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Linearity of minimally superintegrable systems in a static electromagnetic field

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Abstract. Fifteen three-dimensional classical minimally superintegrable systems in a static electromagnetic field are shown to possess hidden symmetries leading to their linearization, and consequently the corresponding subsets of maximally superintegrable subcases are also linearizable. These results are strengthening the conjecture that all three-dimensional minimally superintegrable systems are linearizable by means of hidden symmetries, even in the presence of a magnetic field.

1. Introduction

In classical mechanics [44], Liouville theorem [23] was the starting point for the search of complete integrability [30, 43, 1], and then superintegrability [37], a name that appears for the first time in [45]. While the first steps in the study of superintegrability was made by Bertrand [4] in the 19th century, it was Smorodinsky, Winternitz *et al.* [13, 14, 25] that gave momentum to the field. Subsequently, many papers have been published on this subject by different authors in different countries, see e.g. [10, 19, 12, 40, 2, 42, 24, 29, 28, 8, 9, 39] and references therein. In this paper, our goal is not to find new superintegrable systems, nor to find integrals of motion, but to investigate known superintegrable systems in a static electromagnetic field by means of hidden symmetries and how those can lead to linearizable equations.

The use of Lie symmetries [22] for differential equations has been tremendous, and many textbooks are available, see e.g. [36, 41, 18, 17] and references therein. A major drawback of Lie's method is that it is useless when applied to systems of n first-order equations, e.g. Hamiltonian equations, because they admit an infinite number of Lie symmetries, and there is no systematic way to find even one-dimensional Lie symmetry algebra, apart from trivial groups like translations in time admitted by autonomous systems. However, in [31] it was remarked that any system of n first-order equations

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could be transformed into an equivalent system where at least one of the equations is of second order. Then, the admitted Lie symmetry algebra is no longer infinite dimensional, and hidden symmetries of the original system could be retrieved. Consequently, in [31] hidden symmetries of the Kepler problem were determined by this method. Also, in [34] the well-known linearization of the Kepler problem, as well as the linearity of generalizations of the Kepler problem with and without drag were determined by means of hidden symmetries.

Such hidden symmetries are more general than those considered e.g. in [7], which are just *symmetries of the Hamiltonian, in the sense that they are canonical transformations where both positions and momenta change, and that leave the Hamiltonian function unchanged.*

In [35], it was shown that a two-dimensional superintegrable system [38], such that the corresponding Hamilton-Jacobi equation does not admit the separation of variables in any coordinates, can be transformed into a linear third-order equation by means of hidden symmetries.

In [15], several examples of classical superintegrable systems in two-dimensional Euclidean space [13, 42] were shown to possess hidden symmetries leading to their linearization, and it was conjectured that all classical superintegrable systems in two-dimensional spaces have hidden symmetries that make them linearizable.

In [16], nineteen classical superintegrable systems in two-dimensional non-Euclidean spaces [19, 2, 3] were shown to possess hidden symmetries leading to linearity.

In [32], maximally superintegrable Hamiltonian systems in three-dimensional Euclidean space [10, 11] were also linearized by means of their hidden symmetries, and it was conjectured that three-dimensional minimally superintegrable systems may be similarly linearizable.

In [33], minimally superintegrable Hamiltonian systems in three-dimensional Euclidean space [10] were shown to possess hidden symmetries leading to their linearization.

In [28], a systematic study of integrable and superintegrable systems in the presence of a magnetic field in three-dimensional Euclidean space was initiated, and then continued in several papers [26, 5, 27, 6]. All of those systems are autonomous, integrable and separable in at least one set of coordinates.

The purpose of this work is to to show that all those fifteen nonlinear minimally superintegrable systems are intrinsically linear by determining their hidden Lie symmetries.

The classical Hamiltonian of a particle in Cartesian coordinates $\vec{x} = (x_1, x_2, x_3)$ and with linear momentum $\vec{p} = (p_1, p_2, p_3)$ that moves under the influence of a static electromagnetic field is

$$H = \frac{1}{2}\left((p_1 + A_1(\vec{x}))^2 + (p_2 + A_2(\vec{x}))^2 + (p_3 + A_3(\vec{x}))^2\right) + W(\vec{x}), \tag{1.1}$$

where $W(\vec{x})$ represents the electrostatic potential and the three functions $A_j(\vec{x})$ represent the components of the vector potential $\vec{A}(\vec{x})$ which defines the magnetic field

$\vec{B}(\vec{x}) = \nabla \wedge \vec{A}(\vec{x})$. (In [26, 5, 27, 6], the mass and the charge of the particle have been rescaled to 1 and -1, respectively. We will follow this convention throughout this paper.) The Hamiltonian equations corresponding to the Hamiltonian (1.1) are:

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \qquad (i = 1, 2, 3) \tag{1.2}$$

i.e.:

$$\dot{x}_i = p_i + A_i(\vec{x}), \qquad \dot{p}_i = -\sum_{j=1}^{3} (p_j + A_j(\vec{x})) \frac{\partial A_j}{\partial x_i} - \frac{\partial W}{\partial x_i}. \tag{1.3}$$

If we introduce the covariant momenta $\Pi_i = p_i + A_i(\vec{x})$, then the Hamiltonian equations become invariant under the choice of gauge in the vector potential, i.e.

$$\dot{x}_i = \Pi_i, \qquad \dot{\Pi}_i = \epsilon_{ijk} B_j(\vec{x}) \Pi_k - \frac{\partial}{\partial x_i} W(\vec{x}), \qquad (i, j, k = 1, 2, 3), \tag{1.4}$$

where ϵ_{ijk} is the Levi-Civita symbol. A different choice of gauge in the vector potential A yields a canonical transformation. Consequently, we use the covariant momenta notation throughout this paper, i.e. without having to fix the vector potential.

We will also use cylindrical and spherical coordinates. The cylindrical coordinates are defined as

$$x_1 = r\cos(\theta), \qquad x_2 = r\sin(\theta), \tag{1.5}$$

with the x_3-coordinate unchanged, and their associated covariant momenta are:

$$\Pi_1 = \Pi_r \cos(\theta) - \frac{\sin(\theta)}{r} \Pi_\theta, \qquad \Pi_2 = \Pi_r \sin(\theta) + \frac{\cos(\theta)}{r} \Pi_\theta, \tag{1.6}$$

where r is the polar radius ($r^2 = x_1^2 + x_2^2$). The spherical coordinates are defined as

$$x_1 = R\cos(\theta)\sin(\phi), \qquad x_2 = R\sin(\theta)\sin(\phi), \qquad x_3 = R\cos(\phi) \tag{1.7}$$

and their associated covariant momenta are:

$$\begin{aligned} \Pi_1 &= \cos(\theta)\sin(\phi)\Pi_R + \frac{\cos(\theta)\cos(\phi)}{R}\Pi_\phi - \frac{\sin(\theta)}{R\sin(\phi)}\Pi_\theta, \\ \Pi_2 &= \sin(\theta)\sin(\phi)\Pi_R + \frac{\cos(\phi)\sin(\theta)}{R}\Pi_\phi + \frac{\cos(\theta)}{R\sin(\phi)}\Pi_\theta, \\ \Pi_3 &= \cos(\phi)\Pi_R - \frac{\sin(\phi)}{R}\Pi_\phi, \end{aligned} \tag{1.8}$$

where R is the spherical radius ($R^2 = x_1^2 + x_2^2 + x_3^2$).

2. Two minimally superintegrable Cartesian systems with an additional linear integral of motion

In [26], the authors considered Liouville integrable systems which possess two quadratic integrals of motion (beyond the Hamiltonian) and determined three cases where an additional integral linear in the momenta exists. We show that the two nonlinear minimally superintegrable systems are actually linear. In [20] the linear minimally superintegrable system, namely Case A.2, was studied and the eight-dimensional Lie symmetry algebra of the corresponding linear Lagrangian equations was determined in order to derive integrals of motion by means of a geometrical version Noether theorem.

2.1. Case A.1

The scalar potential and the magnetic field are

$$W(\vec{x}) = \frac{k}{2}\left(x_1^2 + x_2^2\right) - \frac{b^2}{4}\left(x_1^2 + x_2^2\right)^2, \qquad \vec{B}(\vec{x}) = [bx_2, -bx_1, 0]\,, \tag{2.1}$$

respectively, where the parameter b dictates the strength of the magnetic field, while the parameter k appears in the scalar potential only. We use cylindrical coordinates and consequently the Hamiltonian equations are:

$$\dot{r} = \Pi_r,\ \dot{\theta} = \frac{\Pi_\theta}{r^2},\ \dot{x}_3 = \Pi_3,\ \dot{\Pi}_r = \frac{\Pi_\theta^2}{r^3} + b^2 r^3 - (k + b\Pi_3)r,\ \dot{\Pi}_\theta = 0,\ \dot{\Pi}_3 = br\Pi_r. \tag{2.2}$$

This system admits a three-dimensional Abelian Lie symmetry algebra generated by the operators

$$\partial_t, \qquad \partial_\theta, \qquad \partial_{x_3}, \tag{2.3}$$

and consequently the six equations (2.2) can be reduced to the following three equations:

$$\Pi_r' = \frac{\Pi_\theta^2}{y_3^3 \Pi_r} - \frac{(k - b^2 y_3^2) y_3}{\Pi_r} - b y_3 \frac{\Pi_3}{\Pi_r}, \qquad \Pi_\theta' = 0, \qquad \Pi_3' = b y_3, \tag{2.4}$$

where $y_3 \equiv r$ is the new independent variable. We can directly integrate the two last equations and get

$$\Pi_\theta(y_3) = a_1, \qquad \Pi_3(y_3) = \frac{b}{2} y_3^2 + a_2, \tag{2.5}$$

where a_1 and a_2 are arbitrary constants of integration. Substituting these results into (2.4) yields the following first-order nonlinear (but separable) differential equation

$$\Pi_r' = \frac{b^2 y_3^6 - 2(a_2 b + k) y_3^4 + 2a_1^2}{2 y_3^3 \Pi_r}, \tag{2.6}$$

that becomes linear by means of the transformation $\Pi_r(y_3) = \sqrt{u(y_3)}$, i.e.

$$u'(y_3) = b^2 y_3^3 - 2(a_2 b + k) y_3 + \frac{2a_1^2}{y_3^3}. \tag{2.7}$$

This first-order differential equation is invariant under the translation by u, i.e. it admits the Lie point symmetry ∂_u. Hence, we can integrate it to get

$$\Pi_r(y_3) = \frac{\sqrt{b^2 y_3^6 - 4(a_2 b + k) y_3^4 + 4a_3 y_3^2 - 4a_1^2}}{2 y_3}, \tag{2.8}$$

where a_3 is an arbitrary constant of integration.

