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Finitely additive mass transportation

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Some classical mass transportation problems are investigated in a finitely additive setting. Let $\Omega = \prod_{i=1}^n \Omega_i$ and $\mathcal{A} = \otimes_{i=1}^n \mathcal{A}_i$, where $(\Omega_i, \mathcal{A}_i, \mu_i)$ is a (σ-additive) probability space for $i = 1, \ldots, n$. Let $c : \Omega \to [0, \infty]$ be an $\mathcal{A}$-measurable cost function. Let M be the collection of finitely additive probabilities on $\mathcal{A}$ with marginals $\mu_1, \ldots, \mu_n$. If couplings are meant as elements of M, most classical results of mass transportation theory, including duality and attainability of the Kantorovich inf, are valid without any further assumptions. Special attention is devoted to martingale transport. Let $(\Omega_i, \mathcal{A}_i) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for all i and

$$M_1 = \{P \in M : P \ll P^* \text{ and } (\pi_1, \ldots, \pi_n) \text{ is a } P\text{-martingale}\}$$

where P^* is a reference probability on $\mathcal{A}$ and $\pi_1, \ldots, \pi_n$ are the canonical projections on $\Omega = \mathbb{R}^n$. If $M_1 \neq \emptyset$, the Kantorovich inf over M_1 is attained, in the sense that $\int c\, dP = \inf_{Q \in M_1} \int c\, dQ$ for some $P \in M_1$. Conditions for $M_1 \neq \emptyset$ are given as well.

Keywords: Coupling; duality theorem; finitely additive probability; martingale; mass transportation

1. Introduction

Mass transportation is nowadays a dynamic field of research. Its applications range in a number of fields, including probability theory, differential equations, geometric measure theory, economics and finance; see e.g. [1,19,27].

This paper deals with mass transportation problems when couplings are finitely additive probabilities. To be more precise, and to highlight similarities and differences between our approach and the usual one, we need to recall the standard framework where transportation problems are investigated. In the sequel, the abbreviation *f.a.p.* stands for *finitely additive probability* and a *probability measure* is a *σ-additive* f.a.p. Moreover, we use the notation

$$P(f) = \int f\, dP$$

whenever P is a f.a.p. and f a function such that $\int f\, dP$ is well defined.

1.1. The standard framework

Let $I = \{1, \ldots, n\}$ where n is a positive integer. For each $i \in I$, let $(\Omega_i, \mathcal{A}_i)$ be a measurable space and μ_i a probability measure on the σ-field $\mathcal{A}_i$. Define

$$\Omega = \Omega_1 \times \ldots \times \Omega_n \quad \text{and} \quad \mathcal{A} = \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$$

and denote by $\pi_i : \Omega \to \Omega_i$ the i-th canonical projection, namely,

$$\pi_i(\omega) = \omega_i \qquad \text{for all } i \in I \text{ and } \omega = (\omega_1, \ldots, \omega_n) \in \Omega.$$

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Moreover, let $c : \Omega \to [-\infty, \infty]$ be a (cost) function. Various conditions on c can be taken into account. In this paper, *c is $\mathcal{A}$-measurable and takes values in* $[0, \infty]$. *In particular, it may be that $c = \infty$.*

A *coupling* (or a *transport plan*) is a probability measure P on $\mathcal{A}$ having $\mu_1, \ldots, \mu_n$ as marginals, in the sense that

$$P \circ \pi_i^{-1} = \mu_i \quad \text{for all } i \in I.$$

The collection of all couplings, henceforth denoted by Γ, plays a basic role. A few classical issues are:

(i) Give conditions for the existence of $P \in \Gamma$ such that

$$P(c) = \inf_{Q \in \Gamma} Q(c); \tag{1}$$

(ii) Characterize those $P \in \Gamma$ satisfying equation (1) (provided they exist);

(iii) Give conditions for the duality relation

$$\inf_{Q \in \Gamma} Q(c) = \sup_{f_1, \ldots, f_n} \sum_{i=1}^{n} \mu_i(f_i)$$

where sup is over the n-tuple $(f_1, \ldots, f_n)$ such that $f_i \in L_1(\mu_i)$ for all $i \in I$ and $\sum_{i=1}^{n} f_i \circ \pi_i \le c$.

