighborhood of x_*. Since Equation (A.23) fails for $p_0 := 1 - F_\zeta(x_*)$, we conclude that p_0 cannot belong to the range of $F_B(\cdot)$, which in turn implies the existence of some $\varepsilon > 0$ such that $q_B(p) = q_B(p_0)$ for all $p \in (p_0 - \varepsilon, p_0 + \varepsilon)$. Using the continuity of $F_\zeta(\cdot)$ at x_*, we can find some $\delta > 0$ such that

$$\left|\left(1 - F_\zeta(x_*)\right) - \left(1 - F_\zeta(x)\right)\right| = \left|F_\zeta(x) - F_\zeta(x_*)\right| < \varepsilon$$

for all $x \in (x_* - \delta, x_* + \delta)$. Combining these two observations yields that, if $x \in (x_* - \delta, x_* + \delta)$, then $q_B(1 - F_\zeta(x)) = q_B(1 - F_\zeta(x_*))$, whence $w_B^\star(x) = u'(q_B(1 - F_\zeta(x)))/\hat{y} = u'(q_B(1 - F_\zeta(x_*)))/\hat{y} = w_B^\star(x_*)$ as claimed.

It remains only to see that $\underline{w}(x_*) \le w_B^\star(x_*) \le \overline{w}(x_*)$ for all $x_* \in (0,+\infty)$. The result is trivial if $w_B^\star(x_*) = w(x_*)$, so suppose instead that either $w_B^\star(x_*) < w(x_*)$ or $w_B^\star(x_*) > w(x_*)$ hold. Consider the former case (the argument in the other case is identical), and fix an arbitrary $\varepsilon > 0$. By the previous step, $w_B^\star(\cdot)$ is constant on $(x_* - \delta, x_* + \delta)$ for some $\delta > 0$. Since, in addition, $w_B^\star(+\infty) = +\infty$, the set

$$S := \left\{x \in (0,+\infty) : \ w_B^\star(x) = w_B^\star(x_*)\right\}$$

is nonempty and bounded above, therefore $s := \sup S$ exists and satisfies $x_* < s$. It follows from the definition of supremum that there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq S$, strictly increasing to s, which together with the continuity of $w_B^\star(\cdot)$ yields

$$w_B^\star(s) = \lim_{n\to+\infty} w_B^\star(x_n) = \lim_{n\to+\infty} w_B^\star(x_*) = w_B^\star(x_*).$$

On the other hand, $w_B^\star(x) \neq w_B^\star(x_*) = w_B^\star(s)$ for all $x > s$, therefore $w_B^\star(\cdot)$ is not constant on a neighborhood of s, and so $w_B^\star(s) = w(s)$ (recall again the preceding step). Finally, because $w(\cdot)$ is continuous at s, there is some $\rho > 0$ such that $x \in (s-\rho, s+\rho)$ implies $|w(x) - w(s)| < \varepsilon$; in particular,

$$w\left(s + \frac{\rho}{2}\right) < w(s) + \varepsilon = w_B^\star(s) + \varepsilon = w_B^\star(x_*) + \varepsilon.$$

Consequently, $w_B^\star(x_*) \ge \underline{w}(x_*)$; the other inequality $w_B^\star(x_*) < w(x_*) \le \overline{w}(x_*)$ is straightforward.

□

Proof of Proposition 3.3. Since the pricing kernel admits a continuous density by Assumption 2.1(ii), the twist $w(\cdot)$ is continuously differentiable on $(0,+\infty)$ with

$$w'(x) = \frac{H(x)}{\Big(1 - \eta(1-\lambda)\big(1 - F_\zeta(x)\big)\Big)^2} \quad \text{for all } x \in (0,+\infty).$$

(i) Let $X \in \mathrm{PE}(x_0)$. Then $X = I(y^\star w^\star(\zeta))$ for some $w^\star(\cdot) \in \mathscr{W}$, with $y^\star \in (0,+\infty)$ uniquely determined by $\mathbb{E}[y^\star X] = x_0$, so it is immediate that the mapping $\zeta \mapsto X$ inherits its continuity and monotonicity properties from those of $I(\cdot)$ and $w^\star(\cdot)$. Moreover, the limits in Equation (13) follow from the Inada conditions (1).

(ii) Suppose that $w(\cdot)$ is strictly increasing, whence $\mathrm{PE}(x_0) = \{X^*\}$ by Theorem 3.2. Recalling parts (i)–(iii) of Assumption 2.1 gives

$$F_{X^*}(x) = \mathbb{P}\{I(y^* w(\zeta)) \le x\} = \mathbb{P}\left\{\zeta \ge w^{-1}\left(\frac{u'(x)}{y^*}\right)\right\} = 1 - F_\zeta\left(w^{-1}\left(\frac{u'(x)}{y^*}\right)\right)$$

for all $x \in (0,+\infty)$, therefore X^* has a density. Next, let $\eta_1, \eta_2 \in (0,1)$ such that $\eta_1 < \eta_2$, and for $i \in \{1,2\}$ denote the unique personal equilibrium associated with η_i by where $y_i^* \in$

$(0,+\infty)$ is the unique solution of $\mathbb{E}[\zeta X_i^*] = x_0$. Suppose, by contradiction, that $y_1^* \geq y_2^*$. Then,

$$X_2^* \geq I\left(y_1^* \frac{\zeta}{1-\eta_2(1-\lambda)\Big(1-F_\zeta(\zeta)\Big)} \right) > X_1^*,$$

whence $x_0 = \mathbb{E}[\zeta X_2^*] > \mathbb{E}[\zeta X_1^*] = x_0$. An analogous argument yields the monotonicity of y^* with respect to λ and x_0.

(iii) Let $\eta \in (0,1)$. The continuity of $f_\zeta(\cdot)$ on $(0,+\infty)$ together with $\lim_{x\to 0^+} f_\zeta(x)x = 0$ and $\lim_{x\to+\infty} f_\zeta(x)x = 0$ ensures the existence of some $L > 0$ such that $f_\zeta(x)x \leq L$ for all $x \in (0,+\infty)$. As a consequence,

$$H(x) \geq 1-\eta(1-\lambda)(1+L) > 0 \quad \text{for all } x \in (0,+\infty),$$

provided that λ is close enough to 1 (specifically, $1-\lambda < 1/[\eta(1+L)]$). Next, consider a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ converging to 1, and set

$$X_n^* := I\left(y_n^* \frac{\zeta}{1-\eta(1-\lambda_n)\Big(1-F_\zeta(\zeta)\Big)} \right) \quad \text{for all } n \in \mathbb{N},$$

where each $y_n^* \in (0,+\infty)$ is the unique solution of $\mathbb{E}[\zeta X_n^*] = x_0$. We know from Equations (A.25) and (A.26) that, for all $n \in \mathbb{N}$,

$$(1-\eta(1-\lambda_n))\tilde{y} < y_n^* < \tilde{y},$$

hence $\lim_{n\to+\infty} y_n^* = \tilde{y}$ and $\lim_{n\to+\infty} X_n^* = I(\tilde{y}\zeta)$ a.s. A similar reasoning shows that, for fixed $\lambda \in (0,1)$, taking $\eta < 1/[(1-\lambda)(1+L)]$ leads to $w(\cdot)$ strictly increasing; furthermore, $\lim_{\eta\to 0} X^*(\lambda,\eta) = \tilde{X}$.

□

Proof of Proposition 3.5. Since $\operatorname{int}(N) \neq \emptyset$ is open, it can be written uniquely as a countable union of disjoint open intervals. Moreover, $H(0+) := \lim_{x\to 0^+} H(x) = 1-\eta(1-\lambda) > 0$ and $H(+\infty) := \lim_{x\to+\infty} H(x) = 1 > 0$, thus $N \subseteq [\delta, 1/\delta]$ for some $\delta > 0$.