Therefore, Hamiltonian system (2.2) is linearizable using symmetries.

2.2. Case B

The case B in [26] is characterized by the following scalar potential and magnetic field:

$$W(\vec{x}) = V(x_3), \qquad \vec{B}(\vec{x}) = [0, 0, b_z]\,, \tag{2.9}$$

respectively. The function V is an arbitrary function of x_3 and the magnetic field is constant and oriented in the x_3 direction. The Hamiltonian equations in Cartesian coordinates are:

$$\begin{cases} \dot{x}_1 = \Pi_1, \qquad \dot{x}_2 = \Pi_2, \qquad \dot{x}_3 = \Pi_3, \\ \dot{\Pi}_1 = -b_z\Pi_2, \qquad \dot{\Pi}_2 = b_z\Pi_1, \qquad \dot{\Pi}_3 = -V'(x_3). \end{cases} \tag{2.10}$$

This system admits a three-dimensional Abelian Lie symmetry algebra generated by the operators

$$\partial_t, \qquad \partial_{x_1}, \qquad \partial_{x_2}, \tag{2.11}$$

and consequently the six equations (2.10) can be reduced to the following three equations:

$$\Pi_1' = -b_z\frac{\Pi_2}{\Pi_3}, \qquad \Pi_2' = b_z\frac{\Pi_1}{\Pi_3}, \qquad \Pi_3' = -\frac{V'(y_3)}{\Pi_3}, \tag{2.12}$$

where $y_3 \equiv x_3$ is the new independent variable. The third equation (2.12) becomes linear by means of the transformation $\Pi_3(y_3) = \sqrt{u(y_3)}$, and thus we get the general solution

$$\Pi_3(y_3) = \sqrt{a_1 - 2V(y_3)}. \tag{2.13}$$

The two remaining equations in (2.12), i.e.:

$$\Pi_1' = -b_z\frac{\Pi_2}{\sqrt{a_1 - 2V(y_3)}}, \qquad \Pi_2' = b_z\frac{\Pi_1}{\sqrt{a_1 - 2V(y_3)}}, \tag{2.14}$$

become a single linear second-order differential equation by solving the first by Π_2 and substituting its value into the second, i.e.

$$\Pi_1'' = \frac{V'(y_3)}{a_1 - 2V'(y_3)}\Pi_1' + \frac{b_z^2\Pi_1}{2V'(y_3) - a_1}, \tag{2.15}$$

and its general solution is:

$$\Pi_1(y_3) = a_2 \sin\left(b_z \int \frac{dy_3}{\sqrt{a_1 - 2V(y_3)}}\right) + a_3 \cos\left(b_z \int \frac{dy_3}{\sqrt{a_1 - 2V(y_3)}}\right) \tag{2.16}$$

Hence, the Hamiltonian system (2.10) is linearizable for any function $V(y_3)$.

3. Ten minimally superintegrable Cartesian systems with an additional quadratic (or higher-order) integral of motion

In [27], the investigation that began in [26] was continued by searching for additional quadratic (or higher-order) integrals of motion. Eight classes of minimally superintegrable systems were found and summarized in section 9.1 and three examples (one linear) admitting higher-order integrals were presented in section 9.2. We name them 9.2a, 9.2b, and 9.2c, respectively. We show that the two nonlinear minimally superintegrable systems (Case 9.2a and Case 9.2b) hide linearity by means of Lie symmetries. On the contrary, Case 9.2c corresponds to a linear minimally

superintegrable system, and therefore is outside the scope of this paper, although the corresponding three linear Lagrangian equations admits an eight-dimensional Lie symmetry algebra that could be used to determine integrals of motion either by means of Noether's theorem as in [20] or by means of Jacobi last multiplier as in [35].
Here, we consider all the ten classes of nonlinear minimally superintegrable systems, and determine their hidden linearity by means of Lie symmetries.

3.1. Case I.a

The case I.a in [27] is characterized by the following potential and magnetic field:

$$W(\vec{x}) = b_1\left(k_1 + \frac{b_3}{b_2}x_1\right)e^{b_2x_2} - \frac{b_1^2}{2b_2^2}e^{2b_2x_2}, \quad \vec{B}(\vec{x}) = \left[b_1e^{b_2x_2}, b_3, 0\right], \tag{3.1}$$

respectively. The Hamiltonian equations are

$$\begin{cases} \dot{x}_1 = \Pi_1, \qquad \dot{x}_2 = \Pi_2, \qquad \dot{x}_3 = \Pi_3, \\ \dot{\Pi}_1 = b_3\Pi_3 - \frac{b_1b_3}{b_2}e^{b_2x_2}, \qquad \dot{\Pi}_2 = \frac{b_1}{b_2}e^{b_2x_2}\left(b_1e^{b_2x_2} - b_2b_3x_1 - b_2^2k_1\right), \\ \dot{\Pi}_3 = b_1e^{b_2x_2}\Pi_2 - b_3\Pi_1. \end{cases} \tag{3.2}$$

This system admits a two-dimensional Abelian Lie symmetry algebra generated by the operators $\partial_t, \partial_{x_3}$, and consequently the six equations (3.2) can be reduced to the following four equations:

$$\begin{cases} x_1' = \frac{\Pi_1}{\Pi_2}, \qquad \Pi_1' = b_3\frac{\Pi_3}{\Pi_2} - \frac{b_1b_3e^{b_2y_2}}{b_2\Pi_2}, \\ \Pi_2' = -b_1e^{b_2y_2}\frac{\Pi_3}{\Pi_2} - b_1b_3e^{b_2y_2}\frac{x_1}{\Pi_2} + \frac{b_1e^{b_2y_2}}{b_2\Pi_2}\left(b_1e^{b_2y_2} - b_2^2k_1\right), \\ \Pi_3' = b_1e^{b_2y_2} - b_3\frac{\Pi_1}{\Pi_2}, \end{cases} \tag{3.3}$$

with $y_2 \equiv x_2$ the new independent variable. Substituting the ratio Π_1/Π_2 with x_1' into the last equation in (3.3) yields a linear equation that can be integrated, i.e.:

$$\Pi_3(y_2) = \frac{b_1}{b_2}e^{b_2y_2} - b_3x_1 - a_1, \tag{3.4}$$

Then, the three remaining equations in (3.3) become:

$$x_1' = \frac{\Pi_1}{\Pi_2}, \qquad \Pi_1' = -b_3\frac{b_3x_1 + a_1}{\Pi_2}, \qquad \Pi_2' = \frac{b_1(a_1 - b_2k_1)e^{b_2y_2}}{\Pi_2}. \tag{3.5}$$

The third equation becomes linear by means of the transformation $\Pi_2(y_2) = \sqrt{u(y_2)}$, and its general solution is

$$\Pi_2(y_2) = \frac{\sqrt{b_2(-2k_1b_1b_2e^{b_2y_2} + 2a_1b_1e^{b_2y_2} + a_2b_2)}}{b_2}. \tag{3.6}$$

The two remaining equations become a single linear second-order ordinary differential equation by solving the first equation by Π_1 and substituting its value into the second equation, i.e.

$$x_1'' = \frac{-b_1b_2(a_1 - k_1b_2)e^{b_1y_2}}{2b_1(a_1 - k_1b_2)e^{b_2y_2} + a_2b_2}x_1' - \frac{b_2b_3(b_3x_1 + a_1)}{2b_1e^{b_2y_2}(a_1 - k_1b_2) + a_2b_2}. \tag{3.7}$$

Therefore, this minimally superintegrable system (3.2) is linearizable using hidden symmetries.

3.2. Case I.b

The case I.b in [27] is characterized by the following potential and magnetic field:

$$\begin{aligned} W(\vec{x}) = & -\frac{b_1^2}{2}(x_1^2+x_2^2)^2 - \frac{b_2^2}{2x_1^4} - \frac{b_3^2}{2x_2^4} - b_1\left(b_2\frac{x_2^2}{x_1^2} + b_3\frac{x_1^2}{x_2^2}\right) \\ & -\frac{b_2 b_3}{x_1^2 x_2^2} + k_1(x_1^2+x_2^2) + \frac{k_2}{x_1^2} + \frac{k_3}{x_2^2}, \end{aligned} \tag{3.8}$$

$$\vec{B}(\vec{x}) = \left[2b_1 x_2 - 2\frac{b_3}{x_2^3}, -2b_1 x_1 + 2\frac{b_2}{x_1^3}, 0\right], \tag{3.9}$$

respectively. The Hamiltonian equations are:

$$\begin{cases} \dot{x}_1 = \Pi_1, \qquad \dot{x}_2 = \Pi_2, \qquad \dot{x}_3 = \Pi_3, \\ \dot{\Pi}_1 = 2b_1^2 x_1(x_1^2+x_2^2) - 2b_1 b_2\frac{x_2^2}{x_1^3} + 2b_1 b_3\frac{x_1}{x_2^2} - 2b_1 x_1 \Pi_3 \\ \qquad -\frac{2b_2^2}{x_1^5} - \frac{2b_2 b_3}{x_1^3 x_2^2} + 2b_2\frac{\Pi_3}{x_1^3} - 2k_1 x_1 + \frac{2k_2}{x_1^3}, \\ \dot{\Pi}_2 = 2b_1^2 x_2(x_1^2+x_2^2) + 2b_1 b_2\frac{x_2}{x_1^2} - 2b_1 b_3\frac{x_1^2}{x_2^3} - 2b_1 x_2 \Pi_3 - \frac{2b_2 b_3}{x_2^3 x_1^2} \\ \qquad -\frac{2b_3^2}{x_2^5} + 2b_3\frac{\Pi_3}{x_2^3} - 2k_1 x_2 + \frac{2k_3}{x_2^3}, \\ \dot{\Pi}_3 = 2b_1(x_1\Pi_1 + x_2\Pi_2) - 2b_2\frac{\Pi_1}{x_1^3} - 2b_3\frac{\Pi_2}{x_2^3}. \end{cases} \tag{3.10}$$