Here, $L_1(\mu_i) = L_1(\Omega_i, \mathcal{A}_i, \mu_i)$ is the class of $\mathcal{A}_i$-measurable functions $f : \Omega_i \to \mathbb{R}$ such that $\mu_i(|f|) = \int |f| \, d\mu_i < \infty$ (without identifying maps which agree μ_i-a.s.).

A natural development is to fix a subset $\Gamma_0 \subset \Gamma$ and to investigate (i)-(ii)-(iii) (and possibly other problems) with Γ_0 in the place of Γ. Following [28], for instance, Γ_0 could be

$$\Gamma_0 = \{ P \in \Gamma : P(|f|) < \infty \text{ and } P(f) = 0 \text{ for all } f \in F \}$$

where F is a given class of $\mathcal{A}$-measurable functions $f : \Omega \to \mathbb{R}$. As a special case, let $(\Omega_i, \mathcal{A}_i) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for each $i \in I$, where $\mathcal{B}(\cdot)$ denotes the Borel σ-field. Then, a suitable choice of F yields

$$\Gamma_0 = \{ P \in \Gamma : (\pi_1, \ldots, \pi_n) \text{ is a martingale under } P \}.$$

Such a Γ_0, introduced in [3], corresponds to the so-called *martingale transport*. In addition to be theoretically intriguing, martingale transport has solid financial motivations; see e.g. [3,5,15] and references therein.

A last remark is that, for any choice of Γ_0, a preliminary question is whether $\Gamma_0 \neq \emptyset$. In martingale transport, for instance, as a consequence of some results by Strassen [26], one obtains $\Gamma_0 \neq \emptyset$ if and only if

$$\int |x| \, \mu_i(dx) < \infty \quad \text{and} \quad \mu_i(f) \le \mu_{i+1}(f)$$

for all $i \in I$ and all convex functions $f : \mathbb{R} \to \mathbb{R}$.

1.2. Content of this paper

Investigating mass transportation in a finitely additive setting is a quite natural idea, and various hints in this direction are scattered throughout the literature; see e.g. [14,18,19,24] and references therein. Usually, however, f.a.p.'s are only instrumental. Typically, a result concerning f.a.p.'s is nothing but an intermediate step toward the corresponding σ-additive result. Instead, f.a.p.'s have an intrinsic interest in several mass transportation problems; see e.g. the examples of Subsection 3.2. Nevertheless, to the best of our knowledge, no systematic treatment of the finitely additive mass transportation is available to date. This paper aims to fill this gap in the special case where $\mu_1, \ldots, \mu_n$ are probability measures.

Incidentally, we note that various other cases could be dealt with. First of all, $\mu_1, \ldots, \mu_n$ could be f.a.p.'s and not necessarily probability measures. While intriguing, however, this case departs very much from the standard one. We will investigate it in a forthcoming paper but not in the sequel. Another possibility is to fix a field $\mathcal{F}_i$ such that $\mathcal{A}_i = \sigma(\mathcal{F}_i)$, $i \in I$, and to focus on the f.a.p.'s having marginals $\mu_1, \ldots, \mu_n$ on $\mathcal{F}_1, \ldots, \mathcal{F}_n$. Even if only in passing, this case is accounted in this paper; see Example 7.

Let

$$\mathbb{P} = \{\text{all f.a.p.'s on } \mathcal{A}\}$$

and

$$M = \{P \in \mathbb{P} : P \circ \pi_i^{-1} = \mu_i \text{ for all } i \in I\}.$$

In this paper, problems (i)-(ii)-(iii) are investigated with M in the place of Γ. Similarly, the subsets $\Gamma_0 \subset \Gamma$ are replaced by the corresponding subsets $M_0 \subset M$.

Our main result is that, if Γ is replaced by M, each of problems (i)-(ii)-(iii) admits a solution assuming only that c is non-negative and $\mathcal{A}$-measurable. On the contrary, to have a solution in the standard framework, further conditions on c and/or $\mu_1, \ldots, \mu_n$ are needed (such conditions are recalled at the end of Subsection 3.1).