First, observe that, for all $y_0 \in [0,+\infty)$, the set

$$E_{y_0} := \left\{ x \in [0,+\infty) \;:\; w^\star(x) = y_0 \right\}$$

is nonempty and bounded (recall that $w^\star(\cdot)$ is continuous, with $w^\star(0) = 0$ and $w^\star(+\infty) = +\infty$). In particular, $\inf E_{y_0}$ and $\sup E_{y_0}$ both exist and belong to E_{y_0} (again, use the continuity of $w^\star(\cdot)$). Furthermore, $w^\star(\inf E_{y_0}) = w(\inf E_{y_0})$, as $w^\star(x) \neq y_0 = w^\star(\inf E_{y_0})$ for all $x < \inf E_{y_0}$; likewise, $w^\star(\sup E_{y_0}) = w(\sup E_{y_0})$. Lastly, it follows from the increasing property of $w^\star(\cdot)$ that $y_0 =$

$w^\star(a) \le w^\star(z) \le w^\star(b) = y_0$ for all $a, b \in E_{y_0}$ and $a < z < b$, thus E_{y_0} is an interval (possibly a singleton).

Next, we fix $i \in I$ and see that $w^\star(\cdot)$ must be constant on $[a_i, b_i]$. Indeed, if this was not the case, then

$$w\big(\sup E_{w^\star(a_i)}\big) = w^\star\big(\sup E_{w^\star(a_i)}\big) = w^\star(a_i) < w^\star(b_i) = w^\star\big(\inf E_{w^\star(b_i)}\big) = w\big(\inf E_{w^\star(b_i)}\big),$$

contradicting the fact that $w(\cdot)$ is decreasing on (a_i, b_i) (note that $a_i \le \sup E_{w^\star(a_i)} < \inf E_{w^\star(b_i)} \le b_i$).

Third, any E_{y_0} with nonempty interior must contain at least one interval (a_i, b_i). To prove this, suppose that $(a_i, b_i) \nsubseteq E_{y_0}$ for all $i \in I$. As $w^\star(\cdot)$ is constant on every $[a_i, b_i]$ it follows that

$$\operatorname{int}(N) \cap E_{y_0} = \left(\bigcup_{i\in I}(a_i, b_i)\right) \cap E_{y_0} = \bigcup_{i\in I}\big((a_i, b_i) \cap E_{y_0}\big) = \varnothing.$$

Hence, $w(\cdot)$ must be strictly increasing on E_{y_0}, which together with

$$w\big(\inf E_{y_0}\big) = w^\star\big(\inf E_{y_0}\big) = y_0 = w^\star\big(\sup E_{y_0}\big) = w\big(\sup E_{y_0}\big)$$

implies $\inf E_{y_0} = \sup E_{y_0}$.

Lastly, suppose that $|I| = n$ for some $n \in \mathbb{N}$, and denote by $\Delta_{X^\star}$ the set of atoms of the arbitrary (but fixed) personal equilibrium $X^\star := I(y^\star w^\star(\zeta))$, where $w^\star(\cdot) \in \mathscr{W}$ and $y^\star > 0$ is the unique solution of $\mathbb{E}[\zeta X^\star] = x_0$. To show that $\Delta_{X^\star}$ has at most n elements, for each $x_* \in \Delta_{X^\star}$ let $y_0 := u'(x_*)/y^\star$ and define

$$\mathscr{I}_{x_*} := \big\{i \in I \,:\, (a_i, b_i) \subseteq E_{y_0}\big\}.$$

Combining

$$\mathbb{P}\big\{\zeta \in E_{y_0}\big\} = \mathbb{P}\big\{w^\star(\zeta) = y_0\big\} = \mathbb{P}\big\{X^\star = x_*\big\} > 0$$

with Assumption 2.1(ii) and the previous result ensures that $\mathscr{I}_{x_*} \neq \varnothing$. Furthermore, it is easy to check that $\mathscr{I}_{x_*} \cap \mathscr{I}_{x_1} = \varnothing$ for all $x_*, x_1 \in \Delta_{X^\star}$ such that $x_* \neq x_1$. Hence,

$$|\Delta_{X^\star}| \le \sum_{x_*\in\Delta_{X^\star}} \left|\mathscr{I}_{x_*}\right| = \left|\bigcup_{x_*\in\Delta_{X^\star}} \mathscr{I}_{x_*}\right| \le n,$$

as claimed. □

Proof of Corollary 3.6. Let $\lambda, \eta \in (0, 1)$. Because $f_\zeta(\cdot)$ is decreasing, $F_\zeta(x) \ge f_\zeta(x)x$ for all $x \in (0, +\infty)$, thus $\bar{E} \le 1$. Recalling Proposition 3.5 (and the subsequent discussion), we conclude that $w(\cdot)$ is strictly increasing.

Consider now the function $\bar{\Xi} : (0, +\infty) \to [0, +\infty]$ defined by

$$\bar{\Xi}(y) := \mathbb{E}\left[\zeta I\left(\frac{y\zeta}{F_{\zeta}(\zeta)}\right)\right], \quad \text{for all } y \in (0, +\infty).$$

A similar argument to that of Lemma A.1(v) shows that $\bar{\Xi}(\cdot)$ is finite everywhere, continuous and strictly decreasing, with $\bar{\Xi}(0+) := \lim_{y\to 0^+} \bar{\Xi}(y) = +\infty$ and $\bar{\Xi}(+\infty) := \lim_{y\to+\infty} \bar{\Xi}(y) = 0$, thus $\bar{\Xi}(y) = x_0$ admits a unique solution $\bar{y}$ in $(0, +\infty)$. Furthermore, $\bar{\Xi}(\tilde{y}) < \mathbb{E}[\zeta\tilde{X}] = x_0$ implies $\bar{y} < \tilde{y}$.

Next, let $\{(\lambda_n, \eta_n)\}_{n\in\mathbb{N}} \subseteq (0,1) \times (0,1)$ be a sequence converging to $(0, 1)$, and set

$$X_n^* := I\left(y_n^* \frac{\zeta}{1 - \eta_n(1-\lambda_n)\left(1 - F_{\zeta}(\zeta)\right)}\right) \quad \text{for all } n \in \mathbb{N},$$

with each $y_n^* \in (0, +\infty)$ uniquely determined by $\mathbb{E}[\zeta X_n^*] = x_0$. Equations (A.25) and (A.26) give

$$0 < (1 - \eta_n(1 - \lambda_n))\tilde{y} < y_n^* < \tilde{y} \quad \text{for all } n \in \mathbb{N},$$

so in particular $\{y_n^*\}_{n\in\mathbb{N}}$ is a bounded sequence of real numbers. Letting $\{y_{n_k}^*\}_{k\in\mathbb{N}}$ be an arbitrary convergent subsequence, we must have $l := \lim_{k\to+\infty} y_{n_k}^* > 0$, otherwise it would follow from Fatou's lemma that

$$x_0 = \liminf_{k\to+\infty} \mathbb{E}\left[\zeta X_{n_k}^*\right] \geq +\infty.$$

Since $y_{n_k}^* > l/2 > 0$ for all k sufficiently large, we can apply the dominated convergence theorem with dominating random variable $\zeta I(l\zeta/2)$ to obtain

$$\mathbb{E}\left[\zeta I\left(\frac{l\zeta}{F_{\zeta}(\zeta)}\right)\right] = \lim_{k\to+\infty} \mathbb{E}\left[\zeta X_{n_k}^*\right] = x_0,$$

which in turn leads to $\lim_{k\to+\infty} y_{n_k}^* = \bar{y}$.