This system admits a two-dimensional Abelian Lie symmetry algebra generated by the operators $\partial_t, \partial_{x_3}$, and consequently the six equations (3.10) can be reduced to the following four equations:

$$\begin{cases} x_1' = \frac{\Pi_1}{\Pi_2}, \\ \Pi_1' = -2k_1\frac{x_1}{\Pi_2} + \frac{2k_2}{x_1^3\Pi_2} + \frac{2(b_1x_1^4 - b_2)}{y_2^2 x_1^5 \Pi_2}(b_1 y_2^2 x_1 \\ \qquad +(b_1 y_2^4 - y_2^2\Pi_3 + b_3)x_1^2 + b_2 y_2^2), \\ \Pi_2' = -\frac{2k_1 y_2}{\Pi_2} + \frac{2k_3}{y_2^3\Pi_2} + \frac{2(b_1 y_2^4 - b_3)}{y_2^5 x_1^2 \Pi_2}(b_1 y_2^2 x_1^4 \\ \qquad +(b_1 y_2^4 - y_2^2\Pi_3 + b_3)x_1^2 + b_2 y_2^2), \\ \Pi_3' = 2\frac{b_1 x_1^4 - b_2}{x_1^3}\frac{\Pi_1}{\Pi_2} + \frac{2b_1 y_2^4 - 2b_3}{y_2^3}, \end{cases} \tag{3.11}$$

where $y_2 \equiv x_2$ is the new independent variable. Substituting the ratio Π_1/Π_2 with x_1' into the last equation in (3.11) yields a solvable equation which can be easily integrated, i.e.:

$$\Pi_3(y_2) = b_1 x_1^2 + \frac{b_2}{x_1^2} + \frac{b_1 y_2^4 - a_1 y_2^2 + b_3}{y_2^2}, \tag{3.12}$$

Then, the three remaining equations in (3.11) become:

$$\begin{cases} x_1' = \dfrac{\Pi_1}{\Pi_2}, \qquad \Pi_1' = 2K_1\dfrac{x_1}{\Pi_2} - 2\dfrac{K_2}{x_1^3\Pi_2}, \\ \Pi_2' = \dfrac{2K_1y_2^4 - 2K_3}{y_2^3\Pi_2}. \end{cases} \tag{3.13}$$

where $K_1 = a_1b_1 - k_1, K_2 = a_1b_2 - k_2, K_3 = a_1b_3 - k_3$. The third equation in (3.13) becomes linear by means of the transformation $\Pi_2(y_2) = \sqrt{u(y_2)}$, and its general solution is

$$\Pi_2(y_2) = \frac{\sqrt{2K_1y_2^4 + a_2y_2^2 + 2K_3}}{y_2}. \tag{3.14}$$

The two remaining equations become a single nonlinear second-order ordinary differential equation by solving the first equation by Π_1 and substituting its value into the second equation, i.e.

$$x_1'' = 2\frac{x_1'x_1^3(y_2^4K_1 - K_3) - x_1^4y_2^3K_1 + y_2K_2}{x_1^3y_2(a_2y_2^2 - 2y_2^4K_1 - 2K_3)}. \tag{3.15}$$

This equation admits a three-dimensional Lie symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and becomes linear if $K_2 = 0$. Therefore, we use the general method described in [21] and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$. If we solve equation (3.15) with respect to K_2 and derive once with respect to y_2, then the following nonlinear third-order equation is obtained

$$x_1''' = -3\frac{x_1''x_1'}{x_1} - \frac{6(K_1y_2^4 - K_3)(y_2x_1x_1'' - x_1x_1' + y_2(x_1')^2)}{y_2^2x_1(2K_1y_2^4 + a_2y_2^2 + 2K_3)}, \tag{3.16}$$

which is linearizable since it admits a seven-dimensional Lie symmetry algebra and in particular possesses the linearizing symmetry

$$v = \frac{1}{x_1}\partial_{x_1}, \tag{3.17}$$

that yields the linearizing transformation $x_1(y_2) = \sqrt{f(y_2)}$. Consequently, equation (3.16) becomes the following linear equation:

$$f''' = -6\frac{(y_2f'' - f')(K_1y_2^4 - K_3)}{2y_2^2(K_1y_2^4 + K_3) - a_2y_2^4}, \tag{3.18}$$

whose general solution is:

$$f(y_2) = a_3 + a_4(a_2y_2^2 - 4K_3) + a_5\sqrt{a_2y_2^2 - 2y_2^4K_1 - 2K_3}. \tag{3.19}$$

Therefore, the minimally superintegrable system (3.10) is linearizable using hidden symmetries.

3.3. Case I.c

The case I.c in [27], is characterized by the following potential and magnetic field

$$
\begin{aligned}
W(\vec{x}) = & -\frac{b_1^2}{2}\left(4x_1^2+x_2^2\right)^2 - \frac{b_2^2}{2}x_1^2 - \frac{b_3^2}{2x_2^4} - \frac{b_2 b_3 x_1}{x_2^2} - b_1 b_2 x_1(4x_1^2+x_2^2) \\
& - \frac{4b_1 b_3 x_1^2}{x_2^2} + \frac{k_3}{x_2^2} + k_1(4x_1^2+x_2^2) + k_2 x_1,
\end{aligned} \tag{3.20}
$$

$$
\vec{B}(\vec{x}) = \left[2b_1 x_2 - \frac{2b_3}{x_2^3}, -8b_1 x_1 - b_2, 0\right] \tag{3.21}
$$

respectively. The Hamiltonian equations are:

$$
\begin{cases}
\dot{x}_1 = \Pi_1, \qquad \dot{x}_2 = \Pi_2, \qquad \dot{x}_3 = \Pi_3, \\
\dot{\Pi}_1 = 8b_1^2 x_1(4x_1^2+x_2^2) + b_1 b_2(12x_1^2+x_2^2) + \dfrac{8b_1 b_3 x_1}{x_2^2} - 8b_1 x_1 \Pi_3 \\
\qquad + b_2^2 x_2 + \dfrac{b_2 b_3}{x_2^2} - b_2 \Pi_3 - 8k_1 x_1 - k_2, \\
\dot{\Pi}_2 = 2b_1^2 x_2(4x_1^2+x_2^2) + 2b_1 b_2 x_1 x_2 - \dfrac{8b_1 b_3 x_1^2}{x_2^3} - 2b_1 x_2 \Pi_3 - \dfrac{2b_2 b_3 x_1}{x_2^3} \\
\qquad - \dfrac{2b_3^2}{x_2^5} + 2b_3 \dfrac{\Pi_3}{x_2^3} - 2k_1 x_2 + \dfrac{2k_3}{x_2^3}, \\
\dot{\Pi}_3 = 2b_1(4x_1\Pi_1 + x_2\Pi_2) + b_2\Pi_1 - \dfrac{2b_3\Pi_2}{x_2^3}.
\end{cases} \tag{3.22}
$$

This system admits a two-dimensional Abelian Lie symmetry algebra generated by the operators $\partial_t, \partial_{x_3}$, and consequently the six equations (3.22) can be reduced to the following system of four equations:

$$
\begin{cases}
x_1' = \frac{\Pi_1}{\Pi_2}, \\
\Pi_1' = \left(8b_1^2 y_2^2 + b_2^2 - 8k_1 + \dfrac{8b_1 b_3}{y_2^2}\right)\dfrac{x_1}{\Pi_2} + \dfrac{b_1 b_2 y_2^4 - k_2 y_2^2 + b_2 b_3}{y_2^2 \Pi_2} \\
\qquad - \left(8b_1 \dfrac{x_1}{\Pi_2} + \dfrac{b_2}{\Pi_2}\right)\Pi_3 + 32b_1^2 \dfrac{x_1^3}{\Pi_2} + 12b_1 b_2 \dfrac{x_1^2}{\Pi_2}, \\
\Pi_2' = \left(-2b_1 y_2 + \dfrac{2b_3}{y_2^3}\right)\dfrac{\Pi_3}{\Pi_2} + 8b_1\left(b_1 y_2 - \dfrac{b_3}{y_2^3}\right)\dfrac{x_1^2}{\Pi_2} \\
\qquad + 2b_2\left(b_1 y_2 - \dfrac{b_3}{y_2^3}\right)\dfrac{x_1}{\Pi_2} + \dfrac{2b_1^2 y_2^8 - 2k_1 y_2^6 + 2k_3 y_2^2 - 2b_3^2}{y_2^5 \Pi_2}, \\
\Pi_3' = (8b_1 x_1 + b_2)\dfrac{\Pi_1}{\Pi_2} + \dfrac{2b_1 y_2^4 - 2b_3}{y_2^3},
\end{cases} \tag{3.23}
$$

where $y_2 \equiv x_2$ is the new independent variable. Substituting the ratio Π_1/Π_2 with x_1' into the fourth equation of system (3.23) yields an equation that can be easily integrated, i.e.

$$
\Pi_3(y_2) = 4b_1 x_1^2 + b_2 x_1 + \frac{b_1 y_2^4 - a_1 y_2^2 + b_3}{y_2^2}, \tag{3.24}
$$

where a_1 is a constant of integration. Then, the three remaining equations in (3.23) become:

$$\begin{cases} x_1' = \dfrac{\Pi_1}{\Pi_2}, \qquad \Pi_1' = 8(a_1b_1 - k_1)\dfrac{x_1}{\Pi_2} + \dfrac{a_1b_2 - k_2}{\Pi_2}, \\ \Pi_2' = \dfrac{(2a_1b_1 - 2k_1)y_2^4 - 2a_1b_3 + 2k_3}{y_2^3\Pi_2}. \end{cases} \tag{3.25}$$

The third equation (3.25) becomes linear by means of the transformation $\Pi_2(y_2) = \sqrt{u(y_2)}$, and its general solution is

$$\Pi_2(y_2) = \frac{\sqrt{2(a_1b_1 - k_1)y_2^4 + a_2y_2^2 + 2a_1b_3 - 2k_3}}{y_2}, \tag{3.26}$$

where a_2 is a new constant of integration. The two remaining equations, i.e.