Special attention is devoted to martingale transport. To illustrate, suppose $(\Omega_i, \mathcal{A}_i) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for each $i \in I$ and define

$$M_0 = \{P \in M : (\pi_1, \ldots, \pi_n) \text{ is a } P\text{-martingale}\}.$$

If $M_0 \neq \emptyset$, then $P(c) = \inf_{Q \in M_0} Q(c)$ for some $P \in M_0$. Similarly, fix a reference f.a.p. $P^* \in \mathbb{P}$ and define

$$M_1 = \{P \in M : P \ll P^* \text{ and } (\pi_1, \ldots, \pi_n) \text{ is a } P\text{-martingale}\}.$$

Once again, if $M_1 \neq \emptyset$, there is $P \in M_1$ such that $P(c) = \inf_{Q \in M_1} Q(c)$.

A remark on M_1 is in order. Suppose P^* is a probability measure. In the standard martingale transport, to our knowledge, the set

$$M_1 \cap \Gamma = \{P \in \Gamma : P \ll P^* \text{ and } (\pi_1, \ldots, \pi_n) \text{ is a } P\text{-martingale}\}$$

is largely neglected. The only partial exception is [17], which is focused on

$$\Gamma_u = \{P \in \Gamma : P \le u\, P^*\} \qquad \text{where } u > 0 \text{ is a given constant.}$$

There are two main differences between such a Γ_u and $M_1 \cap \Gamma$. The elements of Γ_u need not be martingale probability measures, and $P \ll P^*$ is strengthened into $P \le u\, P^*$ (that is, not only P has a density with respect to P^*, but this density is bounded by a *given* constant u). However, to investigate $M_1 \cap \Gamma$ looks quite natural. In fact, the usual motivation for martingale transport is that martingale probability measures play a role in various financial problems. But, in most such problems, probability measures are also required to be equivalent, or at least absolutely continuous, with respect to some P^*. To sum up, to focus on $M_1 \cap \Gamma$ is reasonable in the standard martingale transport. In turn, in the framework of this paper, it is natural to focus on M_1.

The result quoted above requires $M_1 \neq \emptyset$. This is investigated in the last part of the paper. Suppose P^* is a probability measure with compact support and define

$$\mathcal{U} = \{Q \in \mathbb{P} : Q \ll P^* \text{ and } (\pi_1, \ldots, \pi_n) \text{ is a } Q\text{-martingale}\}.$$

Then, $M_1 \neq \emptyset$ if and only if

$$\sum_{i=1}^n \mu_i(f_i) \geq \inf_{Q \in \mathcal{U}} Q\Big(\sum_{i=1}^n f_i \circ \pi_i\Big)$$

for all bounded Borel functions $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$. This result admits a σ-additive version as well. In fact, if P^* has discrete marginals (except possibly one), the above condition implies the existence of $P \in \Gamma$ such that $P \ll P^*$ and $(\pi_1, \ldots, \pi_n)$ is a P-martingale.

2. Preliminaries

For any measurable space $(S, \mathcal{E})$, we denote by $B(S, \mathcal{E})$ the set of bounded $\mathcal{E}$-measurable functions $f : S \to \mathbb{R}$. If $P, Q \in \mathbb{P}$, we write $P \ll Q$ to mean that $P(A) = 0$ whenever $A \in \mathcal{A}$ and $Q(A) = 0$. Moreover, if $f_i : \Omega_i \to \mathbb{R}$, the map $\sum_i f_i \circ \pi_i$ is denoted by $\oplus_i f_i$. Hence, $\oplus_i f_i$ is the function on Ω defined by

$$\big(\oplus_{i=1}^n f_i\big)(\omega) = \sum_{i=1}^n f_i(\omega_i) \qquad \text{for all } \omega = (\omega_1, \ldots, \omega_n) \in \Omega.$$

2.1. Integrals with respect to f.a.p.'s

If P is a f.a.p. on a σ-field and the integrand function f is measurable with respect to such a σ-field, the integral $\int f\,dP$ is defined essentially in the usual way. To keep the paper self-contained, however, we briefly recall the basic definitions. For more on this subject, we refer to [11].