We conclude the proof by noticing that the mapping $x \mapsto x/F_{\zeta}(x)$ is increasing and

$$\lim_{x\to 0^+} \frac{x}{F_{\zeta}(x)} = \lim_{x\to 0^+} \frac{1}{\frac{F_{\zeta}(x)-F_{\zeta}(0)}{x-0}} = \frac{1}{f_{\zeta}(0)} \in [0, +\infty),$$

where $f_{\zeta}(0) := \lim_{x\to 0^+} f_{\zeta}(x)$ (the existence of the limit, possibly infinite, follows from the monotonicity of the density). Hence, the mapping $\zeta \mapsto \bar{X}$ is decreasing and continuous, with $\lim_{\zeta\to+\infty} \bar{X} = 0$ and

$$\lim_{\zeta\to 0} \bar{X} = \begin{cases} I\left(\bar{y}/f_{\zeta}(0)\right), & \text{if } f_{\zeta}(0) < +\infty, \\ +\infty, & \text{if } f_{\zeta}(0) = +\infty. \end{cases}$$

Additionally, $\bar{X} < \tilde{X}$ a.s. (respectively, $\bar{X} > \tilde{X}$ a.s.) on the event $\{\zeta < q_\zeta(\bar{y}/\tilde{y})\}$ (respectively, $\{\zeta > q_\zeta(\bar{y}/\tilde{y})\}$). □

Proof of Corollary 3.7. Let $\lambda, \eta \in (0,1)$. For all $0 < x_1 < x_2 \leq \theta$,

$$\begin{aligned} H(x_2) - H(x_1) &= \eta(1-\lambda)\Big(F_\zeta(x_2) - F_\zeta(x_1) + f_\zeta(x_1)x_1 - f_\zeta(x_2)x_2\Big) \\ &< \eta(1-\lambda)\Big(f_\zeta(x_1)(x_2 - x_1) + f_\zeta(x_1)x_1 - f_\zeta(x_2)x_2\Big) \\ &= \eta(1-\lambda)\Big(f_\zeta(x_1) - f_\zeta(x_2)\Big)x_2 < 0, \end{aligned}$$

so $H(\cdot)$ is strictly decreasing on $(0,\theta)$; arguing in the same way, we see that $H(\cdot)$ is strictly increasing on $(\theta,+\infty)$. Furthermore, $H(\cdot)$ has the absolute minimum

$$H(\theta) = 1 - \eta(1-\lambda)\Big(1 - F_\zeta(\theta) + f_\zeta(\theta)\theta\Big).$$

Clearly, $w(\cdot)$ is increasing if and only if $H(\theta) \geq 0$, or equivalently Equation (21) is binding; in this case, $w(\cdot)$ is strictly increasing, so there exists the unique personal equilibrium in Equation (14).

Suppose instead that Equation (21) fails. By continuity and monotonicity, $H(\cdot)$ has exactly two zeros ($\bar{x}_1 > \theta$ and $\bar{x}_2 < \theta$), is strictly positive on $(0,\bar{x}_2) \cup (\bar{x}_1,+\infty)$, and strictly negative on $(\bar{x}_2,\bar{x}_1)$, thus $\mathrm{int}(N) = (\bar{x}_2,\bar{x}_1)$. Furthermore, the equation $w(x) = w(\bar{x}_2)$ admits a unique solution x_2 in $(0,+\infty)$, which satisfies $x_2 > \bar{x}_1$. Likewise, there exists a single $x_1 \in (0,+\infty)$ such that $w(x_1) = w(\bar{x}_1)$, and it is straightforward that $x_1 < \bar{x}_2$. Finally, for every $\alpha \in [x_1,\bar{x}_2]$, we can find exactly one $\bar{\alpha} \in [\bar{x}_1, x_2]$ with $w(\alpha) = w(\bar{\alpha})$. We divide the remainder of the proof into three parts.

(i) We show that, for any $\alpha \in [\bar{x}_1, x_2]$, the function

$$w^\star_\alpha(x) := w(\alpha)\mathbb{1}_{[\alpha,\bar{\alpha}]}(x) + w(x)\mathbb{1}_{[0,\alpha)\cup(\bar{\alpha},\infty)}(x), \quad \text{for all } x \in [0,+\infty), \tag{A.27}$$

is a monotonic envelope of $w(\cdot)$. Continuity and monotonicity are easy to check, so are $w^\star_\alpha(0) = 0$ and $w^\star_\alpha(+\infty) := \lim_{x\to+\infty} w^\star_\alpha(x) = +\infty$. Note also that, by construction, $w^\star_\alpha(x_*) \neq w(x_*)$ implies $x_* \in (\alpha,\bar{\alpha})$, thus there exists some $\delta > 0$ such that $(x_* - \delta, x_* + \delta) \subseteq (\alpha,\bar{\alpha})$, which in turn leads to $w^\star_\alpha(x) = w(\alpha)$ for all $x \in (x_* - \delta, x_* + \delta)$ —in other words, $w^\star_\alpha(\cdot)$ is constant around x_*. Finally, it is immediate that $\underline{w}(x_*) \leq w^\star_\alpha(x_*) \leq \overline{w}(x_*)$ if $x_* \notin (\alpha,\bar{\alpha})$, so consider otherwise. Then,

$$\underline{w}(x_*) \leq \underline{w}(\bar{\alpha}) \leq w(\bar{\alpha}) = w^\star_\alpha(x_*) = w(\alpha) \leq \overline{w}(\alpha) \leq \overline{w}(x_*).$$

(ii) Conversely, any element $w^\star(\cdot)$ of $\mathscr{W}$ must be of the form (A.27) for some $\alpha \in [x_1,\bar{x}_2]$.

We know from the proof of Proposition 3.3(ii) that $w^\star(\cdot)$ must be constant on $[\bar{x}_2,\bar{x}_1]$. Take $y_0 := w^\star(\bar{x}_2)$, and $\alpha := \inf E_{y_0} \leq \bar{x}_2$. Because

$$w(\alpha) = w^\star(\alpha) = y_0 = w^\star\big(\sup E_{y_0}\big) = w\big(\sup E_{y_0}\big)$$

and $\sup E_{y_0} \geq \bar{x}_1$, we must have $x_1 \leq \alpha \leq \bar{x}_2$ as well as $\bar{x}_1 \leq \sup E_{y_0} \leq x_2$; in particular, $\sup E_{y_0} = \bar{\alpha}$. Consequently, $w^\star(x) = y_0$ for all $x \in [\alpha,\bar{\alpha}] = E_{y_0}$.

We conclude by showing that $w^\star(\cdot) = w(\cdot)$ outside of $[\alpha, \bar{\alpha}]$. In fact, if $w^\star(x_*) \neq w(x_*)$ for some $x_* \in (\bar{\alpha}, +\infty)$ (a similar reasoning holds if $x_* \in [0, \alpha)$), then $E_{w^\star(x_*)} \supseteq (x_* - \delta, x_* + \delta)$ for some $\delta > 0$. Hence, $\sup E_{w^\star(x_*)} > \inf E_{w^\star(x_*)} > \bar{\alpha}$ (note that $E_{w^\star(\alpha)} \cap E_{w^\star(x_*)} = \emptyset$) and

$$w\big(\inf E_{w^\star(x_*)}\big) = w^\star\big(\inf E_{w^\star(x_*)}\big) = w^\star(x_*) = w^\star\big(\sup E_{w^\star(x_*)}\big) = w\big(\sup E_{w^\star(x_*)}\big),$$

which is absurd (recall that $w(\cdot)$ is strictly increasing on $[\bar{\alpha}, +\infty)$).

