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A general framework for optimal stopping problems with two risk factors and real option applications

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Abstract

A new explicit solution is obtained for a general class of two-dimensional optimal stopping problems arising in real option theory. First, the solvable case of homogeneous and quasi-homogeneous problems is presented in a comprehensive framework. Then the general problem - including the unsolved case of inhomogeneous functions - is considered and an explicit expression for the value function is obtained in terms of a modified Bessel function of second kind. Then we clarify the link between the general solution method and the more elementary one in the specific (quasi-)homogeneous problem. Finally, this article provides some useful formulas and some insights for the one-dimensional case as well.

Keywords: Optimal stopping. Free boundary problems. Bessel functions. Real options.

AMS Classification: 60G40; 91G50; 91B70; 35J05.

1 Introduction

Several research questions in real option theory, investment decisions, valuation of financial securities are formulated as optimal stopping problems and result in free boundary problems for (degenerate) elliptic or parabolic equations. Often, when the underlying stochastic processes are geometric Brownian motions the value-matching and smooth-fit conditions apply. In some special cases (one dimensional problem, homogeneous reward function of degree 1) an explicit solution can be obtained in an elementary way. When there are two risk factors and the reward function is non-homogeneous, only numerical methods are available (e.g. finite difference schemes). On the other hand, a few quasi-analytical solution methods have been proposed to provide an approximate solution (f.e. Adkins and Paxson (2011), Pindyck (2002), Heydari et al. (2012)). Unfortunately, these methods are not rigorous, and the approximation used in the computation is not always justified as the terms that are left-out should not be neglected in some cases (see Compermolle et al., 2021). A recent article by Lange et al. (2020) demonstrates that the quasi-analytical method may yield a suboptimal policy and an inaccurate value function. Although (quasi)-analytical methods are undoubtedly of practical use for many real option problems, they are not founded yet on a more rigorous ground.

In this article, we pursue an analytical approach. The known solvable problems are framed and understood within a general setting and explicit solution methods are presented for some meaningful problems of optimal stopping with two risk factors. Generally speaking, the advantage of analytical methods for boundary-value problems over domain discretization methods is the reduction of problem dimension and a relatively easiness to program. First, we perform suitable change of variables and then we obtain an explicit expression for the solution of the boundary value problem via fundamental solution methods. Second, we provide an expression for the optimal threshold and the value function. Then the effectiveness of the method is tested on a few significant problems arising in economics and corporate finance.

This research question lends itself to extensions to the case where the underlying stochastic processes are not diffusions and the smooth-pasting principle fails. Such a challenging continuation deserves the efforts of future research, although it is unlikely that easy-to-handle results can be obtained.

This article is organized as follows. Section 2 introduces the motivating problems and the main notation. Section 3 is devoted to the two-factor problems which admit an explicit and elementary solution thanks to the homogeneity property of the reward function. We provide a comprehensive framework capturing all the already studied cases from real option theory which usually consider only linear functions; then we propose an extension to quasi-homogeneous reward functions. Our presentation is more general than in the extant literature both in terms of the shape and of the (quasi-)homogeneity property of the reward function. As a by-product, we also correct a misprint in Nunes and Pimentel (2017). Section 4 derives the main result: the general problem -including the unsolved cases- is attacked via fundamental solution methods and suitable changes of variables. Finally, Section 5 applies the results to a couple of meaningful examples. Appendix A surveys the solution of the classical one-factor problems by providing user-friendly formulas which are not published elsewhere and, moreover, it links the results to Wiener-Hopf theory. Appendix B contains the technical definitions for the Bessel functions which are used in Section 4, for readers' convenience.

2 Notation

In several models of economic significance, uncertainty is modelled through a multidimensional geometric Brownian motion, X_t, or a more general multidimensional diffusion, where the dimension depends on the number of risk factors. If X_t is an n-dimensional geometric Brownian motion then its components are of the form

$$X_t^{(i)} = X_0^{(i)} \exp[(\mu_i - \frac{\sigma_i^2}{2})t + \sigma_i W_t^{(i)}] \tag{1}$$

where $(W_t^{(1)}, ..., W_t^{(n)})$ is an n-dimensional Wiener process with respect to a given filtration, $\mathfrak{F}$, and with $E[W_t^{(i)} W_t^{(j)}] = \rho_{ij} t$ where the non-negative definite matrix $(\rho_{ij})_{i,j=1,...n}$ has $\rho_{ii} = 1$. Let σ_{ij} denote $\sigma_i \sigma_j \rho_{ij}$ for all $i, j = 1, ...n$.

The differential operator associated with the process X_t is the following differential operator in $\mathbf{R}_+^n$:

$$L = \frac{1}{2}\sum_{i,j=1}^{n} \sigma_{ij} x_i x_j \partial^2_{x_i x_j} + \sum_{i=1}^{n} \mu_i x_i \partial_{x_i} \tag{2}$$

where $x_j \geq 0$, $1 \leq j \leq n$.

Given a positive reward function (or gain function) $G : \mathbf{R}_+^n \longrightarrow [0, +\infty)$, usually a continuous function, and a discount rate $r > 0$, then the agent's problem is to decide the optimal stopping time, t^*, that maximizes the expected discounted reward, i.e.

$$V(x) = \sup_t E_x[e^{-rt} G(X_t)] = E_x[e^{-rt^*} G(X_{t^*})]$$

where the supremum is taken over all stopping times w.r.t. $\mathfrak{F}$. Here $E_x[.]$ is a short notation for $E[.\,|X_0 = x]$.

In Shiryaev (2008), existence of the solution is obtained when G is lower semicontinuous and $E_x[\sup_{t\geq 0} e^{-rt} G(X_t)] < \infty$. If G is continuous, then t^* is the first entry time of X_t into a stopping region, S, where V equals G, i.e. $t^* = \inf\{t \geq 0; X_t \in S\}$.
A typical interpretation is the optimal investment time in an asset whose instantaneous payoff is $G(X_t)$ and V represents the value of this investment option (See McDonald and D. Siegel, 1986 and Dixit and Pindyck, 1994). Typically, $G(X_t) = \max[\sum_{i=1}^{m} a_i X_t^{(i)} - \sum_{i=m}^{n} c_i X_t^{(i)}, 0]$ where the a_i's are unit revenues and the c_i's are unit costs related to the several risk factors. For this function, the case $n = 2$ and $m = 1$ dates back to McDonald and D. Siegel (1986), while the general case is studied in Gahungu and Smeers (2011), among others, but it is not clear yet how to fix either necessary or sufficient conditions for the proposed solution to hold. In particular, the search for stronger necessary conditions and weaker sufficient conditions and, at the same time, tractable ones, is still an open question.

Here the optimal investment time, t^*, is the first entry time of X_t into the stopping region, S, whereas $V > G$ in the so-called continuation region C. The boundary between the two regions is the optimal stopping boundary[1], ∂C. The boundary-value formulation of this problem is as follows:

[1] In multi-dimensional cases an explicit expression for the unknown stopping bound-

$$\begin{cases} [r-L]V = 0 & \text{in } C \\ V = G & \text{in } S \end{cases} \tag{3}$$

with $V \geq G$ in C and $[r-L]V \geq 0$ in S.

Another relevant decision problem concerns the optimal entry into an investment securing a continuous cash flow $g(X_t)$, that is, the decision maker has to maximize the following expected value:

$$E_x[\int_{t^*}^{+\infty} e^{-rt} g(X_t) dt]$$

Symmetrically, the problem of optimal exit time, so that a continuous cash flow is abandoned when revenues fall below a certain level, consists in maximizing the following expectation:

$$E_x[\int_0^{t^*} e^{-rt} g(X_t) dt]$$

The last expression can be written as

$$E_x[\int_0^{+\infty} e^{-rt} g(X_t) dt] + E_x[\int_{t^*}^{+\infty} e^{-rt}(-g(X_t)) dt],$$

thus reducing the solution of one problem to the other one. The boundary-value formulation of the optimal exit problem is as follows:

$$\begin{cases} [r-L]V = g & \text{in } C \\ V = 0 & \text{in } S \end{cases} \tag{4}$$

with $V \geq 0$ in C and $[r-L]V \geq g$ in S.

Finally, we note that Mayer formulation (3) and Lagrange formulation (4) - and even the Bolza formulation which combines both[2] - are known

ary is hardly found. A representation for the unknown boundary is offered in Christensen and Salminen (2018), equation (13), but the solution of this integral equation remains a non trival task.

[2]The general formulation is:
$$\begin{cases} [r-L]V = g & \text{in } C \\ V = G & \text{in } S \end{cases}$$

to be formally equivalent. For example, if V_0 is a solution of (3) with $G \in \mathcal{C}^2(C)$, then $V = V_0 - G$ satisfies (4) with $g = [L - r]G$. Then, in the next sections, we will focus only on a single problem choosing between (3) and (4).

A general framework is available to attack these problems building on the inversion formula between $L-r$ and the r-Green kernel, $\mathfrak{G}_r$. (See Peskir and Shiryaev, 2006, for example). In particular, an integral equation for the unknown boundary of the stopping set can be written as follow:

$$\int_C \mathfrak{G}_r(x, x')(r - L)G(x')dx' = 0, \;\; x \in \partial S.$$

However, finding a numerical solution for this set equation is not standard and not easy to handle in practice.

