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# FACTORING ISOMETRIES OF QUADRATIC SPACES INTO REFLECTIONS

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Abstract. Let $V$ be a vector space endowed with a non-degenerate quadratic form $Q$. If the base field $\mathbb{F}$ is different from $\mathbb{F}_2$, it is known that every isometry can be written as a product of reflections. In this article, we detail the structure of the poset of all minimal length reflection factorizations of an isometry. If $\mathbb{F}$ is an ordered field, we also study factorizations into positive reflections, i.e., reflections defined by vectors of positive norm. We characterize such factorizations, under the hypothesis that the squares of $\mathbb{F}$ are dense in the positive elements (this includes Archimedean and Euclidean fields). In particular, we show that an isometry is a product of positive reflections if and only if its spinor norm is positive. As a final application, we explicitly describe the poset of all factorizations of isometries of the hyperbolic space.

Let $V$ be a finite-dimensional vector space over a field $\mathbb{F}$. A quadratic form on $V$ is a map $Q\colon V \to \mathbb{F}$ such that: (1) $Q(av) = a^2 Q(v)$ for all $a \in \mathbb{F}$ and $v \in V$; (2) the polar form $\beta(u,v) = Q(u+v) - Q(u) - Q(v)$ is bilinear. When $\mathbb{F}$ is different from the two-element field $\mathbb{F}_2$, every isometry of a non-degenerate quadratic space $(V,Q)$ can be written as a product of at most $\dim V$ reflections and the minimal length of a reflection factorization is determined by geometric attributes of the isometry [Car38, Die48, Sch50, Die55, Cal76, Tay92]. For some applications, e.g. when studying dual Coxeter systems and the associated Artin groups [Bes03, BW08, McC15, MS17, PS21], more fine-grained information is useful: What is the set of all minimal length reflection factorizations? What is the combinatorial structure of the intervals in the orthogonal group $O(V,Q)$, with respect to the metric defined by the reflection length? Answers to these questions have been given for anisotropic quadratic spaces [BW02] and for (affine) Euclidean spaces [BM15]. In the first part of this paper, we give answers for general quadratic spaces. Our treatment is based on Wall's parametrization of the orthogonal group [Wal59, Wal63], which we recall in Section 1.

In the second part of this paper, we turn our attention to the case where $\mathbb{F}$ is an ordered field. We say that a reflection with respect to some vector $v \in V$ is positive if $Q(v)$ is positive. One can ask all the previous questions while restricting to factorizations into positive reflections only. The following are our main motivations for studying this problem: (1) understand reflection factorizations in Coxeter groups (which are discrete groups generated by positive reflections with respect to some quadratic form in $\mathbb{R}^n$); (2) describe reflection factorizations of isometries of the hyperbolic space $\mathbb{H}^n$. We characterize the positive reflection length of all isometries, and we describe the minimal factorizations, under the hypothesis that $\mathbb{F}$ is *square-dense*: the squares of $\mathbb{F}$ are dense in the set of positive elements. Most notably, the class of square-dense ordered fields includes all Archimedean fields (i.e., the subfields of $\mathbb{R}$) and Euclidean fields (i.e., fields where every positive element is a square). In particular, we show that an isometry can be written as a product of positive reflections if and only if its spinor norm is positive.

As an application, we study the reflection factorizations of isometries of the hyperbolic space $\mathbb{H}^n$. In the hyperboloid model $\mathbb{H}^n \subseteq \mathbb{R}^{n+1}$, the isometries of $\mathbb{H}^n$ form an index-two subgroup of the orthogonal group $O(\mathbb{R}^{n+1}, Q)$, where $Q$ is a quadratic form of signature $(n, 1)$. In fact, they are precisely the isometries of $(\mathbb{R}^{n+1}, Q)$ with a positive spinor norm. This observation allows us to give an explicit description of minimal reflection factorizations and intervals in $O(\mathbb{H}^n)$.

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## 1. Wall's parametrization of the orthogonal group

In this section, we recall Wall's parametrization of the orthogonal group of a quadratic space, which was first introduced in [Wal59]. Here and in Section 2, we give proofs for the most important results while omitting the proofs of the lemmas. We largely follow the treatment of [Tay92, Chapter 11], but the reader can also refer to [Wal59, Wal63, Hah79].

Let $V$ be a finite-dimensional vector space over a field $\mathbb{F}$. For now, no hypothesis on $\mathbb{F}$ is required. A *quadratic form* on $V$ is a map $Q\colon V \to \mathbb{F}$ such that:

(1) $Q(av) = a^2 Q(v)$ for all $a \in \mathbb{F}$ and $v \in V$;
(2) the map $\beta(u, v) = Q(u + v) - Q(u) - Q(v)$ is bilinear.

The pair $(V, Q)$ is called a *quadratic space*, and the symmetric bilinear form $\beta$ is called the *polar form* of $Q$. From now on, assume that $(V, Q)$ is a *non-degenerate* quadratic space, i.e., the polar form $\beta$ is non-degenerate: $\beta(u, v) = 0$ for all $v \in V$ implies $u = 0$.

If the characteristic of $\mathbb{F}$ is not 2, the polar form $\beta$ determines $Q$ via the relation $Q(u) = \frac{1}{2}\beta(u, u)$. On the other hand, if the characteristic of $\mathbb{F}$ is 2, $\beta$ is alternating (i.e., $\beta(u, u) = 0$ for all $u \in V$) and does not determine $Q$.

A non-zero vector $u \in V$ is *isotropic* if $\beta(u, u) = 0$ and it is *singular* if $Q(u) = 0$. These two notions coincide when the characteristic of $\mathbb{F}$ is not 2. Given a linear subspace $W \subseteq V$, its orthogonal subspace is defined as $W^\perp = \{v \in V \mid \beta(v, w) = 0 \text{ for all } w \in W\}$. A subspace $W \subseteq V$ is *totally singular* if $Q(u) = 0$ for all $u \in W$, and it is *non-degenerate* if $W \cap W^\perp = \{0\}$ (i.e., if $\beta|_W$ is non-degenerate). Since $\beta$ is non-degenerate, we have that $\dim(W) + \dim(W^\perp) = \dim(V)$ and $(W^\perp)^\perp = W$ for every subspace $W \subseteq V$. However, note that $W \cap W^\perp$ might be non-trivial, so $V$ is not necessarily the direct sum of $W$ and $W^\perp$. If $V = W_1 \oplus W_2$ and $W_1 = W_2^\perp$, we also write $V = W_1 \perp W_2$.

**Definition 1.1** (Orthogonal group)**.** The *orthogonal group* of $(V, Q)$ is

$$O(V, Q) = \{f \in \mathrm{GL}(V) \mid Q(f(u)) = Q(u) \text{ for all } u \in V\}.$$

The elements of the orthogonal group are called *isometries*. We also write $O(V)$ in place of $O(V, Q)$, since the ambient quadratic form $Q$ is always fixed.

By definition, an isometry $f \in O(V)$ also preserves the polar form $\beta$:

$$\begin{aligned}
\beta(f(u), f(v)) &= Q(f(u) + f(v)) - Q(f(u)) - Q(f(v)) \\
&= Q(f(u + v)) - Q(f(u)) - Q(f(v)) \\
&= Q(u + v) - Q(u) - Q(v) \\
&= \beta(u, v).
\end{aligned}$$

Notice that if $f\colon V \to V$ is a linear map that preserves $\beta$, then $f \in \mathrm{GL}(V)$ because $\beta$ is non-degenerate.

Our aim is to characterize the factorizations of isometries as products of reflections. A *reflection* is a non-trivial isometry that fixes every vector in a hyperplane of $V$. Every reflection can be written as

$$r_v(u) = u - \frac{\beta(u,v)}{Q(v)} v \tag{1}$$

for some non-singular vector $v \in V$ [Tay92, Theorem 11.11], and $r_v$ is called the reflection with respect to $v$. Note that $r_v = r_w$ for every non-zero scalar multiple $w$ of $v$. As a consequence of eq. (1), any reflection $r_v$ fixes the hyperplane $\langle v\rangle^\perp$, sends $v$ to $-v$, has order 2 and determinant $-1$. In particular, having order 2 is a consequence of the definition of reflection. The set of reflections is closed under conjugation: $f r_v f^{-1} = r_{f(v)}$ for every $f \in O(V)$.

The following are two important subspaces associated with an isometry.

**Definition 1.2.** Given an isometry $f \in O(V)$, its *fixed space* is $\textsc{Fix}(f) = \ker(\mathrm{id} - f)$ and its *moved space* is $\textsc{Mov}(f) = \mathrm{im}(\mathrm{id} - f)$.

The fixed space is simply the subspace of vectors that are fixed by $f$. The moved space is the subspace of "movement" vectors $f(u) - u$, for $u \in V$. It is also called the *residual space* of $f$. The notation "$\textsc{Fix}(f)$" and "$\textsc{Mov}(f)$" is the one used in [BM15], but several different notations for the moved space have appeared in the literature, including $V_f$, $[V, f]$, and $M(f)$ [Wal59, Wal63, Tay92, BW02].

**Lemma 1.3.** *For every isometry $f \in O(V)$, we have that $\textsc{Fix}(f) = \textsc{Mov}(f)^\perp$.*

Notice that an isometry $f \in O(V)$ is a reflection if and only if $\textsc{Mov}(f)$ is one-dimensional (in which case $f = r_v$ where $\textsc{Mov}(f) = \langle v\rangle$), and this happens if and only if $\textsc{Fix}(f)$ is a hyperplane (in which case $\textsc{Fix}(f) = \langle v\rangle^\perp$).

When $f$ is not a reflection, its moved space $\textsc{Mov}(f)$ does not determine $f$ uniquely. For example, if $V = \mathbb{R}^n$ and $Q$ is the standard (positive definite) quadratic form, a 2-dimensional subspace $W \subseteq V$ is the moved space of infinitely many rotations. By Lemma 1.3, each of $\textsc{Fix}(f)$ and $\textsc{Mov}(f)$ determines the other, so no additional information comes from knowing both of them. The Wall form adds the information needed to determine $f$.

**Definition 1.4** ([Wal59])**.** Let $f \in O(V)$ be an isometry. The *Wall form* of $f$ is the bilinear form $\chi_f$ on $\textsc{Mov}(f)$ defined as $\chi_f(u,v) = \beta(w,v)$, where $w \in V$ is any vector such that $u = w - f(w)$.

**Theorem 1.5.** *The Wall form $\chi_f$ is a well-defined non-degenerate bilinear form on $\textsc{Mov}(f)$, and it satisfies $\chi_f(u,u) = Q(u)$ for all $u \in \textsc{Mov}(f)$.*

*Proof.* Suppose that $u = w - f(w) = w' - f(w')$ for some $w, w' \in V$. Then $w - w' \in \textsc{Fix}(f) = \textsc{Mov}(f)^\perp$ by Lemma 1.3, and therefore $\beta(w,v) - \beta(w',v) = \beta(w - w', v) = 0$, so $\chi_f(u,v)$ is well-defined.

It is immediate to see that $\chi_f$ is a bilinear form. If $\chi_f$ is degenerate, then there is a non-zero vector $v \in \textsc{Mov}(f)$ such that $\chi_f(u,v) = 0$ for all $u \in \textsc{Mov}(f)$. Then $\beta(w,v) = 0$ for all $w \in V$. This is impossible, because $\beta$ is non-degenerate.

Finally, if $u = w - f(w)$, we have $\chi_f(u,u) = \beta(w,u) = -\beta(w,-u) = Q(w) + Q(u) - Q(w - u) = Q(w) + Q(u) - Q(f(w)) = Q(u)$. □

The Wall form $\chi_f$ is not necessarily symmetric. In fact, we show in Lemma 1.7 that $\chi_f$ is symmetric if and only if $f$ is an involution. As anticipated, the Wall form $\chi_f$ carries enough information to recover the isometry $f$.