(iii) Let $\alpha_1, \alpha_2 \in [x_1, \bar{x}_2]$ such that $\alpha_1 < \alpha_2$. Suppose, by contradiction, that $y^\star_{\alpha_1} \leq y^\star_{\alpha_2}$. Straightforward computations give $X^\star_{\alpha_2} \leq X^\star_{\alpha_1}$ a.s. with $\mathbb{P}\{X^\star_{\alpha_2} < X^\star_{\alpha_1}\} \geq \mathbb{P}\{\zeta \in [\alpha_2, \bar{\alpha}_1]\} > 0$, therefore

$$x_0 = \mathbb{E}\big[\zeta X^\star_{\alpha_2}\big] < \mathbb{E}\big[\zeta X^\star_{\alpha_1}\big] = x_0,$$

which is absurd. Hence, $y^\star_\alpha$ is strictly decreasing in α.

We also claim that the mapping $\alpha \mapsto y^\star_\alpha w(\alpha)$ is strictly increasing. Indeed, if $y^\star_{\alpha_1} w(\alpha_1) \geq y^\star_{\alpha_2} w(\alpha_2)$ for some $\alpha_1, \alpha_2 \in [x_1, \bar{x}_2]$ such that $\alpha_1 < \alpha_2$, then $X^\star_{\alpha_1} \leq X^\star_{\alpha_2}$ a.s. with $\mathbb{P}\{X^\star_{\alpha_1} < X^\star_{\alpha_2}\} \geq \mathbb{P}\{\zeta \in (0, \alpha_2) \cup (\bar{\alpha}_1, +\infty)\} > 0$, leading to the contradiction

$$x_0 = \mathbb{E}\big[\zeta X^\star_{\alpha_1}\big] < \mathbb{E}\big[\zeta X^\star_{\alpha_2}\big] = x_0.$$

Combining the two observations above allows us to conclude that $X^\star_{\alpha_1} > X^\star_{\alpha_2}$ if $\zeta \in [\alpha_2, \bar{\alpha}_1]$, and $X^\star_{\alpha_1} < X^\star_{\alpha_2}$ if $\zeta \in (0, \alpha_1] \cup [\bar{\alpha}_2, +\infty)$. The existence of $\underline{\zeta}$ and $\overline{\zeta}$ is then due to the continuity of the mapping $\zeta \mapsto X^\star_\alpha$ for all α.

□

Proof of Example 4. First, recall that $I(y) = y^{-1}$ for all $y > 0$, and

$$F_\zeta(x) = \Phi\left(\frac{\log(x) + \kappa^2 T/2}{\sqrt{\kappa^2 T}}\right)\mathbb{1}_{(0,+\infty)}(x) \quad \text{for all } x \in \mathbb{R}.$$

Moreover, note that ζ_t is lognormally distributed (with parameters $\kappa^2 t/2$ and $\kappa^2 t$) also under the unique equivalent martingale measure $\mathbb{Q}$. We break the rest of the proof into several parts.

(i) Let $t \in [0, T]$. A confidence interval with confidence level $\gamma \in (0,1)$ for ζ_t is

$$\gamma = \mathbb{P}\left\{-\beta \leq \frac{\kappa W_t}{\sqrt{\kappa^2 t}} \leq \beta\right\} = \mathbb{P}\left\{e^{-\beta\sqrt{\kappa^2 t} - \kappa^2 t/2} \leq \zeta_t \leq e^{\beta\sqrt{\kappa^2 t} - \kappa^2 t/2}\right\},$$

where $\beta := \Phi^{-1}(0.5 + \gamma/2)$.

(ii) Let $\lambda, \eta \in (0,1)$. Straightforward computations yield

$$1 - F_\zeta(\theta) + f_\zeta(\theta)\theta = \Phi\left(\sqrt{\kappa^2 T}\right) + \frac{1}{\sqrt{\kappa^2 T}}\phi\left(\sqrt{\kappa^2 T}\right),$$

thus condition (21) can be rewritten as Equation (25), which in turn is equivalent to the inequality

$$\sqrt{\kappa^2 T}\Big(1-\eta(1-\lambda)\Phi\Big(\sqrt{\kappa^2 T}\Big)\Big)-\eta(1-\lambda)\phi\Big(\sqrt{\kappa^2 T}\Big)\geq 0.$$

Since the function $\Theta : \mathbb{R} \to \mathbb{R}$ defined by

$$\Theta(x) \equiv \Theta(\lambda,\eta;x) := x(1-\eta(1-\lambda)\Phi(x))-\eta(1-\lambda)\phi(x), \quad \text{for all } x \in \mathbb{R},$$

is continuous, with $\Theta(0) = -\eta(1-\lambda)/\sqrt{2\pi} < 0$, $\Theta(+\infty) := \lim_{x\to+\infty}\Theta(x) = +\infty$, and

$$\Theta'(x) = 1-\eta(1-\lambda)\Phi(x) > 0 \quad \text{for all } x \in \mathbb{R},$$

it admits a unique root $\varrho \equiv \varrho(\lambda,\eta) \in (0,+\infty)$. Below we investigate the monotonicity and asymptotic behavior of ϱ with respect to preference parameters.

(i) We claim that $\varrho(\lambda,\eta)$ is strictly decreasing in λ, for any $\eta \in (0,1)$ fixed. Indeed, letting $\lambda_1,\lambda_2 \in (0,1)$ such that $\lambda_1 < \lambda_2$, and setting $\varrho_i := \varrho(\lambda_i,\eta)$ for $i \in \{1,2\}$,

$$\begin{aligned}\Theta(\lambda_2,\eta;\varrho_1) &= \varrho_1(1-\eta(1-\lambda_2)\Phi(\varrho_1))-\eta(1-\lambda_2)\phi(\varrho_1)\\ &> \varrho_1(1-\eta(1-\lambda_1)\Phi(\varrho_1))-\eta(1-\lambda_1)\phi(\varrho_1) = \Theta(\lambda_1,\eta;\varrho_1)=0,\end{aligned}$$

so it follows from the strict monotonicity of $\Theta(\lambda_2,\eta;\cdot)$ that $\varrho_1 > \varrho_2$. An identical argument yields that $\varrho(\lambda,\eta)$ is strictly increasing in η, for any fixed $\lambda \in (0,1)$.

(ii) Let $\eta \in (0,1)$, and consider a strictly increasing sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ converging to 1. By the result proved in the previous step, the sequence $\{\varrho(\lambda_n,\eta)\}_{n\in\mathbb{N}}$ is strictly decreasing and bounded below by 0, so it has a limit $\bar{\varrho} \geq 0$. Then,

$$0 = \lim_{n\to+\infty}\Theta(\lambda_n,\eta;\varrho(\lambda_n,\eta)) = \bar{\varrho}(1-\eta(1-1)\Phi(\bar{\varrho}))-\eta(1-1)\phi(\bar{\varrho}) = \bar{\varrho}.$$

Similarly, we obtain $\lim_{\eta\to0}\varrho(\lambda,\eta) = 0$ for any fixed $\lambda \in (0,1)$.

(iii) This step shows that $\lim_{(\lambda,\eta)\to(0,1)}\varrho = +\infty$. Let $\{(\lambda_n,\eta_n)\}_{n\in\mathbb{N}}$ be a sequence in $(0,1)\times(0,1)$ with $\lim_{n\to+\infty}(\lambda_n,\eta_n) = (0,1)$, and set $\varrho_n := \varrho(\lambda_n,\eta_n)$ for all $n \in \mathbb{N}$. Without loss of generality, assume that the sequence of real numbers $\{\eta_n(1-\lambda_n)\}_{n\in\mathbb{N}}$ strictly increases to 1. Arguing as in part (a) we see that $\{\varrho_n\}_{n\in\mathbb{N}}$ is strictly increasing.