3 Homogeneous and Quasi-homogeneous Reward Functions

In this section we compute an explicit solution for a two-dimensional problem of the form (3), by assuming that G is positively homogeneous of degree h. In this situation, the problem can be solved in an elementary way and the special case $h = 1$ is already found in real option literature for several applications. Here we firstly obtain a solution for a generic homogeneous function G. Then we extend the result to quasi-homogeneous reward functions. We use dimension reduction, as is standard in real option literature for the special case of homogeneous functions of degree 1. Let us consider the problem

$$\begin{cases} [L - r]V = 0 & \text{in } C \\ V = G & \text{in } S \end{cases} \tag{5}$$

where $L = \frac{1}{2}[\sigma_1^2 x^2 \partial_x^2 + \sigma_2^2 y^2 \partial_y^2 + 2\rho\sigma_1\sigma_2 xy\partial_{xy}^2] + \mu_1 x\partial_x + \mu_2 y\partial_y$.

As shown below, some sufficient conditions guarantee that the optimal stopping boundary is of the form $\partial C = \{(x, y) \in \mathbf{R}_+^2;\ y = z^* x\}$ for some $z^* > 0$.

We assume G differentiable and use the smooth-pasting principle to perform the calculation.

Let $V(x,y) = x^h v(z)$, where $z = y/x$. Then (5) is transformed into

$$\begin{cases} \widetilde{L}v = 0 & \text{in } \widetilde{C} \\ v(z^*) = G(1, z^*) = \widetilde{G}(z^*) \\ v'(z^*) = \widetilde{G}'(z^*) \end{cases} \tag{6}$$

where the last equation follows from Euler's theorem for homogeneous functions. Here $\widetilde{L}$ denotes the following differential operator:

$$\begin{aligned} \widetilde{L} &= \frac{1}{2}[\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2]z^2\frac{d^2}{dz^2} + [\mu_2 - \mu_1 + (1-h)\sigma_1^2 + (h-1)\rho\sigma_1\sigma_2]z\frac{d}{dz} + \\ &\quad \frac{\sigma_1^2}{2}h(h-1) + h\mu_1 - r \end{aligned} \tag{7}$$

and $\widetilde{C}$ is of the form $(0, z^*)$ or (z^*, ∞).

Assume that $\frac{(h-1)\sigma_1^2}{2} + \mu_1 < \frac{r}{h}$. Then a general solution of $\widetilde{L}v = 0$ is of the form $A_+ z^{\beta_+} + A_- z^{\beta_-}$ where $\beta_- < 0 < \beta_+$ are the roots of the equation associated with $\widetilde{L}$ and $A_\pm$ are generic constants still to be determined. The two identities in z^* yield:

$A_+ = \frac{\widetilde{G}'(z^*)z^* - \beta_-\widetilde{G}(z^*)}{\beta_+ - \beta_-}(z^*)^{-\beta_+}$ and $A_- = \frac{-\widetilde{G}'(z^*)z^* + \beta_+\widetilde{G}(z^*)}{\beta_+ - \beta_-}(z^*)^{-\beta_-}$.

Additional boundary conditions, holding true in the continuation region, allow for obtaining a unique value for z^*. In particular, if $v(0+) < \infty$ then A_- should vanish, and z^* is found by solving $\widetilde{G}'(z^*)z^* = \beta_+\widetilde{G}(z^*)$. Here $G(x, z^*x).\partial_y G(x, z^*x) > 0$ is needed to get a positive threshold z^*. On the other hand, if $v(+\infty) < \infty$ then A_+ should vanish, and z^* is found by solving $\widetilde{G}'(z^*)z^* = \beta_-\widetilde{G}(z^*)$. In this case, $G(x, z^*x).\partial_y G(x, z^*x) < 0$ is needed to get a positive threshold z^*.

Finally, similar conditions can be formulated by interchanging the roles of x and y.

Example 3.1 Consider an investment problem with two risk factors and with $G(x,y) = ax + by$. Assume that a and b have opposite sign, for example, $a < 0 < b$. Assume $\mu_1 < r$ and $\mu_2 \le r$. Then the investment threshold is of the form $y = z^*x$ where $z^* = a\beta_+/[b(1-\beta_+)]$ and β_+ is the positive root of

$\frac{1}{2}[\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2]\beta^2+[\mu_2-\mu_1-\frac{1}{2}\sigma_1^2-\frac{1}{2}\sigma_2^2+\rho\sigma_1\sigma_2]\beta+\mu_1-r=0.$
Note that $\beta_+>1$.
Then, $V(x,y)=[a+bz^*]x^{1-\beta_+}\left(\frac{y}{z^*}\right)^{\beta_+}$.
Note that the condition $G(x,z^*x).\partial_y G(x,z^*x)$ here takes the form $abx/(1-\beta_+)>0$, which holds true under our assumptions on the parameters.

This argument can be extended to quasi-homogeneous reward functions. G is a quasi-homogeneous function of degree h with weights (w_1,w_2) if $G(\lambda^{w_1}x_1,\lambda^{w_2}x_2)=\lambda^h G(x_1,x_2)$ for every positive λ. If G is differentiable, then the generalized Euler identity holds: $\sum_i w_i x_i \partial_{x_i} G = hG$. Note that homogeneous functions are captured by taking the weights $(1,1)$.

In this setting, the argument above can be generalized as follows. Define $V(x,y)=x^{h/w_1}v(z)$, where $z=y/x^{w_2/w_1}$. The operator $\widetilde{L}$ is replaced by:

$$\widetilde{L}=\frac{1}{2}[\sigma_1^2(\tfrac{w_2}{w_1})^2+\sigma_2^2-2\rho\sigma_1\sigma_2\tfrac{w_2}{w_1}]z^2\frac{d^2}{dz^2}+[\mu_2-\mu_1\tfrac{w_2}{w_1}+\tfrac{w_2}{w_1}\sigma_1^2(\tfrac{1}{2}-\tfrac{h}{w_1}+\tfrac{w_2}{2w_1})+\rho\sigma_1\sigma_2\tfrac{h-w_2}{w_1}]z\frac{d}{dz}+\frac{\sigma_1^2}{2}\frac{h}{w_1}(\frac{h}{w_1}-1)+\frac{h}{w_1}\mu_1-r.$$

Finally, the condition $\frac{(h/w_1-1)\sigma_1^2}{2}+\mu_1<\frac{rw_1}{h}$ is sufficient to guarantee that the roots of the equation associated with $\widetilde{L}$ satisfy $\beta_-<0<\beta_+$. Then a general solution of $\widetilde{L}v=0$ is of the form $A_+z^{\beta_+}+A_-z^{\beta_-}$ as above.

Example 3.2 Consider an investment problem where the profit function is quasi-homogeneous and the reward function turns out to be of the form: $G(x,y)=Ky^\theta-x$, where both the investment costs, x, and the demand, y, are uncertain and follow GBMs. (See Nunes and Pimentel, 2017). This problem can be handled in our framework by taking $w_1=1, w_2=1/\theta$ and $h=1$. Defining $z=y/x^{1/\theta}$ and denoting by β_+ the positive root of:

$$\frac{1}{2}[\frac{\sigma_1^2}{\theta^2}+\sigma_2^2-\frac{2}{\theta}\rho\sigma_1\sigma_2]\beta^2+[\mu_2-\frac{\sigma_2^2}{2}-(\mu_1+\frac{\sigma_1^2}{2})\frac{1}{\theta}+\frac{\rho}{\theta}\sigma_1\sigma]\beta+\mu_1-r=0,$$

the threshold value, z^*, is obtained by solving $\widetilde{G}'(z^*)z^*=\beta_+\widetilde{G}(z^*)$, that is, $(z^*)^\theta=\frac{\beta_+}{K(\beta_+-\theta)}$. Finally, the value function is of the form:

$$V(x,y)=\left(\frac{K\theta}{\beta_+}\right)^{(\beta_+/\theta)}\left(\frac{\beta_+}{\theta}-1\right)^{\frac{\beta_+}{\theta}-1}\frac{y^{\beta_+}}{x^{\frac{\beta_+}{\theta}-1}}$$

in the continuation region $y^\theta<\frac{\beta_+ x}{K(\beta_+-\theta)}$.
This solution can be compared with Nunes and Pimentel (2017), where however $\rho=0$ and jumps are added. Note that in Nunes and Pimentel (2017) there is a misprint in equation (19).