**Theorem 1.6** (Wall's parametrization)**.** *The map $f \mapsto (\mathrm{Mov}(f), \chi_f)$ is a one-to-one correspondence between the orthogonal group $O(V)$ and the set of pairs $(W, \chi)$ such that $W$ is a subspace of $V$ and $\chi$ is a non-degenerate bilinear form on $W$ satisfying $\chi(u,u) = Q(u)$ for $u \in W$.*

*Proof.* To prove injectivity, consider two isometries $f, g \in O(V)$ such that $\mathrm{Mov}(f) = \mathrm{Mov}(g) = W$ and $\chi_f = \chi_g = \chi$. By definition of Wall form, $\chi_f(w - f(w), v) = \beta(w,v) = \chi_g(w - g(w), v)$ and therefore $\chi(w - f(w), v) = \chi(w - g(w), v)$, for every $v \in W$ and $w \in V$. Since $\chi$ is non-degenerate, this implies that $w - f(w) = w - g(w)$ for all $w \in V$, thus $f = g$.

To prove surjectivity, given a pair $(W, \chi)$, we want to construct an isometry $f \in O(V)$ such that $\mathrm{Mov}(f) = W$ and $\chi_f = \chi$. For $w \in V$, denote by $\alpha_w \in W^*$ the linear functional given by $\alpha_w(v) = \beta(w,v)$. Since $\chi$ is non-degenerate, the linear map $\varphi\colon W \to W^*$ given by $\varphi(u)(v) = \chi(u,v)$ is an isomorphism. Define $f\colon V \to V$ as follows: $f(w) = w - \varphi^{-1}(\alpha_w)$. By construction, for any $w \in V$ and $v \in W$ we have

$$\beta(w,v) = \alpha_w(v) = \varphi(w - f(w))(v) = \chi(w - f(w), v). \tag{2}$$

This allows us to check that $f$ is an isometry. Indeed, by setting $v = w - f(w)$ in eq. (2) we obtain

$$\begin{aligned}\beta(w, w - f(w)) = \chi(w - f(w), w - f(w)) &= Q(w - f(w)) \\ &= Q(w) + Q(f(w)) - \beta(w, f(w)),\end{aligned}$$

which simplifies to $Q(f(w)) = Q(w)$. By definition of $f$, we immediately see that $\mathrm{Mov}(f) = W$, and eq. (2) implies that $\chi = \chi_f$. ∎

We now list some properties of the Wall form.

**Lemma 1.7.** *For every $f \in O(V)$ and $u, v \in \mathrm{Mov}(f)$, the following properties hold.*

*(i)* $\chi_f(u,v) + \chi_f(v,u) = \beta(u,v)$.
*(ii)* $\chi_f(f(u), v) = -\chi_f(v,u)$.
*(iii)* $\mathrm{Mov}(f) = \mathrm{Mov}(f^{-1})$ *and* $\chi_{f^{-1}}(u,v) = \chi_f(v,u)$.
*(iv)* $\mathrm{Mov}(gfg^{-1}) = g(\mathrm{Mov}(f))$ *and* $\chi_{gfg^{-1}}(g(u), g(v)) = \chi_f(u,v)$ *for every* $g \in O(V)$.
*(v)* $\chi_f$ *is symmetric if and only if* $f$ *is an involution.*

Fix a subspace $W \subseteq V$, and look at all isometries $f \in O(V)$ such that $\mathrm{Mov}(f) = W$. Property (i) of Lemma 1.7 says that the symmetrization of the Wall form $\chi_f$ is necessarily equal to the ambient bilinear form $\beta$ (restricted to $W = \mathrm{Mov}(f)$). In particular, if $W$ is non-degenerate and the characteristic of $\mathbb{F}$ is not 2, there is exactly one isometry $f$ such that $\mathrm{Mov}(f) = W$ and $\chi_f$ is symmetric, and $f$ is an involution by property (v). On the opposite side, if $W$ is totally singular, then $\chi_f$ is alternating by Theorem 1.5. In this case, isometries $f$ with $\mathrm{Mov}(f) = W$ only exist if $\dim W$ is even (otherwise every alternating bilinear form on $W$ is degenerate, as the rank is necessarily even; see for example [Gro02, Theorem 2.10]).

## 2. Factorizations and reflection length

In this section, we continue to follow [Wal59] and [Tay92, Chapter 11] and show how Wall's parametrization leads to a nice procedure to build factorizations of isometries. For a field $\mathbb{F} \neq \mathbb{F}_2$, this allows proving that any isometry $f \in O(V)$ can be written as a product of reflections. It also allows us to characterize the *reflection length*, i.e., the minimal length $k$ of a factorization $f = r_1 r_2 \cdots r_k$ as a product of reflections. We refer to [Tay92, Theorem 11.41] for the case $\mathbb{F} = \mathbb{F}_2$, which we do not treat here. Finally, at the end of this section, we introduce the spinor norm.

**Definition 2.1** (Orthogonal complements)**.** Let $\chi$ be a non-degenerate bilinear form on a finite-dimensional vector space $W$. Define the *left* and *right orthogonal complement* of a subspace $U \subseteq W$ as

$$U^{\triangleleft} = \{v \in W \mid \chi(v, u) = 0 \text{ for all } u \in U\}$$
$$U^{\triangleright} = \{v \in W \mid \chi(u, v) = 0 \text{ for all } u \in U\},$$

respectively.

Since $\chi$ is non-degenerate, we have that $\dim U^{\triangleleft} = \dim U^{\triangleright} = \dim W - \dim U$. As an immediate consequence, $(U^{\triangleright})^{\triangleleft} = (U^{\triangleleft})^{\triangleright} = U$. We will mostly use this notation in the case where $\chi = \chi_f$ is the Wall form of an isometry $f \in O(V)$ and $W = \mathrm{Mov}(f)$.

The following is the basic building block that allows us to construct factorizations of isometries.

**Theorem 2.2** (Factorization theorem)**.** *Let $f \in O(V)$ be an isometry, and let $U_1 \subseteq \mathrm{Mov}(f)$ be a subspace such that the restriction $\chi_1 = \chi_f|_{U_1}$ is non-degenerate. Let $U_2 = U_1^{\triangleright}$ (respectively, $U_2 = U_1^{\triangleleft}$), and $\chi_2 = \chi_f|_{U_2}$. Denote by $f_1$ and $f_2$ the elements of $O(V)$ associated with $(U_1, \chi_1)$ and $(U_2, \chi_2)$ under Wall's parametrization.*

- *(a) $\mathrm{Mov}(f) = U_1 \oplus U_2$, and $f = f_1 f_2$ (respectively, $f = f_2 f_1$).*
- *(b) $f_1 f_2 = f_2 f_1$ if and only if $\mathrm{Mov}(f) = U_1 \perp U_2$. In this case, $f_1$ coincides with $f$ on $U_2^{\perp}$, and $f_2$ coincides with $f$ on $U_1^{\perp}$.*

*Conversely, every factorization $f = f_1 f_2$ with $\mathrm{Mov}(f) = \mathrm{Mov}(f_1) \oplus \mathrm{Mov}(f_2)$ arises in this way.*

*Proof.* We prove part (a) in the case $U_2 = U_1^{\triangleright}$, the case $U_2 = U_1^{\triangleleft}$ being analogous. Since $\chi_1$ is non-degenerate, no non-zero vector of $U_1$ can be right-orthogonal to all of $U_1$. This means that $U_1 \cap U_2 = \{0\}$. We also have $\dim U_1 + \dim U_2 = \dim \mathrm{Mov}(f)$, and therefore $\mathrm{Mov}(f) = U_1 \oplus U_2$.

Notice that $\chi_2$ is non-degenerate because $\chi_f$ is non-degenerate, so $f_2$ is well-defined. To prove that $f = f_1 f_2$, consider the following chain of equalities that holds for every $w \in V$, $u_1 \in U_1$, and $u_2 \in U_2$:

$$\begin{aligned}
\chi_f\big(w - f_1 f_2(w), u_1 + u_2\big) &= \chi_f\big(w - f_2(w) + f_2(w) - f_1 f_2(w), u_1 + u_2\big) \\
&= \chi_f(w - f_2(w), u_1 + u_2) + \chi_f\big((\mathrm{id} - f_1) f_2(w), u_1 + u_2\big) \\
&\overset{(1)}{=} \chi_f\big(w - f_2(w), u_1\big) + \chi_f\big(w - f_2(w), u_2\big) + \chi_f\big((\mathrm{id} - f_1) f_2(w), u_1\big) \\
&\overset{(2)}{=} \beta\big(w - f_2(w), u_1\big) + \beta(w, u_2) + \beta\big(f_2(w), u_1\big)
\end{aligned}$$

$$= \beta(w, u_1 + u_2)$$
$$= \chi_f\big(w - f(w), u_1 + u_2\big).$$

Here (1) follows from bilinearity of $\chi_f$, the term $\chi_f\big((\mathrm{id} - f_1)f_2(w), u_2\big)$ vanishing because $(\mathrm{id} - f_1)f_2(w) \in \mathrm{Mov}(f_1) = U_1$ and $u_2 \in U_2$; in (2), the first term is rewritten using property (i) of Lemma 1.7, whereas the other two terms are rewritten using the definitions of $\chi_1$ and $\chi_2$. From the previous equalities and the fact that $\chi_f$ is non-degenerate, it follows that $w - f_1f_2(w) = w - f(w)$ for all $w \in V$, so $f = f_1f_2$.

We now prove part (b). Suppose that $f_1f_2 = f_2f_1$. By property (iv) of Lemma 1.7, $f$ fixes $\mathrm{Mov}(f_1) = U_1$. Then, by property (ii), we have that $\chi_f(u_2, u_1) = -\chi_f(f(u_1), u_2) = 0$ for all $u_1 \in U_1$ and $u_2 \in U_2$. Therefore $U_2 = U_1^{\triangleright} = U_1^{\triangleleft}$. Property (i) implies that $\mathrm{Mov}(f) = U_1 \perp U_2$.

Conversely, suppose that $\mathrm{Mov}(f) = U_1 \perp U_2$. Since $U_2 = U_1^{\triangleright}$, property (i) of Lemma 1.7 implies that $U_2 = U_1^{\triangleleft}$. By the first part of this theorem, we obtain that $f = f_2f_1$, and therefore $f_1f_2 = f_2f_1$. In addition, $\mathrm{Fix}(f_2) = \mathrm{Mov}(f_2)^{\perp} = U_2^{\perp}$, and thus $f(v) = f_1f_2(v) = f_1(v)$ for every $v \in U_2^{\perp}$. Similarly, $f(v) = f_2f_1(v) = f_2(v)$ for every $v \in U_1^{\perp}$.

Finally, given any factorization $f = f_1f_2$ such that $\mathrm{Mov}(f) = \mathrm{Mov}(f_1) \oplus \mathrm{Mov}(f_2)$, we need to show that $\chi_f|_{\mathrm{Mov}(f_2)} = \chi_{f_2}$. Let $u, v \in \mathrm{Mov}(f_2)$. By definition of $\chi_f$, we have that $\chi_f(u, v) = \beta(w, v)$, where $w \in V$ is a vector such that $u = w - f(w)$. Now write $u = w - f_2(w) + f_2(w) - f_1f_2(w)$, and notice that $w - f_2(w) \in \mathrm{Mov}(f_2)$ and $f_2(w) - f_1f_2(w) \in \mathrm{Mov}(f_1)$. Since $u \in \mathrm{Mov}(f_2)$ and $\mathrm{Mov}(f) = \mathrm{Mov}(f_1) \oplus \mathrm{Mov}(f_2)$, we have that $u = w - f_2(w)$. Then $\chi_{f_2}(u, v) = \beta(w, v) = \chi_f(u, v)$. □

From the definition of moved space, it is easy to see that $\mathrm{Mov}(f_1f_2) \subseteq \mathrm{Mov}(f_1) + \mathrm{Mov}(f_2)$ for any two isometries $f_1, f_2 \in O(V)$. Theorem 2.2 allows to construct factorizations $f = f_1f_2$ where the equality $\mathrm{Mov}(f_1f_2) = \mathrm{Mov}(f_1) \oplus \mathrm{Mov}(f_2)$ holds. These are called *direct factorizations* in [Wal59]. More generally, we give the following definition.