$$\begin{cases} x_1' = \dfrac{y_2\Pi_1}{\sqrt{2(a_1b_1 - k_1)y_2^4 + a_2y_2^2 + 2a_1b_3 - 2k_3}}, \\ \Pi_1' = \dfrac{y_2(8(a_1b_1 - k_1)x_1 + a_1b_2 - k_2)}{\sqrt{2(a_1b_1 - k_1)y_2^4 + a_2y_2^2 + 2a_1b_3 - 2k_3}}, \end{cases} \tag{3.27}$$

become a single linear second-order ordinary differential equation by solving the first equation by Π_1 and substituting its value into the second equation, i.e.

$$\begin{aligned} x_1'' = {} & \frac{2(k1 - a_1b_1)y_2^4 + 2a_1b_3 - 2k_3}{y_2(2(a_1b_1 - k_1)y_2^4 + a_2y_2^2 + 2a_1b_3 - 2k_3)}x_1' + \frac{8(a_1b_1 - k_1)y_2^2}{2(a_1b_1 - k_1)y_2^4 + a_2y_2^2 + 2a_1b_3 - 2k_3}x_1 \\ & + \frac{(a_1b_2 - k_2)y_2^2}{2(a_1b_1 - k_1)y_2^4 + a_2y_2^2 + 2a_1b_3 - 2k_3}, \end{aligned} \tag{3.28}$$

an its general solution is

$$\begin{aligned} x_1(y_2) = {} & a_3\sqrt{2(a_1b_1 - k_1)y_2^4 + a_2y_2^2 + 2a_1b_3 - 2k_3} \\ & + a_4(4(a_1b_1 - k_1)y_2^2 + a_2) + \frac{(a_1b_2 - k_2)y_2^2}{2a_2}. \end{aligned} \tag{3.29}$$

Thus, the minimally superintegrable system (3.22) is linearizable using hidden symmetries.

3.4. Case I.d

The case I.d in [27], is characterized by the following potential and magnetic field

$$W(\vec{x}) = k_1(x_1^2 + x_2^2) + k_2x_1 + k_3x_2 - \frac{1}{2}\left(b_1x_1^2 + b_1x_2^2 + b_2x_1 + b_3x_2\right)^2, \tag{3.30}$$

$$\vec{B}(\vec{x}) = [2b_1x_2 + b_3, -2b_1x_1 - b_2, 0], \tag{3.31}$$

respectively. The Hamiltonian equations are:

$$\begin{cases} \dot{x}_1 = \Pi_1, \qquad \dot{x}_2 = \Pi_2, \qquad \dot{x}_3 = \Pi_3, \\ \dot{\Pi}_1 = (b_1x_1^2 + b_1x_2^2 + b_2x_1 + b_3x_2 - \Pi_3)(2b_1x_1 + b_2) - 2k_1x_1 - k_2, \\ \dot{\Pi}_2 = (b_1x_1^2 + b_1x_2^2 + b_2x_1 + b_3x_2 - \Pi_3)(2b_1x_2 + b_3) - 2k_1x_2 - k_3, \\ \dot{\Pi}_3 = (2b_1x_2 + b_3)\Pi_2 + (2b_1x_1 + b_2)\Pi_1. \end{cases} \tag{3.32}$$

This system admits a two-dimensional Abelian Lie symmetry algebra generated by the operators $\partial_t, \partial_{x_3}$, and consequently the six equations (3.32) can be reduced to the following system of four equations:

$$\begin{cases} x_1' = \frac{\Pi_1}{\Pi_2}, \\ \Pi_1' = (b_1x_1^2 + b_1y_2^2 + b_2x_1 + b_3y_2 - \Pi_3)\dfrac{2b_1x_1 + b_2}{\Pi_2} - \dfrac{2k_1x_1 + k_2}{\Pi_2}, \\ \Pi_2' = (b_1x_1^2 + b_1y_2^2 + b_2x_1 + b_3y_2 - \Pi_3)\dfrac{2b_1y_2 + b_3}{\Pi_2} - \dfrac{2k_1y_2 + k_3}{\Pi_2}, \\ \Pi_3' = (2b_1y_2 + b_3) + (2b_1x_1 + b_2)\dfrac{\Pi_1}{\Pi_2}, \end{cases} \tag{3.33}$$

where $y_2 \equiv x_2$ is the new independent variable. Substituting the ratio Π_1/Π_2 with x_1' into the fourth equation of system (3.33) yields an equation that can be easily integrated, i.e.

$$\Pi_3(y_2) = b_1x_1^2 + b_2x_1 + b_1y_2^2 + b_3y_2 - a_1, \tag{3.34}$$

where a_1 is a constant of integration. Then, the three remaining equations in (3.33) become:

$$\begin{cases} x_1' = \dfrac{\Pi_1}{\Pi_2}, \qquad \Pi_1' = 2(a_1b_1 - k_1)\dfrac{x_1}{\Pi_2} + \dfrac{a_1b_2 - k_2}{\Pi_2}, \\ \Pi_2' = \dfrac{2(a_1b_1 - k_1)y_2 + a_1b_3 - k_3}{\Pi_2}. \end{cases} \tag{3.35}$$

The third equation in (3.35) becomes linear by means of the transformation $\Pi_2(y_2) = \sqrt{u(y_2)}$, and its general solution is

$$\Pi_2(y_2) = \sqrt{2a_1b_1y_2^2 + 2a_1b_3y_2 - 2k_1y_2^2 - 2k_3y_2 + a_2} \tag{3.36}$$

The two remaining equations, i.e.

$$\begin{cases} x_1' = \dfrac{\Pi_1}{\sqrt{2(a_1b_1 - k_1)y_2^2 + 2a_1b_3y_2 - 2k_3y_2 + a_2}}, \\ \Pi_1' = \dfrac{2(a_1b_1 - k_1)x_1 + a_1b_2 - k_2}{\sqrt{2(a_1b_1 - k_1)y_2^2 + 2a_1b_3y_2 - 2k_3y_2 + a_2}}, \end{cases} \tag{3.37}$$

become a single linear second-order ordinary differential equation by solving the first equation by Π_1 and substituting its value into the second equation, i.e.

$$\begin{aligned} x_1'' = &\frac{2(k_1 - a_1b_1)y_2 - a_1b_3 + k_3}{2(a_1b_1 - k_1)y_2^2 + 2(a_1b_3 - k_3)y_2 + a_2}x_1' \\ &+ \frac{2(a_1b_1 - k_1)x_1 + a_1b_2 - k_2}{2(a_1b_1 - k_1)y_2^2 + 2(a_1b_3 - k_3)y_2 + a_2} \end{aligned} \tag{3.38}$$

and its general solution is

$$\begin{aligned} x_1(y_2) = a_3&\sqrt{2a_1b_1y_2^2 + 2a_1b_3y_2 - 2k_1y_2^2 - 2k_3y_2 + a_2} \\ &+ a_4(2(a_1b_1 - k_1)y_2 + a_1b_3 - k_3) + \frac{k_2 - a_1b_2}{2(a_1b_1 - k_1)}. \end{aligned} \tag{3.39}$$

Consequently, the minimally superintegrable system (3.32) is linearizable using hidden symmetries.

3.5. Cases II

All four systems of type II in [27] can be treated in the same manner. The scalar potential and the magnetic field of each case are as follow:

- Case II.a

$$W(\vec{x}) = k_1 x_1 + k_2 e^{b_2 x_1} - \frac{b_1^2}{2b_2^2} e^{2b_2 x_1}, \qquad \vec{B}(\vec{x}) = \left[0, 0, b_1 e^{b_2 x_1}\right]. \tag{3.40}$$

- Case II.b

$$W(\vec{x}) = -\frac{b_1^2}{2} x_1^{2(b_2-2)} + b_1(b_2-2) k_1 x_1^{b_2-2} + \frac{k_2}{x_1^2}, \tag{3.41}$$

$$\vec{B}(\vec{x}) = \left[0, 0, b_1(b_2-2) x_1^{b_2-3}\right]. \tag{3.42}$$

- Case II.c

$$W(\vec{x}) = -\frac{b^2}{2} (\ln|x_1|)^2 + k_1 \ln|x_1| + \frac{k_2}{x_1^2}, \qquad \vec{B}(\vec{x}) = \left[0, 0, \frac{b}{x_1}\right]. \tag{3.43}$$

- Case II.d

$$W(\vec{x}) = -b k_1 \frac{\ln|x_1|}{x_1^2} - \frac{k_1^2}{8 x_1^4} + \frac{k_2}{x_1^2}, \qquad \vec{B}(\vec{x}) = \left[0, 0, \frac{b}{x_1^3}\right]. \tag{3.44}$$

We notice that all potentials depend on x_1 only, and all magnetic fields have only one component along the x_3-axis, $B_3(x_1)$, that depends on x_1 only. Therefore, the Hamiltonian equations are:

$$\begin{cases} \dot{x}_1 = \Pi_1, \qquad \dot{x}_2 = \Pi_2, \qquad \dot{x}_3 = \Pi_3, \\ \dot{\Pi}_1 = -B_3(x_1)\Pi_2 - \dfrac{\mathrm{d}W(x_1)}{\mathrm{d}x_1}, \\ \dot{\Pi}_2 = B_3(x_1)\Pi_1, \\ \dot{\Pi}_3 = 0. \end{cases} \tag{3.45}$$

The sixth and third equation can be immediately solved ($\Pi_3 = a_0 \Rightarrow x_3 = a_0 t + a_1$), and consequently the equations of motion are reduced to the remaining four equations (i.e., a system in two-dimensional space) that admit a two-dimensional Abelian Lie symmetry algebra generated by the operators $\partial_t, \partial_{x_2}$, and consequently the four equations can be reduced to the following system of two equations (N.B. Equation $x_2' = \Pi_2/\Pi_1$ is easy to integrate once system (3.46) is solved.):

$$\begin{cases} \Pi_1'(y_1) = \dfrac{-f_3'(y_1)\Pi_2 - W'(y_1)}{\Pi_1}, \\ \Pi_2'(y_1) = f_3'(y_1), \end{cases} \tag{3.46}$$

where $y_1 \equiv x_1$ is the new independent variable, $B_3(y_1) = f_3'(y_1)$, and prime denotes the total derivative with respect to y_1. The second equation can be easily integrated, i.e.

$$\Pi_2(y_1) = f_3(y_1) + a_2 \tag{3.47}$$

and then the first equation in (3.46) becomes

$$\Pi_1'(y_1) = \frac{-f_3'(y_1)(f_3(y_1) + a_2) - W'(y_1)}{\Pi_1}, \tag{3.48}$$

which can be linearized by the transformation $\Pi_1(y_1) = \sqrt{2u(y_1)}$, and its general solution is

$$\Pi_1(y_1) = \sqrt{a_3 - a_2 f_3(y_1) - f_3(y_1)^2/2 - W(y_1)}. \tag{3.49}$$

Consequently, the four minimally superintegrable systems of type II in [27] are all linearizable using hidden symmetries.

