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ORIGINAL ARTICLE

Frege: A fusion of horizontals

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Abstract

In *Die Grundgesetze der Arithmetik* (I, §48), Frege introduces his rule of the fusion of horizontals, according to which if an occurrence of the horizontal stroke is followed by another occurrence of the same stroke, either in isolation or "contained" in a propositional connective, the two occurrences can be *fused* with each other. However, the role of this rule, and of the horizontal sign more generally, is controversial; Michael Dummett notoriously claimed, for instance, that the horizontal is "wholly superfluous" in Frege's logical system. In this paper, we challenge Dummett's view by providing a comprehensive analysis of the significance of the horizontal stroke. After some preliminary remarks, we argue that even if Frege's connectives in some sense "contain" the horizontal, yet they are total functions. Then, we take up the question of the *sense* expressed by the horizontal, and we claim that, unlike other sentential operators, the horizontal is not sense-compositional. Finally, we consider the semantic and pragmatic aspects of Frege's horizontal in connection to his judgment stroke and the double judgment stroke. *Contra* Dummett, we argue that the horizontal is a special and indispensable element of Frege's logic.

KEYWORDS

assertion, Frege, fusion of horizontals, horizontal, inference, logical notation

1 | INTRODUCTION

In *Begriffsschrift* §2,[1] Frege introduces the "horizontal stroke" (*wagerechte Strich*) as a part of a composite sign for judgment. A judgment is the recognition of the truth of a content and is

[1]In this paper, the following abbreviations are used: BGS for Frege (1879), GL for Frege (1884), GG for Frege (1893–1903); PW for Frege (1979), CP for Frege (1984), PMC for Frege (1980).

always expressed in Frege's notation by means of a sign "which stands to the left of the signs, or the combination of signs, indicating the content of the judgment". Supposing the "signs, or the combination of signs" that indicate the content of the judgment to be expressed by "Δ", then the sign that represents the judgment that Δ is the following:

"$\vdash$ Δ"

This sign is composite: besides "Δ", which represents the content judged, there is a vertical stroke (*senkrechte Strich*) that appears to the left of a horizontal stroke (*wagerechte Strich*), thus:

"$\vdash$ "

The horizontal stroke combines the signs to its right into a totality, and the vertical stroke always relates to such a totality, that is, has such totality as its scope. The horizontal stroke is also called *Inhaltsstrich*, "content stroke", the vertical stroke is also called *Urtheilsstrich*, "judgment stroke". Frege says that if the *Urtheilsstrich* is omitted, as in

"— Δ"

we no longer have the expression of the judgment that Δ, but the mere expression of a content, a content that would constitute a judgment if it were recognised as true.

Frege then adds that "[n]ot every content becomes a judgment when $\vdash$ is written before its sign; for example, the idea 'house' does not. We therefore distinguish contents that can become a judgment from those that cannot" (BGS §2). If "Δ" is the content of a possible judgment, then "$\vdash$ Δ" expresses the judgment that Δ; but if "Δ" is not the content of a possible judgment, then "$\vdash$ Δ" expresses no judgment at all. The distinction between contents that can become judgments and those that cannot is in fact a restriction over the kind of things to which the content stroke can be prefixed: it can only be prefixed to signs of contents of possible judgments. Frege is explicit about this at the end of BGS §2: "whatever follows the content stroke must have a content that can become a judgment".

With the distinction between *Sinn* and *Bedeutung*, and the identification of the *Bedeutung* of a sentence with its truth-value, this description becomes inappropriate. What in the earlier *Begriffsschrift* was the content of a sentence, in the later *Grundgesetze* is split into its *Sinn*, which is the thought the sentence expresses, and its *Bedeutung*, which is its truth-value. The "content stroke" must therefore change name: it is now simply called the "horizontal" (*Wegerecht*). Like in *Begriffsschrift*, the sign "$\vdash$" is still composite: "I regard '$\vdash$' as composed of the vertical stroke, which I call the judgement-stroke, and the horizontal stroke, which I now propose to label simply the horizontal" (GG §5). Unlike in *Begriffsschrift*, the horizontal is now regarded as a function in its own right, whose explanation precedes that of the judgment stroke. The horizontal is the name of a function[2] such that "— Δ" refers to the True when "Δ" refers to the True, and refers to the False when "Δ" refers to something other than the True. The horizontal is unique to Frege and a major difference between the system presented in *Grundgesetze* and modern logic (Panza, 2021).

While its explanation is relatively clear and coherent with Frege's logical system in GG, the question has often arisen in the literature whether the horizontal and the function it names have any non-superfluous role in the system. In this paper, we propose to address this question, and

[2]Even though in GG §5 Frege speaks of the horizontal as the name of a function, this name should include, strictly speaking, also the Greek letter working as a placeholder for the argument. Therefore, the term "horizontal" should refer either to the function — ξ itself or to a part of its name, namely the horizontal stroke, although we will use "horizontal" as a shorthand for "horizontal stroke" throughout the paper. We thank an anonymous reviewer for pointing this out.

to connect it to another one that has, by contrast, rarely been addressed: the question of the sense of the horizontal, and in particular the issue of the contribution that the horizontal makes to the sense of any sentence in which it occurs.

The paper is structured as follows. In the first section we discuss the idea that the horizontal is a superfluous ingredient of the system of GG. This idea probably originated with Dummett (1973), but some recent commentators have argued otherwise; thus, it is necessary to begin our analysis of the horizontal with a discussion of this point. In the second section, we argue that Frege's connectives are total functions, and therefore that the horizontal cannot be taken to function as a "filter" from total functions to truth functions. The third section, then, inquiries into the problems connected to the sense dimension of the horizontal. Here we show that if the compositionality of the horizontal is construed according to the function/argument relationship rather than according to the part/whole relationship, and is thus in some sense related to Dummett's idea of "decomposition" (as opposed to "analysis"), then it emerges that, unlike the other truth-functional connectives, the horizontal does not contribute to the sense of any sentence in which it occurs, but is the notational place where the decomposition of the sentence into function-names and argument-names may occur in order for the decomposition to yield only function-names for Frege's connectives and horizontalized argument-names that express judgeable contents. Finally, in Section 4, we consider the semantic and pragmatic aspects of Frege's horizontal in connection to his judgment stroke and the double judgment stroke. Section 5 concludes the paper.

2 | THE FUSION OF HORIZONTALS

In Frege's logical system, the horizontal is the name of a function such that "— Δ" denotes the True when Δ is the True and denotes the False when Δ is any object other than the True. In "Function and Concept", which contains a presentation that closely parallels that in *Grundgesetze*, the following qualification is added: "in all other cases the value of this function is the False – that is, both when the argument is the False and when it is not a truth-value at all" (CP 149).[3] The same point, that is, that the value of the horizontal is the False when either the argument is the False or something not a truth-value, is fully explained in the sequel of GG §5. This definition can be represented in a table like that in Figure 1.