Finally, we obtain the solution for the problem of acquiring or abandoning a continuous cash flow which is represented through a homogeneous function, $g(x, y)$. Let us consider problem (4), where g is homogeneous of degree h. Changing to variables $V(x, y) = x^h v(z)$, where $z = y/x$, we are reduced to consider the following problem:

$$\begin{cases} \widetilde{L}v = \widetilde{g} & \text{in } \widetilde{C} \\ v(z^*) = v'(z^*) = 0 \end{cases} \tag{8}$$

where $g(x, y) = x^h \widetilde{g}(z)$. Here $\widetilde{L}$ denotes the same differential operator as (7) and z^* will be determined below. Let us focus on the case where $\widetilde{C}$ is of the form $]0, z^*[$ and $\lim_{z\to 0+} v(z) < \infty$. Let $\beta_- < 0 < \beta_+$ be the roots of the equation:

$$\tfrac{1}{2}S^2\beta^2 + [\mu_2 - \mu_1 + (1-h)\sigma_1^2 + h\rho\sigma_1\sigma_2 - \tfrac{1}{2}(\sigma_1^2 + \sigma_2^2)]\beta + \tfrac{\sigma_1^2}{2}h(h-1) + h\mu_1 - r = 0,$$

with $S^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$. Using Lagrange formula of the variations of constants we get

$$v(z) = \frac{2z^{\beta_+}}{S^2(\beta_+ - \beta_-)} \int_z^{z*} \frac{\widetilde{g}(z)}{z^{\beta_+ + 1}} dz + \frac{2z^{\beta_-}}{S^2(\beta_+ - \beta_-)} \int_{z*}^{z} \frac{\widetilde{g}(z)}{z^{\beta_- + 1}} dz$$

where z^* is determined by solving

$$\int_0^{z*} \frac{\widetilde{g}(z)}{z^{\beta_+ + 1}} dz = 0.$$

Going back to the coordinates (x, y), the following solution is obtained:

$$V(x, y) = \frac{2}{S^2(\beta_+ - \beta_-)} \left\{ y^{\beta_+} \int_y^{z^* x} \frac{g(x,t)}{t^{\beta_+ + 1}} dt + y^{\beta_-} \int_{z^* x}^{y} \frac{g(x,t)}{t^{\beta_- + 1}} dt \right\}$$

where the threshold value z^* is obtained from $\int_0^{z*} \frac{g(1,z)}{z^{\beta_+ + 1}} dz = 0$. We refer to Appendix A, Remark A1, for the issue of existence of z^*.
Of course, a similar result can be obtained by interchanging the role of x and y.

Remark 3.1 In the case where g and G are homogeneous functions of the same degree, we can use the procedure above to solve the general problem:

$$\begin{cases} [r - L]V = g & \text{in } C \\ V = G & \text{in } S \end{cases} \tag{9}$$

One can go through the following steps: reduce the problem dimension by changing variables as above, compute the fundamental solution of the one-dimensional problem as in Appendix A, and finally use the smooth-pasting conditions to match G and find the free boundary in the form $y^*(x) = z^*x$. We omit the details of the proof and report the final formula below. For the case *i)* the value function can be written as:

$$V(x,y) = \frac{2y^{\beta_+}}{\Sigma^2(\beta_+-\beta_-)}\int_y^{z^*x}\frac{g(x,z)}{z^{\beta_++1}}dz + \frac{2y^{\beta_-}}{\Sigma^2(\beta_+-\beta_-)}\int_0^y\frac{g(x,z)}{z^{\beta_-+1}}dz + \left(\frac{y}{z^*x}\right)^{\beta_+}\frac{G'(1,z^*)z^*-\beta_-G(1,z^*)}{\beta_+-\beta_-}$$

where $\Sigma^2 = \sigma_1^2(\frac{w_2}{w_1})^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\frac{w_2}{w_1}$ and β_+, β_- are as above. Here z^* is found by solving

$$\frac{2(z^*)^{\beta_+}}{\Sigma^2(\beta_+-\beta_-)}\int_0^{z^*}\frac{g(1,z)}{z^{\beta_-+1}}dz = \beta_+G(1,z^*) - G'(1,z^*)z^*.$$

The value function can be rewritten in the equivalent form:

$$V(x,y) = \frac{2}{\Sigma^2(\beta_+-\beta_-)}[y^{\beta_+}\int_y^{z^*x}\frac{g(x,z)}{z^{\beta_++1}}dz + y^{\beta_-}\int_{z^*x}^y\frac{g(x,z)}{z^{\beta_-+1}}dz] + \left(\frac{y}{z^*x}\right)^{\beta_+}\frac{G'(1,z^*)z^*-\beta_-G(1,z^*)}{\beta_+-\beta_-} + \left(\frac{y}{z^*x}\right)^{\beta_-}\frac{\beta_+G(1,z^*)-G'(1,z^*)z^*}{\beta_+-\beta_-}.$$

4 Exact Solution to the Free Boundary Problem with two Risk Factors

In this section, we start with an optimal exit problem with instantaneous reward g. Assume that the economic problem under consideration is driven by two stochastic variables X_i, $i = 1, 2$, following (1) and such that there exists $c_0 = const > 0$ s.t.

$$\sum_{i,j=1}^{2}\sigma_{ij}\xi_i\xi_j \geq c_0|\xi|^2, \text{ for all } \xi = (\xi_1, \xi_2) \in \mathbf{R}^2 \tag{10}$$

where $\sigma_{ii} = \sigma_i^2$, $i = 1, 2$ and $\sigma_{12} = \rho\sigma_1\sigma_2$. Denote $\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$ by Σ and let $m_i = \mu_i - \frac{\sigma_i^2}{2}$, $i = 1, 2$.

Assume that the optimal boundary for problem (4) is of the form $x_1 = x_1^*(x_2)$ or $x_2 = x_2^*(x_1)$. We will show that this assumption is legitimate in most cases of practical interest.

Typically, the continuation region C can be of the form:

i) $\{(x_1, x_2);\ 0 \le x_1 < x^*(x_2))\}$ for exit problems when costs become too high;

ii) $\{(x_1, x_2);\ x_1 > x^*(x_2))\}$ for exit problems when returns become too low.

4.1 Boundary Data on the Axes x_i=0

Suppose that $g \in C(\overline{\mathbf{R}}_+^2)$ and $g(.,0)$ changes sign in its domain. On the ray $x_2 = 0$ the equation $[r - L]V = g$ becomes:

$$[\tfrac{1}{2}\sigma_1^2 x_1^2 \tfrac{d^2}{dx_1^2} + \mu_1 x_1 \tfrac{d}{dx_1} - r]V = -g(x_1, 0).$$

To simplify the notation, in this subsection, by $V(x_1)$ we mean $V(x_1, 0)$. Let $\beta_- < 0 < \beta_+$ denote the roots of the equation

$$\tfrac{1}{2}\sigma_1^2 \beta_\pm^2 + (\mu_1 - \tfrac{1}{2}\sigma_1^2)\beta_\pm - r = 0.$$

If we assume $\mu_1 < r$, then $\beta_+ > 1$.

Note that $\frac{g(x_1,0)}{x_1^{\beta_-+1}}$ is locally integrable in a neighborhood of 0. As we are reduced to a one-dimensional problem, we can rely on the explicit solutions available which are reported in Appendix A.

In most problems of interest, some meaningful boundary conditions are added.

In the case *i)*, let us require $\lim_{x_1 \to 0+} V(x_1, 0) < \infty$.

In the case *ii)*, the no-bubble condition takes the form $V(x_1) = O(g(x_1, 0))$ for $x_1 \to \infty$.

Then the solution in case *i)* can be written as:

$$V(x_1) = \frac{2x_1^{\beta_+}}{\sigma_1^2(\beta_+ - \beta_-)} \int_{x_1}^{x_1^*} \frac{g(t,0)}{t^{\beta_+ + 1}} dt + \frac{2x_1^{\beta_-}}{\sigma_1^2(\beta_+ - \beta_-)} \int_{x_1^*}^{x_1} \frac{g(t,0)}{t^{\beta_- + 1}} dt,$$

where $x_1^* > 0$ denotes the optimal stopping point, which is found by solving the following equation:

$$\int_0^{x_1^*} \frac{g(t,0)}{t^{\beta_- + 1}} dt = 0.$$

We refer to Appendix A for comments on existence and uniqueness of x_1^*.

In the case *ii)*, we write the solution in the form:

$$V(x_1) = \frac{2x_1^{\beta_+}}{\sigma_1^2(\beta_+ - \beta_-)} \int_{x_1}^{x_1^*} \frac{g(t,0)}{t^{\beta_+ + 1}} dt + \frac{2x_1^{\beta_-}}{\sigma_1^2(\beta_+ - \beta_-)} \int_{x_1^*}^{x_1} \frac{g(t,0)}{t^{\beta_- + 1}} dt$$

where x_1^* can be determined by solving

$$\int_{x_1^*}^{+\infty} \frac{g(t,0)}{t^{\beta_+ + 1}} dt = 0.$$

Remark 4.1 The argument above can be replicated on the ray $x_1 = 0$ and an optimal threshold point x_2^* can be defined along the same lines.