**Definition 2.3** (Direct factorization)**.** A factorization $f = f_1 \cdots f_k$ is called a direct factorization if $\mathrm{Mov}(f) = \mathrm{Mov}(f_1) \oplus \cdots \oplus \mathrm{Mov}(f_k)$ and no $f_i$ is the identity.

Recall that the reflections are precisely the isometries with a one-dimensional moved space. The relation $\mathrm{Mov}(f_1f_2) \subseteq \mathrm{Mov}(f_1) + \mathrm{Mov}(f_2)$ yields a lower bound on the reflection length of an isometry $f \in O(V)$: if $f = r_1 \cdots r_k$ is a product of $k$ reflections, then $\mathrm{Mov}(f) \subseteq \mathrm{Mov}(r_1) + \cdots + \mathrm{Mov}(r_k)$, so $k \geq \dim \mathrm{Mov}(f)$. This lower bound is attained precisely when the factorization is direct. In the rest of this section, we are going to see that most isometries admit a direct factorization, but not all of them.

**Lemma 2.4.** *Let $\chi$ be a non-degenerate bilinear form on a finite-dimensional vector space $W$ over a field $\mathbb{F} \neq \mathbb{F}_2$. If $\chi$ is not alternating, then $W$ has a basis $e_1, \ldots, e_m$ such that $\chi(e_i, e_i) \neq 0$ for all $i$, and $\chi(e_i, e_j) = 0$ for $i < j$.*

**Remark 2.5.** It is worth noting that Lemma 2.4 is false for $\mathbb{F} = \mathbb{F}_2$. See [Tay92, Chapter 11] for additional details.

The following lemma describes how the moved space changes when multiplying an isometry by a reflection.

**Lemma 2.6.** *Let $f \in O(V)$ be an isometry, and let $v \in V$ be a non-singular vector.*

*(a) If $v \in \textsc{Mov}(f)$, then $\textsc{Mov}(r_v f) = \langle v \rangle^{\triangleright}$, where the right orthogonal complement is taken inside $\textsc{Mov}(f)$ with respect to the Wall form $\chi_f$. In particular, $\dim \textsc{Mov}(r_v f) = \dim \textsc{Mov}(f) - 1$.*
*(b) If $v \notin \textsc{Mov}(f)$, then $\textsc{Mov}(r_v f) = \textsc{Mov}(f) \oplus \langle v \rangle$. In particular, $\dim \textsc{Mov}(r_v f) = \dim \textsc{Mov}(f) + 1$.*

*As a consequence, if $f$ is a product of $k$ reflections, then $\dim \textsc{Mov}(f) \equiv k \pmod 2$.*

We are now ready to give a simple formula for the reflection length of any isometry. In the case of fields of characteristic $\neq 2$, this result was first proved by Scherk [Sch50].

**Theorem 2.7** (Reflection length)**.** *Assume $\mathbb{F} \neq \mathbb{F}_2$, and let $f \in O(V)$ be an isometry different from the identity. The reflection length of $f$ is equal to $\dim \textsc{Mov}(f)$ if $\textsc{Mov}(f)$ is not totally singular, and to $\dim \textsc{Mov}(f) + 2$ otherwise. In particular, every isometry can be written as a product of at most $\dim V$ reflections.*

*Proof.* If $\textsc{Mov}(f)$ is not totally singular, then Lemma 2.4 applies to the Wall form $\chi_f$ and yields a basis of $\textsc{Mov}(f)$ consisting of non-singular vectors $e_1, \dots, e_m$ such that $\chi(e_i, e_j) = 0$ for $i < j$. By a repeated application of Theorem 2.2, we get a direct factorization $f = r_{e_1} \cdots r_{e_m}$ of length $m = \dim \textsc{Mov}(f)$.

Suppose now that $\textsc{Mov}(f)$ is totally singular. Choose any non-singular vector $v \in V$, and consider $g = r_v f$. By Lemma 2.6, we have $\textsc{Mov}(g) = \textsc{Mov}(f) \oplus \langle v \rangle$. In particular, $\textsc{Mov}(g)$ contains the non-singular vector $v$, so by the previous part $g$ can be written as a product of $\dim \textsc{Mov}(g)$ reflections. Then $f$ can be written as a product of $\dim \textsc{Mov}(g) + 1 = \dim \textsc{Mov}(f) + 2$ reflections. It is not possible to use less than $\dim \textsc{Mov}(f) + 2$ reflections: a factorization into $\dim \textsc{Mov}(f)$ reflections would be a direct factorization, which does not exist because $\textsc{Mov}(f)$ is totally singular; a factorization into $\dim \textsc{Mov}(f) + 1$ reflections does not exist by the last part of Lemma 2.6.

Finally, we want to show that the reflection length is always at most $\dim V$. This is immediate if $\textsc{Mov}(f)$ is not totally singular, so assume now that $\textsc{Mov}(f)$ is totally singular. Since $\chi_f$ is non-degenerate, we have $\dim \textsc{Mov}(f) \geq 2$. On the other hand, $\dim \textsc{Mov}(f)$ is bounded above by the Witt index of $\beta$, which is at most $\frac{1}{2} \dim V$. Therefore the reflection length is $\dim \textsc{Mov}(f) + 2 \leq 2 \dim \textsc{Mov}(f) \leq \dim V$. □

In the final part of this section, we introduce the spinor norm following [Wal59, Section 4]. See also [Zas62, Hah79, Sch12]. Let $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$.

**Definition 2.8** (Wall's spinor norm)**.** The *spinor norm* is the map $\theta \colon O(V) \to \mathbb{F}^\times/(\mathbb{F}^\times)^2$ defined as $\theta(f) = [\det(A)]$, where $A$ is the matrix of $\chi_f$ with respect to any basis of $\textsc{Mov}(f)$. Here $[a]$ indicates the class of $a \in \mathbb{F}^\times$ in the quotient group $\mathbb{F}^\times/(\mathbb{F}^\times)^2$.

Note that $\det(A) \neq 0$ because $\chi_f$ is non-degenerate, and $\theta(f)$ does not depend on the choice of the basis. For example, we have $\theta(\mathrm{id}) = 1$ and $\theta(r_v) = [Q(v)]$ for every non-singular vector $v \in V$. The following lemma follows immediately from Theorem 2.2.

**Lemma 2.9.** *Given a direct factorization $f = f_1 f_2$, we have $\theta(f) = \theta(f_1)\theta(f_2)$.*

**Theorem 2.10.** *The spinor norm is a group homomorphism.*

*Proof.* If $\mathbb{F} = \mathbb{F}_2$, the spinor norm is trivial, so we can assume from now on that $\mathbb{F} \neq \mathbb{F}_2$. Then $O(V)$ is generated by reflections by Theorem 2.7. Therefore it is enough to show that, for every factorization $f = r_1 \cdots r_k$ into reflections, we have $\theta(f) = \theta(r_1) \cdots \theta(r_k)$. We prove this by induction on $k$, the cases $k = 0$ and $k = 1$ being trivial.

Fix a length $k$ reflection factorization $f = r_1 \cdots r_k$ with $k \geq 2$. Let $g = r_1 f = r_2 \cdots r_k$. If $f = r_1 g$ is a direct factorization, then $\theta(f) = \theta(r_1)\theta(g)$ by Lemma 2.9. If $f = r_1 g$ is not a direct factorization, then $g = r_1 f$ is a direct factorization by Lemma 2.6, and $\theta(g) = \theta(r_1)\theta(f)$ by Lemma 2.9. Since all non-trivial elements of $\mathbb{F}^\times/(\mathbb{F}^\times)^2$ have order 2, we have $\theta(f) = \theta(r_1)\theta(g)$ in both cases. By induction, $\theta(g) = \theta(r_2) \cdots \theta(r_k)$ and thus $\theta(f) = \theta(r_1)\theta(g) = \theta(r_1) \cdots \theta(r_k)$. □

## 3. Partial order on the orthogonal group

In this section, we introduce the partial order on $O(V)$ naturally induced by minimal reflection factorizations. It generalizes the partial order of [BW02]. We show that for most isometries $f \in O(V)$, the interval $[\mathrm{id}, f]$ naturally includes into the poset (i.e., partially ordered set) of subspaces of $\mathrm{Mov}(f)$. We assume throughout this section that $\mathbb{F} \neq \mathbb{F}_2$, so that Theorem 2.7 applies.

**Definition 3.1** (Partial order on $O(V)$)**.** Given two isometries $f, g \in O(V)$, define $g \leq f$ if and only if $f$ admits a minimal length reflection factorization that starts with a minimal length reflection factorization of $g$. Equivalently, $g \leq f$ if and only if $l(f) = l(g) + l(g^{-1}f)$, where $l\colon O(V) \to \mathbb{N}$ denotes the reflection length.

Since the set of reflections is closed under conjugation, it is equivalent to require that $f$ admits a minimal factorization that *ends* with a minimal factorization of $g$. Notice that $O(V)$ is ranked (in the sense of posets) by the reflection length $l$, and it has the identity as the unique $\leq$-minimal element. This partial order was studied in [BW02] for isometries of an anisotropic bilinear form $\beta$, and in [BM15] for isometries of the affine Euclidean space.

Although the global combinatorics of $O(V)$ is complicated, most of the intervals

$$[g, f] = \{h \in O(V) \mid g \leq h \leq f\} \quad \text{for } g \leq f$$

have a structure that we can explicitly describe. Notice that the interval $[g, f]$ is isomorphic (as a poset) to the interval $[\mathrm{id}, g^{-1}f]$ via the isomorphism $h \mapsto g^{-1}h$. Therefore, the combinatorial study of all intervals in $O(V)$ reduces to the study of the intervals of the form $[\mathrm{id}, f]$.

Recall from Section 2 that the reflection length of an isometry $f \in O(V)$ is at least $\dim \mathrm{Mov}(f)$, and the reflection factorizations of length $\dim \mathrm{Mov}(f)$ (if they exist) are the direct factorizations. In light of Theorem 2.7, we can characterize in a couple of different ways the isometries $f$ with reflection length equal to $\dim \mathrm{Mov}(f)$.

**Definition 3.2.** An isometry $f \in O(V)$ is *minimal* if any of the following equivalent conditions hold:

(i) $f$ admits a direct factorization as a product of reflections;
(ii) its reflection length is equal to $\dim \mathrm{Mov}(f)$;
(iii) $f = \mathrm{id}$, or $\mathrm{Mov}(f)$ is not totally singular.

Roughly speaking, condition (iii) tells us that most isometries are minimal. There are many simple sufficient conditions for an isometry to be minimal: if $\dim \mathrm{Mov}(f) > \frac{1}{2} \dim V$, then $f$ is minimal; if $\dim \mathrm{Mov}(f)$ is odd, then $f$ is minimal (because all alternating forms are degenerate, so $\chi_f$ is not alternating); if $V$ contains no singular vectors, then all isometries are minimal.