The table in Figure 1 differs from ordinary truth-tables in that it does not represents a function from truth-values to truth-values, as ordinary truth-tables do. It represents a function from any object to a truth-value, that is, a "total function".[4] The horizontal function may have any object as its argument but has only truth-values as values. When something that is not a truth-value is the argument of the function, the value of the function for that argument is the False. In case the argument of the function is a truth-value, the value of the function for that argument is *the same* truth-value.

According to Dummett, the horizontal "appears to be wholly superfluous" (1973, p. 315). The reason for this, Dummett argues, is that in *Grundgesetze* all functional expressions, including negation and the conditional, are defined for all arguments of the appropriate type, that is, are total functions. If negation and the conditional were defined only for truth-values, the horizontal might be used to restrict them to truth-values alone. But since they are defined for all arguments, the horizontal is superfluous.

[3]In GG, truth-values are objects (see GG §2). We follow the convention of using capitalised "the True" and "the "False" to indicate them.
[4]Cf. Berg and Cook (2017, p. 5).

Δ	— Δ
T	T
F	F
Other	F

FIGURE 1 Table for the horizontal.

Here are the passages from "Function and Concept" and *Grundgesetze* where negation and the negation sign are introduced and defined. (For convenience of reference, we shall refer to them with the aid of labels).

T1. The next simplest function, we may say, is the one whose value is the False for just those arguments for which the value of — x is the True, and, conversely, is the True for the arguments for which the value of — x is the False. I symbolize it thus:

$$\top x$$

and here I call the little vertical stroke the stroke of negation. I conceive of this as a function with the argument — x:

$$(\top x) = (\top (— x))$$

where I imagine the two horizontal strokes to be fused together. But we also have:

$$(— (\top x)) = (\top x)$$

since the value of $\top x$ is always a truth-value. I thus regard the bits of the stroke in "$\top x$" to the right and the left of the stroke of negation as horizontals, in the sense of the word that I defined previously. (CP 150).

T2. The value of the function

$$\top \xi$$

is to be the False for every argument for which the value of the function

$$— \xi$$

is the True; and it is to be the True for all other arguments. We thus have in

$$\top \xi$$

a function whose value is always a truth-value: it is a concept under which all objects fall with the sole exception of the True. From this it follows that "$\top$ Δ" always refers to (*bedeutet*) the same as "$\top$ (— Δ)", as "—$\top$ Δ", and as "—$\top$ (— Δ)". We therefore regard "$\top$" as composed of the small vertical stroke, the *negation-stroke*, and the two parts of the horizontal stroke each of which can be regarded as a *horizontal* in our sense. The transition from "$\top$ (— Δ)" or from "— ($\top$ Δ)" to "$\top$ Δ", as well as that from "— — Δ" to "— Δ", I will call the *fusion* of horizontals. (GG §6)

In both T1 and T2 negation is explicitly defined for any argument, not just for truth-values. The value of the negation function is the False for just those arguments for which the value of the horizontal is the True, and, conversely, is the True for the arguments for which the value of the horizontal is the False. In other words, the negation function reverses the truth-value of the horizontal function. It is, like the horizontal, a total function.

The same is true of the conditional function. Here are the passages from "Function and Concept" and *Grundgesetze* where the conditional and its sign are introduced and defined:

> T3. The value of the function
>
> $$\begin{array}{l} \top\, x \\ \llcorner\, y \end{array}$$
>
> is to be the False if we take the True as the y-argument and at the same time some object that is not the True as the x-argument; in all other cases, the value of this function is to be the True. The lower horizontal stroke, and the two bits that the upper one is split into by the vertical, are to be regarded as horizontals in our sense. Consequently, we can always regard as the arguments of our function $— x$ and $— y$, that is, truth-values. (CP 154).
>
> T4. In order to be able to designate the subordination of concepts and other important relations, I introduce the function with two arguments
>
> $$\begin{array}{l} \top\, \xi \\ \llcorner\, \zeta \end{array}$$
>
> by means of the specification that its value shall be the False if the True is taken as the ζ-argument, while any object that is not the True is taken as ξ-argument; that in all other cases the value of the function shall be the True. According to this and the previous stipulations, the value of this function is also determined for value-ranges as arguments. It follows that
>
> $$\begin{array}{l} \top\, \Gamma \\ \llcorner\, \Delta \end{array}$$
>
> is the same as
>
> $$—\left(\begin{array}{l} \top\, (—\Gamma) \\ \llcorner\, (—\Delta) \end{array}\right)$$
>
> and therefore that in
>
> $$`\begin{array}{l} \top\, \Gamma \\ \llcorner\, \Delta' \end{array}$$
>
> we can regard the horizontal stroke before "Δ" as well as each of the two parts of the upper horizontal stroke partitioned by the vertical, as *horizontals* in our particular sense. We speak here, just as previously, of the *fusion of horizontals*. (GG §12).

Again, in both T3 and T4 the conditional function is defined for all arguments, that is, is a total function.

We mentioned that Dummett's argument for the superfluity of the horizontal is that both negation and the conditional are total functions. In the face of T1–T4, this is hardly deniable. Yet, it would seem from what Frege says in T1–T4 that the negation and the conditional *signs* only operate on horizontalized contents, that is, on contents prefixed by the horizontal. This is a point that has often been raised: "truth-functional operators such as the negation stroke and the conditional stroke cannot grammatically operate directly on sentences, but only as concatenated with horizontals into complex truth-functional signs" (Heck & Lycan, 1979, p. 486); "Though rarely remarked upon, the negation stroke, the condition stroke, and Frege's concavity can only operate on expressions prefaced by the horizontal" (Taschek, 2008, pp. 392–392). Now, since the value of the horizontal function is always a truth-value, that the negation sign only operates on horizontalized contents would imply that negation does in fact only have truth-values as arguments, that is, that negation is not in fact a total function. (*Mutatis mutandis*, the same applies to the conditional.)

It may be thought that the problem lies in a distinction, which Frege does not explicitly make, between the function itself and its sign in the notation. This is the proposal made by Heck and Lycan (1979). The reason why Dummett (1973) — like Berg and Cook (2017) — can think that negation and the conditional are defined for all arguments is that they do not distinguish Frege's primitive truth-functors as such, such as negation and the conditional functors, from the truth-functional *signs*, which are the results of amalgamating the truth-functors with horizontals. If this distinction is made, it is argued, then the claim that all functional expressions are defined for all arguments may be taken to apply to truth-functional signs (which are composite because they enclose the horizontals) but not to truth-functors themselves: negation and the conditional are total functions only because they already contain the horizontal. In this view, the negation functor would be the simple vertical stroke in "$\top$", which indeed only operates on horizontalized contents, while the two horizontal strokes that flank the negation functor would be horizontals in Frege's sense. *Mutatis mutandis*, the same would apply to the conditional. Frege himself suggests as much when he says that he calls "the little vertical stroke the stroke of negation" (CP 150), thus implying that "$\top$" *contains* the negation stroke along with occurrences of the horizontal.

What is the source of the idea that negation and the conditional contain the horizontal, that is, only operate on horizontalized contents? Take the following two parallel excerpts from T1 to T3.