3.6. Case 9.2a

The potential and magnetic field of Case 9.2a are

$$W(\vec{x}) = \frac{k_1}{x_2^2} + k_2 x_2^2, \qquad \vec{B}(\vec{x}) = [0, b, 0], \tag{3.50}$$

respectively. The Hamiltonian equations are:

$$\dot{x}_1 = \Pi_1, \qquad \dot{x}_2 = \Pi_2, \qquad \dot{x}_3 = \Pi_3, \tag{3.51}$$

$$\dot{\Pi}_1 = b\Pi_3, \qquad \dot{\Pi}_2 = \frac{2k_1}{x_2^3} - 2k_2 x_2, \qquad \dot{\Pi}_3 = -b\Pi_1. \tag{3.52}$$

Case 9.2a is actually a subcase of Case B by exchanging x_2 with x_3. In the following, we show another way to determine the hidden linearity of Case 9.2a. If we derive the three covariant momenta $\Pi_i (i = 1, 2, 3)$ from equations (3.51) and replace them into equations (3.52), then we obtain the following system of three second-order equations, i.e.

$$\ddot{x}_1 = b\dot{x}_3, \qquad \ddot{x}_2 = \frac{2k_1}{x_2^3} - 2k_2 x_2, \qquad \ddot{x}_3 = -b\dot{x}_1. \tag{3.53}$$

The second equation in x_2 admits a three-dimensional Lie symmetry algebra $\mathfrak{sl}(2, \mathbb{R})$ generated by the following operators:

$$\begin{aligned} &\partial_t, \qquad \sin(\sqrt{8k_2}t)\partial_t + \sqrt{2k_2}x_2 \cos(\sqrt{8k_2}t)\partial_{x_1}, \\ &\cos(\sqrt{8k_2}t)\partial_t - \sqrt{2k_2}x_2 \sin(\sqrt{8k_2}t)\partial_{x_1}. \end{aligned} \tag{3.54}$$

However, if $k_1 = 0$, then the same equation admits an eight-dimensional Lie symmetry algebra $\mathfrak{sl}(3, \mathbb{R})$ and thus it is linearizable. Therefore, we use the general method described in [21] and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra $\mathfrak{sl}(2, \mathbb{R})$. If we solve the second-order equation with respect to k_1 and derive once with respect to y_2, then the following nonlinear third-order equation is obtained

$$\dddot{x}_2 = -\frac{3\dot{x}_2 \ddot{x}_2}{x_2} - 8k_2 \dot{x}_2. \tag{3.55}$$

which admits a seven-dimensional Lie symmetry algebra, and therefore is linearizable. Indeed, the new dependent variable $u(t) = x_2^2$ transforms equation (3.55) into the linear equation

$$\dddot{u} = -8k_2 \dot{u} \qquad \Rightarrow \qquad u(t) = c_1 + a_1 \sin(\sqrt{8k_2}t) + a_2 \cos(\sqrt{8k_2}t), \tag{3.56}$$

where c_1, a_1 and a_2 are integration constant. However, since $v = x_2^{-1}\partial_{x_2}$ is not a symmetry of the second-order differential equation (3.53), an additional constraint on the integration is needed. By substituting u in (3.53), we get

$$c_1 = \sqrt{\frac{a_1^2 k_2 + a_2^2 k_2 + k_1}{k_2}}, \tag{3.57}$$

$$x_2(t) = \sqrt{a_1 \sin(\sqrt{8k_2}t) + a_2 \cos(\sqrt{8k_2}t) + \sqrt{\frac{a_1^2 k_2 + a_2^2 k_2 + k_1}{k_2}}}. \tag{3.58}$$

The two remaining equations in (3.53) are linear, i.e.

$$\ddot{x}_1 = b\dot{x}_3, \qquad \ddot{x}_3 = -b\dot{x}_1. \tag{3.59}$$

If we derive $\dot{x}_3$ from the first equation and replace it into the second equation, then the second equation becomes a linear third-order equation in the dependent variable x_1, i.e.

$$\dddot{x}_1 = -b^2\dot{x}_1, \tag{3.60}$$

and its general solution is

$$x_1(t) = a_3 \sin(bt) + a_4 \cos(bt) + a_5. \tag{3.61}$$

Consequently, we have shown that system Case 9.2a can be linearized in two different ways by means of hidden symmetries.

3.7. Case 9.2b

The potential and magnetic field of case 9.2b are

$$\begin{aligned} W(\vec{x}) = & -\frac{b_3^2}{2}\left(l_1x_1^2 + m_1x_2^2\right)^2 + b_3\left(l_2x_1^2 + m_2x_2^2 - b_2m_1\frac{x_2^2}{x_1^2} - b_1l_1\frac{x_1^2}{x_2^2}\right) \\ & + \frac{k_1}{x_1^2} + \frac{k_2}{x_2^2} - \frac{1}{2}\left(\frac{b_2}{x_1^2} + \frac{b_1}{x_2^2}\right)^2 \end{aligned} \tag{3.62}$$

$$\vec{B}(\vec{x}) = \left[2b_3m_1x_2 - \frac{2b_1}{x_2^3}, -2b_3l_1x_1 + \frac{2b_2}{x_1^3}, 0\right], \tag{3.63}$$

respectively. (This system is integrable but not superintegrable in general. By imposing some constraints on the parameters, the system becomes minimally superintegrable. However, we will not impose those constraints.) The Hamiltonian equations are

$$\begin{cases} \dot{x}_1 = \Pi_1, \qquad \dot{x}_2 = \Pi_2, \qquad \dot{x}_3 = \Pi_3, \\ \dot{\Pi}_1 = -\frac{2b_2^2}{x_1^5} + b_2\left(\frac{-2m_1b_3x_2^2}{x_1^3} + \frac{2x_2^2\Pi_3 - 2b_1}{x_1^3x_2^2}\right) + 2l_1b_3^2x_1(l_1x_1^2 + m_1x_2^2) \\ \qquad + \frac{2b_3x_1}{x_2^2}\left(b_1l_1 - (l_1\Pi_3 + l_2)x_2^2\right) + \frac{2k_1}{x_1^3}, \\ \dot{\Pi}_2 = b_2\left(2m_1b_3\frac{x_2}{x_1^2} - \frac{2b_1}{x_1^2x_2^3}\right) + 2b_3^2m_1x_2\left(l_1x_1^2 + m_1x_2^2\right) \\ \qquad -2\frac{b_3}{x_2^3}((m_1\Pi_3 + m_2)x_2^4 + b_1l_1x_1^2) + \frac{2k_2}{x_2^3} - \frac{2b_1}{x_2^5}(b_1 - x_2^2\Pi_3), \\ \dot{\Pi}_3 = -2b_1\frac{\Pi_2}{x_2^3} - 2b_2\frac{\Pi_1}{x_1^3} + 2b_3(l_1x_1\Pi_1 + m_1x_2\Pi_2). \end{cases} \tag{3.64}$$

This system admits a two-dimensional Abelian Lie symmetry algebra generated by the operators $\partial_t, \partial_{x_3}$, and consequently the six equations (3.64) can be reduced to the following system of four equations:

$$\begin{cases} x_1' = & \dfrac{\Pi_1}{\Pi_2}, \\ \Pi_1' = & \left(\dfrac{2b_2}{x_1^3} - 2b_3l_1x_1\right)\dfrac{\Pi_3}{\Pi_2} + \dfrac{2b_3^2l_1^2x_1^3}{\Pi_2} + \dfrac{2b_3x_1}{y_2^2\Pi_2}(b_3l_1m_1y_2^4 - l_2y_2^2 + b_1l_1) \\ & + \dfrac{-2b_2b_3m_1y_2^4 + 2k_1y_2^2 - 2b_1b_2}{y_2^2x_1^3\Pi_2} - \dfrac{2b_2^2}{x_1^5\Pi_2}, \\ \Pi_2' = & (b_1 - b_3m_1y_2^4)\dfrac{2\Pi_3}{y_2^3\Pi_2} - \dfrac{2b_3l_1x_1^2}{y_2^3\Pi_2}(b_1 - b_3m_1y_2^4) \\ & + \dfrac{2b_3^2m_1^2y_2^8 - 2b_3m_2y_2^6 + 2k_2y_2^2 - 2b_1^2}{y_2^5\Pi_2} + 2b_2\dfrac{b_3m_1y_2^4 - b_1}{y_2^3x_1^2\Pi_2}, \\ \Pi_3' = & \left(2b_3l_1x_1 - \dfrac{2b_2}{x_1^3}\right)\dfrac{\Pi_1}{\Pi_2} + \dfrac{2b_3m_1y_2^4 - 2b_1}{y_2^3}. \end{cases} \tag{3.65}$$

where $y_2 \equiv x_2$ is the new independent variable. Substituting the ratio Π_1/Π_2 with x_1' into the fourth equation of system (3.65) yields an equation that can be easily integrated, i.e.

$$\Pi_3(y_2) = b_3l_1x_1^2 + \frac{b_2}{x_1^2} + \frac{b_3m_1y_2^4 - a_1y_2^2 + b_1}{y_2^2}, \tag{3.66}$$

Then, the three remaining equations in (3.65) become:

$$\begin{cases} x_1' = & \frac{\Pi_1}{\Pi_2}, \\ \Pi_1' = & 2b_3(a_1l_1 - l_2)\dfrac{x_1}{\Pi_2} + 2\dfrac{k_1 - a_1b_2}{x_1^3\Pi_2}, \\ \Pi_2' = & \dfrac{2b_3(a_1m_1 - m_2)y_2^4 - 2a_1b_1 + 2k_2}{y_2^3\Pi_2}. \end{cases} \tag{3.67}$$