By contradiction, suppose that the sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ is bounded above. Then we can extract a subsequence $\{\varrho_{n_k}\}_{k\in\mathbb{N}}$ converging to some limit $\bar{\varrho} \geq 0$. This leads to

$$0 = \lim_{k\to+\infty}\Theta\big(\lambda_{n_k},\eta_{n_k};\varrho_{n_k}\big) = \bar{\varrho}(1-1(1-0)\Phi(\bar{\varrho}))-1(1-0)\phi(\bar{\varrho}) = \bar{\varrho}(1-\Phi(\bar{\varrho}))-\phi(\bar{\varrho}),$$

which is absurd because, for all $x \in [0,+\infty S)$,

$$x(1-\Phi(x)) = \int_x^{+\infty} x\phi(t)\,dt < \int_x^{+\infty} t\phi(t)\,dt = \phi(x).$$

Hence, being a strictly increasing and unbounded-above sequence, $\{\varrho_n\}_{n\in\mathbb{N}}$ diverges to $+\infty$.

(iii) Let $\lambda, \eta \in (0,1)$ such that Equation (25) holds, and let $t \in [0,T]$.

(a) Applying Corollary 3.7, the terminal value of the unique personal equilibrium is

$$X_T^* = \frac{1-\eta(1-\lambda)\Big(1-F_\zeta(\zeta)\Big)}{y^*\zeta}.$$

Here, $y^* \in (0,+\infty)$ is the unique solution of

$$x_0 = \frac{1}{y^*}\Big(1-\eta(1-\lambda)\mathbb{E}\Big[1-F_\zeta(\zeta)\Big]\Big) = \frac{2-\eta(1-\lambda)}{2y^*}, \tag{A.28}$$

the last equality being a consequence of the standard uniform law of $1-F_\zeta(\zeta)$ under $\mathbb{P}$.

(b) Bayes' formula and straightforward computations yield

$$X_t^* = \mathbb{E}_\mathbb{Q}\Big[X_T^*\Big|\mathcal{F}_t\Big] = \frac{1}{y^*\zeta_t}\mathbb{E}\left[1-\eta(1-\lambda)\Phi\left(-\frac{\log(\zeta)+\kappa^2T/2}{\sqrt{\kappa^2 T}}\right)\middle|\mathcal{F}_t\right] = \frac{1}{y^*\zeta_t}\chi^*(t,\zeta_t),$$

where

$$\chi^*(t,z) := \int_{-\infty}^{+\infty}\left(1-\eta(1-\lambda)\Phi\left(-\frac{\log(z)+\kappa^2t/2}{\sqrt{\kappa^2T}}+\sqrt{\frac{T-t}{T}}x\right)\right)\phi(x)\,dx = 1-\eta(1-\lambda)\Phi(-d_1(z,t,T)).$$

Plugging into the above equation the expression for y^* from Equation (A.28) gives the desired result.

(c) Because the mapping $x \mapsto \sqrt{\kappa^2(2T-t)}(1-\eta(1-\lambda)\Phi(x)) - \eta(1-\lambda)\phi(x)$ attains its global minimum at $\sqrt{\kappa^2(2T-t)} \geq \sqrt{\kappa^2 T} \geq \varrho$,

$$\begin{aligned}\frac{\partial X_t^*}{\partial \zeta_t} &= -\frac{2x_0}{(2-\eta(1-\lambda))\zeta_t^2}\left(1-\eta(1-\lambda)\Phi(-d_1(\zeta_t,t,T)) - \frac{1}{\sqrt{\kappa^2(2T-t)}}\eta(1-\lambda)\phi(-d_1(\zeta_t,t,T))\right)\\ &\leq -\frac{2x_0}{(2-\eta(1-\lambda))\zeta_t^2\sqrt{\kappa^2(2T-t)}}\Theta\Big(\sqrt{\kappa^2(2T-t)}\Big) \leq 0.\end{aligned}$$

In addition, by Proposition 3.3(ii), we have $X_T^* < \tilde{X}_T$ a.s. if and only if

$$\zeta < \tilde{\zeta} = q_\zeta\left(\frac{(2-\eta(1-\lambda))/(2x_0) - (1-\eta(1-\lambda))/x_0}{\eta(1-\lambda)/x_0}\right) = q_\zeta\left(\frac{1}{2}\right).$$

More generally, $X_t^* < \tilde{X}_t$ if and only if $\Phi(-d_1(\zeta_t,t,T)) > 1/2$, or equivalently $\zeta_t < e^{-\kappa^2 t/2} = q_{\zeta_t}(1/2)$.

Furthermore, straightforward calculations yield

$$\begin{aligned}\frac{\partial X_t^*}{\partial \lambda} &= \frac{2x_0\eta}{(2-\eta(1-\lambda))^2\zeta_t}(2\Phi(-d_1(\zeta_t,t,T))-1),\\ \frac{\partial X_t^*}{\partial \eta} &= \frac{2x_0(1-\lambda)}{(2-\eta(1-\lambda))^2\zeta_t}(1-2\Phi(-d_1(\zeta_t,t,T))),\end{aligned}$$

thus $\partial X_t^*/\partial\lambda > 0$ and $\partial X_t^*/\partial\eta < 0$ are each equivalent to $\zeta_t < e^{-\kappa^2 t/2} = q_{\zeta_t}(1/2)$.

Finally,

$$\frac{\partial X_t^*}{\partial T} = \frac{x_0}{\zeta_t}\left(\frac{2\eta(1-\lambda)}{2-\eta(1-\lambda)}\right)\phi(d_1(\zeta_t,t,T))\frac{\partial d_1(\zeta_t,t,T)}{\partial T};$$

in particular, for all $t \in [0,T)$,

$$\frac{\partial d_1(x,t,T)}{\partial T} = -\left(\log(x) + \frac{\kappa^2 t}{2}\right)\left(\kappa^2(2T-t)\right)^{-3/2}\kappa^2,$$

whence $\frac{\partial X_t^*}{\partial T} > 0$ if and only if $\zeta_t < q_{\zeta_t}(1/2)$.

(d) Since $X_t^* = f(t,\zeta_t)$ with

$$f(t,z) := \frac{2x_0}{2-\eta(1-\lambda)} z^{-1}\chi^*(t,z),$$

it follows from Itô's lemma that

$$dX_t^* = \left(\frac{\partial f}{\partial t}(t,\zeta_t) + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}(t,\zeta_t)\kappa^2\zeta_t^2\right)dt - \frac{\partial f}{\partial z}(t,\zeta_t)\kappa\zeta_t\, dW_t.$$

Combining this equation with

$$\frac{\partial f}{\partial z}(t,z) = -\frac{1}{z}f(t,z)\left(1 - \frac{\eta(1-\lambda)\phi(-d_1(z,t,T))}{\chi^*(t,z)\sqrt{\kappa^2(2T-t)}}\right)$$

and $dX_t^* = \pi_t^* X_t^*(\mu\, dt + \sigma\, dW_t)$ yields

$$\sigma\pi_t^* X_t^* = \kappa X_t^*\left(1 - \frac{\eta(1-\lambda)\phi(-d_1(\zeta_t,t,T))}{\sqrt{\kappa^2(2T-t)}(1-\eta(1-\lambda)\Phi(-d_1(\zeta_t,t,T)))}\right).$$

(e) The inequality $|\pi_t^*| < |\mu|/\sigma^2$ is trivial. On the other hand, as seen in step (c),

$$\sqrt{\kappa^2(2T-t)}(1-\eta(1-\lambda)\Phi(-d_1(\zeta_t,t,T))) - \eta(1-\lambda)\phi(-d_1(\zeta_t,t,T)) \geq \Theta\left(\sqrt{\kappa^2(2T-t)}\right) \geq 0,$$

which in turn implies $\operatorname{sgn}(\pi_t^*) = \operatorname{sgn}(\mu)$. [11]

Furthermore,

$$\frac{\partial \pi_t^*}{\partial \zeta_t} = -\frac{\mu\eta(1-\lambda)\phi(-d_1(\zeta_t,t,T))}{\sigma^2\kappa^2(2T-t)\zeta_t(1-\eta(1-\lambda)\Phi(-d_1(\zeta_t,t,T)))^2}\Theta(-d_1(\zeta_t,t,T)) > 0$$

if and only if $-d_1(\zeta_t, t, T) < \varrho$, that is $\zeta_t > \exp\{-\kappa^2 t - \varrho\sqrt{\kappa^2(2T-t)}\} =: \check{\zeta}_t$. In particular, the minimum proportion of wealth is

$$\frac{\mu}{\sigma^2}\left(1 - \frac{\eta(1-\lambda)\phi(\varrho)}{\sqrt{\kappa^2(2T-t)}(1-\eta(1-\lambda)\Phi(\varrho))}\right) = \frac{\mu}{\sigma^2}\left(1 - \frac{\varrho}{\sqrt{\kappa^2(2T-t)}}\right) > 0.$$

Note also that $\lim_{\zeta_t \to 0} \pi_t^* = \mu/\sigma^2$ and $\lim_{\zeta_t \to 0} \pi_t^* = \mu/\sigma^2$.