> T1. …I conceive of [negation] as a function with the argument — x:
>
> $$(\top\, x) = (\top\, (—\, x))$$
>
> where I imagine the two horizontal strokes to be fused together. …
>
> T3. …we can always regard as the arguments of our function [the conditional] — x and — y, that is, truth-values. (CP 154).

Here Frege seems to be saying that negation (T1) and the conditional (T3) "contain" as it were the horizontal, which in turn is taken to mean that they only operate on horizontalised contents. In "$\top\, x$" one can imagine the right-hand horizontal stroke to be composed of two successive horizontals, which are "fused together". This is the "fusion of horizontals" mentioned in T2 and T4. The idea that grounds the fusion is that negation and the conditional "contain" the horizontal, that is, have a horizontalised content as argument: since both the function (expressed by the negation sign, "$\top$") and the argument (expressed by the horizontalised argument mark, "— x") contain horizontal strokes, they can be regarded as "fused" into one single horizontal. The fusion is, as it were, the notational embodiment of the equivalence.

This, however, is problematic. It is far from clear in what sense "$\top$" is a composite sign. Berg and Cook (2017, p. 7) justly observe that compoundness in this case cannot be taken literally. For if "$\top$" were composite in the same sense in which, say, "$\top\top$" is, then we should allow the "little vertical stroke" to occur before an argument without the two horizontals flanking it. But this is excluded, since "$|\ \alpha$" is not a well-formed GG formula. "$\top$" is a composite sign only in a particular sense. Berg and Cook say: "Frege is suggesting that the horizontal, negation, and the conditional can be treated *as if* they contain occurrences of the horizontal" (2017, p. 7). But the sense of this "as-if-containment" is not spelt out.[5] As we argue in the next section, a solution to this problem becomes possible if we appeal to the Dummettian idea of decomposition.

3 | *SINNE* AND *BEDEUTUNGEN*

In T1 the following equivalencies are presented as illustrations of the "fusion of horizontals" in formulas involving negation:

$$1)\ (\top\, x) = (\top\,(\text{—}\ x))$$
$$2)\ (\text{—}\ (\top\, x)) = (\top\, x)$$

In T2 Frege says that the transition from "— (— Δ)" to "— Δ" illustrates the "fusion of horizontals". Since the parallel with the examples involving negation is very strong, and since the transition is in this case also valid in the reverse order, we are justified in expressing it as follows:

$$3)\ (\text{—}\ (\text{—}\ \Delta)) = (\text{—}\ \Delta)$$

The same applies to T4 about the conditional, which can likewise be interpreted as illustrating the "fusion of horizontals" in terms of equivalencies of the form:

$$4)\ \begin{array}{l} \top\ \Gamma \\ \llcorner\ \Delta \end{array} = \text{—}\left(\begin{array}{l} \top\ (\text{—}\Gamma) \\ \llcorner\ (\text{—}\Delta) \end{array}\right)$$

Frege introduces the identity sign "=" as follows:

> We have already used the equality-sign rather casually to form examples but it is necessary to stipulate something more precise regarding it.
>
> "Γ = Δ"
>
> refers to the True, if Γ is the same as Δ; in all other cases it is to refer to the False. (GG §7)

The identity sign denotes a two-places function from objects to truth-value that takes as value the True if "Γ" and "Δ" denote the same object and takes as value the False otherwise. From a contemporary point of view, however, Frege's identity sign admits of a dual interpretation. Concept-script formulas are names denoting truth-values; therefore, the identity sign can be flanked not only by individual terms (as in "1 = 1"), but also by complete sentences. If the

[5]Frege argued that also the *concavity sign* contains an occurrence of the horizontal (GG §8). The concavity sign is one way in which generality is expressed in Frege's system. In this paper we focus on the propositional fragment of Frege's system, but similar considerations apply to the concavity sign as well.

Δ	$—\,\Delta$	$\top\,\Delta$	$\top\,(—\,\Delta)$	$—\,(\top\,\Delta)$	$—\,(\top\,(—\,\Delta))$
T	T	F	F	F	F
F	F	T	T	T	T
Other	F	T	T	T	T

FIGURE 2 Harmlessness of the horizontal.

identity sign is flanked by individual terms, the identity sign can be taken to express the standard identity relation in first-order logic. If the identity sign is flanked by complete sentences, by contrast, it can be interpreted as a propositional connective such that "$\alpha = \beta$" is the True if "α" and "β" denote the same truth-value and is the False otherwise. In this second case, Frege's identity sign is equivalent to the modern biconditional.[6] The identity sign can also be flanked by an individual constant and a sentence, in which case the identity is always the False unless the individual constant denotes a truth-value (see also Berg & Cook, 2017, p. 9).

Now, we noticed in the previous section that in T2 Frege says that "$\top\,\Delta$" always refers to (*bedeutet*) the same as "$\top\,(—\,\Delta)$". Taken together with the fact that the identity sign, as defined in GG §7, operates on the *Bedeutung* of sentences, that is, their truth-values, this can only mean that the equivalencies (1)–(4) express the identity of *Bedeutung*of the sentences flanking the identity sign, that is, that the sentences flanking the identity sign have the same truth-value. Take the equivalence:

$$5)\ \ \Delta = (—\,\Delta)$$

The *Bedeutung* of (5) is the True if Δ is a truth-value, and it is the False if Δ is not a truth-value. (For this reason, Frege says (*ibid.*) that the assertion of (5) amounts to the assertion that "Δ is a truth-value").

Since (1)–(4) express identity of truth-values (*Bedeutung*), the occurrence of one horizontal in a sentence does not necessarily contribute to the determination of its *Bedeutung*. This is unlike what happens with standard sentential operators: in the case of negation, for example, the occurrence of the negation sign in a formula does contribute to its *Bedeutung*. With the horizontal, on the contrary, this does not necessarily happen. *Bedeutungweise*, the horizontal is harmless when the name of the argument of the horizontal function is already horizontalised itself (see Figure 2; notice that, for Frege, the second row is redundant, since the False is just an object other than the True). A different question is whether fused horizontalised sentences have the same sense, that is, if fusing the horizontals is *Sinne*-preserving or not.[7]

Frege scholars know too well that Frege quite clearly endorsed a principle of *sense compositionality*, that is, the principle that the sense of a part of a sentence is a part of the thought that the sentence expresses. This principle is enunciated at various junctures in Frege's philosophical career and is commonly considered a cornerstone of his entire philosophy of logic. The principle is clearly formulated in *Grundgesetze* (GG §I.32), in the "Einführung in die Logik" of 1906 (PW 191–192), in Carnap's Jena notes (Reck & Awodey, 2004, p. 87), in "Logik in der Mathematik" of 1914 (PW 225), in "Die Verneinung" of 1918 (CP 378), in the notes

[6]It is worth stressing that this is true only from a contemporary perspective. In Frege's formal language, the identity sign is always flanked by names of objects (or Latin marks for objects). Moreover, Frege did not appear to have any notion of a sentence as a well-formed closed formula taken as such, and not for its asserting something. He directly transforms a proper name, or Latin mark for objects, into a statement, that is, in the assertion that this name is a name of the True, or this mark is transformed into a name of True for all admitted replacements. We are grateful to an anonymous reviewer for pointing this out.