**Remark 3.3.** If the characteristic of $\mathbb{F}$ is not 2, there are several additional conditions equivalent to Definition 3.2. In fact, the moved space $\operatorname{Mov}(f)$ is totally singular if and only if $\beta$ vanishes on $\operatorname{Mov}(f)$, which happens if and only if the Wall form $\chi_f$ is skew-symmetric (by property (i) of Lemma 1.7). In addition, it is noted in [Gro02, Corollary 6.3] that $\operatorname{Mov}(f)$ is totally singular if and only if $(f-\mathrm{id})^2 = 0$ (i.e., the unipotency index of $f$ is 2), or equivalently $\operatorname{Mov}(f) \subseteq \operatorname{Fix}(f)$. See also [Nok17].

In what follows, we aim to describe the combinatorics of the interval $[\mathrm{id}, f]$ associated with a minimal isometry $f$.

**Lemma 3.4.** *Let $f \in O(V)$ be a minimal isometry, and let $g \leq f$. Then:*

*(a)* $\operatorname{Mov}(g) \subseteq \operatorname{Mov}(f)$*;*
*(b) $g$ is minimal;*
*(c) $\chi_g$ is the restriction of $\chi_f$ to* $\operatorname{Mov}(g)$*.*

*Proof.* Let $k = \dim \operatorname{Mov}(f)$. Since $f$ is minimal, its reflection length is equal to $k$, and $\operatorname{Mov}(f) = \operatorname{Mov}(r_1) \oplus \cdots \oplus \operatorname{Mov}(r_k)$ for every minimal length factorization $f = r_1 \cdots r_k$ of $f$ as a product of reflections. Then there is one such factorization for which $g = r_1 \cdots r_m$ for some $m \leq k$, and the reflection length of $g$ is equal to $m$. By a repeated application of part (b) of Lemma 2.6, we get that $\operatorname{Mov}(g) = \operatorname{Mov}(r_1) \oplus \cdots \oplus \operatorname{Mov}(r_m) \subseteq \operatorname{Mov}(f)$. In addition, the reflection factorization $g = r_1 \cdots r_m$ is a direct factorization, so $g$ is minimal.

If $g = f$, then $\chi_g = \chi_f$ and we are done. Suppose now that $g \neq f$, i.e., $m < k$. Since $\operatorname{Mov}(r_k)$ is 1-dimensional, the property $\chi(u, u) = Q(u)$ (Theorem 1.5) implies that $\chi_{r_k}$ is the restriction of $\chi_f$ to $\operatorname{Mov}(r_k)$. By Theorem 2.2, $\chi_{r_1 \cdots r_{k-1}}$ is the restriction of $\chi_f$ to $\operatorname{Mov}(r_1 \cdots r_{k-1})$. Now, $f' := r_1 \cdots r_{k-1}$ is minimal by part (b), and $g \leq f'$, so we are done by induction on $k$. □

In the full group $O(V)$, there can be many isometries with the same moved space. However, once we restrict to an interval $[\mathrm{id}, f]$ where $f$ is minimal, an isometry is completely determined by its moved space.

**Theorem 3.5** (Minimal intervals)**.** *Let $f \in O(V)$ be a minimal isometry. Then $g \mapsto \operatorname{Mov}(g)$ is an order-preserving bijection between the interval $[\mathrm{id}, f]$ and the poset of linear subspaces $U \subseteq \operatorname{Mov}(f)$ that satisfy the following conditions:*

*(i) $U = \{0\}$ or $U$ is not totally singular;*
*(ii) $U^{\triangleright} = \{0\}$ or $U^{\triangleright}$ is not totally singular;*
*(iii) $\chi_f|_U$ is non-degenerate.*

*In addition, the rank of $g \in [\mathrm{id}, f]$ is equal to* $\dim \operatorname{Mov}(g)$*.*

*Proof.* Let $g \in [\mathrm{id}, f]$, and let $U = \operatorname{Mov}(g)$. We have that $g$ is minimal by Lemma 3.4, so $U$ satisfies condition (i). In addition, we have $U^{\triangleright} = \operatorname{Mov}(g^{-1}f)$ by Theorem 2.2, and $g^{-1}f \in [\mathrm{id}, f]$ is also minimal, so condition (ii) is satisfied. Finally, condition (iii) is a consequence of Theorem 2.2.

We now explicitly construct the inverse map $\phi$. Suppose that $U \subseteq \operatorname{Mov}(f)$ satisfies all three conditions. By Theorem 2.2 and condition (iii), there is a direct factorization $f = f_1 f_2$ where $f_1$ is the isometry associated with $(U, \chi_f|_U)$. By conditions (i) and (ii), both $f_1$ and $f_2$ are minimal. Then their reflection lengths are $\dim \operatorname{Mov}(f_1)$ and $\dim \operatorname{Mov}(f_2)$, which add up to $\dim \operatorname{Mov}(f)$. Therefore $f_1 \in [\mathrm{id}, f]$. Define $\phi(U) = f_1$.

We now check that $\phi$ is indeed the inverse of $\text{Mov}$. For any isometry $g \in [\mathrm{id}, f]$, we have that $g' = \phi(\text{Mov}(g))$ is an isometry such that $\text{Mov}(g') = \text{Mov}(g)$, and $\chi_{g'} = \chi_f|_{\text{Mov}(g)}$. By Lemma 3.4, we also have that $\chi_g = \chi_f|_{\text{Mov}(g)}$. This means that $g'$ and $g$ have the same moved space and the same Wall form, so $g' = g$ by Theorem 1.6. In addition, for any subspace $U \subseteq \text{Mov}(f)$ satisfying conditions (i)-(iii), we have that $\text{Mov}(\phi(U)) = U$ by construction of $\phi$.

If $g \leq g'$ in $[\mathrm{id}, f]$, then $g'$ is minimal by part (b) of Lemma 3.4, and $\text{Mov}(g) \subseteq \text{Mov}(g')$ by part (a) of Lemma 3.4. This means that the bijection $g \mapsto \text{Mov}(g)$ is order-preserving. Finally, the rank of an isometry $g$ in $[\mathrm{id}, f]$ is given by its reflection length, which is equal to $\dim \text{Mov}(g)$ because $g$ is minimal. $\square$

For every $U \subseteq \text{Mov}(f)$, we have that $U^{\triangleleft} = f(U^{\triangleright})$ by property (ii) of Lemma 1.7, so $U^{\triangleleft}$ and $U^{\triangleright}$ are isometric. In particular, $U^{\triangleleft}$ is totally singular if and only if $U^{\triangleright}$ is totally singular, and this gives an equivalent way to write condition (ii) of Theorem 3.5. Note that condition (ii) is not redundant, due to the following example.

**Example 3.6.** Consider an isometry $f$ with a 3-dimensional moved space and a Wall form given by the following matrix, with respect to some basis $e_1, e_2, e_3$ of $\text{Mov}(f)$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

If $U_1 = \langle e_1 \rangle$ and $U_2 = U_1^{\triangleright} = \langle e_2, e_3 \rangle$, then Theorem 2.2 yields a direct factorization $f = f_1 f_2$ such that $\chi_{f_1} = \chi_f|_{U_1}$ is not alternating, whereas $\chi_{f_2} = \chi_f|_{U_2}$ is alternating. Then $f_1$ is minimal, and $f_2$ is not. As a consequence, we have $f_1 \not\leq f$ despite the inclusion $\text{Mov}(f_1) \subseteq \text{Mov}(f)$.

Notice that the bijection $g \mapsto \text{Mov}(g)$ of Theorem 3.5 is not a poset isomorphism. Indeed, it is possible to have elements $g, g' \in [\mathrm{id}, f]$ with $g \not\leq g'$ but $\text{Mov}(g) \subseteq \text{Mov}(g')$. We construct such a case in the following example.

**Example 3.7.** Consider an isometry $f$ with a 4-dimensional moved space and a Wall form given by the following matrix, with respect to some basis $e_1, e_2, e_3, e_4$ of $\text{Mov}(f)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Theorem 3.5, the subspaces $U = \langle e_1 \rangle$ and $U' = \langle e_1, e_2, e_3 \rangle$ have associated isometries $g, g' \in [\mathrm{id}, f]$ with $\text{Mov}(g) = U$ and $\text{Mov}(g') = U'$. Then $\text{Mov}(g) \subseteq \text{Mov}(g')$, but $g \not\leq g'$ as seen in Example 3.6.

In the case where the bilinear form $\beta$ is anisotropic, we recover the description of the intervals in $O(V)$ given in [BW02]. In fact, the same description is obtained in the more general setting where $V$ contains no singular vectors.

**Corollary 3.8.** *Suppose that $V$ contains no singular vectors, and let $f \in O(V)$ be any isometry. Then $f$ is minimal, and $g \mapsto \operatorname{Mov}(g)$ is an isomorphism between the interval $[\mathrm{id}, f]$ and the poset of all linear subspaces $U \subseteq \operatorname{Mov}(f)$.*

*Proof.* We already noted that every isometry $f$ is minimal if $V$ contains no singular vectors. To prove that $g \mapsto \operatorname{Mov}(g)$ is an order-preserving bijection, it is enough to apply Theorem 3.5 and show that conditions (i)–(iii) are satisfied by every subspace $U \subseteq \operatorname{Mov}(f)$. Conditions (i) and (ii) are trivially satisfied because $\{0\}$ is the only totally singular subspace of $V$. For condition (iii), $\chi_f(u, u) = Q(u) \neq 0$ for any non-zero vector $u \in U$, so $\chi_f|_U$ is non-degenerate. To conclude the proof, we need to show that $\operatorname{Mov}(g) \subseteq \operatorname{Mov}(g')$ implies $g \leq g'$ for every $g, g' \in [\mathrm{id}, f]$. If we define $h = g^{-1}g'$, we obtain that $g' = gh$ is a direct factorization by Theorem 2.2. Since $h$ is minimal, we deduce that $l(g') = l(g) + l(h)$ and therefore $g \leq g'$. ☐

In the last part of this section, we turn our attention to non-minimal isometries, which behave in a substantially different way.

**Theorem 3.9.** *Let $f \in O(V)$ be a non-minimal isometry.*

*(a) For every reflection $r \in O(V)$, we have $r < f$ and $rf < f$.*
*(b) Every isometry $g < f$ is minimal.*
*(c) $f$ is $\leq$-maximal in $O(V)$.*

*Proof.* In the proof of Theorem 2.7, it is shown that any reflection $r \in O(V)$ is part of some minimal length reflection factorization of $f$. This implies both $r \leq f$ and $rf \leq f$. Note that $r \neq f$ because every reflection is minimal, and clearly $rf \neq f$, so the strict relations of part (a) hold. From that proof it is also clear that $rf$ is minimal, so every isometry $g < f$ is minimal by Lemma 3.4, proving part (b). Part (c) follows from Lemma 3.4 and part (b). ☐

In the following, we give a coarse description of the structure of $[\mathrm{id}, f]$ for a non-minimal isometry $f$. Note that $[\mathrm{id}, f]$ contains multiple isometries with the same moved space, so a bijection like the one of Theorem 3.5 does not exist. Denote by $(\mathrm{id}, f) = [\mathrm{id}, f] \setminus \{\mathrm{id}, f\}$ the open interval between the identity and $f$. Let $\mathcal{W}_f$ be the set of all subspaces $W \subseteq V$ containing $\operatorname{Mov}(f)$ as a codimension-one subspace and not totally singular. For any subspace $W \in \mathcal{W}_f$, let $P_{f,W} = \{g \in (\mathrm{id}, f) \mid \operatorname{Mov}(g) \subseteq W\}$.