Let $P \in \mathbb{P}$ and $f : \Omega \to \mathbb{R}$ a real-valued $\mathcal{A}$-measurable function. If f is simple, $\int f\,dP$ is defined in the usual way. If f is bounded, f is the uniform limit of a sequence f_k of simple functions, and we let $\int f\,dP = \lim_k \int f_k\,dP$. If $f \geq 0$, then f is P-integrable if and only if

$$\sup_k \int f \wedge k\,dP < \infty,$$

and we let

$$\int f\,dP = \sup_k \int f \wedge k\,dP. \tag{2}$$

In general, f is P-integrable if and only if f^+ and f^- are both P-integrable, or equivalently $\int |f|\,dP < \infty$, and in this case $\int f\,dP = \int f^+\,dP - \int f^-\,dP$. Observe now that equation (2) makes sense whenever $0 \leq f \leq \infty$, merely $\int f\,dP = \infty$ if f is real-non-negative but not P-integrable or if $P(f = \infty) > 0$. Hence, $\int f\,dP$ is always defined by (2) whenever f is $\mathcal{A}$-measurable and takes values in $[0, \infty]$. In a nutshel, this is a concise summary on finitely additive integration. As already noted, we will use the notation $P(f) = \int f\,dP$ whenever $\int f\,dP$ is well defined.

2.2. Finitely additive martingales

Let $(\Omega_i, \mathcal{A}_i) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for all $i \in I$. Define a class H of real-valued functions on $\Omega = \mathbb{R}^n$ as

$$H = \{\pi_1\} \cup \left\{ \left(\pi_{i+1} - \pi_i\right) g(\pi_1, \ldots, \pi_i) : 1 \le i < n,\ g \in B(\mathbb{R}^i, \mathcal{B}(\mathbb{R}^i)) \right\}.$$

If P is a probability measure on $\mathcal{A}$, then $(\pi_1, \ldots, \pi_n)$ is a P-martingale if

(a) $P(|\pi_i|) < \infty$ for all $i \in I$,
(b) $P(\pi_1) = 0$ and

$$E_P\left(\pi_{i+1} \mid \pi_1, \ldots, \pi_i\right) = \pi_i,\ P\text{-a.s., for all } i < n.$$

However, under (a), condition (b) is equivalent to

(c) $P(f) = 0$ for each $f \in H$.

Therefore, according to us, it is reasonable to state the following definition. For any f.a.p. $P \in \mathbb{P}$, we say that $(\pi_1, \ldots, \pi_n)$ is a P-martingale if conditions (a) and (c) are satisfied; see also [7,8].

2.3. The product topology on $[0,1]^{\mathcal{A}}$

We first recall some basic notions. Let S be any topological space. A *net* in S is a map from a directed set J into S. Such a map is usually denoted by $(x_\alpha : \alpha \in J)$ or merely by (x_α) if J is understood. A point $x \in S$ is a limit of a net (x_α) (or $x_\alpha \to x$, or (x_α) converges to x) if, for any neighborhood U of x, there is $\alpha_0 \in J$ such that $x_\alpha \in U$ whenever $\alpha \ge \alpha_0$. Moreover, a subset $A \subset S$ is closed if and only if $x \in A$ provided $x_\alpha \to x$ for some net $(x_\alpha) \subset A$.

Next, let $[0,1]^{\mathcal{A}}$ be the collection of all functions $P : \mathcal{A} \to [0,1]$. The product topology on $[0,1]^{\mathcal{A}}$ is the weakest topology which makes continuous the maps $P \mapsto P(A)$ for all $A \in \mathcal{A}$. Under this topology, $[0,1]^{\mathcal{A}}$ is a compact Hausdorff space and convergence is setwise convergence. Precisely, a net (P_α) in $[0,1]^{\mathcal{A}}$ converges to $P \in [0,1]^{\mathcal{A}}$ if and only if

$$P(A) = \lim_\alpha P_\alpha(A) \text{ for all } A \in \mathcal{A}.$$

In particular, when $[0,1]^{\mathcal{A}}$ is equipped with the product topology, $\mathbb{P}$ and M are compact while Γ is not. Moreover, if $f \in B(\Omega, \mathcal{A})$, the map $P \mapsto P(f)$ is continuous on $\mathbb{P}$. In the sequel, $[0,1]^{\mathcal{A}}$ *is given the product topology and all its subsets are equipped with the corresponding relative topology.*