Example 4.1 Consider an exit problem and let $g(x_1, x_2) = ax_1 + bx_2 + c$. Assume that $a > 0$, $b > 0$ and $c < 0$. This is the case with a problem where ax_1 and bx_2 model positive cash flows from two factors and c the current costs. Let us restrict the problem to $x_2 = 0$. Then the solution on $x_2 = 0$ is

$V(x_1, 0) = \frac{2}{\sigma_1^2(\beta_+ - \beta_-)}(\frac{x_1}{x_1^*})^{\beta_-}\left(\frac{ax_1^*}{\beta_- - 1} + \frac{c}{\beta_-}\right) - \frac{ax_1}{\mu_1 - r} + \frac{c}{r},\ x_1 > x_1^*,$

where $\beta_\pm$ are the roots of $\frac{1}{2}\sigma_1^2\beta_\pm^2 + (\mu_1 - \frac{1}{2}\sigma_1^2)\beta_\pm - r = 0$ and x_1^* is obtained as follows:

$\int_{x_1^*}^{+\infty} \frac{g(t,0)}{t^{\beta_+ + 1}} dt = 0 \Leftrightarrow \frac{ax_1^*}{\beta_+ - 1} + \frac{c}{\beta_+} = 0 \Leftrightarrow x_1^* = \frac{c(1-\beta_+)}{a\beta_+}.$

When we restrict the problem to $x_1 = 0$ we get

$V(0, x_2) = \frac{2}{\sigma_1^2(\widehat{\beta}_+ - \widehat{\beta}_-)}(\frac{x_2}{x_2^*})^{\widehat{\beta}_-}\left(\frac{bx_2^*}{\widehat{\beta}_- - 1} + \frac{c}{\widehat{\beta}_-}\right) - \frac{bx_2}{\mu_2 - r} + \frac{c}{r},\ x_2 > x_2^*,$

where $\widehat{\beta}_\pm$ are the roots of $\frac{1}{2}\sigma_2^2\widehat{\beta}_\pm^2 + (\mu_2 - \frac{1}{2}\sigma_2^2)\widehat{\beta}_\pm - r = 0$ and x_2^* is obtained as follows:

$\int_{x_2^*}^{+\infty} \frac{g(t,0)}{t^{\beta_+ + 1}} dt = 0 \Leftrightarrow \frac{bx_2^*}{\widehat{\beta}_+ - 1} + \frac{c}{\widehat{\beta}_+} = 0 \Leftrightarrow x_2^* = \frac{c(1-\widehat{\beta}_+)}{b\widehat{\beta}_+}.$

4.2 Properties of the Optimal Boundary

A typical assumption on g is that g is continuous and satisfies an integrability condition of the form:

$$E_x[\int_0^{+\infty} e^{-rt}\, |g(X_t^1, X_t^2)|\, dt] < \infty \text{ for any } x.$$

Note that if $g \geq 0$ then the optimal stopping time is ∞, while if $g \leq 0$ it is optimal to stop immediately. To avoid such trivial situations we assume that g changes sign. Moreover, if g is monotonic some regularity properties of the optimal stopping boundary are easily obtained. We refer to Peskir (2019) for the continuity property of the optimal stopping boundary within a more general framework and to De Angelis and Peskir (2020) for the issue of regularity of the value function up to the free boundary.

In this Subsection we focus on a specific situation which is satisfied, for example, in exit problems where a cash flow is abandoned when costs

become too high. In this case, some properties of the optimal threshold can be established in a more elementary way. Other possible situations (e.g. entry problems) can be treated in a similar way. In particular, let us assume that g satisfies:
(H1) $g(0+, x_2) > 0$, $g(+\infty, x_2) < 0$ and $g(., x_2)$ is decreasing $\forall x_2 \in I$, where I is of the form $(0, x_2^*)$, $(x_2^*, +\infty)$ or $(0, +\infty)$.
Then the optimal boundary can be written in the form $x_1 = x^*(x_2)$ in I and the continuation region is of the form $\{(x_1, x_2);\ x_1 < x^*(x_2))\}$. We can obtain the following

Proposition 4.1 *In addition to (H1), let us assume that g is monotonic in x_2 for any fixed x_1. Then V is continuous and has the same monotonicity as g. If g is convex then V is convex and the stopping set is convex, the function x^* is convex and continuous and it increases (decreases) whenever g increases (decreases) in x_2 for any fixed x_1.*

Proof. Continuity of V follows from general theory (see, f.e. Peskir and Shiryaev, 2006). Let us show that V decreases in x_1 in the continuation set. Let $(x_1, x_2) \in C$ and $(x_1 + \varepsilon, x_2) \in C$ with $\varepsilon > 0$. Let τ denote any stopping time and let $X_t^i(x)$ denote $xe^{(\mu_i - \sigma_i^2)t + \sigma W_t^i}$. Then

$$E[\int_0^\tau e^{-rt} g(X_t^1(x_1 + \varepsilon), X_t^2(x_2))dt] \leq E[\int_0^\tau e^{-rt} g(X_t^1(x_1), X_t^2(x_2))dt]$$

because g is decreasing in the first component. If τ is the optimal stopping time for the problem with initial state $(x_1 + \varepsilon, x_2)$ it follows that $V(x_1 + \varepsilon, x_2) \leq E[\int_0^\tau e^{-rt} g(X_t^1(x_1), X_t^2(x_2))dt]$. For that specific stopping time τ, we know that $E[\int_0^\tau e^{-rt} g(X_t^1(x_1), X_t^2(x_2))dt] \leq V(x_1, x_2)$, and thus $V(x_1 + \varepsilon, x_2) \leq V(x_1, x_2)$. The other monotonicity properties of V can be proved in a similar fashion.
Let g be convex. Let us prove that V is convex in C. Let (x_1', x_2') and $(x_1'', x_2'') \in C$, $\theta \in [0, 1]$ and denote $\theta x_i' + (1 - \theta)x_i''$ by $\overline{x}_i$, $i = 1, 2$. In view of convexity of g for any stopping time τ we have:

$$E[\int_0^\tau e^{-rt} g(X_t^1(\overline{x}_1), X_t^2(\overline{x}_2))dt] \leq$$

$$\theta E[\int_0^\tau e^{-rt} g(X_t^1(x_1'), X_t^2(x_2'))dt] + (1-\theta) E[\int_0^\tau e^{-rt} g(X_t^1(x_1''), X_t^2(x_2''))dt] \leq$$

$$\theta V(x_1', x_2') + (1-\theta) V(x_1'', x_2'').$$

Then taking the *sup* of the left-hand side we get:

$$V(\overline{x}_1, \overline{x}_2) \leq \theta V(x_1', x_2') + (1-\theta) V(x_1'', x_2'').$$

It follows that the stopping set is convex. Let us argue by contradiction and suppose that (x_1', x_2') and $(x_1'', x_2'') \in S$, but $(\overline{x}_1, \overline{x}_2) \in C$. Then

$$V(\overline{x}_1, \overline{x}_2) \leq \theta V(x_1', x_2') + (1-\theta) V(x_1'', x_2'') = 0 < V(\overline{x}_1, \overline{x}_2)$$

which yields a contradiction.

The function x^* is convex because its epigraph is the convex set S; it is continuous because it is a convex function on an open convex set.

Finally, x^* is monotonic because $x^*(x_2) = \inf \{x_1; V(x_1, x_2) = 0\}$ and thus x^* increases (decreases) whenever V increases (decreases) in x_2.

4.3 Explicit Solution via Fundamental Solution

Let us now solve problem (4). Our first step is to make the following change of variables:

$$y_j = \ln x_j,\ j = 1, 2,\ \ V(x_1, x_2) = e^{\alpha_1 y_1 + \alpha_2 y_2} v(y_1, y_2) \Longrightarrow$$

$$g(x_1, x_2) = -f(y_1, y_2) e^{\alpha_1 y_1 + \alpha_2 y} \text{ and } \partial\widetilde{C} = \{(y^*(y_2), y_2)\,; y_2 \in I \subset \mathbb{R}\}$$

where $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = -\Sigma^{-1} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = -\Sigma^{-1} m.$

Then the continuation region $\widetilde{C}$ is either of the form:

i) $\{(y_1, y_2);\ y_1 < y^*(y_2))\}$ for exit problems when costs become too high, or

ii) $\{(y_1, y_2);\ y_1 > y^*(y_2))\}$ for exit problems when returns become too low.

Thus, (4) takes the form:

$$\begin{cases} \widetilde{L} v = f & \text{in } \widetilde{C} \\ v = 0 & \text{in } \widetilde{S} \end{cases} \tag{11}$$

where

$$\widetilde{L} = \sum_{i,j=1}^{2} \frac{\sigma_{ij}}{2} \frac{\partial^2}{\partial y_i \partial y_j} + \widetilde{k}, \tag{12}$$

with $\widetilde{k} = \frac{1}{2}[(\alpha_1\sigma_1)^2 + (\alpha_2\sigma_2)^2 + 2\rho\alpha_1\alpha_2\sigma_1\sigma_2] + \alpha_1 m_1 + \alpha_2 m_2 - r$. Note that $\widetilde{k} + r = -\frac{1}{2} m^t \Sigma m \leq 0$, and therefore $\widetilde{k}$ can be written as $-k^2$ for some real k.

Changing to variables $\begin{cases} \sigma_1^{-1} y_1 = z_1\sqrt{1-\rho^2} + \rho z_2 \\ \sigma_2^{-1} y_2 = z_2 \end{cases}$ and $v(y_1, y_2) = u(z_1, z_2)$ the differential equation $\widetilde{L}v = f$ is transformed into:

$$\frac{1}{2}[\partial^2_{z_1} u + \partial^2_{z_2} u] - k^2 u = \varphi(z_1, z_2), \tag{13}$$

This is a modified Helmholtz type equation. Recall that a fundamental solution, $\mathcal{E}_2$, for the modified Helmholtz equation in $\mathbf{R}^2$ is written as follows.