**Theorem 3.10** (Non-minimal intervals)**.** *Let $f \in O(V)$ be a non-minimal isometry. As a poset, the open interval $(\mathrm{id}, f)$ is the disjoint union (also called "parallel composition") of the subposets $P_{f,W}$:*

$$(\mathrm{id}, f) = \bigsqcup_{W \in \mathcal{W}_f} P_{f,W}.$$

*Proof.* Let $g \in (\mathrm{id}, f)$. Then $g \leq rf$ for some reflection $r$, and $rf$ is minimal by Theorem 3.9. Since $f$ is non-minimal, $\operatorname{Mov}(f)$ is a codimension-one subspace of $W = \operatorname{Mov}(rf)$ by part (b) of Lemma 2.6. Then $W \in \mathcal{W}_f$ because $rf$ is minimal, and $g \in P_{f,W}$ by Lemma 3.4.

Let $W' \in \mathcal{W}_f$ be any subspace such that $g \in P_{f,W'}$. Note that $g$ is minimal by Theorem 3.9, so $\operatorname{Mov}(g) \not\subseteq \operatorname{Mov}(f)$. Since $\operatorname{Mov}(f)$ is a codimension-one subspace of $W'$, we have that $W' = \operatorname{Mov}(f) + \operatorname{Mov}(g)$. Therefore $W'$ is uniquely determined by $f$ and $g$. In other words, $g$ is contained in exactly one $P_{f,W'}$.

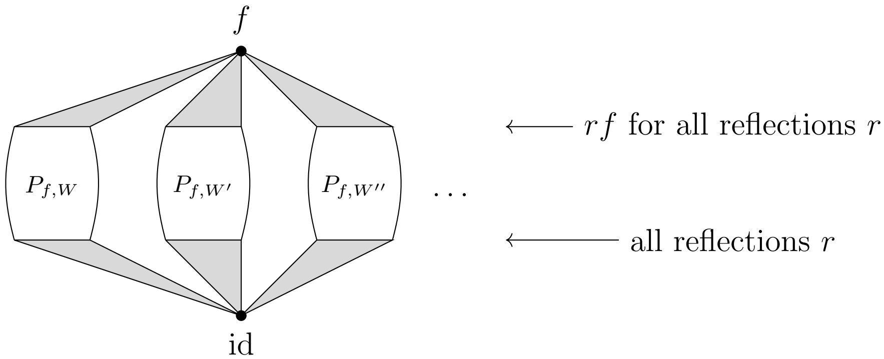


FIGURE 1. Coarse structure of an interval $[\mathrm{id}, f]$ for a non-minimal isometry $f$, as described by Theorem 3.10.

Finally, if $g \in P_{f,W}$ and $g' \leq g$, then $\mathrm{Mov}(g') \subseteq \mathrm{Mov}(g)$ by Lemma 3.4 and therefore $g' \in P_{f,W} \cup \{\mathrm{id}\}$. This means that there is no order relation between $P_{f,W}$ and $P_{f,W'}$ if $W \neq W'$. □

Figure 1 shows the Hasse diagram of a non-minimal interval $[1, f]$, as described by the previous theorem. Note that each subposet $P_{f,W}$ is self-dual: the map $g \mapsto g^{-1}f$ is an order-reversing bijection from $P_{f,W}$ to itself.

## 4. Positive factorizations

Let $(V, Q)$ be a non-degenerate quadratic space over an ordered field $\mathbb{F}$. In particular, $\mathbb{F}$ has characteristic 0. A non-singular vector $v \in V$ is said to be *positive* if $Q(v) > 0$, and *negative* if $Q(v) < 0$. In this section we focus on the factorizations of isometries into *positive reflections*, i.e., reflections with respect to positive vectors. We refer to these factorizations as *positive reflection factorizations*. Under the hypothesis that $\mathbb{F}$ is *square-dense* (the squares are dense in the positive elements), we obtain a clean description of the minimal length of a positive reflection factorization of any isometry $f \in O(V)$. In particular, we show that $f$ admits a positive reflection factorization if and only if its spinor norm is positive.

Recall that a subspace $W \subseteq V$ is *positive definite* (resp. *negative definite*) if $Q(v) > 0$ (resp. $< 0$) for every non-zero vector $v \in W$. It is *positive semi-definite* (resp. *negative semi-definite*) if $Q(v) \geq 0$ (resp. $\leq 0$) for all $v \in W$. By the inertia theorem of Jacobi and Sylvester [Sch12, Theorem 4.4], $V$ can be decomposed as an orthogonal direct sum $V^+ \perp V^-$, where $V^+$ is a positive definite subspace and $V^-$ is a negative definite subspace. The dimensions of $V^+$ and $V^-$ do not depend on the chosen decomposition, and the pair $(\dim V^+, \dim V^-)$ is called the *signature* of $(V, Q)$. We refer to [Sch12] for additional theory on quadratic spaces over ordered fields. We assume from now on that $V$ is not negative definite, because otherwise there are no positive vectors.

Denote by $\mathbb{F}^+ \subseteq \mathbb{F}$ the subset of all positive elements of $\mathbb{F}$. Since $(\mathbb{F}^\times)^2 \subseteq \mathbb{F}^+$, there is a well-defined quotient map $\pi\colon \mathbb{F}^\times/(\mathbb{F}^\times)^2 \to \mathbb{F}^\times/\ \mathbb{F}^+ \cong \mathbb{Z}_2$. In other words, every element of $\mathbb{F}^\times/(\mathbb{F}^\times)^2$ is either positive or negative, and this notion is well-defined.

**Definition 4.1.** An isometry $f \in O(V)$ is *positive* (resp. negative) if its spinor norm $\theta(f)$ is positive (resp. negative).

Notice that this definition is compatible with the previous definition of positive reflection: a reflection $r_v$ is positive if and only if $Q(v) > 0$. The positive isometries form a subgroup $O_+(V)$ of $O(V)$, being the kernel of the composition

$$O(V) \xrightarrow{\theta} \mathbb{F}^\times/(\mathbb{F}^\times)^2 \xrightarrow{\pi} \mathbb{Z}_2.$$

In particular, if an isometry $f \in O(V)$ can be written as a product of positive reflections, then it is positive. The subgroup $O_+(V)$ has index 2 in $O(V)$ unless $V$ is positive definite, in which case $O_+(V) = O(V)$.

**Example 4.2** (Isometries over the real numbers)**.** If $\mathbb{F} = \mathbb{R}$ and $V$ is not (positive or negative) definite, then $O(V)$ has four connected components. They are detected by the surjective group homomorphism $O(V) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ defined as $f \mapsto (\pi(\theta(f)), \det(f))$. The connected component of the identity is $O_+(V) \cap SO(V)$.

We are interested in determining the *positive reflection length* of a positive isometry $f \in O_+(V)$, i.e., the minimal length of a positive reflection factorization of $f$. A lower bound for the positive reflection length is given by the reflection length, which is computed in Theorem 2.7. The following example shows that this lower bound is not always attained.

**Example 4.3.** Suppose that $W \subseteq V$ is a 2-dimensional negative definite subspace, and let $\chi = \frac{1}{2}\beta|_W$. Let $f \in O(V)$ be the isometry with $\textsc{Mov}(f) = W$ and $\chi_f = \chi$. Then $f$ is positive and minimal (in the sense of Definition 3.2), but all the reflections $r \leq f$ are negative. Therefore $f$ is a product of 2 negative reflections, but it cannot be written as a product of 2 positive reflections. Note that $f$ is an involution, by property (v) of Lemma 1.7.

More generally, if $f$ is an involution, we have $\chi_f = \frac{1}{2}\beta|_{\textsc{Mov}(f)}$ by properties (i) and (v) of Lemma 1.7. Then a triangular basis (as in Lemma 2.4) of positive vectors exists if and only if $\textsc{Mov}(f)$ is positive definite. In other words, an involution $f$ admits a direct factorization into positive reflections if and only if $\textsc{Mov}(f)$ is positive definite.

We aim to show that all positive non-involutions admit a direct factorization into positive reflections provided that $\textsc{Mov}(f)$ contains at least one positive vector. To prove this, in the rest of this section, we are going to assume that the field $\mathbb{F}$ satisfies the following property.

**Definition 4.4.** An ordered field $\mathbb{F}$ is *square-dense* if the set of squares $(\mathbb{F}^\times)^2$ is dense in the set of positive elements $\mathbb{F}^+$. In other words, for every $0 < a < b$, there exists a square $c^2$ such that $a < c^2 < b$.

The class of square-dense fields includes all *Archimedean fields* (i.e., the subfields of $\mathbb{R}$) and *Euclidean fields* (i.e., ordered fields where every positive element is a square), which include all *real closed fields*. See [Sch12, Chapter 3] for the definitions and properties of these classes of fields, particularly in relation to the theory of quadratic forms. An example of an ordered field that is not square-dense is the field of rational functions $\mathbb{Q}(X)$, with the order determined by $a < X$ for all $a \in \mathbb{Q}$ (this is a typical example of a non-Archimedean field).

Our reason to choose the square-dense property as our working hypothesis is that it is quite general, but at the same time, it allows us to obtain the same characterization of the positive reflection length (Theorem 4.11) that we would obtain over the real numbers.

We start by proving a variant of Lemma 2.4.

**Lemma 4.5.** *Let $\chi$ be a non-degenerate bilinear form on a finite-dimensional vector space $W$ over an ordered field $\mathbb{F}$, with $\dim W \geq 2$. Suppose that there is at least one vector $u \in W$*

*with $\chi(u,u) > 0$. Then there is a basis $e_1, \dots, e_m$ such that $\chi(e_1,e_1) > 0$, $\chi(e_i,e_i) \neq 0$ for $i \geq 2$, and $\chi(e_i,e_j) = 0$ for $i < j$.*

*Proof.* Proceed as in the proof of Lemma 2.4, starting with a vector $u$ such that $\chi(u,u) > 0$. Choose $a \in \mathbb{F}^\times$ such that $\chi(u,u) + a\chi(v,u) > 0$, for example by taking $a = \chi(v,u)$. Then the first basis vector $e_1$ satisfies $\chi(e_1,e_1) > 0$. The rest of the proof is unchanged. $\square$

Next, we prove a technical lemma in dimension 3. This is the building block that allows us to construct triangular bases of positive vectors when the Wall form is not symmetric.

**Lemma 4.6.** *Let $W$ be a 3-dimensional vector space over a square-dense field $\mathbb{F}$. Let $\chi$ be a non-degenerate bilinear form on $W$. Suppose that $\chi$ is not symmetric, and that there is at least one vector $u \in W$ with $\chi(u,u) > 0$. Then there exist two vectors $v_1, v_2 \in W$ such that $\chi(v_1,v_1) > 0$, $\chi(v_2,v_2) > 0$, and $\chi(v_1,v_2) = 0$.*

*Proof.* By Lemma 4.5, there exists a vector $e_1 \in W$ such that $\chi(e_1,e_1) > 0$ and $\chi|_{\langle e_1\rangle^\triangleright}$ is not alternating. Fix any non-zero vector $e_2 \in \langle e_1\rangle^\triangleleft \cap \langle e_1\rangle^\triangleright$. If $\chi(e_2,e_2) > 0$, we are done by choosing $v_1 = e_1$ and $v_2 = e_2$. So we may assume that $\chi(e_2,e_2) \leq 0$.

*Case 1:* $\chi(e_2,e_2) = 0$. Since $\chi|_{\langle e_1\rangle^\triangleright}$ is not alternating, there exists a vector $e_3 \in \langle e_1\rangle^\triangleright$ such that $\chi(e_3,e_3) \neq 0$. If $\chi(e_3,e_3) > 0$, we are done by choosing $v_1 = e_1$ and $v_2 = e_3$. So we can assume that $\chi(e_3,e_3) < 0$. Note that $e_3$ is not a scalar multiple of $e_2$, so $e_2, e_3$ is a basis of $\langle e_1\rangle^\triangleright$. Therefore $e_1, e_2, e_3$ is a basis of $W$, and in this basis the matrix of $\chi$ has the following form:

$$\begin{pmatrix} \gamma & 0 & 0 \\ 0 & 0 & c \\ a & b & -\delta \end{pmatrix},$$

with $\gamma, \delta > 0$, and $b, c \neq 0$ (otherwise $\chi$ is degenerate). We may also assume $a \neq 0$ since otherwise we can exchange $e_2$ and $e_3$ and reduce to the case 2 below.