3. Finitely additive couplings

In this section, $c : \Omega \to [0, \infty]$ is an $\mathcal{A}$-measurable map (the value ∞ is admissible for c). Recall also that $\mathbb{P}$ denotes the set of all f.a.p.'s on $\mathcal{A}$ and M the collection of those $P \in \mathbb{P}$ with marginals $\mu_1, \ldots, \mu_n$.

3.1. Results

We begin with a preliminary lemma. Fix a collection F of real-valued $\mathcal{A}$-measurable functions on Ω and define

$$\mathcal{K}(F) = \left\{ P \in M : P(|f|) < \infty \text{ and } P(f) = 0 \text{ for all } f \in F \right\}.$$

Lemma 1. *$\mathcal{K}(F)$ is compact provided*

$$\lim_k \sup_{P\in M} P\{|f|\,1(|f|>k)\} = 0 \qquad \text{for each } f \in F. \tag{3}$$

Moreover, $P \mapsto P(c)$ is a lower semicontinuous map from $\mathbb{P}$ into $[0,\infty]$.

Proof. To make the notation easier, write $\mathcal{K}$ instead of $\mathcal{K}(F)$. Since $\mathcal{K}\subset M$ and M is compact, it suffices to show that $\mathcal{K}$ is closed. Let (P_α) be a net in $\mathcal{K}$ such that $P_\alpha \to P$ for some $P\in[0,1]^{\mathcal{A}}$. Since M is closed, $P\in M$. Hence, we have to show that $P(|f|)<\infty$ and $P(f)=0$ for all $f\in F$. Fix $f\in F$. By (3), there is k such that $Q\{|f|\,1(|f|>k)\}\le 1$ for each $Q\in M$. Hence, $P\in M$ implies

$$P(|f|)\le k + P\{|f|\,1(|f|>k)\}\le k+1.$$

Next, for fixed α, one obtains $P_\alpha(f)=0$ (due to $P_\alpha\in\mathcal{K}$) and

$$\begin{aligned} 0 = P_\alpha(f) &= P_\alpha\{f\,1(|f|\le k)\} + P_\alpha\{f\,1(|f|>k)\} \\ &\le P_\alpha\{f\,1(|f|\le k)\} + \sup_{Q\in M} Q\{|f|\,1(|f|>k)\} \qquad \text{for all } k. \end{aligned}$$

Moreover, for fixed k,

$$\lim_\alpha P_\alpha\{f\,1(|f|\le k)\} = P\{f\,1(|f|\le k)\}.$$

It follows that

$$0\le P\{f\,1(|f|\le k)\} + \sup_{Q\in M} Q\{|f|\,1(|f|>k)\} \le P(f) + 2\sup_{Q\in M} Q\{|f|\,1(|f|>k)\}.$$

Hence, $P(f)\ge 0$ because of (3). Similarly, one obtains $P(f)\le 0$. Thus $P(f)=0$, and this proves that $\mathcal{K}$ is closed. Finally, to show that $P\mapsto P(c)$ is lower semicontinuous on $\mathbb{P}$, it suffices to note that $P\mapsto P(c\wedge k)$ is continuous for fixed k. Therefore, lower semicontinuity of $P\mapsto P(c)$ follows from

$$P(c)=\sup_k P(c\wedge k) \qquad \text{for all } P\in\mathbb{P}.$$

□

Condition (3) is a form of uniform integrability. Since

$$P\{|f|\,1(|f|>k)\}\le k^{-\epsilon}\,P\{|f|^{1+\epsilon}\},$$

a sufficient condition for (3) is that, for each $f\in F$, there is $\epsilon>0$ such that $\sup_{P\in M} P\{|f|^{1+\epsilon}\}<\infty$. In particular, condition (3) holds whenever $F\subset B(\Omega,\mathcal{A})$. In any case, since $P\mapsto P(c)$ is lower semicontinuous, for each non-empty compact set $K\subset M$ one obtains

$$P(c)=\inf_{Q\in K} Q(c) \qquad \text{for some } P\in K. \tag{4}$$

Thus, with reference to problem (i), the inf is attained for *any* non-negative $\mathcal{A}$-measurable cost function c provided Γ is replaced by a non-empty compact $K\subset M$. The next result highlights some meaningful special cases.