The third equation in (3.67) becomes linear by means of the transformation $\Pi_2(y_2) = \sqrt{u(y_2)}$, and its general solution is

$$\Pi_2(y_2) = \frac{\sqrt{2b_3(a_1m_1 - m_2)y_2^4 + a_2y_2^2 + 2a_1b_1 - 2k_2}}{y_2}. \tag{3.68}$$

The two remaining equations become a single nonlinear second-order ordinary differential equation by solving the first equation by Π_1 and substituting its value into the second equation, i.e.

$$\begin{aligned} x_1'' = & \frac{-2b_3(a_1m_1 - m_2)y_2^4 + 2a_1b_1 - 2k_2}{y_2(2b_3(a_1m_1 - m_2)y_2^4 + a_2y_2^2 + 2a_1b_1 - 2k_2)}x_1' \\ & + \frac{2b_3y_2^2(a_1l_1 - l_2)}{2b_3(a_1m_1 - m_2)y_2^4 + a_2y_2^2 + 2a_1b_1 - 2k_2}x_1 \\ & - \frac{2y_2^2(a_1b_2 - k_1)}{2b_3(a_1m_1 - m_2)y_2^4 + a_2y_2^2 + 2a_1b_1 - 2k_2}x_1^{-3}. \end{aligned} \tag{3.69}$$

This equation admits a three-dimensional Lie symmetry algebra $\mathfrak{sl}(2, \mathbb{R})$. However, if $K_1 \equiv k_1 - a_1b_2 = 0$, then the same equation admits an eight-dimensional Lie symmetry algebra $\mathfrak{sl}(3,\mathbb{R})$ and thus it is linearizable. Therefore, we use again the general method described in [21]. If we solve the second-order equation with respect to K_1 and derive once with respect to y_2, then the following nonlinear third-order equation is obtained

$$\begin{aligned} x_1''' = & -3\frac{x_1''x_1'}{x_1} - 6\frac{b_3(a_1m_1-m_2)y_2^4 - a_1b_1 + k_2}{y_2(2b_3(a_1m_1-m_2)y_2^4 + a_2y_2^2 + 2a_1b_1 - 2k_2}x_1'' \\ & - 6\frac{b_3(a_1m_1-m_2)y_2^4 - a_1b_1 + k_2}{y_2(2b_3(a_1m_1-m_2)y_2^4 + a_2y_2^2 + 2a_1b_1 - 2k_2)}\frac{(x_1')^2}{x_1} \\ & + 2\frac{((4l_1-m_1)a_1 - 4l_2 + m_2)b_3y_2^4 - 3a_1b_1 + 3k_2}{y_2^2(2b_3(a_1m_1-m_2)y_2^4 + a_2y_2^2 + 2a_1b_1 - 2k_2)}x_1', \end{aligned} \tag{3.70}$$

which possesses a seven-dimensional Lie symmetry algebra, hence it is linearizable and in particular possesses the linearizing symmetry

$$\frac{1}{x_1}\partial_{x_1}, \tag{3.71}$$

that yields the linearizing transformation $x_1(y_2) = \sqrt{u(y_2)}$ that turns equation (3.70) into the following linear equation:

$$\begin{aligned} u''' = 6 & \frac{-b_3(a_1m_1-m_2)y_2^4 + a_1b_1 - k_2}{y_2(2b_3(a_1m_1-m_2)y_2^4 + a_2y_2^2 + 2a_1b_1 - 2k_2)}u'' \\ & + \frac{2((4l_1-m_1)a_1 - 4l_2 + m_2)b_3y_2^4 - 6a_1b_1 + 6k_2}{y_2^2(2b_3(a_1m_1-m_2)y_2^4 + a_2y_2^2 + 2a_1b_1 - 2k_2)}u', \end{aligned} \tag{3.72}$$

and its general solution can be written in terms of hypergeometric functions. Consequently, we have shown that system (3.64) is linearizable by means of hidden symmetries.

4. Three minimally superintegrable system of non-subgroup type admitting non-zero magnetic fields and an axial symmetry

In [5] the authors studied three-dimensional integrable systems of non-subgroup type admitting non-zero magnetic fields and an axial symmetry. The systems correspond to the circular parabolic, oblate and prolate spheroidal cases. In addition to those integrable cases, one minimally superintegrable system was found with an additional quadratic integral of motion. This system represents the intersection between the circular parabolic case and the spherical case with a magnetic field. In [6] the authors continued the study of three-dimensional integrable systems of non-subgroup type admitting non-zero magnetic fields and an axial symmetry. Two new minimally superintegrable systems admitting an additional quadratic integral were presented and they represent the intersection of more than one integrable case.

We do not consider the superintegrable systems admitting an additional linear integral as determined in [5] and [6] since they are subcases of the systems we have already investigated in our present paper.

Here, we consider all the three classes of nonlinear minimally superintegrable systems, and determine their hidden linearity by means of Lie symmetries.

4.1. The intersection of the circular parabolic and spherical cases

The scalar potential and the magnetic field are

$$W(\vec{x}) = \frac{k_1}{r^2} + \frac{k_2}{R} + \frac{k_3 x_3}{r^2 R} + \frac{b_m^2}{2R^2} + \frac{b_z b_m x_3}{2R} - \frac{b_z b_n r^2}{2R} + \frac{b_m b_n x_3}{R^2} - \frac{b_n^2 r^2}{2R^2} - \frac{b_z^2}{8} r^2, \tag{4.1}$$

$$\vec{B}(\vec{x}) = \left[\frac{(b_m + b_n x_3)x_1}{R^3}, \frac{(b_m + b_n x_3)x_2}{R^3}, \frac{b_m x_3 + b_n(R^2 + x_3^2)}{R^3} + b_z\right]. \tag{4.2}$$

We will use a more natural set of coordinates, the spherical coordinates, as defined in equations (1.7) and (1.8). The Hamiltonian equations are

$$\begin{cases} \dot{R} = & \Pi_R, \qquad \dot{\phi} = \dfrac{\Pi_\phi}{R^2}, \qquad \dot{\theta} = \dfrac{\Pi_\theta}{R^2 \sin^2(\phi)}, \\ \dot{\Pi}_R = & \dfrac{\Pi_\phi^2}{R^3} + \dfrac{\Pi_\theta^2}{R^3 \sin^2(\phi)} + \dfrac{2k_1}{R^3 \sin^2(\phi)} + \dfrac{k_2}{R^2} + \dfrac{2k_3 \cos(\phi)}{R^3 \sin^2(\phi)} - \dfrac{b_n \Pi_\theta}{R^2} - \dfrac{b_z \Pi_\theta}{R} \\ & + \dfrac{b_m^2}{R^3} + \dfrac{b_n b_m \cos(\phi)}{R^2} + \dfrac{b_z b_n}{2} \sin^2(\phi) + \dfrac{b_z^2}{4} R \sin^2(\phi), \\ \dot{\Pi}_\phi = & \dfrac{\Pi_\theta^2 \cos(\phi)}{R^2 \sin^3(\phi)} + \dfrac{2k_1 \cos(\phi)}{R^2 \sin^3(\phi)} + \dfrac{k_3(\cos^2(\phi) + 1)}{R^2 \sin^3(\phi)} - \dfrac{b_m \Pi_\theta}{R^2 \sin(\phi)} \\ & - \dfrac{b_z \Pi_\theta \cos(\phi)}{\sin(\phi)} - \dfrac{2b_n \Pi_\theta \cos(\phi)}{R \sin(\phi)} + b_n^2 \cos(\phi) \sin(\phi) + \dfrac{b_n b_m \sin(\phi)}{R} \\ & + b_z b_n R \cos(\phi) \sin(\phi) + \dfrac{b_m b_z}{2} \sin(\phi) + \dfrac{b_z^2}{4} R^2 \cos(\phi) \sin(\phi), \\ \dot{\Pi}_\phi = & b_n \Pi_R \sin^2(\phi) + b_z R \Pi_R \sin^2(\phi) + b_z \Pi_\phi \cos(\phi) \sin(\phi) \\ & + \dfrac{b_m \Pi_\phi \sin(\phi)}{R^2} + \dfrac{2b_n \Pi_\phi \cos(\phi) \sin(\phi)}{R} \end{cases} \tag{4.3}$$

This system admits a two-dimensional Abelian Lie symmetry algebra generated by the operators $\partial_t, \partial_\theta$, and consequently the six equations (4.3) can be reduced to the following system of four equations:

$$\begin{cases} R' = & \dfrac{R^2 \Pi_R}{\Pi_\phi}, \\ \Pi_R' = & \dfrac{\Pi_\phi}{R} + \dfrac{\Pi_\theta^2}{R \Pi_\phi \sin^2(y_2)} + \dfrac{2k_1}{R \Pi_\phi \sin^2(y_2)} + \dfrac{k_2}{\Pi_\phi} + \dfrac{2k_3 \cos(y_2)}{R \Pi_\phi \sin^2(y_2)} - \dfrac{b_n \Pi_\theta}{\Pi_\phi} \\ & - \dfrac{b_z R \Pi_\theta}{\Pi_\phi} + \dfrac{b_m^2}{R \Pi_\phi} + \dfrac{b_m b_n \cos(y_2)}{\Pi_\phi} + \dfrac{b_z^2 R^3 \sin^2(y_2)}{4\Pi_\phi} + \dfrac{b_n b_z R^2 \sin^2(y_2)}{2\Pi_\phi} \\ \Pi_\phi' = & \dfrac{\Pi_\theta^2 \cos(y_2)}{\Pi_\phi \sin^3(y_2)} + \dfrac{2k_1 \cos(y_2)}{\Pi_\phi \sin^3(y_2)} + \dfrac{k_3(\cos^2(y_2) + 1)}{\Pi_\phi \sin^3(y_2)} - \dfrac{b_m \Pi_\theta}{\Pi_\phi \sin(y_2)} \\ & - \dfrac{2b_n R \Pi_\theta \cos(y_2)}{\Pi_\phi \sin(y_2)} - \dfrac{b_z R^2 \Pi_\theta \cos(y_2)}{\Pi_\phi \sin(y_2)} + \dfrac{b_n^2 R^2 \cos(y_2) \sin(y_2)}{\Pi_\phi} + \dfrac{b_m b_n R \sin(y_2)}{\Pi_\phi} \\ & + \dfrac{b_n b_z R^3 \sin(y_2) \cos(y_2)}{\Pi_\phi} + \dfrac{b_m b_z R^2 \sin(y_2)}{2\Pi_\phi} + \dfrac{b_z^2 R^4 \sin(y_2) \cos(y_2)}{4\Pi_\phi} \\ \Pi_\theta' = & R^2 \sin^2(y_2)(b_n + b_z R) \dfrac{\Pi_R}{\Pi_\phi} + \sin(y_2)(b_z R^2 \cos(y_2) + 2b_n R \cos(y_2) + b_m), \end{cases} \tag{4.4}$$

where $y_2 \equiv \phi$ is the new independent variable. If we take the ratio Π_R/Π_ϕ from the first equation and substitute it into the fourth equation, then it can be integrated directly by expressing Π_θ as a function of R and and y_2, i.e.