If $b + c \neq 0$, then set $v_1 = e_1$ and $v_2 = 2\delta e_2 + (b+c)e_3$. We have that $\chi(v_1,v_2) = 0$, and $\chi(v_2,v_2) = \delta(b+c)^2 > 0$, so we are done. Suppose now that $b + c = 0$, so the matrix of $\chi$ becomes

$$\begin{pmatrix} \gamma & 0 & 0 \\ 0 & 0 & -b \\ a & b & -\delta \end{pmatrix}.$$

Let $v_1 = abe_1 + \gamma\delta e_2$ and $v_2 = \delta e_1 + ae_3$. Then

$$\begin{aligned}
\chi(v_1,v_1) &= \gamma(ab)^2 > 0 \\
\chi(v_1,v_2) &= \gamma \cdot ab \cdot \delta - b \cdot \gamma\delta \cdot a = 0 \\
\chi(v_2,v_2) &= \gamma\delta^2 + a \cdot \delta \cdot a - \delta a^2 = \gamma\delta^2 > 0.
\end{aligned}$$

*Case 2:* $\chi(e_2,e_2) < 0$. Then $\chi|_{\langle e_1,e_2\rangle}$ is non-degenerate, and $\langle e_1,e_2\rangle \cap \langle e_1,e_2\rangle^\triangleright = \{0\}$. Let $e_3 \in \langle e_1,e_2\rangle^\triangleright$ be any non-zero vector. Note that $\chi(e_3,e_3) \neq 0$, because $\chi$ is non-degenerate. If $\chi(e_3,e_3) > 0$, we are done by setting $v_1 = e_1$ and $v_2 = e_3$, so we can assume that $\chi(e_3,e_3) < 0$.

Then the matrix of $\chi$ with respect to the basis $e_1, e_2, e_3$ has the following form:

$$\begin{pmatrix} \gamma & 0 & 0 \\ 0 & -\delta & 0 \\ a & b & -\epsilon \end{pmatrix},$$

where $\gamma, \delta, \epsilon > 0$, and at least one of $a$ and $b$ is non-zero (because $\chi$ is not symmetric). Define

$$\begin{aligned} v_1 &= qe_1 + e_2 \\ v_2 &= e_1 + \frac{\gamma}{\delta} qe_2 + \frac{1}{2\epsilon}\left(a + \frac{\gamma}{\delta} bq\right) e_3, \end{aligned}$$

where $q \in \mathbb{F}$ is yet to be determined. Then

$$\begin{aligned} \chi(v_1, v_1) &= \gamma q^2 - \delta \\ \chi(v_1, v_2) &= \gamma q - \delta \cdot \frac{\gamma}{\delta} q = 0 \\ \chi(v_2, v_2) &= \gamma - \frac{\gamma^2}{\delta} q^2 + \frac{1}{4\epsilon}\left(a + \frac{\gamma}{\delta} bq\right)^2. \end{aligned}$$

We are going to show how to choose $q$ so that $\chi(v_1, v_1) > 0$ and $\chi(v_2, v_2) > 0$. The first condition is

$$q^2 > \frac{\delta}{\gamma}. \tag{3}$$

Now fix the sign of $q$ so that $abq \geq 0$. Then

$$\chi(v_2, v_2) \geq \gamma - \frac{\gamma^2}{\delta} q^2 + \frac{1}{4\epsilon}\left(a^2 + \left(\frac{\gamma}{\delta} b\right)^2 q^2\right).$$

In order to have $\chi(v_2, v_2) > 0$, it is enough to have that the right hand side of the previous equation is positive, and this condition can be rewritten as

$$\left(1 - \frac{b^2}{4\delta\epsilon}\right) q^2 < \left(1 + \frac{a^2}{4\gamma\epsilon}\right) \frac{\delta}{\gamma}. \tag{4}$$

If $b^2 \geq 4\delta\epsilon$, then eq. (4) is always satisfied, and eq. (3) is satisfied for

$$q = \pm\left(\frac{\delta}{\gamma} + 1\right).$$

If $b^2 < 4\delta\epsilon$, then eqs. (3) and (4) are satisfied if

$$\frac{\delta}{\gamma} < q^2 < \frac{1 + a^2/4\gamma\epsilon}{1 - b^2/4\delta\epsilon} \cdot \frac{\delta}{\gamma}$$

Recall that at least one of $a$ and $b$ is non-zero, so these inequalities define a non-empty interval in $\mathbb{F}^+$. Since $\mathbb{F}$ is square-dense, this interval contains at least one square $q^2$. $\square$

It is worth mentioning that Lemma 4.6 does not hold over a general ordered field $\mathbb{F}$, as we show in the next example.

**Example 4.7.** Let $\mathbb{F} = \mathbb{Q}(X)$, with the non-Archimedean order determined by $a < X$ for all $a \in \mathbb{Q}$. On $W = \mathbb{F}^3$, consider the non-symmetric bilinear form $\chi$ defined by the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -X & 0 \\ 0 & 1 & -X \end{pmatrix}.$$

Let $v = (p, q, r) \in W$ be any vector satisfying $\chi(v, v) > 0$. Then we have $p^2 - Xq^2 - Xr^2 + qr > 0$. Note that $\deg(qr) < \max\{\deg(Xq^2), \deg(Xr^2)\}$, unless both $q$ and $r$ are zero. Therefore we must have $\deg(p^2) \geq \max\{\deg(Xq^2), \deg(Xr^2)\}$, which can be rewritten as $\deg(p) > \deg(q)$ and $\deg(p) > \deg(r)$. Now, suppose to have two vectors $v_1 = (p_1, q_1, r_1)$, $v_2 = (p_2, q_2, r_2)$ with $\chi(v_1, v_1) > 0$ and $\chi(v_2, v_2) > 0$. Then $\chi(v_1, v_2) = p_1p_2 - Xq_1q_2 - Xr_1r_2 + r_1q_2$, and here the degree of $p_1p_2$ is greater than the degree of all other terms. Therefore $\chi(v_1, v_2) \neq 0$.

We are going to need some flexibility in the choice of the vectors $v_1, v_2$ given by Lemma 4.6. The following two easy lemmas allow us to modify a pair $(v_1, v_2)$ while maintaining the properties we need.

**Lemma 4.8.** *Let $W$ be a finite-dimensional vector space over an ordered field $\mathbb{F}$, with $\dim W \geq 2$. Let $\chi$ be a non-degenerate bilinear form on $W$, and suppose to have two non-zero vectors $v_1, v_2 \in W$ with $\chi(v_1, v_2) = 0$. For every $u \in W$, there exists a vector $w \in W$ such that $\chi(v_1 + au, v_2 + aw) = 0$ for all $a \in \mathbb{F}$.*

*Proof.* If $u \in \langle v_1 \rangle$, then we can simply choose $w = 0$. Suppose now that $u \notin \langle v_1 \rangle$. Then $\langle v_1 \rangle^{\triangleright}$ and $\langle u \rangle^{\triangleright}$ are two distinct hyperplanes of $W$. The set $H = \{w \in W \mid \chi(u, v_2) + \chi(v_1, w) = 0\}$ is an affine translate of $\langle v_1 \rangle^{\triangleright}$, and so it intersects the linear hyperplane $\langle u \rangle^{\triangleright}$. Let $w \in H \cap \langle u \rangle^{\triangleright}$. Then

$$\chi(v_1 + au, v_2 + aw) = \chi(v_1, v_2) + a\big(\chi(u, v_2) + \chi(v_1, w)\big) + a^2\chi(u, w) = 0$$

for all $a \in \mathbb{F}$. □

**Lemma 4.9.** *Let $W$ be a finite-dimensional vector space over an ordered field $\mathbb{F}$. Let $\chi$ be a non-degenerate bilinear form on $W$, and suppose to have a vector $v \in W$ with $\chi(v, v) > 0$. For every $u \in W$, there exists $\delta \in \mathbb{F}^+$ such that $\chi(v + au, v + au) > 0$ for all $a$ in the open interval $(-\delta, \delta)$.*

*Proof.* We have

$$\chi(v + au, v + au) = \chi(v, v) + a\chi(u, v) + a\chi(v, u) + a^2\chi(u, u).$$

The absolute value of the last three summands can be made smaller than $\frac{1}{3}\chi(v, v)$, for a sufficiently small $a$. □

We are finally able to refine Lemma 4.5, and obtain a whole triangular basis of positive vectors.

**Proposition 4.10.** *Let $W$ be a finite-dimensional vector space over a square-dense field $\mathbb{F}$. Let $\chi$ be a non-degenerate bilinear form on $W$ with $\det(\chi) > 0$. Suppose that $\chi$ is not symmetric, and that there is at least one vector $u \in W$ with $\chi(u, u) > 0$. Then $W$ has a basis $e_1, \ldots, e_m$ such that $\chi(e_i, e_i) > 0$ for all $i$, and $\chi(e_i, e_j) = 0$ for $i < j$.*

*Proof.* The proof is by induction on $m = \dim W$, the case $m = 1$ being trivial. By Lemma 4.5, there is a basis $e_1, \ldots, e_m$ such that $\chi(e_1, e_1) > 0$, $\chi(e_i, e_i) \neq 0$ for $i \geq 2$, and $\chi(e_i, e_j) = 0$ for $i < j$. If $m = 2$, since $\det(\chi) > 0$, we deduce that $\chi(e_2, e_2) > 0$ and we are done. Assume from now on that $m \geq 3$.

Since $\chi$ is not symmetric, there exist two indices $2 \leq i < j \leq m$ such that at least one of $\chi(e_i, e_1)$, $\chi(e_j, e_1)$, $\chi(e_j, e_i)$ is not zero. Apply Lemma 4.6 to the restriction of $\chi$ to the 3-dimensional subspace $U = \langle e_1, e_i, e_j \rangle$ and get two positive vectors $v_1, v_2 \in U$ such that $\chi(v_1, v_2) = 0$. In particular, the subspace $\langle v_1 \rangle^{\triangleright}$ contains the positive vector $v_2$ (here the right orthogonal complement is taken in the entire space $W$ with respect to the bilinear form $\chi$).

By Lemmas 4.8 and 4.9, there exists $a \in \mathbb{F}^{\times}$ such that for all $i = 1, \ldots, m$ we have: (1) $\chi(v_1 + ae_i, v_1 + ae_i) > 0$; (2) the subspace $\langle v_1 + ae_i \rangle^{\triangleright}$ contains some positive vector $v_2 + ae_i'$. Let $N = \{v_1, v_1 + ae_1, \ldots, v_1 + ae_n\}$, and notice that $\langle N \rangle = W$. We are going to prove that there is at least one vector $u \in N$ such that $\chi|_{\langle u \rangle^{\triangleright}}$ is not symmetric. Then we are done by applying the induction hypothesis on $\chi|_{\langle u \rangle^{\triangleright}}$.