Theorem 2. *Let $P^* \in \mathbb{P}$ and*

$$K_1 = \{P \in M : P \ll P^*\}.$$

If $(\Omega_i, \mathcal{A}_i) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for each $i \in I$, define also

$$K_2 = \{P \in M : (\pi_1, \dots, \pi_n) \text{ is a } P\text{-martingale}\} \qquad \text{and}$$
$$K_3 = \{P \in M : P \ll P^* \text{ and } (\pi_1, \dots, \pi_n) \text{ is a } P\text{-martingale}\}.$$

Then, equation (4) holds with $K = M$. Moreover, for each $j = 1, 2, 3$, equation (4) holds with $K = K_j$ provided $K_j \neq \emptyset$.

Proof. As noted in Subsection 2.3, M is compact (and obviously non-empty). Moreover, $K_1 = \mathcal{K}(F)$ where

$$F = \{1_A : A \in \mathcal{A},\ P^*(A) = 0\}.$$

Since $F \subset B(\Omega, \mathcal{A})$, condition (3) holds. Hence, K_1 is compact.

Next, we let $(\Omega_i, \mathcal{A}_i) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for all $i \in I$ and we prove that K_2 is compact (provided it is non-empty). To this end, we first note that $K_2 = \mathcal{K}(H)$ where

$$H = \{\pi_1\} \cup \{(\pi_{i+1} - \pi_i)\, g(\pi_1, \dots, \pi_i) : 1 \le i < n,\ g \in B(\mathbb{R}^i, \mathcal{B}(\mathbb{R}^i))\};$$

see Subsection 2.2. Hence, it suffices to check condition (3) for each $f \in H$.

Since $K_2 \neq \emptyset$, there is $Q \in K_2$ and this implies

$$\int |x|\, \mu_i(dx) = Q(|\pi_i|) < \infty \qquad \text{for each } i \in I.$$

Let $f \in H$. If $f = \pi_1$, then

$$P\{|\pi_1|\, 1(|\pi_1| > k)\} = \int |x|\, 1(|x| > k)\, \mu_1(dx) \qquad \text{for each } P \in M.$$

Hence, (3) holds since μ_1 is σ-additive and $\int |x|\, \mu_1(dx) < \infty$. Suppose now that

$$f = (\pi_{i+1} - \pi_i)\, g(\pi_1, \dots, \pi_i)$$

for some $1 \le i < n$ and some $g \in B(\mathbb{R}^i, \mathcal{B}(\mathbb{R}^i))$. To prove (3), it can be assumed $\sup|g| \le 1$. Fix $\epsilon > 0$, and take a constant $a > 0$ such that

$$\int |x|\, 1(|x| > a)\, \mu_i(dx) < \epsilon \qquad \text{for each } i \in I.$$

Then, for each $P \in M$, one obtains

$$\begin{aligned}
P\{|f|\, 1(|f| > 2a)\} &\le P\{|\pi_{i+1} - \pi_i|\, 1(|\pi_{i+1} - \pi_i| > 2a)\} \\
&\le P\Big\{\big(|\pi_{i+1}| + |\pi_i|\big)\, \big(1(|\pi_{i+1}| > a) + 1(|\pi_i| > a)\big)\Big\} \\
&= P\{|\pi_i|\, 1(|\pi_i| > a)\} + P\{|\pi_{i+1}|\, 1(|\pi_{i+1}| > a)\} + \\
&\quad + P\{|\pi_{i+1}|\, 1(|\pi_i| > a)\} + P\{|\pi_i|\, 1(|\pi_{i+1}| > a)\}.
\end{aligned}$$