Let K_ν denote the modified Bessel function of second kind (see Appendix B) and let $\mathcal{E}_2(x) = \frac{K_0(h|x|)}{-2\pi}$, $x \in R^2$, $h > 0$. Then $[\Delta_2 - h^2]\mathcal{E}_2 = \delta(x)$, i.e. $\mathcal{E}_2$ is a fundamental solution of the Helmholtz operator $\Delta_2 - h^2$. (See

Budak et al., 1972).

The continuation region is either of the form:

i) $\{(z_1, z_2);\ z_1 < z^*(z_2))\}$ or

ii) $\{(z_1, z_2);\ z_1 > z^*(z_2))\}$

where $z^*(z_2) = [\sigma_1 y^*(z_2/\sigma_2) + \rho z_2]/\sqrt{1-\rho^2}$.

Then the solution of the boundary value problem for *(11)* in case *i)* and with the threshold function defined on $(0, x_2^*)$ is given by

$$u(z_1, z_2) = \frac{-1}{\pi} \int_{-\infty}^{\sigma_2^{-1} y_2^* - z_2} d\eta_2 \int_{-\infty}^{z^*(z_2+\eta_2)-z_1} K_0(\sqrt{2}k\sqrt{\eta_1^2 + \eta_2^2})\varphi(z_1+\eta_1, z_2+\eta_2) d\eta_1$$

where $y_2^* = \ln(x_2^*)$ and x_2^* can be found as in Subsection 4.1. Going back to the old coordinates we obtain:

$$V(x_1,x_2) \;=\; \frac{1}{\pi}\int_{-\infty}^{\frac{1}{\sigma_2}\ln\frac{x_2^*}{x_2}}\int_{-\infty}^{\zeta^*(x_1,x_2;\eta_2)} K_0(\sqrt{2}k\sqrt{\eta_1^2+\eta_2^2})e^{-\alpha_1\sigma_1[\eta_1\sqrt{1-\rho^2}+\rho\eta_2]-\alpha_2\sigma_2\eta_2}$$
$$g(x_1e^{(\eta_1\sqrt{1-\rho^2}+\rho\eta_2)\sigma_1}, x_2e^{\eta_2\sigma_2})d\eta_1 d\eta_2 \qquad (14)$$

with $\zeta^*(x_1,x_2;\eta_2) = [\frac{1}{\sigma_1}(\ln x^*(x_2e^{\eta_2\sigma_2}) - \ln x_1) - \rho\eta_2]/\sqrt{1-\rho^2}$.

Changing to variables $\begin{cases} \xi_1 = \eta_1\sqrt{1-\rho^2}+\rho\eta_2 \\ \xi_2 = \eta_2 \end{cases}$ and setting $k_\rho = k/\sqrt{1-\rho^2}$
we can also write:

$$V(x_1,x_2) = \frac{1}{\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\frac{1}{\sigma_2}\ln\frac{x_2^*}{x_2}}\int_{-\infty}^{\frac{1}{\sigma_1}\ln[x^*(x_2e^{\xi_2\sigma_2})/x_1]/\sqrt{1-\rho^2}} K_0(\sqrt{2}k_\rho\sqrt{\xi_1^2+\xi_2^2-2\rho\xi_1\xi_2})$$
$$e^{-\alpha_1\sigma_1\xi_1-\alpha_2\sigma_2\xi_2}g(x_1e^{\sigma_1\xi_1}, x_2e^{\sigma_2\xi_2})d\xi_1 d\xi_2$$

The other possible cases can be treated in a similar way. For example, in the case *(ii)* and with the threshold function defined on $(x_2^*, +\infty)$ one gets

$$V(x_1,x_2) = \frac{1}{\pi\sqrt{1-\rho^2}}\int_{\frac{1}{\sigma_2}\ln\frac{x_2^*}{x_2}}^{+\infty}\int_{\frac{1}{\sigma_1}\ln[x^*(x_2e^{\xi_2\sigma_2})/x_1]/\sqrt{1-\rho^2}}^{+\infty} K_0(\sqrt{2}k_\rho\sqrt{\xi_1^2+\xi_2^2-2\rho\xi_1\xi_2})$$
$$e^{-\alpha_1\sigma_1\xi_1-\alpha_2\sigma_2\xi_2}g(x_1e^{\sigma_1\xi_1}, x_2e^{\sigma_2\xi_2})d\xi_1 d\xi_2$$

4.4 An Equation for the Optimal Threshold

If we insert the optimal boundary $x_1 = x^*(x_2)$ into the above-obtained expression for the value function then we get $V(x^*(x_2), x_2) = 0$ for all x_2, that is, we get an equation for the unknown function $x_2 \to x^*(x_2)$. In principle, this can be solved through numerical iteration. However the presence of the unknown in an endpoint of the double integral makes the procedure rather unpleasant. In what follows we obtain a more easy-to-handle expression.
Changing to variables $\sigma_1\xi_1 = Z_1 + \ln[x^*(x_2e^{\xi_2\sigma_2})/x^*(x_2)]$ and $Z_2 = x_2e^{\xi_2\sigma_2}$, in the case *(i)* above the unknown threshold is the solution of the integral equation:

$$\int_0^{x_2^*} \frac{1}{Z_2^{\alpha_2+1}}\left(\frac{x^*(x_2)}{x^*(Z_2)}\right)^{\alpha_1} \int_{\mathbf{0}}^1 K_0\left(\sqrt{2}k_\rho\sqrt{[\ln \frac{Z_1 x^*(Z_2)}{x^*(x_2)}]^2 \frac{1}{\sigma_1^2} + [\frac{1}{\sigma_2}\ln\frac{Z_2}{x_2}]^2 - \frac{2\rho}{\sigma_1\sigma_2}\ln\frac{Z_1 x^*(Z_2)}{x^*(x_2)}\ln\frac{Z_2}{x_2}}\right)$$
$$\frac{1}{Z_1^{\alpha_1+1}} g(Z_1 x^*(Z_2), Z_2) dZ_1 dZ_2 = 0.$$

In order to avoid singularity in the integral some restrictions on the parameters needs to be posed. A sufficient condition which guarantees that α_1 and α_2 are negative is the following:

$m_1 > 0$ and $m_2 > 0$; moreover $\rho^2 \frac{m_2}{\sigma_2} < \rho\frac{m_1}{\sigma_1} < \frac{m_2}{\sigma_2}$ for $0 < \rho < 1$.

For the case of an integral equation of the form:

$$\int_{x_2^*}^{\infty} \frac{1}{Z_2^{\alpha_2+1}}\left(\frac{x^*(x_2)}{x^*(Z_2)}\right)^{\alpha_1} \int_{\mathbf{0}}^1 K_0\left(\sqrt{2}k_\rho\sqrt{[\ln \frac{Z_1 x^*(Z_2)}{x^*(x_2)}]^2 \frac{1}{\sigma_1^2} + [\frac{1}{\sigma_2}\ln\frac{Z_2}{x_2}]^2 - \frac{2\rho}{\sigma_1\sigma_2}\ln\frac{Z_1 x^*(Z_2)}{x^*(x_2)}\ln\frac{Z_2}{x_2}}\right)$$
$$\frac{1}{Z_1^{\alpha_1+1}} g(Z_1 x^*(Z_2), Z_2) dZ_1 dZ_2 = 0$$

one needs to guarantee that $\alpha_1 < 0$ and $\alpha_1 + \alpha_2 > 0$, and similar conditions should be posed on the parameter values in all other possible cases.

5 Applications

In this Section some examples are presented to illustrate how to find the optimal boundary. First we show that the homogeneous case of Section 3 can be recovered from our general expression. Our computation serves as a sanity check and, at the same time, provides a method to reduce the dimensionality of the integral, which can be used also in other cases.