Suppose by contradiction that $\chi|_{\langle u \rangle^{\triangleright}}$ is symmetric for every $u \in N$. In other words, the alternating form $\gamma(v, w) := \chi(v, w) - \chi(w, v)$ vanishes on the hyperplane $\langle u \rangle^{\triangleright}$ for every $u \in N$. In particular, the rank of $\gamma$ is at most 2. However, the rank of $\gamma$ is even (because $\gamma$ is alternating) and non-zero (because $\chi$ is not symmetric), so it is equal to 2. For $u \in W$, denote by $\alpha_u, \alpha_u' \in W^*$ the linear forms defined by $\alpha_u(w) = \chi(u, w)$ and $\alpha_u'(w) = \gamma(u, w)$. Let $\phi, \psi \colon W \to W^*$ be the linear maps given by $\phi(u) = \alpha_u$ and $\psi(u) = \alpha_u'$. Note that $\phi$ is a vector space isomorphism because $\chi$ is non-degenerate, whereas $\psi$ has rank 2 because $\gamma$ has rank 2. For every $u \in N$ we have $\gamma|_{\langle u \rangle^{\triangleright}} = 0$, which can be written as: $w \in \ker \alpha_v'$ for every $v, w \in \langle u \rangle^{\triangleright}$. By definition of $\alpha_u$, we have $\langle u \rangle^{\triangleright} = \ker \alpha_u$. Therefore, for every $u \in N$ and $v \in \ker \alpha_u$, we have $\ker \alpha_u \subseteq \ker \alpha_v'$ and thus $\alpha_v'$ is a scalar multiple of $\alpha_u$. This means that, for every $u \in N$, the image of the restriction of $\psi$ to the hyperplane $\ker \alpha_u$ is contained in the 1-dimensional subspace $\langle \alpha_u \rangle$. Since $\psi$ has rank 2, $\alpha_u$ must be in the image of $\psi$. Then the isomorphism $\phi$ sends $N$ inside the image of $\psi$, which is a 2-dimensional subspace of $V^*$. This is a contradiction, because $N$ spans $W$, whereas the image of $\psi$ has codimension $m - 2 \geq 1$ in $W^*$. $\square$

We are now ready to compute the positive reflection length of any positive isometry. In the case $\mathbb{F} = \mathbb{R}$, this was done by Malzan [Mal82] and Djoković [Djo83].

**Theorem 4.11** (Positive reflection length)**.** *Let $(V, Q)$ be a non-degenerate quadratic space over a square-dense field $\mathbb{F}$. Assume that $V$ is not negative definite, and let $f \in O_+(V)$ be a positive isometry with $f \neq \mathrm{id}$. If at least one of the following conditions holds:*

*(i)* $\textsc{Mov}(f)$ *is positive definite,*

*(ii)* $f$ *is not an involution and* $\textsc{Mov}(f)$ *is not negative semi-definite,*

*then the positive reflection length of $f$ is equal to* $\dim \textsc{Mov}(f)$. *Otherwise, it is equal to* $\dim \textsc{Mov}(f) + 2$. *In particular, every positive isometry is a product of positive reflections.*

*Proof.* Let $m = \dim \textsc{Mov}(f) \geq 1$. If (i) holds, then $\textsc{Mov}(f)$ is not totally singular and $f$ has a direct factorization as a product of reflections by Theorem 2.7. These reflections are positive, because $\textsc{Mov}(f)$ is positive definite.

If (ii) holds, then $\chi_f$ is not symmetric by property (v) of Lemma 1.7, and Proposition 4.10 yields a basis $e_1, \ldots, e_m$ such that $\chi_f(e_i, e_i) > 0$ for all $i$ and $\chi(e_i, e_j) = 0$ for $i < j$. By

Theorem 2.2, we have $f = r_1 \cdots r_m$ where $r_i$ is the reflection with respect to $e_i$. Therefore, $f$ is a product of $m$ positive reflections.

Conversely, if $f$ can be written as a product of $m$ positive reflections with respect to some positive vectors $e_1, \ldots, e_m$, then by Theorem 2.2 we have $\chi(e_i, e_i) > 0$ for all $i$ and $\chi(e_i, e_j) = 0$ for $i < j$. In particular, $\mathrm{Mov}(f)$ contains at least one positive vector. If $\chi_f$ is symmetric, then $\mathrm{Mov}(f)$ is positive definite and (i) holds. If $\chi_f$ is not symmetric, then (ii) holds. Therefore, if both (i) and (ii) do not hold, then every factorization of $f$ as a product of positive reflections requires at least $m + 2$ reflections.

Finally, we are going to show that any positive isometry $f$ can be written as a product of $\leq m + 2$ positive reflections. We do this by induction on $m$, the case $m = 0$ being trivial. Let $m \geq 1$. If $\mathrm{Mov}(f)$ contains at least one positive vector $u$, then we can write $f = r_u f'$ where $\dim \mathrm{Mov}(f') = m - 1$ by Lemma 2.6, and proceed by induction. Therefore we may assume that $\mathrm{Mov}(f)$ is negative semi-definite. We are going to show that there is at least one positive vector $v \in V$ such that $\chi_{r_v f}$ is not symmetric. Notice that $\mathrm{Mov}(r_v f) = \mathrm{Mov}(f) \oplus \langle v \rangle$ by Lemma 2.6, so $\mathrm{Mov}(r_v f)$ contains the positive vector $v$. Then Proposition 4.10 can be applied to $\chi = \chi_{r_v f}$, yielding a factorization of $r_v f$ as a product of $m + 1$ positive reflections, and thus allowing us to write $f$ as a product of $m + 2$ positive reflections.

We only need to show that, if $\mathrm{Mov}(f) \neq \{0\}$ is negative semi-definite, then there is at least one positive vector $v \in V$ such that $\chi_{r_v f}$ is not symmetric. Let $v$ be any positive vector. Recall that $\mathrm{Mov}(f) = \langle v \rangle^{\triangleright}$, where the right orthogonal complement is taken in $\mathrm{Mov}(r_v f) = \mathrm{Mov}(f) \oplus \langle v \rangle$ with respect to the bilinear form $\chi_{r_v f}$. If $\chi_{r_v f}$ is symmetric, then $\mathrm{Mov}(r_v f) = \mathrm{Mov}(f) \perp \langle v \rangle$. Therefore $v \in \mathrm{Mov}(f)^{\perp} = \mathrm{Fix}(f)$. The set of positive vectors of $V$ is non-empty because $V$ is not negative definite, and it spans $V$ by Lemma 4.9. If $\chi_{r_v f}$ is symmetric for all positive vectors $v \in V$, then $v \in \mathrm{Fix}(f)$ for all positive vectors $v$, so $\mathrm{Fix}(f) = V$ and thus $f = \mathrm{id}$, which is a contradiction. $\square$

We say that an isometry $f \in O_+(V)$ is *positive-minimal* if it is a product of $\dim \mathrm{Mov}(f)$ positive reflections. Theorem 4.11 provides a characterization of positive-minimal isometries: an involution is positive-minimal if and only if its moved space is positive definite; a non-involution is positive-minimal if and only if its moved space is not negative semi-definite (i.e., it contains at least one positive vector).

If we replace reflection factorizations with *positive* reflection factorizations in Definition 3.1, we obtain a partial order on the group $O_+(V)$. This is not simply the restriction to $O_+(V)$ of the partial order on $O(V)$. Indeed, if $f \in O_+(V)$ is minimal but not positive-minimal, then there is a minimal positive factorization $f = r_1 r_2 g$ with $l(g) = l(f) = \dim \mathrm{Mov}(f)$, and we have $g \leq f$ in $O_+(V)$ but $g \not\leq f$ in $O(V)$. For the same reason, the rank function of $O_+(V)$ is not the restriction of the rank function of $O(V)$.

If $f \in O_+(V)$ is a positive-minimal isometry, then Theorem 4.11 allows us to include the interval $[\mathrm{id}, f]$ in $O_+(V)$ into the poset of linear subspaces of $\mathrm{Mov}(f)$, in the same spirit as Theorem 3.5.

## 5. Isometries of the hyperbolic space

In this section, we describe reflection length and intervals in the isometry group of the hyperbolic space $\mathbb{H}^n$. We follow the notation of [CFK+97].

Let $V = \mathbb{R}^{n+1}$, with the quadratic form $Q(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$. Then $(V, Q)$ is a real quadratic space of signature $(n, 1)$. The *hyperboloid model* of the hyperbolic space is

$$\mathbb{H}^n = \{x \in V \mid Q(x) = -1 \text{ and } x_{n+1} > 0\}.$$

The quadratic form $Q$ induces a (positive definite) Riemannian metric on $\mathbb{H}^n$. The condition $x_{n+1} > 0$ selects the upper sheet of the hyperboloid $\{Q(x) = -1\}$. Every isometry of $\mathbb{H}^n$ uniquely extends to an isometry of $(V, Q)$; conversely, every isometry of $(V, Q)$ that fixes $\mathbb{H}^n$ (as a set) restricts to an isometry of $\mathbb{H}^n$.

**Lemma 5.1.** *The subgroup of $O(V)$ that fixes $\mathbb{H}^n$ (as a set) coincides with the index-two subgroup $O_+(V)$ of the positive isometries.*

*Proof.* Both subgroups have index 2, so it is enough to show one containment. By Theorem 4.11, the subgroup $O_+(V)$ is generated by the positive reflections $r \in O(V)$, and therefore it is enough to show that every positive reflection fixes $\mathbb{H}^n$. If $v \in V$ is a positive vector, then $\langle v \rangle^\perp$ has signature $(n-1, 1)$, so it intersects $\mathbb{H}^n$. Therefore $r_v$ fixes at least one point of $\mathbb{H}^n$. Note that any isometry $f \in O(V)$ fixes the hyperboloid $\{Q(x) = -1\}$, and $\mathbb{H}^n$ is one of the two connected components of this hyperboloid. Then the reflection $r_v$ fixes $\mathbb{H}^n$ as a set. $\square$

Reflections in the hyperbolic space $\mathbb{H}^n$ are restrictions of positive reflections of $(V, Q)$. Therefore, the study of reflection length and intervals in the isometry group of $\mathbb{H}^n$ reduces to the study of positive reflection length and intervals in $O_+(V)$. This is exactly the setting of Section 4. It turns out that every isometry of $\mathbb{H}^n$ is positive-minimal.

**Theorem 5.2.** *The positive reflection length of an isometry $f \in O_+(V)$ is equal to* $\dim \operatorname{Mov}(f)$.

*Proof.* We prove this by induction on $k = \dim \operatorname{Mov}(f)$, the case $k = 0$ (the identity) being trivial. If $k = 1$, then $f$ is a positive reflection. If $k \geq 2$, then $\operatorname{Mov}(f)$ intersects the hyperplane $\{x_{n+1} = 0\}$ non-trivially, so it contains at least one positive vector $v$. By Theorem 2.2, there is a direct factorization $f = r_v g$. Then $\dim \operatorname{Mov}(g) = k - 1$, and $g$ can be written as a product of $k - 1$ positive reflections by induction. $\square$

We are then able to obtain a clean description of all intervals $[\mathrm{id}, f]$ in $O_+(V)$.

**Theorem 5.3.** *Let $f \in O_+(V)$. The interval $[\mathrm{id}, f]$ in $O_+(V)$ is isomorphic to the poset of linear subspaces $U \subseteq \operatorname{Mov}(f)$ such that $\det(\chi_f|_U) > 0$.*

*Proof.* By Theorem 5.2, we have that $f$ is positive-minimal. Therefore, all minimal length factorizations of $f$ into positive reflections are direct factorizations. In particular, the interval $[\mathrm{id}, f]$ in $O_+(V)$ is contained in the interval $[\mathrm{id}, f]$ in the whole group $O(V)$. To avoid confusion, denote by $[\mathrm{id}, f]_+$ the interval in $O_+(V)$. If $g \in [\mathrm{id}, f]$ is a positive isometry, then $h = g^{-1} f$ is also positive, and $g$ and $h$ are positive-minimal by Theorem 5.2. Therefore $g \in [\mathrm{id}, f]_+$. This shows that $[\mathrm{id}, f]_+ = [\mathrm{id}, f] \cap O_+(V)$.