$$\Pi_\theta = \frac{b_z}{2}R^2\sin^2(y_2) + b_n R\sin^2(y_2) - b_m\cos(y_2) + a_1, \tag{4.5}$$

where a_1 is a constant of integration. Substituting this result into (4.4), we are left with the following three nonlinear equations

$$R' = \frac{R^2\Pi_R}{\Pi_\phi}, \tag{4.6}$$

$$\Pi_R' = \frac{a_1 b_n + k_2}{\Pi_\phi} + \frac{\Pi_\phi}{R} + \frac{(-2a_1 b_m + 2k_3)\cos(y_2) + a_1^2 + b_m^2 + 2k_1}{R\Pi_\phi\sin^2(y_2)}, \tag{4.7}$$

$$\Pi_\phi' = \frac{(-a_1 b_m + k_3)\cos^2(y_2) + (a_1^2 + b_m^2 + 2k_1)\cos(y_2) - a_1 b_m + k_3}{\Pi_\phi\sin^3(y_2)}. \tag{4.8}$$

The equation (4.8) is separable and linearizable by setting $\Pi_\phi(y_2) = \sqrt{u(y_2)}$. Hence, we obtain

$$\Pi_\phi(y_2) = \sqrt{\frac{-a_2\cos^2(y_2) + (2a_1 b_m - 2k_3)\cos(y_2) - a_1^2 - b_m^2 + a_2 - 2k_1}{\sin^2(y_2)}}, \tag{4.9}$$

where a_2 is a constant of integration. The remaining two nonlinear equations are

$$R' = \frac{R^2\Pi_R\sin(y_2)}{\sqrt{-a_2\cos^2(y_2) + (2a_1 b_m - 2k_3)\cos(y_2) - a_1^2 - b_m^2 + a_2 - 2k_1}}, \tag{4.10}$$

$$\Pi_R' = \frac{(a_2 + (a_1 b_n + k_2)R)\sin(y_2)}{R\sqrt{-a_2\cos^2(y_2) + (2a_1 b_m - 2k_3)\cos(y_2) - a_1^2 - b_m^2 + a_2 - 2k_1}}. \tag{4.11}$$

If we derive Π_R from (4.10) and substitute it into (4.11), then we obtain the nonlinear second-order equation

$$R'' = 2\frac{(R')^2}{R} + \alpha(y_2)R' - \beta(y_2)R^2 - \gamma(y_2)R, \tag{4.12}$$

where $\alpha(y_2)$, $\beta(y_2)$ and $\gamma(y_2)$ are given by

$$\alpha(y_2) = \frac{(k_3 - a_1 b_m)\cos^2(y_2) + (a_1^2 + b_m^2 + 2k_1)\cos(y_2) - a_1 b_m + k_3}{\sin(y_2)(a_2\cos^2(y_2)^2 + 2(k_3 - a_1 b_m)\cos(y_2) + a_1^2 + b_m^2 - a_2 + 2k_1)}, \tag{4.13}$$

$$\beta(y_2) = \frac{(a_1 b_n + k_2)\sin^2(y_2)}{a_2\cos^2(y_2)^2 + 2(k_3 - a_1 b_m)\cos(y_2) + a_1^2 + b_m^2 - a_2 + 2k_1}, \tag{4.14}$$

$$\gamma(y_2) = \frac{a_2\sin^2(y_2)}{a_2\cos^2(y_2)^2 + 2(k_3 - a_1 b_m)\cos(y_2) + a_1^2 + b_m^2 - a_2 + 2k_1}. \tag{4.15}$$

However, this equation admits an eight-dimensional Lie symmetry algebra, and therefore it is linearizable by the transformation of the dependent variable, $R(y_2) = 1/v(y_2)$ that yields the linear equation

$$v'' = \alpha(y_2)v' + \gamma(y_2)v + \beta(y_2), \tag{4.16}$$

and its general solution is

$$\begin{aligned} v(y_2) = a_3 \sqrt{a_2 \cos^2(y_2) + 2(k_3 - a_1 b_m)\cos(y_2) + a_1^2 + b_m^2 - a_2 + 2k_1} \\ - \frac{(a_1 b_n + k_2)((k_3 - a_1 b_m)\cos(y_2) + a_1^2 + b_m^2 - a_2 + 2k_1)}{a_1^2(a_2 - b_m^2) + 2a_1 b_m k_3 - a_2^2 + a_2(b_m^2 + 2k_1) - k_3^2} \\ + a_4(a_1 b_m - k_3 - a_2 \cos(y_2)) \end{aligned} \tag{4.17}$$

where a_3 and a_4 are constants of integration.

We can conclude that the minimally superintegrable system (4.3) is linearizable by means of hidden symmetries.

4.2. The intersection between the cylindrical, spherical, oblate and prolate spheroidal cases

Now let us consider the minimally superintegrable systems coming from section 5 in [6]. This Hamiltonian system is at the intersection of four integrable cases: the cylindrical, spherical, oblate spheroidal and prolate spheroidal cases. The scalar potential and the magnetic field are

$$\begin{aligned} W(\vec{x}) &= \frac{k_1}{r^2} + \frac{k_2}{x_3^2} - k_3 R^2 - \frac{b_p b_s R^4}{4x_3^2} - \frac{b_z b_p r^2}{4x_3^2} - \frac{b_z b_s}{4} r^2 R^2 - \frac{b_s^2}{8} r^2 R^4 + \frac{b_z^2}{8} x_3^2 - \frac{b_p^2 r^2}{8x_3^4}, \\ \vec{B}(\vec{x}) &= \left[\frac{b_p x_1}{x_3^3} - b_s x_1 x_3, \frac{b_p x_2}{x_3^3} - b_s x_2 x_3, \frac{b_p}{x_3^2} + b_s(r^2 + R^2) + b_z \right]. \end{aligned} \tag{4.18}$$

We use the cylindrical coordinates as defined in (1.5) and (1.6), and in those coordinates, the Hamiltonian equations are

$$\begin{cases} \dot{r} = \Pi_r, \qquad \dot{\theta} = \dfrac{\Pi_\theta}{r^2}, \qquad \dot{x}_3 = \Pi_3, \\ \dot{\Pi}_r = \dfrac{2k_1}{r^3} + 2k_3 r + \left(\dfrac{b_z}{2} r^2 - \Pi_\theta\right) \dfrac{b_p}{r x_3^2} + \left(2r^2 + x_3^2\right)\left(\dfrac{b_z}{2} r^2 - \Pi_\theta\right) \dfrac{b_s}{r} \\ \quad -(b_z r^2 - \Pi_\theta)\dfrac{\Pi_\theta}{r^3} + \dfrac{b_p^2 r}{4x_3^4} + \dfrac{b_s b_p r}{x_3^2}(r^2 + x_3^2) + \dfrac{b_s^2 r}{4}(3r^4 + 4r^2 x_3^2 + x_3^4), \\ \dot{\Pi}_\theta = \dfrac{b_p r}{x_3^3}(x_3 \Pi_r - r\Pi_3) + b_s(r(2r^2 + x_3^2)\Pi_r + r^2 x_3 \Pi_3) + b_z r \Pi_r, \\ \dot{\Pi}_3 = \dfrac{2k_2}{x_3^3} + 2k_3 x3 + \left(\Pi_\theta - \dfrac{b_z r^2}{2}\right)\dfrac{b_p}{x_3^2} + \dfrac{b_s}{2}(b_z r^2 - 2\Pi_\theta)x_3 \\ \quad -\dfrac{b_p^2 r^2}{2x_3^5} + \dfrac{b_s b_p}{2x_3^3}(x_3^4 - r^4) + \dfrac{b_s^2 r^2 x_3}{2}(x_3^2 + r^2) - \dfrac{b_z^2}{4} x_3. \end{cases} \tag{4.19}$$

This system admits a two-dimensional Abelian Lie symmetry algebra generated by the operators $\partial_t, \partial_\theta$, and consequently the six equations (4.19) can be reduced to the following system of four equations:

$$x_3' = \frac{\Pi_3}{\Pi_r}, \tag{4.20}$$

$$\Pi_r' = \frac{b_p^2 y_2}{4x_3^4 \Pi_r} + \frac{b_s b_p y_2}{x_3^2 \Pi_r}(x_3^2 + y_2^2) + \frac{b_p(b_z y_2^2 - 2\Pi_\theta)}{2x_3^2 y_2 \Pi_r} + \frac{b_s^2 y_2}{4\Pi_r}(x_3^4 + 4x_3^2 y_2^2 + 3y_2^4)$$

$$+\left(\frac{b_z}{2}y_2^2-\Pi_\theta\right)\frac{b_s(2y_2^2+x_3^2)}{y_2\Pi_r}+\frac{2k_1}{y_2^3\Pi_r}+\frac{2k_3y_2}{\Pi_r}+(\Pi_\theta-b_zy_2^2)\frac{\Pi_\theta}{y_2^3\Pi_r}, \quad (4.21)$$

$$\Pi_\theta'=(x_3\Pi_r-y_2\Pi_3)\frac{b_py_2}{x_3^3\Pi_r}+((x_3^2+2y_2^2)\Pi_r+y_2x_3\Pi_3)\frac{b_sy_2}{\Pi_r}+b_zy_2, \quad (4.22)$$

$$\Pi_3'=-\frac{b_p^2y_2^2}{2x_3^5\Pi_r}+(x_3^4-y_2^4)\frac{b_sb_p}{2x_3^3\Pi_r}+(-b_zy_2^2+2\Pi_\theta)\frac{b_p}{2x_3^3\Pi_r}+(x_3^2+y_2^2)\frac{b_s^2y_2^2x_3}{2\Pi_r}$$
$$+(b_zy_2^2-2\Pi_\theta)\frac{b_sx_3}{2\Pi_r}+\frac{2k_2}{x_3^3\Pi_r}+\frac{2k_3x_3}{\Pi_r}-\frac{b_z^2x_3}{4\Pi_r}, \quad (4.23)$$

where $y_2\equiv r$ is the new independent variable. If we derive Π_3 from (4.20) and substitute it into (4.22), then it can be integrated directly by expressing Π_θ as a function of x_3 and y_2, i.e.