In addition,

$$P\left\{|\pi_i|\,1(|\pi_{i+1}|>a)\right\} \le a\,P(|\pi_{i+1}|>a) + P\left\{|\pi_i|\,1(|\pi_i|>a)\right\}$$
$$\le P\left\{|\pi_{i+1}|\,1(|\pi_{i+1}|>a)\right\} + P\left\{|\pi_i|\,1(|\pi_i|>a)\right\}$$

and similarly

$$P\left\{|\pi_{i+1}|\,1(|\pi_i|>a)\right\} \le P\left\{|\pi_i|\,1(|\pi_i|>a)\right\} + P\left\{|\pi_{i+1}|\,1(|\pi_{i+1}|>a)\right\}.$$

Therefore,

$$P\left\{|f|\,1(|f|>2a)\right\} \le 6 \max_{i\in I} P\left\{|\pi_i|\,1(|\pi_i|>a)\right\}.$$

Finally, since $P\left\{|\pi_i|\,1(|\pi_i|>a)\right\} = \int |x|\,1(|x|>a)\,\mu_i(dx)$ for each $P \in M$, one obtains

$$\sup_{P\in M} P\left\{|f|\,1(|f|>k)\right\} \le 6 \max_{i\in I} \int |x|\,1(|x|>a)\,\mu_i(dx) < 6\,\epsilon$$

whenever $k \ge 2a$. Hence, condition (3) holds and this proves that K_2 is compact. Finally, K_3 is compact since $K_3 = K_1 \cap K_2$. □

Whether or not the sets K_1 and K_3 are non-empty is investigated in Section 4.

Let us turn to problem (iii). Similarly to (i), everything goes smoothly if Γ is replaced by M.

Theorem 3. *Duality always holds with M in the place of Γ, namely*

$$\inf_{Q\in M} Q(c) = \sup_{f_1,\dots,f_n} \sum_{i=1}^n \mu_i(f_i)$$

where sup *is over the n-tuple $(f_1,\dots,f_n)$ such that $f_i \in L_1(\mu_i)$ for each $i \in I$ and $\oplus_{i=1}^n f_i \le c$.*

Proof. If $Q \in M$, $f_i \in L_1(\mu_i)$ for all $i \in I$ and $\oplus_{i=1}^n f_i \le c$, then

$$Q(c) \ge Q\Big(\oplus_{i=1}^n f_i\Big) = \sum_{i=1}^n Q\big(f_i \circ \pi_i\big) = \sum_{i=1}^n \mu_i(f_i).$$

Hence, it suffices to show that

$$\sup_{f_1,\dots,f_n} \sum_{i=1}^n \mu_i(f_i) \ge \inf_{Q\in M} Q(c).$$

Let

$$D = \left\{ \oplus_{i=1}^n f_i : f_i \in B(\Omega_i,\mathcal{A}_i),\, i \in I \right\}.$$

Then, D is a linear subspace of $B(\Omega,\mathcal{A})$ including the constants. If

$$\oplus_{i=1}^n f_i = \oplus_{i=1}^n g_i \qquad \text{with } f_i,\, g_i \in B(\Omega_i,\mathcal{A}_i) \text{ for all } i,$$

then $f_i = g_i + a_i$ for all $i \in I$ where $a_1, \dots, a_n$ are constants satisfying $\sum_{i=1}^n a_i = 0$. Thus,

$$\sum_{i=1}^n \mu_i(f_i) = \sum_{i=1}^n \mu_i(g_i).$$

As a consequence, for each $\oplus_{i=1}^n f_i \in D$, one can define

$$T\left(\oplus_{i=1}^n f_i\right) = \sum_{i=1}^n \mu_i(f_i).$$

Then, $T : D \to \mathbb{R}$ is a linear positive functional such that $T(1) = 1$.