[7]This question is made under the presupposition that incomplete expressions such as function-terms have a sense. Frege talks about the sense of incomplete expressions in "Gedankengefüge" (1923) and in some of his posthumous writings, especially in "Comments on Sense and Meaning" (1895).

composed for L. Darmstaedter in 1919 (PW 255), and in the *ouverture* of "Gedankengefüge" (CP 390). At some point, Frege seems to have toyed with the idea that a parallel principle holds for the *Bedeutung* of the part of a sentence. But he subsequently expressly withdraws it (Mars is not part of the truth-value of the sentence "Mars is half the diameter of the Earth"), while continuing to affirm that the principle holds for senses.[8]

It is important to distinguish the *strong* claim that the sense of a part of a sentence is part of the sense of the sentence from the *weaker* claim that the sense of a part of a sentence contributes to the determination of the sense of the sentence (see Heck & May, 2011). It does not follow from the fact that the senses of the parts of a sentence contribute to determining the sense of the sentence that a difference in sense between two sentences can only be explained by appealing to the difference in sense between the parts of which they are composed. That there is a conceptual difference between the two claims is indeed evidenced by way of comparison with the case of reference: although the references of the parts of a sentence determine the reference of the sentence (i.e., its truth-value), yet the references of the parts of a sentence are not parts of the reference of the sentence (Mars is not part of the truth-value of the sentence "Mars is half the diameter of the Earth"). The many formulations of the principle (cited above) suggest that Frege, at least after 1906, was committed to the strong version of the principle.

The strong version of the principle of the compositionality of senses (the sense of a part of a sentence is part of the sense of the sentence) implies what Dummett (1991) has called "Principle K", namely that if one sentence involves a concept that another sentence does not involve, the two sentences cannot express the same thought. By Principle K, a sentence that involves a given sentential operator cannot have the same sense as a sentence in which that operator does not occur. Since Frege maintained the strong version of the principle of the compositionality of sense, and since this principle implies Principle K, it follows that for Frege truth-functionally equivalent sentences that contain different sentential operators must have different senses. This is precisely what he says in some of his later papers: in "Die Verneinung" (1918), the second part of the series titled *Logische Untersuchungen*, he refers to "A" and to "the negation of the negation of A" as "two thoughts" that possess the same truth-value (CP, 389), and later in the same paper he refers to sentences of the form "$P \supset Q$" and "$\sim Q \supset \sim P$" as two distinct "complex thoughts" (CP 376).[9]

The principle of sense compositionality is enunciated by Frege in the strong version (the sense of a part of a sentence is part of the sense of the sentence); the strong version implies Dummett's Principle K; and in various places (including "Die Verneinung" of 1918) Frege seems to accept Principle K; accepting these points is sufficient for our purpose of discussing the way, if any, in which the horizontal may figure as a part of the sense of the whole of which it is part.

The question is, does "— Δ" have the same sense as "— (— Δ)"? We saw in the previous section that the passage from "— (— Δ)" to "— Δ" is made by that operation that Frege calls "fusion of the horizontals". Let us consider two Begriffsschrift terms such that one is the result of the fusion of the horizontals of the other. The fusion of the horizontals, then, implies that of these terms, one "always refers to the same" as the other (GG §I, 6). This, we saw, perfectly works at the level of the *Bedeutung*. But what about the sense of the two formulas? Do they have the same sense or different senses?

[8]On Frege's principle of sense compositionality see van Heijenoort (1977); Bermúdez (2001); Penco (2002); Penco (2003); Heck and May (2011); Schellenberg (2012); and Bronzo (2017).

[9]In "Gedankengefüge", the third part of the series, published five years after the second (1923), he seems to have a different opinion: here "not (not *B*)" is said to have the same sense as "*B*" (CP 399); he also says that in a conditional sentence we can replace the antecedent with the negation of the consequent and the consequent with the negation of the antecedent "without altering the sense of the whole" (CP 403); perhaps Frege changed his mind between the first and the third paper in the *Logische Untersuchungen*; see Künne (2009, p. 42), who detects a possible influence of Wittgenstein on the logical doctrines of the series. We ignore this difficulty here and simply assume that Frege accepted the principle K, which is consistent with the treatment of complex senses in "Die Verneinung" and inconsistent with that of "Gedankengefüge".

One possible strategy to answer this question is by reference to the Dummettian idea of decomposition. As is well known, Dummett (1973, 1981). thought that Frege was under the influence of a distinction, which he never makes explicitly but which is needed to interpret his work, between two kinds of analysis, which Dummett calls "analysis" proper and "decomposition". Analysis is the process by which we break down a sentence into its "constituents". To grasp the sense of a sentence (the thought that the sentence expresses) involves knowing the way in which the sense is determined by the senses of its constituents. By the strong version of the principle of sense compositionality, the way in which that sense of the whole is determined by the senses of its constituents is by these latters being *parts* of the sense of the whole. Analysis is unique: the constituents of the thought are the senses of the constituents of the sentence, and the sense of the sentence is determined by the senses of its constituents; thus if one and the same sentence were capable of distinct analyses, which involves that it would be analysable into *distinct* constituents, then, since distinct constituents have distinct senses, the sense of the sentence would *not* be determined by the senses of its constituents. Take the sentence "Cain kills Abel". It has three constituents: the proper names "Cain" and "Abel", and the dyadic predicate "kills". In order to grasp the sense of this sentence, one has to grasp the sense of the three constituents of it (plus the manner of their composition).

By contrast, decomposition is achieved by removing from a complete sentence one or more occurrences of each of one or more proper names or other expressions, thus leaving an incomplete expression. In Dummett's terminology, the parts revealed by the decomposition of a sentence are not constituents of that sentence, but "components" of it. One of the purposes of decomposition is to construct quantified sentences. For example, to construct the sentence "For every x, $F(x) \supset G(x)$" we first need to extract the complex predicate "$F(\xi) \supset G(\zeta)$", and this is done, by decomposition, by omitting the argument-names in a sentence of the form "$F(a) \supset G(a)$" (Dummett, 1973, pp. 27–32). The decomposition of a sentence differs from its analysis in that its purpose is not to reveal how the sense of the sentence is dependent on the senses of its constituents, but to construct quantified sentences in which complex predicates occur and to explain the validity of the inferences involving that sentence. Thus, the sentence "Cain kills Abel" can be decomposed in following ways:

d1) "Cain", "ξ kills Abel"
d2) "Cain kills ζ", "Abel"
d3) "Cain", "Abel", "ξ kills ζ"
d4) "kills", "ϕ(Cain, Abel)"

d1) is necessary if we want to construct the quantified sentence "For any x, x kills Abel" = "Everyone kills Abel";

d2) is necessary if we want to construct the quantified sentence "For any x, Cain kills x" = "Cain kills everyone";

d3) is necessary if we want to construct the quantified sentence "For any x, for some y, x kills y" = "Everyone kills someone";

d4) is necessary if we want to construct the quantified sentence "For any R, Cain and Abel stand in relation R" = "Cain and Abel stand in any relationship".