$$\Pi_\theta(y_2)=y_2^2\left(\frac{b_sx_3^2}{2}+\frac{b_p}{2x_3^2}\right)+\frac{b_s}{2}y_2^4+\frac{b_z}{2}y_2^2+a_1, \quad (4.24)$$

where a_1 is an constant of integration. We are left with the following three equations

$$x_3'=\frac{\Pi_3}{\Pi_r}, \quad (4.25)$$

$$\Pi_r'=\frac{(-4a_1b_s+2b_pb_s-b_z^2+8k_3)y_2^4+4a_1^2+8k_1}{4y_2^3\Pi_r}, \quad (4.26)$$

$$\Pi_3'=((-4a_1+2b_p)b_s-b_z^2+8k_3)\frac{x_3}{4\Pi_r}+\frac{a_1b_p+2k_2}{x_3^3\Pi_r}. \quad (4.27)$$

The equation (4.26) is separable and linearizable by setting $\Pi_r(y_2)=\sqrt{u(y_2)}$. Its general solution is

$$\Pi_r(y_2)=\sqrt{\frac{(-4a_1b_s+2b_pb_s-b_z^2+8k_3)y_2^2}{4}-\frac{4a_1^2+8k_1}{4y_2^2}+a_2}, \quad (4.28)$$

where a_2 is a constant of integration. Consequently, we are left with the following two nonlinear equations

$$x_3'=\frac{2y_2\Pi_3}{\sqrt{(-4a_1b_s+2b_pb_s-b_z^2+8k_3)y_2^4+4a_2y_2^2-4a_1^2-8k_1}}, \quad (4.29)$$

$$\Pi_3'=\frac{y_2}{2x_3^3}\frac{(-4a_1b_s+2b_pb_s-b_z^2+8k_3)x_3^4+4a_1b_p+8k_2}{\sqrt{(-4a_1b_s+2b_pb_s-b_z^2+8k_3)y_2^4+4a_2y_2^2-4a_1^2-8k_1}}. \quad (4.30)$$

If we derive Π_3 from (4.29) and substitute it in the equation (4.30), then we obtain the following nonlinear second-order equation

$$x_3''=\frac{\alpha y_2^4-\alpha_1}{y_2(-\alpha y_2^4-4a_2y_2^2-\alpha_1)}x_3'+\frac{y_2^2(\alpha_2-x_3^4\alpha)}{x_3^3(-\alpha y_2^4-4a_2y_2^2-\alpha_1)}, \quad (4.31)$$

where

$$\alpha=(-4a_1+2b_p)b_s-b_z^2+8k_3, \quad \alpha_1=-4a_1^2-8k_1, \quad \alpha_2=-4a_1b_p-8k_2. \quad (4.32)$$

This equation admits a three-dimensional Lie symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and becomes linear if $\alpha_2=0$. Therefore, we use the general method described in [21] and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$. If we solve equation (4.31) with respect to α_2 and derive once with respect to y_2, then the nonlinear third-order equation that is obtained admits a seven-dimensional Lie symmetry algebra, and is therefore linearizable.

Consequently, we have shown that system (4.19) is linearizable by means of hidden symmetries.

4.3. The intersection between the cylindrical and circular parabolic cases

Here, we consider the Hamiltonian system at the intersection of the circular parabolic case and the cylindrical case with a non-zero magnetic field. This system is minimally superintegrable and has been investigated in Section 6 in [6]. The associated scalar potential and magnetic field are

$$W(\vec{x}) = k_1 x_3 + \frac{k_2}{r^2} + k_3(r^2+4x_3^2) - \frac{r^2}{32}\left(2b_z + b_q(r^2+4x_3^2)\right)^2, \tag{4.33}$$

$$\vec{B}(\vec{x}) = \left[-2b_q x_1 x_3, -2b_q x_2 x_3, b_z + b_q(r^2+2x_3^2)\right]. \tag{4.34}$$

We use the cylindrical coordinates as defined in (1.5) and (1.6), and in those coordinates, the Hamiltonian equations are

$$\begin{cases} \dot{r} = \Pi_r, \qquad \dot{\theta} = \dfrac{\Pi_\theta}{r^2}, \qquad \dot{x}_3 = \Pi_3, \\ \dot{\Pi}_r = \dfrac{\Pi_\theta^2}{r^3} + \dfrac{2k_2}{r^3} - 2k_3 r - (b_z + b_q(r^2+2x_3^2))\dfrac{\Pi_\theta}{r} \\ \qquad +(b_q(r^2+4x_3^2)+2b_z)(b_q(3r^2+4x_3^2)+2b_z)\dfrac{r}{16}, \\ \dot{\Pi}_\theta = (b_q(r^2+2x_3^2)+b_z)r\Pi_r + 2b_q r^2 x_3 \Pi_3, \\ \dot{\Pi}_3 = -k_1 - 8k_3 x_3 - 2b_q x_3 \Pi_\theta + \dfrac{b_q r^2 x_3}{2}(b_q(r^2+4x_3^2+2b_z), \end{cases} \tag{4.35}$$

This system admits a two-dimensional Abelian Lie symmetry algebra generated by the operators $\partial_t, \partial_\theta$, and consequently the six equations (4.35) can be reduced to the following system of four equations:

$$x_3'(y_2) = \frac{\Pi_3}{\Pi_r}, \tag{4.36}$$

$$\begin{aligned} \Pi_r'(y_2) = {} & \frac{\Pi_\theta^2}{y_2^3\Pi_r} - (b_q(y_2^2+2x_3^2)+b_z)\frac{\Pi_\theta}{y_2\Pi_r} + \frac{b_q^2 y_2 x_3^4}{\Pi_r} + \frac{b_q y_2(b_q y_2^2+b_z)x_3^2}{\Pi_r} \\ & + \frac{3b_q^2 y_2^8 + 8b_z b_q y_2^6 + (4b_z^2 - 32k_3)y_2^4 + 32k_2}{16y_2^3\Pi_r}, \end{aligned} \tag{4.37}$$

$$\Pi_\theta'(y_2) = y_2(b_q y_2^2 + 2b_q x_3^2 + b_z) + 2b_q y_2^2 x_3 \frac{\Pi_3}{\Pi_r}, \tag{4.38}$$

$$\Pi_3'(y_2) = -2b_q x_3\frac{\Pi_\theta}{\Pi_r} + \frac{4b_q^2 y_2^2 x_3^3 + (b_q^2 y_2^4 + 2b_q b_z y_2^2 - 16k_3)x_3 - 2k_1}{2\Pi_r}, \tag{4.39}$$

where $y_2 \equiv r$ is the new independent variable. If we derive Π_3 from (4.36) and substitute it into (4.38), then it can be integrated directly by expressing Π_θ as a function of x_3 and

y_2, i.e.

$$\Pi_\theta(y_2) = b_q y_2^2 x_3(y_2)^2 + \frac{b_q}{4} y_2^4 + \frac{b_z}{2} y_2^2 + a_1, \tag{4.40}$$

where a_1 is a constant of integration. Using this result, we are left with the following nonlinear equations:

$$x_3'(y_2) = \frac{\Pi_3}{\Pi_r}, \tag{4.41}$$

$$\Pi_r'(y_2) = \frac{(-a_1 b_q - 4k_3)y_2^4 + 2a_1^2 + 4k_2}{2y_2^3 \Pi_r}, \tag{4.42}$$

$$\Pi_3'(y_2) = -(2a_1 b_q + 8k_3)\frac{x_3}{\Pi_r}. \tag{4.43}$$

The equation (4.42) is separable and linearizable by means of the transformation $\Pi_r(y_2) = \sqrt{u(y_2)}$, and consequently we have

$$u(y_2) = a_2 - \left(\frac{a_1 b_q}{2} + 2k_3\right) y_2^2 - \frac{a_1^2 + 2k_2}{y_2^2}, \tag{4.44}$$

where a_2 is a constant of integration. Consequently, we are left with the following two linear first-order differential equations

$$x_3'(y_2) = \frac{2\Pi_3}{\sqrt{4a_2 - 2(a_1 b_q + 4k_3)y_2^2 - 4(a_1^2 + 2k_2)y_2^{-2}}}, \tag{4.45}$$

$$\Pi_3'(y_2) = \frac{(4a_1 b_q + 16k_3)x_3 + 2k_1}{\sqrt{4a_2 - 2(a_1 b_q + 4k_3)y_2^2 - 4(a_1^2 + 2k_2)y_2^{-2}}}. \tag{4.46}$$

We can conclude that the minimally superintegrable system (4.35) is also linearizable.

5. Final remarks

In this paper, fifteen three-dimensional nonlinear minimally superintegrable systems in a static electromagnetic field are shown to possess hidden symmetries leading to their linearization, and consequently the corresponding subsets of maximally superintegrable subcases are also linearizable.

We underline that in each case none of the known first integrals have been used.

Our results are strengthening the conjecture that all three-dimensional minimally superintegrable systems are linearizable by means of hidden symmetries.

It is worth noting that Case 9.2b, namely Hamiltonian system (3.64), is just integrable, not superintegrable. Some parameters need to be commensurable for the system to be superintegrable, constraints that we did not impose. This example hints that also integrable systems may possess hidden symmetries leading to linearization.

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