Example 5.1 *(Homogeneous reward function).*
Assume that g is homogeneous of degree h and consider case *(i)*. The optimal threshold is of the form $x_2 = z^* x_1$, where z^* will be determined through (14). In view of homogeneity of g the equation for the unknown threshold can be written in the form:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{s\eta_2} K_0(\sqrt{2}k\sqrt{\eta_1^2+\eta_2^2}) e^{(h-\alpha_1)\sigma_1[\eta_1\sqrt{1-\rho^2}+\boldsymbol{\rho\eta}_2]-\alpha_2\sigma_2\eta_2}$$
$$g(1, z^* e^{-\sigma_1[\eta_1\sqrt{1-\rho^2}+\boldsymbol{\rho\eta}_2]+\sigma_2\eta_2}) d\eta_1 d\eta_2 = 0$$

where $s = (\frac{\sigma_2}{\sigma_1} - \rho)\frac{1}{\sqrt{1-\rho^2}}$. Now we use the following identity (see Erdely et al., 1953):

$$K_0(\sqrt{2}k\xi) = \int_0^\infty e^{-k^2 t} e^{-\xi^2/(2t)} \frac{dt}{t}$$

to replace $K_0(\sqrt{2}k\sqrt{\eta_1^2+\eta_2^2})$ with an integral in dt. Then, changing to variables $\sqrt{1-\rho^2}[s\eta_2 - \eta_1] = -z\sqrt{t}$ and $\eta_2 = \eta\sqrt{t}$, we get:

$$\int_0^\infty e^{-k^2 t} e^{\frac{t(\alpha_1+\alpha_2-h)^2\sigma_2^2}{2(s^2+1)}} \int_{-\infty}^0 g(1, z^* e^{-\sigma_1 z\sqrt{t}}) e^{\frac{-z^2}{2(s^2+1)(1-\rho^2)}} e^{Bz\sqrt{t}}$$
$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}[\sqrt{s^2+1}\eta+\omega(z,t)]} d\eta dz dt = 0$$

where $\quad B = (h-\alpha_1)\sigma_1 + \frac{(\alpha_1+\alpha_2-h)s\sigma_2}{(s^2+1)\sqrt{1-\rho^2}} = \frac{\sigma_1}{\Xi}[(h-\alpha_1)\Delta_1 + \alpha_2\Delta_2]$,

$\Delta_1 = \sigma_1^2 - \rho\sigma_1\sigma_2$, $\Delta_2 = \sigma_2^2 - \rho\sigma_1\sigma_2$, $\Xi = \Delta_1 + \Delta_2$;

moreover $\omega(z,t) = \frac{sz}{(s^2+1)\sqrt{1-\rho^2}} + \frac{(\alpha_1+\alpha_2-h)\sigma_2\sqrt{t}}{(s^2+1)}$.

The integral in $d\eta$ is equal to $\sqrt{2\pi}N(+\infty) = \sqrt{2\pi}$ and thus, after changing variables $\sigma_1 z\sqrt{t} = \zeta$ in the integral in dz, we are reduced to consider the equation:

$$\int_0^\infty \frac{1}{\sqrt{t}} e^{-\frac{A^2 t}{2\Xi}} \int_{-\infty}^0 g(1, z^* e^{-\zeta}) e^{\frac{-\zeta^2}{2t\Xi}} e^{B\zeta/\sigma_1} d\zeta dt = 0$$

where $A^2 = (2r + \alpha^t\Sigma\alpha)\Xi - (\alpha_1+\alpha_2-h)^2(\sigma_1\sigma_2)^2(1-\rho^2)$ is obtained from the explicit expression of k^2 provided in Subsection 4.2.

Now changing to variables $A\sqrt{t} + \zeta/\sqrt{t} = \sqrt{\Xi}\tau$ in the integral in dt we obtain the following equation:

$$\int_{-\infty}^0 g(1, z^* e^{-\zeta}) \exp[\zeta((h-\alpha_1)\Delta_1 + \alpha_2\Delta_2 + A)/\Xi] \int_{-\infty}^{+\infty} e^{-\tau^2/2}[\tfrac{\sqrt{\Xi}}{A} d\tau +$$
$$\tfrac{2\Xi\tau d\tau}{A\sqrt{\Xi\tau - 4\zeta A}}] d\zeta = 0.$$

The integral in $d\tau$ is equal to $\sqrt{2\pi\Xi}/A$. Therefore the equation for z^* can be finally written in the form:

$$\int_{-\infty}^0 g(1, z^* e^{-\zeta}) \exp[\zeta((h-\alpha_1)\Delta_1 + \alpha_2\Delta_2 + A)/\Xi] d\zeta = 0.$$

Note that $((h-\alpha_1)\Delta_1 + \alpha_2\Delta_2 + A)/\Xi = \beta_+$ is the positive root of

$$\tfrac{\Xi}{2}\beta^2 + [m_2 - m_1 + h(\sigma_1^2 - \rho\sigma_1\sigma_2)]\beta + hm_1 + \tfrac{(h\sigma_1)^2}{2} - r = 0,$$

which is the equation for the h-homogeneous case studied in Section 3. Changing to variables $z^* e^{-\zeta} = z$ we find again that z^* solves the integral equation:

$$\int_{z^*}^{\infty} g(1,z) z^{-\beta_+ -1} dz = 0$$

which is consistent with the results of Section 3.

Example 5.2 *(Affine reward function).* Let us consider the problem of abandoning a cash flow $g(x_1, x_2) = 1 - x_1 - x_2$, where x_1 and x_2 represents current costs driven by two stochastic sources, while the return flow is deterministic. Figure 1 plots an approximated optimal boundary when the following parameter values are adopted: μ_1=0.02, μ_2=0.04, σ_1=0.1, σ_2=0.15, ρ=0 and r=0.005. From Subsection 4.1 we know that the optimal boundary crosses the axes at about $x_1^* = 0.833$ and $x_2^* = 0.789$ and the picture is obtained by improving the approximation starting from the straight line joining $(x_1^*, 0)$ and $(0, x_2^*)$. As this straight line is, in general, a rough representation of the optimal threshold, we improve the approximation by replacing it with two segments joining $(0, x_2^*)$ with $(\frac{x_1^*}{2}, \widehat{x})$ and $(\frac{x_1^*}{2}, \widehat{x})$ with $(x_1^*, 0)$, respectively, where $\widehat{x}$ is chosen as to minimize the distance between 0 and the integral of Subsection 4.4, that is to yield a better approximate solution of the integral equation. The procedure is iterated by inserting further points till the integral equation of Subsection 4.4 obtains a better solution than in the previous step. We point out that a more accurate numerical solution would require a more efficient choice of the interpolating points, which is a challenging task and is beyond the scope of our study.

6 Conclusions

An explicit expression has been obtained for the value function of a general class of two-dimensional optimal stopping problems arising in real option theory. Our formula applies also to the unsolved case of inhomogeneous reward functions. Although our solution is expressed in terms of non-elementary functions, that is, modified Bessel function of second kind, such functions are included in built-in-function packages in various numerical softwares, and thus our formula can be easily computed numerically. Our paper also clarifies why the "quasi-analytical" method - which employs solutions of power type- does not rely on the correct form for the value function, while functions of power type provide exact solutions for

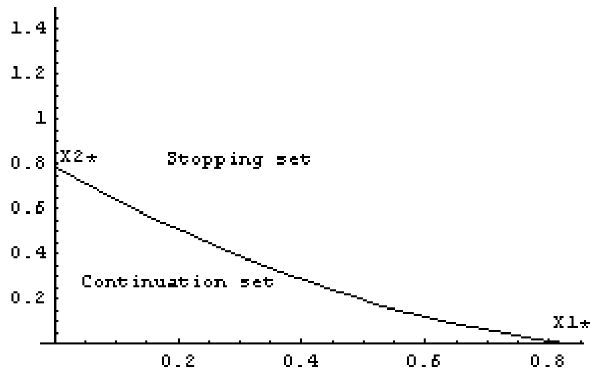

Figure 1: Optimal threshold for the problem of Example 5.2.

the case of homogeneous reward functions. For the latter case, we present a general framework: the method of reduction of dimensionaly has been applied many times in the real option literature, but usually only to linear functions, while we provide general formulas and, more importantly, show how to extend the method to quasi-homogeneous reward functions. Finally, in the special homogeneous case (see Example 5.1), we reconcile our general non-elementary solution with the elementary method based on power functions. For the unsolved inhomogeneous case, we provide an integral equation to determine the optimal threshold. Although an analytical expression for the optimal threshold remains a challenge, our equation allows to control for numerical approximations of the unknown threshold, which is of practical significance.

Finally, this paper includes some formulas for the one-dimensional case which are reported for readers' convenience: we trust that they can be useful for scholars working in real option theory as well, as these mathematical expressions are not completely new, but they are dispersed or hardly found in the literature.

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7 Appendix A: Option to Abandon or to Acquire a Perpetual Cash Flow

In this section we review the one-dimensional case of an option to abandon or to acquire a cash flow which is modelled through a function, $g(X_t)$, where X_t is of the form (1), that is, $X_t = x\exp[(\mu - \frac{\sigma^2}{2})t + \sigma W_t]$.
Let us assume that $g \in C(\overline{\mathbf{R}}_+)$ and g changes sign in its domain. For the option to abandon the stream $g(X_t)$, in the continuation region the differential equation (4) holds, that is:
$[r - L]V = g,$
while V becomes 0 in the stopping region where the flow is abandoned. Specifically, the differential equation is:

$$[\frac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} + \mu x \frac{d}{dx} - r]V + g(x) = 0 \tag{A1}$$

and the related algebraic equation is:

$$\frac{1}{2}\sigma^2\beta^2 + (\mu - \frac{1}{2}\sigma^2)\beta - r = 0. \tag{A2}$$

Let $\beta_- < 0 < \beta_+$ denote the roots of this equation. A standard assumption is $\mu < r$, which yields $\beta_+ > 1$.

(A1) is a Euler type differential equation and its classical solution is obtained through Lagrange formula of the variations of constants.