It may be that a component of a sentence is also a constituent of it. But it need not be: to grasp the sense of the sentence one does not need to grasp the sense of the components "ξ kills Abel" or "Cain kills ζ".

The model of analysis is the part/whole relationship, while the model of decomposition is the function/argument relationship; a constituent of a sentence is a part of that sentence; a component of a sentence, by contrast, is something obtained by that sentence by a certain decomposition of it into function- and argument-names. Here is Frege's illustration of the function/argument relationship in "Function and Concept":

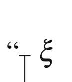
> Statements in general, just like equations or inequalities or expressions in Analysis, can be imagined to be split up into two parts; one complete in itself, and the other in need of supplementation, or “unsaturated”. Thus, for example, we split up the sentence “Caesar conquered Gaul” into “Caesar” and “conquered Gaul”. The second part is “unsaturated” – it contains an empty place; only when this place is filled up with a proper name, or with an expression that replaces a proper name, does a complete sense appear. Here too I give the name “function” to what is meant by this “unsaturated” part. In this case the argument is Caesar. (CP 146–147)

The prototype of Frege’s function/argument analysis, that is, of Dummettian decomposition, is the atomic sentence, which is split into a monadic or polyadic function-names (referring to concepts and relations, respectively) and one or more argument-names (proper names or expressions that replace proper names). But the idea works perfectly well with molecular sentences, too, as Frege shows in GG §12. An atomic sentence is one in which no sentential operator or quantifier occurs; a molecular sentence is one that is not atomic. Take the compound conditional in Figure 3A. Figure 3B,C illustrate two distinct decompositions of it: Figure 3B takes the R-consequent as argument and the remaining as function, while Figure 3C takes the QR-conditional as argument and the remaining as function.

The sense of a sentence is the thought that it expresses; when we decompose a sentence into function- and argument-names, we are thereby decomposing the thought that the sentence expresses; the components so obtained are not parts or constituents of the thought, and thus that a sentence is susceptible of multiple decompositions by no means entails that to each decomposition a distinct sense can be attributed. What is decomposed is one and the same thought.

This may help us solve the problem of the horizontal. We saw in the first section that in T1 Frege says that he regards “$_{\top}\,\Delta$” to contain “$—\,\Delta$”, that is, to have this horizontalized content as argument. This may be taken to mean that the sentence expressed by “$_{\top}\,\Delta$” can be decomposed (in the Dummettian sense) into the function-name “$_{\top}\,\xi$” and the argument-name “$—\,\Delta$”. The same is true of the conditional. In T3 Frege says that we can always regard as the arguments of the conditional function $—\,\xi$ and $—\,\zeta$. This may be taken to mean that the sentence expressed by

can be decomposed into the function-name



and the argument-names “$—\,\Delta$” and “$—\,\Gamma$”. In T1 Frege also envisages the possibility of decomposing “$_{\top}\,\Delta$” into the function-name “$—\,\xi$” and the argument-name “$_{\top}\,\Delta$”. This suggests that also the sentence expressed by “$—\,\Delta$” can be decomposed into the function-name “$—\,\xi$”

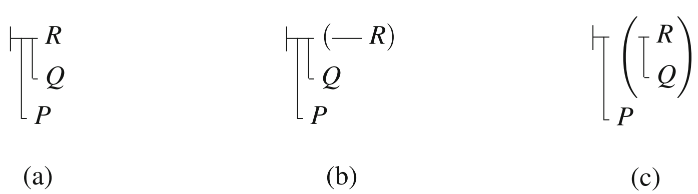


FIGURE 3 A conditional and distinct decompositions of it.

and the argument-name "— Δ", which in turn entails that a corresponding decomposition is possible in the case of the negation and the conditional if indeed the horizontal strokes at the right of both the negation stroke and the conditional stroke are horizontals in the defined sense.

Let us indeed consider the sentential fragment of Frege's Begriffsschrift. In fact, Frege never isolated the sentential fragment of his logical system either in BG or in GG. However, it is legitimate to focus on this fragment because of the particular way that Frege devises to decompose sentential expressions. We will rely on the idea that the horizontal is a device to indicate that a specific content is judgeable. The key idea will be that *by taking the horizontal to be "contained" in Frege's connectives, the decomposition of expressions that correspond to judgeable contents yields either these connectives or other expressions with a judgeable content*.

Consider, for example, Frege's commentary on the following formula, that asserts, informally, that both $3>2$ and $2+3=5$:[10]

$$\text{(1) “}\vdash\!\top\!\!\top\begin{array}{l} 3>2 \\ \llcorner\, 2+3=5 \end{array}\text{”}$$

Frege writes:

> $\top\!\!\top\begin{array}{l} 3>2 \\ \llcorner\, 2+3=5 \end{array}$ is the value of the function $\top\!\!\top\begin{array}{l} \xi \\ \llcorner\, \zeta \end{array}$, when $\top\, 3>2$ is the ξ-argument and $2+3=5$ is the ζ-argument (GG I, §12).

In this passage, Frege is referring to a function and its arguments, that is, to the *Bedeutungen*of these terms. At any rate, at least two different decompositions of the formula (1) can be envisaged. On the one hand, this formula can be decomposed into a function-name, that is,

$$\text{“}\top\!\!\top\begin{array}{l} \xi \\ \llcorner\, \zeta \end{array}\text{”,}$$

and two horizontalized formulas, that is, "— $3>2$" and "— $2+3=5$". On the other hand, the formula can be decomposed into the same function-name and two complex terms without the horizontal, that is, "$3>2$" and "$2+3=5$". From the syntactic point of view, both decompositions are legitimate. However, there is a *pragmatic* difference between the two. Indeed, the first decomposition yields, in Frege's term, two proper-names that *refer to the True*. The second decomposition yields two horizontalized formulas that can be *judged as true*.

Using this distinction, we can isolate the sentential fragment of the Begriffsschrift as follows. The fragment consists of Frege's sentential connectives, namely the horizontal, the negation stroke, the conditional stroke and atomic formulas.[11] In this fragment, formulas are decomposed only up to the horizontal. Since the horizontal can be indefinitely 'de-fused', atomic formulas are those for which decomposition only leads to a horizontal and the formula itself. This restriction on the possibility of decomposing sentences mirrors the idea that, in sentential logic, decomposing an expression with a judgeable content results in at least one expression whose content can itself be judged.

[10]This example is similar to one suggested to us by an anonymous reviewer and that involves the negation of a mathematical falsehood, for example "$\top\, 2+1=4$".
[11]An analysis of the propositional fragment of Frege's GG is in Berg and Cook (2017).