In addition,

$$\frac{\partial \pi_t^*}{\partial \lambda} = \mu \frac{\eta\phi(-d_1(\zeta_t, t, T))}{\sigma^2\sqrt{\kappa^2(2T-t)}(1-\eta(1-\lambda)\Phi(-d_1(\zeta_t, t, T)))^2},$$
$$\frac{\partial \pi_t^*}{\partial \eta} = -\mu \frac{(1-\lambda)\phi(-d_1(\zeta_t, t, T))}{\sigma^2\sqrt{\kappa^2(2T-t)}(1-\eta(1-\lambda)\Phi(-d_1(\zeta_t, t, T)))^2},$$

therefore $\partial\pi_t^*/\partial\lambda > 0$ and $\partial\pi_t^*/\partial\eta < 0$ are each equivalent to $\mu > 0$. It is straightforward that $\lim_{\eta\to 0} \pi_t^* = \mu/\sigma^2$ and $\lim_{\lambda \to 1} \pi_t^* = \mu/\sigma^2$.

Lastly,

$$\pi_0^* = \frac{\mu}{\sigma^2}\left(1 - \eta(1-\lambda)\frac{1}{\sqrt{\pi\kappa^2 T}(2-\eta(1-\lambda))}\right)$$

is strictly increasing and concave with respect to T, with

$$\lim_{T\to T^*} \pi_0^* = \frac{\mu}{\sigma^2}\left(1 - \eta(1-\lambda)\frac{1}{\varrho\sqrt{\pi}(2-\eta(1-\lambda))}\right).$$

That the above limit has the same sign as μ is due the fact that the mapping $x \mapsto \varrho\sqrt{2}(1 - \eta(1-\lambda)\Phi(x)) - \eta(1-\lambda)\phi(x)$ attains its global minimum at $\varrho\sqrt{2}$, whence

$$\varrho(2-\eta(1-\lambda)) - \eta(1-\lambda) = \frac{1}{\sqrt{2\pi}}\Big(\varrho\sqrt{2}(1-\eta(1-\lambda)\Phi(0)) - \eta(1-\lambda)\phi(0) \geq \Theta\Big(\varrho\sqrt{2}\Big)\Big) > 0.$$

(iv) Now, consider $\lambda, \eta \in (0,1)$ for which Equation (25) fails, and let $t \in [0, T]$.

(a) By Corollary 3.7, the personal equilibrium associated with each $\alpha \in [x_1, \bar{x}_2]$ has terminal value

$$X_{\alpha,T}^\star = \frac{1-\eta(1-\lambda)\Big(1-F_\zeta(\alpha)\Big)}{y_\alpha^\star \alpha}\mathbb{1}_{\{\zeta\in[\alpha,\bar{\alpha}]\}} + \frac{1-\eta(1-\lambda)\Big(1-F_\zeta(\zeta)\Big)}{y_\alpha^\star \zeta}\mathbb{1}_{\{\zeta\notin[\alpha,\bar{\alpha}]\}},$$

where $y_\alpha^\star$ is uniquely determined by

$$x_0 = \frac{1-\eta(1-\lambda)\Big(1-F_\zeta(\alpha)\Big)}{y_\alpha^\star \alpha}\mathbb{E}\big[\zeta\mathbb{1}_{\{\zeta\in[\alpha,\bar{\alpha}]\}}\big] + \frac{1}{y_\alpha^\star}\mathbb{E}\Big[\Big(1-\eta(1-\lambda)\Big(1-F_\zeta(\zeta)\Big)\Big)\mathbb{1}_{\{\zeta\notin[\alpha,\bar{\alpha}]\}}\Big].$$

Recalling that the random variable $1 - F_\zeta(\zeta)$ is uniformly distributed on $(0,1)$ under $\mathbb{P}$ results in

$$\mathbb{E}\left[\left(1-\eta(1-\lambda)\left(1-F_\zeta(\zeta)\right)\right)\mathbb{1}_{\left\{1-F_\zeta(\zeta)\notin\left[1-F_\zeta(\bar\alpha),1-F_\zeta(\alpha)\right]\right\}}\right]$$
$$= 1-\frac{\eta(1-\lambda)}{2}+\left(F_\zeta(\alpha)-F_\zeta(\bar\alpha)\right)\left(1-\eta(1-\lambda)\frac{2-F_\zeta(\alpha)-F_\zeta(\bar\alpha)}{2}\right).$$

(b) We have

$$X^\star_{\alpha,t} = \mathbb{E}_\mathbb{Q}\left[X^\star_{\alpha,T}\middle|\mathcal{F}_t\right] = \frac{1-\eta(1-\lambda)\Phi(-d_1(\alpha,T,T))}{y^\star_\alpha \alpha}\mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{\{\zeta\in[\alpha,\bar\alpha]\}}\middle|\mathcal{F}_t\right]$$
$$+\frac{1}{y^\star_\alpha}\mathbb{E}_\mathbb{Q}\left[\frac{1-\eta(1-\lambda)\Phi(-d_1(\zeta,T,T))}{\zeta}\mathbb{1}_{\{\zeta\notin[\alpha,\bar\alpha]\}}\middle|\mathcal{F}_t\right].$$

It is immediate that $\mathbb{E}_\mathbb{Q}[\mathbb{1}_{\{\zeta\in[\alpha,\bar\alpha]\}}|\mathcal{F}_t] = \chi^\star_{\alpha,1}(t,\zeta_t)$, where

$$\chi^\star_{\alpha,1}(t,z) := \mathbb{Q}\left\{\alpha \le z\exp\left\{-\kappa(\tilde W_T-\tilde W_t)+\frac{1}{2}\kappa^2(T-t)\right\}\le\bar\alpha\right\}$$
$$=\Phi\left(\frac{\log(z/\alpha)+\kappa^2(T-t)/2}{\sqrt{\kappa^2(T-t)}}\right)-\Phi\left(\frac{\log(z/\bar\alpha)+\kappa^2(T-t)/2}{\sqrt{\kappa^2(T-t)}}\right).$$

In particular,

$$\mathbb{E}_\mathbb{Q}\left[\mathbb{1}_{\{\zeta\in[\alpha,\bar\alpha]\}}\right]=\Phi\left(-\frac{\log(\alpha)-\kappa^2T/2}{\sqrt{\kappa^2T}}\right)-\Phi\left(-\frac{\log(\bar\alpha)-\kappa^2T/2}{\sqrt{\kappa^2T}}\right)=\Phi(d_2(\bar\alpha,0,T))-\Phi(d_2(\alpha,0,T)).$$