Note that $\frac{g(x)}{x^{\beta_-+1}}$ is locally integrable in a neighborhood of 0. Moreover, we assume that $\frac{g(x)}{x^{\beta_++1}}$ is integrable in a neighborhood of ∞.[3]
Then the general solution is of the form

$$V(x) = A_+ x^{\beta_+} + A_- x^{\beta_-} + \frac{2x^{\beta_+}}{\sigma^2(\beta_+-\beta_-)} \int_x^{\infty} \frac{g(z)}{z^{\beta_++1}} dz + \frac{2x^{\beta_-}}{\sigma^2(\beta_+-\beta_-)} \int_0^x \frac{g(z)}{z^{\beta_-+1}} dz$$

with arbitrary constants A_+, A_- to be determined.
In most problems of interest, some meaningful boundary conditions are added. Let us fix two cases.

i) The continuation region is of the form: $[0, x^*)$;

ii) the continuation region is of the form: $(x^*, +\infty)$,

where the threshold point, $x^* > 0$, is the optimal exit point.

In the case *i)*, let us require $\lim_{x\to 0+} V(x) < \infty$.[4]
In the case *ii),* the no-bubble condition may take the form $V(x) = O(g(x))$ for $x \to \infty$.
Then the general solution in case *i)* can be written as:

$$V(x) = A_+ x^{\beta_+} + \frac{2x^{\beta_+}}{\sigma^2(\beta_+-\beta_-)} \int_x^{+\infty} \frac{g(z)}{z^{\beta_++1}} dz + \frac{2x^{\beta_-}}{\sigma^2(\beta_+-\beta_-)} \int_0^x \frac{g(z)}{z^{\beta_-+1}} dz.$$

In view of the smooth-pasting condition one can compute A_+ and x^* by solving the following equations:

$$\begin{cases} A_+(x^*)^{\beta_+} + \frac{2x^{*\beta_+}}{\sigma^2(\beta_+-\beta_-)} \int_{x^*}^{+\infty} \frac{g(z)}{z^{\beta_++1}} dz + \frac{2x^{*\beta_-}}{\sigma^2(\beta_+-\beta_-)} \int_0^{x^*} \frac{g(z)}{z^{\beta_-+1}} dz = 0 \\ A_+\beta_+(x^*)^{\beta_+} + \frac{2x^{*\beta_+}\beta_+}{\sigma^2(\beta_+-\beta_-)} \int_{x^*}^{+\infty} \frac{g(z)}{z^{\beta_++1}} dz + \frac{2x^{*\beta_-}\beta_-}{\sigma^2(\beta_+-\beta_-)} \int_0^{x^*} \frac{g(z)}{z^{\beta_-+1}} dz = 0 \end{cases}$$

In particular, we obtain x^* by solving $\int_0^{x^*} \frac{g(z)}{z^{\beta_-+1}} dz = 0$. If g is not monotonic we consider the smallest solution. Finally, $A_+ = \frac{-2}{\sigma^2(\beta_+-\beta_-)} \int_{x^*}^{+\infty} \frac{g(z)}{z^{\beta_++1}} dz$, which gives

$V(x) = \frac{2x^{\beta_+}}{\sigma^2(\beta_+-\beta_-)} \int_x^{x^*} \frac{g(z)}{z^{\beta_++1}} dz + \frac{2x^{\beta_-}}{\sigma^2(\beta_+-\beta_-)} \int_0^x \frac{g(z)}{z^{\beta_-+1}} dz,$

which can be written as

$V(x) = \frac{2x^{\beta_+}}{\sigma^2(\beta_+-\beta_-)} \int_x^{x^*} \frac{g(z)}{z^{\beta_++1}} dz + \frac{2x^{\beta_-}}{\sigma^2(\beta_+-\beta_-)} \int_{x^*}^x \frac{g(z)}{z^{\beta_-+1}} dz,$

as well.

Remark A.1 If g never changes sign, then $\int_0^{x^*} \frac{g(z)}{z^{\beta_-+1}} dz = 0$ does not achieve any positive solution. However, this equation may happen to admit no solution even in the case of g with variable sign.

[3] A sufficient condition is that $|g(x)| \leq Cx^\gamma$ for large x, where $C \geq 0$ and $\gamma < \beta_+$.

[4] For example, if x represents the current cost variable in an exit problem, it is reasonable to assume that the value function is zero when $x = 0$, because the option to abandon operations is worthless in this extreme situation.

In the case *ii)*, we look for a general solution of the form:

$V(x) = A_{-}x^{\beta_{-}} + \frac{2x^{\beta_{+}}}{\sigma^2(\beta_{+}-\beta_{-})}\int_x^{+\infty}\frac{g(z)}{z^{\beta_{+}+1}}dz + \frac{2x^{\beta_{-}}}{\sigma^2(\beta_{+}-\beta_{-})}\int_0^x\frac{g(z)}{z^{\beta_{-}+1}}dz$

where the constant A_{-} is determined through the smooth pasting conditions at some x^*. Here x^* can be determined by solving

$\int_{x^*}^{+\infty}\frac{g(z)}{z^{\beta_{+}+1}}dz = 0$. Then we obtain:

$V(x) = \frac{2x^{\beta_{+}}}{\sigma^2(\beta_{+}-\beta_{-})}\int_x^{x^*}\frac{g(z)}{z^{\beta_{+}+1}}dz + \frac{2x^{\beta_{-}}}{\sigma^2(\beta_{+}-\beta_{-})}\int_{x^*}^x\frac{g(z)}{z^{\beta_{-}+1}}dz.$

The value of an option to acquire a cash flow, $g(X_t)$, can be obtained from the formula for the option to abandon. Indeed, if t^* denotes the optimal stopping time for this problem, then we have

$$E_x[\int_{t^*}^{+\infty} e^{-rt}g(X_t)dt] = E_x[\int_0^{+\infty} e^{-rt}g(X_t)dt] + E_x[\int_0^{t^*} e^{-rt}(-g(X_t))dt],$$

which can be rephrased as follows.

The option to acquire a perpetual stream, $g(X_t)$, at no cost, can be regarded as the expected value of the full stream plus the option to abandon a stream $-g(X_t)$, at no cost. Preliminarly we compute the expected value of the entire perpetual flow.

Lemma A1. Let $g \in C(\overline{\mathbf{R}}_{+})$ and let $E_x[\int_0^{+\infty} e^{-rt}\left|g(X_t)\right|dt] < \infty$. Then, for any $x > 0$, $E_x[\int_0^{+\infty} e^{-rt}g(X_t)dt]$ can be written as:

$$\frac{2x^{\beta_{+}}}{\sigma^2(\beta_{+}-\beta_{-})}\int_x^{\infty}\frac{g(z)}{z^{\beta_{+}+1}}dz + \frac{2x^{\beta_{-}}}{\sigma^2(\beta_{+}-\beta_{-})}\int_0^x\frac{g(z)}{z^{\beta_{-}+1}}dz. \qquad \text{(A3)}$$

Proof. $E_x[\int_0^{+\infty} e^{-rt}g(X_t)dt] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\varepsilon^2/2}\int_0^{+\infty} e^{-rt}g(xe^{\nu t+\sigma\varepsilon\sqrt{t}})dtd\varepsilon$ with $\nu = \mu - \frac{\sigma^2}{2}$. Denote $xe^{\nu t+\sigma\varepsilon\sqrt{t}}$ by z, $\ln(\frac{z}{x})$ by z_0 and let $\beta\pm$ denote the roots of (A2) with $\beta_{-} < 0 < \beta_{+}$. Then the integral above can be written as:

$\frac{1}{\sqrt{2\pi}\sigma}\int_0^{+\infty}\int_0^{+\infty}\exp[-rt - \frac{(z_0-\nu t)^2}{2\sigma^2 t}]\frac{g(z)}{\sqrt{t}z}dtdz.$

Let $\sqrt{2r + \left(\frac{\nu}{\sigma}\right)^2}$ be denoted by Δ, which is also equal to $\frac{\sigma}{2}(\beta_+ - \beta_-)$. Let us split the integral in dz into two integrals $\int\limits_0^x + \int\limits_x^{+\infty}$. In the first integral z_0 is negative and the change of variables $\frac{z_0}{\sigma\sqrt{t}} + \Delta\sqrt{t} = \tau$ yields:

$$\frac{\sqrt{2}}{\sqrt{\pi}\sigma^2(\beta_+ - \beta_-)} \int\limits_0^x \int\limits_{-\infty}^{+\infty} \exp[-z_0\beta_- - \frac{\tau^2}{2}] \frac{g(z)}{z} (1 + \frac{\tau}{\sqrt{\tau^2 - 4z_0\Delta/\sigma}}) d\tau dz = \frac{2x^{\beta_-}}{\sigma^2(\beta_+ - \beta_-)} \int_0^x \frac{g(z)}{z^{\beta_- + 1}} dz.$$

As for the other integral, bearing in mind that z_0 is positive, the change of variables $\frac{z_0}{\sigma\sqrt{t}} - \Delta\sqrt{t} = \tau$ yields:

$$\frac{\sqrt{2}}{\sqrt{\pi}\sigma^2(\beta_+ - \beta_-)} \int\limits_x^{+\infty} \int\limits_{-\infty}^{+\infty} \exp[-z_0\beta_+ - \frac{\tau^2}{2}] \frac{g(z)}{z} (-1 + \frac{\tau}{\sqrt{\tau^2 + 4z_0\Delta/\sigma}}) d\tau dz =$$

$$\frac{2x^{\beta_+}}{\sigma^2(\beta_+ - \beta_-)} \int_x^{+\infty} \frac{g(z)}{z^{\beta_+ + 1}} dz.$$