If one considers this sentential fragment of Frege's Begriffsschrift, one sees that in this fragment *Begriffsschrift formulas can be decomposed only at the level of the horizontal. Conversely, any fusion of horizontals in a formula occurs at a place where a decomposition of that formula can also occur*. In the sentential fragment, only the horizontal, the negation stroke, and the conditional stroke can be used to construct molecular sentences out of atomic sentences. Now, for whatever Δ, the sentence expressed by "$\top\,\Delta$" can be decomposed into the function-name "$\top\,\xi$" and the argument-name "$—\,\Delta$", or into the function-name "$—\,\xi$" and the argument-name "$\top\,\Delta$". There is no third possibility if decomposition must yield only function-names for Frege's connectives and horizontalised argument-names. Decomposition must occur at the level of the horizontal, that is, it is one of the two horizontals that the sentence contains that is split, one part of it remaining with the function sign and the other with the argument sign. In the same way, the sentence expressed by

$$``\begin{array}{l}\top\,\Gamma\\ \llcorner\,\Delta\end{array}"$$

may be decomposed into the function-name

$$``\begin{array}{l}\top\,\xi\\ \llcorner\,\zeta\end{array}"$$

and the argument-names "$—\,\Delta$" and "$—\,\Gamma$" (case 1), or into the function-name $``\begin{array}{l}\top\,\xi"\\ \llcorner\,\Delta\end{array}$ and the -argument-name "$—\,\Gamma$" (case 2), or into the function-name $``\begin{array}{l}\top\,\Gamma"\\ \llcorner\,\zeta\end{array}$ and the argument-name "$—\,\Delta$" (case 3), or, finally, into the function "$—\,\xi$" and the argument

$$``\begin{array}{l}\top\,\Gamma\\ \llcorner\,\Delta\end{array}"$$

(case 4)

There is no fifth possibility, again, if decomposition must yield only connectives and horizontalised formulas. Again, these decompositions must occur at the level of the horizontal. When the decomposition is into a monadic function-name and one argument-name (cases 2, 3 and 4), it is one of the horizontals that the sentence contains that is split, one part of it remaining with the function-name and the other with the argument-name. When the decomposition is into a dyadic function and two arguments (case 1), it is two of the horizontals that the sentence contains that are split. In like manner, any sentence containing only the horizontal as a sentential connective can by definition be decomposed only at the level of the horizontal. In general, for any Begriffsschrift term of whatever complexity that belongs to the sentential fragment of the system, since it will contain the horizontal, the negation stroke, or the conditional stroke, or some combination of these, any decomposition of that formula will occur at the level of the horizontal.

This is not true if the first-order fragment of Frege's system is considered. For example, let's consider Basic Law IIa:

$$\vdash\!\begin{array}{l}\top\, fa\\ \llcorner\overset{\mathfrak{a}}{\smile}\, f\mathfrak{a}\end{array}$$

Any instance of this law where the Latin mark f is substituted by function-name can be decomposed into the name for a higher-level function:

$$\text{“}\begin{array}{l} \top\, \phi a \\ \llcorner \overset{\mathfrak{a}}{\smile}\, \phi\mathfrak{a}\text{”} \end{array}$$

and the name of the first-level function. Here, the horizontal plays no role in the decomposition; decomposition does not occur at the level of the horizontal.[12] But this is to be expected, since quantifiers bind function-names (rather than horizontalized formulas) to form horizontalised formulas. By contrast, when the sentential fragment is considered, any decomposition of a Begriffsschrift formula must occur at the level of the horizontal.

In this perspective, that operation that Frege calls the “fusion of horizontals” is the notational counterpart of decomposition: whenever horizontals in a Begriffsschrift term can be fused, that is a (notational) place where a decomposition of that formula can occur.

The formula “— Δ”, then, must have the same sense as “— (— Δ)”, because the latter formula is simply a way of regarding the former formula as being decomposed into the function-name “— ξ” and the argument-name “— Δ”. When this function-name and this argument-name are “fused”, they become the formula “— Δ”, which is the formula that is the object of decomposition. And since decomposition is always of one and the same thought into components (which are not “constituents” or “parts” of the thought), the two Begriffsschrift formulas “— Δ” and “— (— Δ)” must have the same sense.

This may help us solve the problem of the horizontal. We saw that in both T1 and T3 Frege suggests that negation and the conditional “contain”, as it were, the horizontal, which in turn is taken to mean that they only operate on horizontalised contents. We saw that this is incompatible with the fact that both negation and the conditional are total functions, that is, are defined for all objects and not just for truth-values. The idea of decomposition reconciles these two dimensions. The sentence “$\top \Delta$” may be decomposed into the function-name “$\top \xi$” and the argument-name “— Δ”; this is what Frege means when he says that he conceives negation as a function with argument — Δ. But that this is a possible decomposition of negation does not mean that negation always has a horizontalised content as argument, just as decomposition (c) in Figure 3 does not imply that the conditional always has a conditional as argument. The fusion of the horizontal is not a consequence of the fact that negation and the conditional only have horizontalised contents as arguments; it is a consequence of the fact that Begriffsschrift terms can be decomposed only at the level of the horizontal so that any fusion of horizontals in a formula occurs at a place where a decomposition of that formula can also occur.

The horizontal, then, is a sentential operator *sui generis*. Its peculiarity is not in the kind of function it is; like negation and the conditional, it is a total function. Its peculiarity rather lies in the way it contributes to the sense of any sentence in which it is included. Unlike the other sentential operators, the horizontal is not compositional, and its sense is not part of the sense of the whole molecular sentence in which it occurs.[13] The horizontal is the meaningless (in the sense of *Sinnlos*) connector of meaningful functors; since it does not contribute to the sense of the whole, its sense may be as it was assigned to either or the other or both the components into which the whole is decomposed. Only such “sense-inert” function can be the (notational) place of the whole where a decomposition of the whole may be effected to yield other expressions with judgeable content.

4 | THE ASSERTION SIGN, THE DOUBLE JUDGMENT STROKE AND THE LINGUISTIC MEANING OF THE HORIZONTAL

We argued that the horizontal sign is sense-inert. However, our argument requires that this sign is not inert from the pragmatic point of view, since, as we saw, a fully satisfactory interpretation

[12]We owe this point to an anonymous referee of this journal.

[13]Frege claims that also his concavity sign for generality contains an occurrence of the horizontal stroke (GG §8).

of the fusion of horizontals requires taking into account the role of the horizontal sign as a device to form expressions with judgeable contents. We must therefore consider the role of this sign as a component of the assertion sign, including its use in the double judgment stroke for definitions.

Frege's assertion sign "⊢" is a key element of his logical notation. The assertion sign is not intended by Frege to be primitive because it is composed by the vertical "|" and the horizontal[14] "—" strokes.[15] The vertical stroke is also known as the judgment stroke. The interpretation of the assertion sign, its two constituents and in its relation to the asserted content, is not without problems.[16]

In the *Begriffsschrift*, "— Δ" represents a content that can be judged or asserted,[17] and "⊢ Δ" indicates the full judgment of that content. The sign "⊢" was assumed by Frege to be the common predicate of all judgments stating: "it is a fact". The idea is that a possible content "Δ" is nominalised by the horizontal stroke (a proposition "α" is converted into an expression of the form "that α") and then by means of the vertical stroke one may predicate that the judgment expresses a fact ("it is a fact that α"). Frege observed that "The content-stroke is horizontal, it is always prefixed to the expression of a content of possible judgment" (PW 9 [1880–1881]). As noted by Geach (1965), the Fregean notation allows differentiating asserted from non-asserted occurrences of one and the same formula.