By Theorem 3.5, the map $g \mapsto \operatorname{Mov}(g)$ is a bijection between $[\mathrm{id}, f]_+$ and the poset of linear subspaces $U \subseteq \operatorname{Mov}(f)$ such that: $U$ satisfies conditions (i)-(iii) of Theorem 3.5; (iv) $\det(\chi_f|_U) > 0$ (this is the same as saying that the preimage of $U$ is a positive isometry). Since the signature of $V$ is $(n, 1)$, the totally singular subspaces have dimension 0 or 1, so conditions (i) and (ii) are implied by condition (iii). In addition, we can disregard condition (iii) as it is implied by (iv). Putting everything together, the map $g \mapsto \operatorname{Mov}(g)$ is a bijection between $[\mathrm{id}, f]_+$ and the poset of linear subspaces $U \subseteq \operatorname{Mov}(f)$ satisfying $\det(\chi_f|_U) > 0$.

If $g \leq g'$ in $[\mathrm{id}, f]_+$, then $g \leq g'$ in $[\mathrm{id}, f]$, and thus $\mathrm{Mov}(g) \subseteq \mathrm{Mov}(g')$ by Theorem 3.5. Conversely, suppose that we have $g, g' \in [\mathrm{id}, f]_+$ such that $\mathrm{Mov}(g) \subseteq \mathrm{Mov}(g')$. By Lemma 3.4, $\chi_g$ and $\chi_{g'}$ are the restrictions of $\chi_f$ to $\mathrm{Mov}(g)$ and $\mathrm{Mov}(g')$, respectively. Then $\chi_g = \chi_{g'}|_{\mathrm{Mov}(g)}$, so there is a direct factorization $g' = gh$ and $h$ is positive-minimal by Theorem 5.2. Therefore $g \leq g'$ in $[\mathrm{id}, f]_+$. This shows that the bijection $g \mapsto \mathrm{Mov}(g)$ is a poset isomorphism. $\square$

Notice that Theorem 5.3 gives a poset isomorphism, whereas Theorem 3.5 only gives an order-preserving bijection. A counterexample like the one in Example 3.6 cannot occur in this context, since all positive isometries are positive-minimal. Indeed, for Example 3.6 to arise, the Witt index of the ambient space $V$ needs to be at least 2 (in other words, over an ordered field, the signature needs to be $(p, q)$ with $p, q \geq 2$).

It is also true that all isometries of $O(V)$ are minimal, by Theorem 2.7. Indeed, the only non-trivial totally singular subspaces are one-dimensional, and they do not arise as moved spaces of any isometry, because the Wall form would be identically zero.

Recall that, if we interpret the hyperboloid model as lying in the projective space $\mathbb{P}(V)$, the singular lines $\langle v \rangle \subseteq \{Q(x) = 0\}$ can be interpreted as "points at infinity" of the hyperbolic space $\mathbb{H}^n$. Then the isometries of $\mathbb{H}^n$ can be classified into three types: *elliptic* isometries, that fix at least one point of $\mathbb{H}^n$; *parabolic* isometries, that fix no point of $\mathbb{H}^n$ and fix exactly one point at infinity; *hyperbolic* isometries, that fix no point of $\mathbb{H}^n$ and fix two points at infinity. See [CFK+97, Section 12]. We now rewrite this classification in terms of fixed space and moved space.

**Definition 5.4.** An isometry $f \in O_+(V)$ is

- *elliptic* if $\mathrm{Fix}(f)$ contains a negative vector (i.e., it is not positive semi-definite);
- *parabolic* if $\mathrm{Fix}(f)$ is positive semi-definite but not positive definite;
- *hyperbolic* if $\mathrm{Fix}(f)$ is positive definite.

**Lemma 5.5.** *Let $f \in O_+(V)$. We have that $\mathrm{Fix}(f) \cap \mathrm{Mov}(f) = \{0\}$ if $f$ is elliptic or hyperbolic, whereas $\mathrm{Fix}(f) \cap \mathrm{Mov}(f)$ is a singular line if $f$ is parabolic. In addition:*

- *$f$ is elliptic if and only if $\mathrm{Mov}(f)$ is positive definite;*
- *$f$ is parabolic if and only if $\mathrm{Mov}(f)$ is positive semi-definite but not positive definite;*
- *$f$ is hyperbolic if and only if $\mathrm{Mov}(f)$ contains a negative vector.*

*Proof.* We have that $\mathrm{Mov}(f) = \mathrm{Fix}(f)^\perp$ by Lemma 1.3. Therefore $\mathrm{Fix}(f) \cap \mathrm{Mov}(f)$ is a totally singular subspace, so its dimension is at most 1. If $\mathrm{Fix}(f) \cap \mathrm{Mov}(f)$ contains a non-trivial singular vector $v$, then $\mathrm{Fix}(f)$ is not positive definite, so $f$ is elliptic or parabolic.

If $f$ is elliptic, then up to conjugating by an isometry in $O_+(V)$ we may assume that $f$ fixes the point $e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{H}^n$. Then $f$ is an isometry also with respect to the standard (positive definite) Euclidean quadratic form $Q_E(x) = x_1^2 + \ldots + x_{n+1}^2$. Therefore $\mathrm{Fix}(f)$ and $\mathrm{Mov}(f)$ are $Q_E$-orthogonal by Lemma 1.3, and in particular $\mathrm{Fix}(f) \cap \mathrm{Mov}(f) = \{0\}$. If $f$ is parabolic, then $\mathrm{Fix}(f)$ contains a singular line, so $\mathrm{Fix}(f) \cap \mathrm{Mov}(f)$ is a singular line. This finishes the proof of the first part of the statement.

We now prove the classification in terms of the moved space. If $f$ is elliptic, then $\mathrm{Fix}(f)$ contains a negative vector and $V = \mathrm{Fix}(f) \perp \mathrm{Mov}(f)$, so $\mathrm{Mov}(f)$ is positive definite. Similarly, if $f$ is hyperbolic, then $\mathrm{Fix}(f)$ is positive definite and $V = \mathrm{Fix}(f) \perp \mathrm{Mov}(f)$, so $\mathrm{Mov}(f)$ contains a negative vector. If $f$ is parabolic, then $\mathrm{Mov}(f)$ contains a singular vector and so it is not positive definite. Finally, if $\textsc{Mov}(f)$ contains a negative vector $w$, then $\langle w\rangle^\perp$ is positive definite and $\textsc{Fix}(f) = \textsc{Mov}(f)^\perp \subseteq \langle w\rangle^\perp$, so $f$ is not parabolic. □

For elliptic isometries, the description of the intervals given by Theorem 5.3 becomes particularly simple thanks to the following observation.

**Lemma 5.6.** *Let $f \in O_+(V)$. If $U \subseteq \textsc{Mov}(f)$ is a positive definite subspace, then $\det(\chi_f|_U) > 0$.*

*Proof.* The restriction $\chi_f|_U$ is non-degenerate, because $\chi(u,u) = Q(u) > 0$ for all $u \in U$. Applying Lemma 2.4 to $\chi_f|_U$, we obtain a basis $e_1, \dots, e_m$ of $U$ such that $\chi_f(e_i, e_i) \neq 0$ for all $i$, and $\chi(e_i, e_j) = 0$ for $i < j$. Additionally, we have $\chi_f(e_i, e_i) = Q(e_i) > 0$ for all $i$. Therefore, $\det(\chi_f|_U) > 0$. □

**Theorem 5.7** (Elliptic intervals)**.** *Let $f \in O_+(V)$ be an elliptic isometry. Then the interval $[\mathrm{id}, f]$ is isomorphic to the poset of all linear subspaces of $\textsc{Mov}(f)$. In particular, the isomorphism type of $[\mathrm{id}, f]$ only depends on the dimension of $\textsc{Mov}(f)$, and not on the Wall form $\chi_f$.*

*Proof.* This follows immediately from Theorem 5.3 and Lemma 5.6. □

The description of Theorem 5.3 can be simplified also for parabolic intervals.

**Lemma 5.8.** *Let $f \in O_+(V)$ be a positive isometry, and $U \subseteq \textsc{Mov}(f)$ a subspace. The restriction $\chi_f|_U$ is degenerate if and only if there is a singular vector $v \in \textsc{Mov}(f) \setminus \{0\}$ such that $\langle v\rangle \subseteq U \subseteq \langle v\rangle^\triangleright$. Note that $\langle v\rangle^\triangleright = \langle w\rangle^\perp$ where $w$ is any vector such that $w - f(w) = v$.*

*Proof.* The restriction $\chi_f|_U$ is degenerate if and only if there is a non-zero vector $v \in U$ such that $\chi_f(v,u) = 0$ for all $u \in U$, or equivalently $\langle v\rangle \subseteq U \subseteq \langle v\rangle^\triangleright$. Since $\chi_f(v,v) = Q(v)$, the containment $\langle v\rangle \subseteq \langle v\rangle^\triangleright$ holds if and only if $v$ is singular. Finally, by definition of $\chi_f$, we have $\chi_f(v,u) = \beta(w,u)$ for all $u \in U$, and therefore $\langle v\rangle^\triangleright = \langle w\rangle^\perp$. □

**Theorem 5.9** (Parabolic intervals)**.** *Let $f \in O_+(V)$ be a parabolic isometry which pointwise fixes the singular line $\langle v\rangle$. Then the interval $[\mathrm{id}, f]$ is isomorphic to the poset of linear subspaces $U \subseteq \textsc{Mov}(f)$ that do not satisfy $\langle v\rangle \subseteq U \subseteq \langle v\rangle^\triangleright$. In particular, the isomorphism type of $[\mathrm{id}, f]$ only depends on the dimension of $\textsc{Mov}(f)$, and not on the Wall form $\chi_f$.*

*Proof.* Let $U \subseteq \textsc{Mov}(f)$ be a subspace. If $\langle v\rangle \nsubseteq U$, then $U$ is positive definite and thus $\det(\chi_f|_U) > 0$ by Lemma 5.6. Since $\langle v\rangle$ is the only singular line in $\textsc{Mov}(f)$, the restriction $\chi_f|_U$ is degenerate if and only if $\langle v\rangle \subseteq U \subseteq \langle v\rangle^\triangleright$ by Lemma 5.8. Finally, if $\langle v\rangle \subseteq U \nsubseteq \langle v\rangle^\triangleright$, then Lemma 2.4 yields a basis $e_1, \dots, e_m$ of $U$ such that $\chi_f(e_i, e_i) \neq 0$ for all $i$ and $\chi_f(e_i, e_j) = 0$ for $i < j$. Since $f$ is parabolic, $\textsc{Mov}(f)$ is positive semi-definite by Lemma 5.5 and therefore $\chi(e_i, e_i) = Q(e_i) > 0$ for all $i$. Thus $\det(\chi_f|_U) > 0$ also in this case. We conclude by applying Theorem 5.3. □

The subgroup of $O_+(V)$ that fixes a singular line $\langle v\rangle$ is isomorphic to the isometry group of the affine Euclidean space $\mathbb{R}^n$. This is easily seen in the *half-space model* of the hyperbolic space (see [CFK+97, Section 12]). In particular, parabolic intervals are isomorphic to intervals in the group of affine Euclidean isometries, which have been explicitly described in [BM15]. Our description is more compact than the one of [BM15], where the elliptic and the parabolic portions of an interval are described separately.

The results of this section leave open the following natural question: if $f \in O_+(V)$ is a hyperbolic isometry, does the isomorphism type of $[\mathrm{id}, f]$ depend only on the dimension of $\mathrm{Mov}(f)$?

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