Let us now turn to the valuation of the option to acquire a cash flow. If the continuation region is of the form $[0, x^*)$, then $V(x) = E_x[\int\limits_0^{+\infty} e^{-rt} g(X_t) dt]$ in the stopping region $x > x^*$, where x^* is obtained as the optimal threshold of an exit problem for $-g$ in the case *(i)*. Thus we obtain x^* by solving $\int_0^{x^*} \frac{g(z)}{z^{\beta_- + 1}} dz = 0$. Moreover, the value function can be obtained as follows.

$$V(x) = E_x[\int\limits_0^{+\infty} e^{-rt} g(X_t) dt] +$$

$$\begin{cases} 0 & \text{if } x > x^* \\ \frac{2x^{\beta_+}}{\sigma^2(\beta_+ - \beta_-)} \int_x^{x^*} \frac{-g(z)}{z^{\beta_+ + 1}} dz + \frac{2x^{\beta_-}}{\sigma^2(\beta_+ - \beta_-)} \int_0^x \frac{-g(z)}{z^{\beta_- + 1}} dz & \text{if } x < x^* \end{cases}.$$

In view of Lemma A1 the value of $V(x)$ for $x < x^*$ can be written as:
$V(x) = \frac{2x^{\beta_+}}{\sigma^2(\beta_+ - \beta_-)} \int_{x^*}^{\infty} \frac{g(z)}{z^{\beta_+ + 1}} dz.$

Similarly, the option to acquire a cash flow when the continuation region is of the form (x^*, ∞) can be obtained as

$$V(x) = E_x[\int\limits_0^{+\infty} e^{-rt} g(X_t) dt] +$$

$$\begin{cases} 0 & \text{if } x < x^* \\ \frac{2x^{\beta_+}}{\sigma^2(\beta_+-\beta_-)}\int_x^{\infty}\frac{-g(z)}{z^{\beta_++1}}dz + \frac{2x^{\beta_-}}{\sigma^2(\beta_+-\beta_-)}\int_{x^*}^{x}\frac{-g(z)}{z^{\beta_-+1}}dz & \text{if } x > x^* \end{cases}$$

where x^* is determined by solving $\int_{x^*}^{+\infty}\frac{g(z)}{z^{\beta_++1}}dz = 0$. For $x > x^*$ the option value is:
$V(x) = \frac{2x^{\beta_-}}{\sigma^2(\beta_+-\beta_-)}\int_0^{x^*}\frac{g(z)}{z^{\beta_-+1}}dz.$

We can summarize all formulas in the following

Proposition A1. Let $g(X_t)$ represent a cash flow satisfying all the assumptions listed above. Then the option value, V, in the continuation region has the following expression.

For the problem to abandon the cash flow

$V(x) = \frac{2}{\sigma^2(\beta_+-\beta_-)}[x^{\beta_+}\int_x^{x^*}\frac{g(z)}{z^{\beta_++1}}dz + x^{\beta_-}\int_{x^*}^{x}\frac{g(z)}{z^{\beta_-+1}}dz],$

where x^* is obtained by solving $\int_0^{x^*}\frac{g(z)}{z^{\beta_-+1}}dz = 0$ for a continuation region of the form $x < x^*$,

while x^* is obtained by solving $\int_{x^*}^{+\infty}\frac{g(z)}{z^{\beta_++1}}dz = 0$ for a continuation region of the form $x > x^*$.

For the problem to acquire the cash flow

$V(x) = \frac{2}{\sigma^2(\beta_+-\beta_-)}x^{\beta_+}\int_{x^*}^{+\infty}\frac{g(z)}{z^{\beta_++1}}dz$ for a continuation region of the form $x < x^*$, where x^* is obtained by solving $\int_0^{x^*}\frac{g(z)}{z^{\beta_-+1}}dz = 0$;

$V(x) = \frac{2}{\sigma^2(\beta_+-\beta_-)}x^{\beta_-}\int_0^{x^*}\frac{g(z)}{z^{\beta_-+1}}dz$ for a continuation region of the form $x > x^*$, where x^* is obtained by solving $\int_{x^*}^{+\infty}\frac{g(z)}{z^{\beta_++1}}dz = 0$.

The results above can be linked to optimal stopping theory through Wiener-Hopf factorization as in Boyarchenko and S. Z. Levendorskiĭ (2007). Let us define the following convolution operators with exponentially decaying kernel:

$(\mathcal{E}^+f)(x) = \int_0^{\infty}\beta_+e^{-\beta_+y}f(x+y)dy$

and

$(\mathcal{E}^-f)(x) = \int_{-\infty}^{0}\beta_-e^{-\beta_-y}f(x+y)dy.$

These operators are related to the Wiener-Hopf factors, since $\mathcal{E}^+\mathcal{E}^- = \mathcal{E}^-\mathcal{E}^+ = \mathcal{E}$, where $\mathcal{E}$ is the expected present value (EPV) operator under the Brownian motion $B_t = (\mu - \frac{\sigma^2}{2})t + \sigma W_t$, that is,

$$(\mathcal{E}f)(x) = rE_x[\int_0^{+\infty}e^{-rt}f(B_t)dt].$$

On the other hand, $\mathcal{E}^+$ ($\mathcal{E}^-$) is the EPV- operator under the supremum process $\sup_{0\leq s\leq t} B_s$ (the infimum process $\inf_{0\leq s\leq t} B_s$, respectively).

Denoting $X = X_0 e^x$ and $g(X) = \widetilde{g}(x)$, we note that $(\mathcal{E}^+\widetilde{g})(X) = X^{\beta_+}\int_X^\infty \beta_+ g(Y)\frac{dY}{Y^{\beta_+ +1}}$ and $(\mathcal{E}^-\widetilde{g})(X) = X^{\beta_-}\int_0^X \beta_- g(Y)\frac{dY}{Y^{\beta_- +1}}$. Thus the results above can be rewritten in terms of these EPV-operators as in the following Proposition, where we set $X^* = X_0 e^{x^*}$.

Proposition A2.

For an exit problem, the option value is obtained as

$V(X) = \frac{1}{r}(\mathcal{E}^+\, 1_{(-\infty,x^*)}\mathcal{E}^-)\widetilde{g}(x)$ where $(\mathcal{E}^-\widetilde{g})(x^*) = 0$, for a continuation region of the form $X < X^*$,

$V(X) = \frac{1}{r}(\mathcal{E}^-\, 1_{(x*,+\infty)}\mathcal{E}^+)\widetilde{g}(x)$ where $(\mathcal{E}^+\widetilde{g})(x^*) = 0$, for a continuation region of the form $X > X^*$.

For an entry problem, the option value is obtained as

$V(X) = \frac{1}{r}(\mathcal{E}^+\, 1_{(x^*,+\infty)}\mathcal{E}^-)\widetilde{g}(x)$ where $(\mathcal{E}^-\widetilde{g})(x^*) = 0$, for a continuation region of the form $X < X^*$,

$V(X) = \frac{1}{r}(\mathcal{E}^-\, 1_{(-\infty,x*)}\mathcal{E}^+)\widetilde{g}(x)$ where $(\mathcal{E}^+\widetilde{g})(x^*) = 0$, for a continuation region of the form $X > X^*$.

8 Appendix B : Bessel Functions

Consider the modified Bessel equation:

$$y'' + \frac{1}{x}y' - (1 + \frac{\nu^2}{x^2})y = 0. \tag{B1}$$

It is known that (B1) has the solution

$$I_\nu(x) = \sum_{k=0}^{\infty}(\frac{x}{2})^{2k+\nu}\frac{1}{\Gamma(k+1)\Gamma(k+\nu+1)}$$

I_ν, $I_{-\nu}$, with ν non-integer, are linearly independent , $I_n = I_{-n}$. Note that

$$I_0(x) = 1 + (\frac{x}{2})^2 + \frac{1}{(2!)^2}(\frac{x}{2})^4 + \frac{1}{(3!)^2}(\frac{x}{2})^6 + \dots$$

is a solution of $y'' + \frac{1}{x}y' - y = 0$.

The function $K_\nu(x) = \frac{\pi}{2sin(\pi\nu)}(I_{-\nu}(x) - I_\nu(x))$ (modified Bessel function of second kind) is also called McDonald's function. De L'Hospital rule yields:

$$K_n(x) = \frac{(-1)^n}{2}[\frac{\partial I_{-\nu}}{\partial \nu}|_{\nu=n} - \frac{\partial I_\nu}{\partial \nu}|_{\nu=n}]$$

Moreover, K_n is a solution of $(B1)$ with $\nu = n$. Finally, I_ν, K_ν are linearly independent for each $\nu \geq 0$.

Moreover the following properties hold:

a) $K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}, x \to +\infty$

b) $\nu > 0 \Rightarrow K_\nu(x) \sim const. x^{-\nu}, x \to +0$ (pole)

c) $K_0(x) \sim ln\frac{1}{x}, x \to +0.$

d) $K_0(x) = \int_0^\infty e^{-xch\xi} d\xi,\ x > 0.$

Then K_0 satisfies $y'' + \frac{1}{x}y' - y = 0.$

DECLARATION OF INTERESTS: NONE