However, this proposal does not clarify the use of formulas in the context of indirect proofs that are *assumed* rather than asserted as true.[18] Moreover, the interpretation of the assertion sign as a predicate seems to violate the distinction between the pragmatic and the semantic aspects of language: assertion, being a pragmatic notion, cannot be reduced to the semantic notion of a predicate that designates a fact.

In *Grundgesetze* and in "Function and Concept", Frege modified his views on the assertion sign. In a footnote to "Function and Concept" he says:

> [T]he judgement stroke cannot be used to construct a functional expression; for it does not serve, in conjunction with other signs, to designate an object. "⊢ $2 + 3 = 5$" does not designate [*bezeichnet*] anything; it asserts something. (CP 199)

In this case, the horizontal stroke is a function-name denoting a concept under which the True falls, while the judgment stroke is the acknowledgement of the truth of the Thought. In light of this, "— Δ" means that "Δ" can be presented without asserting its being true.

It is well known that in the beginning of "Der Gedanke" and other works, Frege defines logic as the science that is specifically concerned with the notion of truth. However, some of his posthumous writings make it clear that the notion of assertion is closely associated with the nature of logic:

> [W]hat logic is really concerned with is not contained in the word "true" at all but in the assertoric force with which a sentence is uttered. (PW 323)

Indeed, the pragmatic notion of assertoric force ends up being a crucial element for logicality in Frege's mature reflections.

In spite of this, there is not complete agreement in the literature on the nature of the assertion sign in Frege. We can distinguish between a slightly more semantic and a slightly more

[14]As mentioned in the introduction, in Frege's early works the horizontal stroke is called "content stroke".

[15]On the notion of judgment in Frege, see van der Schaar (2018) and Smith (2009).

[16]Frege's views on the assertion sign, in fact, in his early writings suffer from some problems, as discussed, among others, by Dudman (1970) and Bell (1979).

[17]In BGS, Frege (1879) stated that "When the vertical stroke is omitted, we express ourselves *paraphrastically*, using the words 'the circumstance that' or 'the proposition that'". However, in a letter to Husserl in PMC, Frege acknowledges that "Instead of 'circumstance' and 'proposition' I would simply say 'thought'".

[18]It is not simple to understand what Frege's attitude to indirect proofs was. They cannot be formulated in his logical language but he had different attitudes toward their use in mathematics. See Bell (1979, pp. 91-92).

pragmatic interpretation of the assertion sign (with the majority of scholars falling on the pragmatic side; see e.g., Greimann, 2014). An example of the former is provided by Cadet and Panza (2015), who suggest that assertions in Frege's system can receive a slightly more semantic interpretation; in their words,

> the act of asserting something, or claiming truth or falsehood is rendered, within [Frege's *Grundgesetze* system], as the act of holding that a certain appropriate term or all the terms of a class of appropriate terms refer to the True (Cadet & Panza, 2015, p. 21).

In particular, Cadet and Panza claim that the assertion sign is used by Frege as a means to form the statement that whatever is preceded by that sign is a name for the True. The alternative interpretation has been recently defended by Ruffino et al. (2020). as follows:

> Frege actually regards illocutionary force indicators as something essential to the formal language as well. The sign for assertion ("⊢", which is, in fact, a composition of the horizontal or content-stroke "—" with vertical or judgement stroke "|") is a central element in Frege's notation. Indeed, it is the very first formal symbol introduced both in the Begriffsschrift and later in the Basic Laws of Arithmetic. If one looks at the formal part of both works, there is no single sentence that is not preceded by the assertion sign. There are no logical inferences except those made from asserted premises to asserted conclusions. (Ruffino et al., 2020, p. 10066).

According to them, the judgment stroke expresses a specific speech act (i.e., putting a thought forward as true) that is distinct from merely considering the content of a sentence. The inclusion of a sign with illocutionary force into his logical system would set Frege apart from most of his coeval and later logicians.

Let us now consider the interplay between the horizontal stroke and the assertion stroke. The horizontal refers to a function that always has truth values as values. Only if $— \Delta$ is the True can it then be asserted. The main idea behind the differences between the horizontal and the assertion signs is not something associated with the possibility to filter the acceptable truth-values by means of a specific logical sign but to acknowledge that not all kinds of contents can be justifiably asserted.

The following is an example by Frege of a content that cannot be judged because of its indeterminateness:

> For example, let F be the property of being a heap of beans; let f be the procedure of removing one bean from a heap of beans; so that $f(a,b)$ means the circumstance that b contains all beans of the heap a except one and does not contain anything else. Then by means of our proposition we would arrive at the result that a single bean, or even none at all, is a heap of beans if the property of being a heap of beans is hereditary in the f-sequence. This is not the case in general, however, since there are certain z for which $F(z)$ cannot become a judgment on account of the indeterminateness of the notion "heap". (BGS §27).

The previous example involves a type of horizontalized formula whose content cannot be asserted. Of course, falsehood cannot be asserted, while truth can. Yet, a content cannot be asserted for reasons other than because it is false.[19] What connects contents expressed by

[19]There may be horizontalised contents of which we know under what conditions the content is True or False but we do not know if these conditions (i) are actually obtained or (ii) even if they can be obtained in line of principle. The situation (i) shows a contingent unassertability, whereas situation (ii) implies an intrinsic unassertability. A similar distinction regarding contingent and intrinsic knowability was proposed by Dummett (2009). On the limits of asserting, see Haaparanta (2019).

horizontalised formulas with the possibility of being asserted or not is given by the possibility to show that the judgment is *justified*. These justification conditions depend on the very nature of the judgment under consideration: (a) analytic judgments are justified by general logical laws and definitions; (b) synthetic judgments are justified by the laws of a specific discipline; (c) *a posteriori* judgments are justified when it is not possible to give a proof without appealing to facts; (d) *a priori* judgments are justified by general laws not requiring any proof (GL §3). Only those contents (expressed by horizontalised formulas) that fulfil the specific justification conditions can be finally asserted. Frege observed that:

> It not uncommonly happens that we first discover the content of a proposition, and only later give the rigorous proof of it, on other and more difficult lines; and often this same proof also reveals more precisely the conditions restricting the validity of the original proposition. In general, therefore, the question of how we arrive at the content of a judgement should be kept distinct from the other question, whence do we derive the justification for its assertion? (GL §3).