On the other hand, another application of Bayes' formula leads to

$$\mathbb{E}_\mathbb{Q}\left[\frac{1-\eta(1-\lambda)\Phi(-d_1(\zeta,T,T))}{\zeta}\mathbb{1}_{\{\zeta\notin[\alpha,\bar\alpha]\}}\middle|\mathcal{F}_t\right]$$
$$=\frac{1}{\zeta_t}\mathbb{E}\left[(1-\eta(1-\lambda)\Phi(-d_1(\zeta,T,T)))\mathbb{1}_{\{\zeta\notin[\alpha,\bar\alpha]\}}\middle|\mathcal{F}_t\right]=\frac{1}{\zeta_t}\chi^\star_{\alpha,2}(t,\zeta_t),$$

with

$$\chi^\star_{\alpha,2}(t,z) := \int_{-\infty}^{\frac{\log(z/\bar\alpha)-\kappa^2(T-t)/2}{\sqrt{\kappa^2(T-t)}}}\left(1-\eta(1-\lambda)\Phi\left(-\frac{\log(z)+\kappa^2t/2}{\sqrt{\kappa^2T}}+\sqrt{\frac{T-t}{T}}x\right)\right)\phi(x)\,dx$$
$$+\int_{\frac{\log(z/\alpha)-\kappa^2(T-t)/2}{\sqrt{\kappa^2(T-t)}}}^{+\infty}\left(1-\eta(1-\lambda)\Phi\left(-\frac{\log(z)+\kappa^2t/2}{\sqrt{\kappa^2T}}+\sqrt{\frac{T-t}{T}}x\right)\right)\phi(x)\,dx$$
$$=\Phi\left(\frac{\log(z/\bar\alpha)-\kappa^2(T-t)/2}{\sqrt{\kappa^2(T-t)}}\right)-\eta(1-\lambda)\mathrm{M}\left(\frac{\log(z/\bar\alpha)-\kappa^2(T-t)/2}{\sqrt{\kappa^2(T-t)}},-d_1(z,t,T);-\sqrt{\frac{T-t}{2T-t}}\right)$$

$$+1-\Phi\left(\frac{\log(z/\alpha)-\kappa^2(T-t)/2}{\sqrt{\kappa^2(T-t)}}\right)-\eta(1-\lambda)\mathrm{M}\left(-\frac{\log(z/\alpha)-\kappa^2(T-t)/2}{\sqrt{\kappa^2(T-t)}},-d_1(z,t,T);\sqrt{\frac{T-t}{2T-t}}\right).$$

(c) Let

$$g_\alpha(t,z) := \frac{x_0}{\ell_\alpha}\left(\frac{1-\eta(1-\lambda)\Phi(-d_1(\alpha,T,T))}{\alpha}\chi^\star_{\alpha,1}(t,z)+\frac{1}{z}\chi^\star_{\alpha,2}(t,z)\right),$$

and set

$$\rho := -\sqrt{\frac{T-t}{2T-t}}.$$

Observe that

$$\frac{\partial g_\alpha}{\partial z}(t,z)=\frac{x_0}{\ell_\alpha}\left(\frac{1-\eta(1-\lambda)\Phi(-d_1(\alpha,T,T))}{\alpha}\frac{\partial\chi^\star_{\alpha,1}}{\partial z}(t,z)-\frac{1}{z^2}\chi^\star_{\alpha,2}(t,z)+\frac{1}{z}\frac{\partial\chi^\star_{\alpha,2}}{\partial z}(t,z)\right),$$

with

$$\frac{\partial\chi^\star_{\alpha,1}}{\partial z}(t,z)=\frac{1}{z\sqrt{\kappa^2(T-t)}}\left(\phi\left(d_2\left(\frac{\alpha}{z},t,T\right)\right)-\phi\left(d_2\left(\frac{\bar\alpha}{z},t,T\right)\right)\right),$$

and

$$\begin{aligned}\frac{\partial\chi^\star_{\alpha,2}}{\partial z}(t,z)=&\frac{1}{z\sqrt{\kappa^2(T-t)}}\phi\left(d_2\left(\frac{z}{\bar\alpha},t,T\right)\right)\left(1-\eta(1-\lambda)\Phi\left(\frac{-d_1(z,t,T)-\rho d_2(z/\bar\alpha,t,T)}{\sqrt{1-\rho^2}}\right)\right)\\&-\frac{1}{z\sqrt{\kappa^2(T-t)}}\phi\left(d_2\left(\frac{z}{\alpha},t,T\right)\right)\left(1-\eta(1-\lambda)\Phi\left(\frac{-d_1(z,t,T)+\rho d_2(z/\alpha,t,T)}{\sqrt{1-\rho^2}}\right)\right)\\&+\eta(1-\lambda)\frac{1}{z\sqrt{\kappa^2(2T-t)}}\phi(-d_1(z,t,T))\Phi\left(\frac{d_2(z/\bar\alpha,t,T)+\rho d_1(z,t,T)}{\sqrt{1-\rho^2}}\right)\\&+\eta(1-\lambda)\frac{1}{z\sqrt{\kappa^2(2T-t)}}\phi(-d_1(z,t,T))\Phi\left(\frac{-d_2(z/\alpha,t,T)+\rho d_1(z,t,T)}{\sqrt{1-\rho^2}}\right).\end{aligned}$$

Applying Itô's lemma to $X^\star_{\alpha,t}=g_\alpha(t,\zeta_t)$, and recalling that $dX^\star_{\alpha,t}=\pi^\star_{\alpha,t}X^\star_{\alpha,t}(\mu\,dt+\sigma\,dW_t)$ leads to

$$\sigma\pi^\star_{\alpha,t}X^\star_{\alpha,t}=-\frac{\partial g_\alpha}{\partial z}(t,\zeta_t)\kappa\zeta_t,$$

or, equivalently,

$$\pi^\star_{\alpha,t} = \frac{\kappa x_0}{\sigma \ell_\alpha X^\star_{\alpha,t}} \left(\frac{1}{\zeta_t} \chi^\star_{\alpha,2}(t,\zeta_t) - \frac{\partial \chi^\star_{\alpha,2}}{\partial z}(t,\zeta_t) - \frac{1-\eta(1-\lambda)\Phi(-d_1(\alpha,T,T))}{\alpha\sqrt{\kappa^2(T-t)}} \left(\phi\left(d_2\left(\frac{\alpha}{\zeta_t}, t, T\right)\right) - \phi\left(d_2\left(\frac{\bar{\alpha}}{\zeta_t}, t, T\right)\right)\right)\right).$$

Taking $t = 0$ yields

$$\begin{aligned}
\pi^\star_{\alpha,0} = \frac{\kappa}{\sigma\ell_\alpha} \Bigg[& \Phi(-d_1(\bar{\alpha},T,T)) - \eta(1-\lambda)\frac{\Phi(-d_1(\bar{\alpha},T,T))^2}{2} \\
& + \Phi(d_1(\alpha,T,T)) - \eta(1-\lambda)\frac{1-\Phi(d_1(\alpha,T,T))^2}{2} \\
& - \frac{1}{\sqrt{\kappa^2 T}}\phi(d_1(\bar{\alpha},T,T))(1-\eta(1-\lambda)\Phi(-d_1(\bar{\alpha},T,T))) \\
& + \frac{1}{\sqrt{\kappa^2 T}}\phi(d_1(\alpha,T,T))(1-\eta(1-\lambda)\Phi(d_1(\alpha,T,T))) \\
& - \eta(1-\lambda)\frac{1}{2\sqrt{\pi\kappa^2 T}}\Phi\left(-\sqrt{2}d_1(\bar{\alpha},T,T)\right) - \eta(1-\lambda)\frac{1}{2\sqrt{\pi\kappa^2 T}}\Phi\left(\sqrt{2}d_1(\alpha,T,T)\right) \\
& - \frac{1-\eta(1-\lambda)\Phi(-d_1(\alpha,T,T))}{\alpha\sqrt{\kappa^2 T}}(\phi(d_2(\alpha,0,T)) - \phi(d_2(\bar{\alpha},0,T)))\Bigg].
\end{aligned}$$