The relation between horizontalised content and asserted sentence (*Begriffsschriftsatz*) is a crucial feature of Frege's logic and its understanding is related to his idea that the task of logic is the study of inference based on justified assertions, not consequence (Sundholm, 2012). According to Frege, logical inferences occur among asserted sentences that express justified judgments. Therefore, asserted sentences have a major inferential role in his logic, unlike unasserted horizontalised contents, which cannot take part in an inference unless as parts of asserted horizontalised contents. Conversely, the assertion sign cannot fall under the scope of a horizontal, since a judgment is not a component or constituent of a thought but the acknowledgement of its truth.[20] Thus, a *Begriffsschriftsatz* cannot be horizontalised: "— (⊢ Δ)" is not a well-formed formula of the system. In Frege's early writings, the horizontal (then called the "content-stroke") is a notational device indicating that a specific content is judgeable. It could be considered as a *predicate of judgeability*, that is, "— Δ" means that Δ fulfils the semantic conditions for being judged. The horizontal can be interpreted as a predicate of judgeability, that is a predicate expressing the *possibility* for a thought to be judged as true. Still, this fact is coherent with the idea that truth is a necessary but insufficient requirement for a justified assertion. In order to fully justify an assertion, we need to know if a sentence is analytic or synthetic, a priori or a posteriori, since different conditions of assertability may be involved. Once these conditions are fulfilled, we can justifiably assert a content. Following this line of reasoning, one thus concludes that there may be true contents whose corresponding judgments cannot be justified.

In his mature system, the horizontal is a specific kind of predicative expression that forms a name for the True when completed by an expression naming the True and forms a name for the False when completed by an expression naming any object different from the True. We might say that the content stroke of his earlier system was a notational expression clearly working at the interface between the pragmatic and the semantic aspects of language, whereas the horizontal of the mature system shows a more semantic characterisation since it does not explicitly refer to the pragmatic conditions of justifiability. Still, in both cases, the horizontal is a predicate that occurs in every asserted formula and the connection between the semantic and the pragmatic aspects of the Begriffsschrift is an essential aspect of the system.

Quite interestingly, the horizontal is also required for creating a definition, which is indicated by a specific sign "||-", which can be viewed as a "double judgement stroke" connected to the horizontal (Schirn, 1989). According to Frege:

[20]For a critical discussion of a similar point, see Taschek (2008).

> In order now to introduce new signs in terms of those already familiar, we require the double-stroke of definition, which appears as an iterated judgment-stroke coupled with a horizontal:
>
> $$\Vdash$$
>
> which is used in place of the judgment-stroke where something is to be, not judged, but abbreviated by definition. We introduce a new name by means of a definition by stipulating that it is to have the same sense and the same denotation as some name composed of signs that are familiar. Thereby the new sign becomes the same in meaning as that being used to define it; and thus the definition goes over directly into a proposition. Hence we may cite a definition in the same way as a proposition, in the process replacing the stroke of definition by the judgment-stroke. A definition is always presented here in the form of an identity with "$\Vdash$" prefixed. To the left of the identity-sign we will always write the definiens, and to the right the definiendum. The definiens will be composed of familiar signs. (GG §27)

Thus, the *definiens* and the *definiendum* of a definition must have the same sense and the same denotation[21]; moreover, since a definition is an act, "$\Vdash$" must be applied to a horizontalised content (in which there is an identity sign) but not *vice versa*.

In *Begriffsschrift*, Frege points out that

> [t]his proposition [expressed with the double stroke of definition] differs from the judgments considered up to now in that it contains signs that have not been defined before; it itself gives the definition. It does not say "The right side of the equation has the same content as the left", but "It is to have the same content". Hence this proposition is not a judgment, and consequently not a synthetic judgment either, to use the Kantian expression (BGS §24).

However, he clarifies that

> [a]lthough originally [the formula expressed with the double stroke of definition] is not a judgment, it is immediately transformed into one; for, once the meaning of the new signs is specified, it must remain fixed, and therefore [this] formula also holds as a judgment, but as an analytic one, since it only makes apparent again what was put into the new signs. This dual character of the formula is indicated by the use of a double judgment stroke (BGS §24).

So, the dual use of the double judgment stroke "$\Vdash$" implies that the first vertical bar indicates a stipulation that is required in order to conclude a judgment of identity. This judgment is analytic and, in fact, can be obtained, as we have seen, by general laws and definitions and used as a premise of an inference and adopted in a proof as an axiom or as an already proven theorem.

According to Frege, a definition is justified, from a logical perspective, when at least the following conditions hold: (i) the contents must have sense and meaning[22] and (ii) the definiens occurring in the horizontal should be composed by familiar signs.[23] When these conditions are fulfilled, then we can conclude to a definition which is basically a judgment of identity between

[21]Before the introduction of the distinction between sense and reference, Frege considered a definition as an association of a sign to a content or meaning (notions combining sense and reference), resulting in a stipulation. See (Shieh, 2008).

[22]In "Fuction and Concept", Frege writes: "in definition it is always a matter of associating with a sign a sense or a reference. Where sense and reference are missing, we cannot properly speak either of a sign or of a definition" (CP 139, footnote 4).

[23]According to Frege, definitions must be fruitful and eliminable. For the analysis of definitions in Frege, see (Horty, 2007).

the *definiens* and the *definiendum* not simply acknowledged but stipulated. Its notational counterpart is an iterated judgment-stroke coupled again with a horizontal.

5 | CONCLUSIONS

The "horizontal stroke" (*wagerechte Strich*) is both a peculiar and a constant feature of Frege's logic. In BGS §2, the horizontal in isolation indicates a judgeable content (*Inhalt*), and in combination with the vertical stroke (*senkrechte Strich*) it indicates a judgment, that is, the recognition of that content as true. In GG §5, the horizontal sign is introduced to name a function such that its value is the True if what is preceded by the horizontal is a name of the True and is the False otherwise. In Frege's logical system, the use of this sign is governed by the rule of the *fusion of horizontals*, according to which if an occurrence of the horizontal is followed by another occurrence, either on its own or "contained" in a propositional connective, these two occurrences can be fused into a single one. Despite Frege's insistence on the use of the horizontal, its role remains controversial. Michael Dummett famously argued that the horizontal is "wholly superfluous" in Frege's system. As we saw, Dummett's main point seems to be that in *Grundgesetze* the primitive functions are total. Dummett's point would be correct if the *only* use for the horizontal was to "filter" the arguments of specific functions that can take only truth-values as their arguments. But even conceding Dummett's point about Frege's primitive functions, it would remain to explain (i) the compositionality of Frege's horizontal stroke and (ii) its interplay with the vertical stroke to form the assertion sign. As regards (i), we argued that, unlike other sentential operators, the horizontal sign is not compositional. In particular, the horizontal does not appear to abide by Dummett's "Principle K", which requires that if one sentence involves a concept that another sentence does not involve, the two sentences cannot express the same thought (*Gedanke*). We argued on the contrary that Frege's horizontal stroke should be considered as "sense-inert", that is, it contributes nothing to the sense of the whole in which it occurs. At the same time, this sign is not inert from a pragmatic point of view, since it is a device to form expressions with a judgeable content. Indeed, as regards (ii), the horizontal stroke is part of Frege's assertion sign. Recently, the assertion sign has received two slightly different interpretations: according to the slightly semantic interpretation, the assertion sign is a means to express that the truth-value named by what is proceeded by that sign is the True; according to the slightly pragmatic interpretation, the assertion sign is a signpost for a specific speech act, that is, putting forward a thought as true. We have highlighted the mutual relation between these two aspects that are essential to understand the intended meanings of the horizontal/content stroke and the assertion sign in the different phases of Frege's logical research. We finally connected our considerations to Frege's use of the horizontal as part of his "double judgment stroke".

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