(d) To determine the preferred personal equilibrium, first note that

$$I_1(\alpha) := \mathbb{E}\left[X^\star_{\alpha,T}\right] = \int_0^\alpha u\left(I\left(y^\star_\alpha w(z)\right)\right) f_\zeta(z)\,dz + \int_\alpha^{\bar{\alpha}} u\left(I\left(y^\star_\alpha w(\alpha)\right)\right) f_\zeta(z)\,dz + \int_{\bar{\alpha}}^{+\infty} u\left(I\left(y^\star_\alpha w(z)\right)\right) f_\zeta(z)\,dz$$

for all $\alpha \in [x_1, \bar{x}_2]$; therefore,

$$\begin{aligned}
I_1'(\alpha) = {} & y^\star_\alpha \int_0^\alpha I'\left(y^\star_\alpha w(z)\right)\frac{dy^\star_\alpha}{d\alpha} w(z)^2 f_\zeta(z)\,dz + y^\star_\alpha \int_\alpha^{\bar{\alpha}} I'\left(y^\star_\alpha w(\alpha)\right)\left(\frac{dy^\star_\alpha}{d\alpha} + y^\star_\alpha \frac{y^\star_\alpha w'(\alpha)}{w(\alpha)}\right) w(\alpha)^2 f_\zeta(z)\,dz \\
& + y^\star_\alpha \int_{\bar{\alpha}}^{+\infty} I'\left(y^\star_\alpha w(z)\right)\frac{dy^\star_\alpha}{d\alpha} w(z)^2 f_\zeta(z)\,dz = -\frac{1}{y^\star_\alpha}\left[\frac{dy^\star_\alpha}{d\alpha} + \frac{y^\star_\alpha w'(\alpha)}{w(\alpha)}\left(F_\zeta(\bar{\alpha}) - F_\zeta(\alpha)\right)\right],
\end{aligned}$$

where the last equality uses $I'(y) = -y^{-2}$ for all $y \in (0, +\infty)$. On the other hand, it follows from Fubini's theorem that

$$
\begin{aligned}
I_2(\alpha) &:= \int_{\mathbb{R}} \int_{\mathbb{R}} \nu(u(z)-u(b))\, d\mathbb{P}_{X^\star_{\alpha,T}}(b) d\mathbb{P}_{X^\star_{\alpha,T}}(z) \\
&= -\frac{\eta(1-\lambda)}{1-\eta} \Bigg[\int_0^{\alpha} \int_0^{z} \big(u\big(I\big(y^\star_\alpha w(b)\big)\big) - u\big(I\big(y^\star_\alpha w(z)\big)\big)\big) f_\zeta(b) f_\zeta(z) db\, dz \\
&\qquad + \int_{\alpha}^{\bar\alpha} \int_0^{\alpha} \big(u\big(I\big(y^\star_\alpha w(b)\big)\big) - u\big(I\big(y^\star_\alpha w(\alpha)\big)\big)\big) f_\zeta(b) f_\zeta(z) db\, dz \\
&\qquad + \int_{\bar\alpha}^{+\infty} \int_0^{\alpha} \big(u\big(I\big(y^\star_\alpha w(b)\big)\big) - u\big(I\big(y^\star_\alpha w(z)\big)\big)\big) f_\zeta(b) f_\zeta(z) db\, dz \\
&\qquad + \int_{\bar\alpha}^{+\infty} \int_{\alpha}^{\bar\alpha} \big(u\big(I\big(y^\star_\alpha w(\alpha)\big)\big) - u\big(I\big(y^\star_\alpha w(z)\big)\big)\big) f_\zeta(b) f_\zeta(z) db\, dz \\
&\qquad + \int_{\bar\alpha}^{+\infty} \int_{\bar\alpha}^{z} \big(u\big(I\big(y^\star_\alpha w(b)\big)\big) - u\big(I\big(y^\star_\alpha w(z)\big)\big)\big) f_\zeta(b) f_\zeta(z) db\, dz \Bigg],
\end{aligned}
$$

and further cumbersome but straightforward computations yield

$$
\begin{aligned}
I_2'(\alpha) &= -\frac{\eta(1-\lambda)}{1-\eta} \Bigg[\int_0^{\alpha} \int_0^{z} y^\star_\alpha \left(I'\big(y^\star_\alpha w(b)\big) \frac{dy^\star_\alpha}{d\alpha} w(b)^2 - I'\big(y^\star_\alpha w(z)\big) \frac{dy^\star_\alpha}{d\alpha} w(z)^2 \right) f_\zeta(b) f_\zeta(z) db\, dz \\
&\quad + \int_{\alpha}^{\bar\alpha} \int_0^{\alpha} y^\star_\alpha \left(I'\big(y^\star_\alpha w(b)\big) \frac{dy^\star_\alpha}{d\alpha} w(b)^2 - I'\big(y^\star_\alpha w(\alpha)\big) \left(\frac{dy^\star_\alpha}{d\alpha} + y^\star_\alpha \frac{y^\star_\alpha w'(\alpha)}{w(\alpha)} \right) w(\alpha)^2 \right) f_\zeta(b) f_\zeta(z) db\, dz \\
&\quad + \int_{\bar\alpha}^{+\infty} \int_0^{\alpha} y^\star_\alpha \left(I'\big(y^\star_\alpha w(b)\big) \frac{dy^\star_\alpha}{d\alpha} w(b)^2 - I'\big(y^\star_\alpha w(z)\big) \frac{dy^\star_\alpha}{d\alpha} w(z)^2 \right) f_\zeta(b) f_\zeta(z) db\, dz \\
&\quad + \int_{\bar\alpha}^{+\infty} \int_{\alpha}^{\bar\alpha} y^\star_\alpha \left(I'\big(y^\star_\alpha w(\alpha)\big) \left(\frac{dy^\star_\alpha}{d\alpha} + y^\star_\alpha \frac{y^\star_\alpha w'(\alpha)}{w(\alpha)} \right) w(\alpha)^2 - I'\big(y^\star_\alpha w(z)\big) \frac{dy^\star_\alpha}{d\alpha} w(z)^2 \right) f_\zeta(b) f_\zeta(z) db\, dz \\
&\quad + \int_{\bar\alpha}^{+\infty} \int_{\bar\alpha}^{z} y^\star_\alpha \left(I'\big(y^\star_\alpha w(b)\big) \frac{dy^\star_\alpha}{d\alpha} w(b)^2 - I'\big(y^\star_\alpha w(z)\big) \frac{dy^\star_\alpha}{d\alpha} w(z)^2 \right) f_\zeta(b) f_\zeta(z) db\, dz \Bigg] \\
&= -\frac{\eta(1-\lambda)}{1-\eta} \frac{1}{y^\star_\alpha} \frac{y^\star_\alpha w'(\alpha)}{w(\alpha)} \Big(F_\zeta(\bar\alpha) - F_\zeta(\alpha) \Big) \Big(F_\zeta(\bar\alpha) + F_\zeta(\alpha) - 1 \Big).
\end{aligned}
$$

Finally, combine the expressions above results in

$$
\frac{d}{d\alpha} U\Big(X^\star_{\alpha,T} \Big| X^\star_{\alpha,T} \Big) = -\frac{1}{y^\star_\alpha} \left[\frac{dy^\star_\alpha}{d\alpha} + \frac{y^\star_\alpha w'(\alpha)}{w(\alpha)} \Big(F_\zeta(\bar\alpha) - F_\zeta(\alpha) \Big) \left(1 - \frac{\eta(1-\lambda)}{1-\eta} \Big(1 - F_\zeta(\bar\alpha) - F_\zeta(\alpha) \Big) \right) \right].
$$

To find the value of α whose associated anchor leads to the highest overall utility, we evaluate the sign of the above derivative numerically.

□