For each integer $k \geq 1$, define

$$D_k = \left\{ f \in D : f \leq c \wedge k \right\}.$$

By Hahn-Banach theorem, T can be extended to a linear positive functional T_k on $B(\Omega, \mathcal{A})$ such that

$$T_k(c \wedge k) = \sup_{f \in D_k} T(f);$$

see e.g. [10, Lemma 2]. Moreover, by standard arguments regarding de Finetti's coherence theory (see e.g. [7, Sect. 2] and [10, Sect. 2]), there is a f.a.p. $P_k \in \mathbb{P}$ such that

$$T_k(f) = \int f\, dP_k \qquad \text{for all } f \in B(\Omega, \mathcal{A}).$$

For $i \in I$ and $B \in \mathcal{A}_i$, one obtains

$$P_k(\pi_i \in B) = T_k\big(1(\pi_i \in B)\big) = T\big(1(\pi_i \in B)\big) = \mu_i(B).$$

Hence, $P_k \in M$. Moreover, letting

$$D_\infty = \left\{ f \in D : f \leq c \right\},$$

one obtains

$$\sup_{f \in D_\infty} T(f) \geq \sup_{f \in D_k} T(f) = \int c \wedge k\, dP_k = P_k(c \wedge k).$$

Next, consider the sequence $(P_k : k \geq 1)$. Since M is compact,

$$P = \lim_\alpha P_{k_\alpha}$$

for some $P \in M$ and some subnet (P_{k_α}) of (P_k). Given an integer $m \geq 1$, there is α^* such that $k_\alpha \geq m$ whenever $\alpha \geq \alpha^*$. Therefore,

$$\sup_{f \in D_\infty} T(f) \geq P_{k_\alpha}(c \wedge k_\alpha) \geq P_{k_\alpha}(c \wedge m) \qquad \text{for each } \alpha \geq \alpha^*.$$

It follows that

$$\sup_{f \in D_\infty} T(f) \geq \lim_\alpha P_{k_\alpha}(c \wedge m) = P(c \wedge m).$$

Since $P \in M$, this in turn implies

$$\sup_{f \in D_\infty} T(f) \geq \sup_m P(c \wedge m) = P(c) \geq \inf_{Q \in M} Q(c).$$

Finally, fix $\epsilon > 0$ and $f_i \in L_1(\mu_i)$ such that $\oplus_{i=1}^n f_i \leq c$. It is easily seen that there are $g_1, \ldots, g_n$ satisfying

$$g_i \in B(\Omega_i, \mathcal{A}_i) \text{ for each } i \in I, \quad \oplus_{i=1}^n g_i \leq c, \quad \sum_{i=1}^n \mu_i(g_i) + \epsilon > \sum_{i=1}^n \mu_i(f_i).$$

Therefore,

$$\sup_{f \in D_\infty} T(f) = \sup_{f_1,\ldots,f_n} \sum_{i=1}^n \mu_i(f_i)$$

where the sup on the right is over the n-tuple $(f_1, \ldots, f_n)$ such that $f_i \in L_1(\mu_i)$ for each $i \in I$ and $\oplus_{i=1}^n f_i \leq c$. This concludes the proof. □

Finally, we focus on problem (ii). Let

$$L = \left\{ \oplus_{i=1}^n f_i : f_i \in L_1(\mu_i), i \in I \right\}.$$

Theorem 4. *Suppose $c \leq f^*$ for some $f^* \in L$. Then, for each $P \in M$,*

$$P(c) = \inf_{Q \in M} Q(c)$$

if and only if

$$\text{There is } f \in L \text{ such that } f \leq c \text{ and } P(c > f + \epsilon) = 0 \text{ for all } \epsilon > 0. \tag{5}$$

It is worth noting that, since $P \in M$ is not necessarily σ-additive, $P(c > f + \epsilon) = 0$ for all $\epsilon > 0$ does not imply $P(c > f) = 0$.

Proof of Theorem 4. First note that, for all $Q \in M$ and $g = \oplus_{i=1}^n g_i \in L$,

$$Q(g) = \sum_{i=1}^n Q(g_i \circ \pi_i) = \sum_{i=1}^n \mu_i(g_i).$$

Hence, for fixed $g \in L$, the map $Q \mapsto Q(g)$ is constant on M. A second (and basic) remark is that, since $c \leq f^*$ for some $f^* \in L$, there is $f : \Omega \to \mathbb{R}$ such that

$$f \in L, \; f \leq c \quad \text{and} \quad Q(f) = \sup_{g \in L,\, g \leq c} Q(g) \quad \text{for each } Q \in M. \tag{6}$$

The existence of such f is a well known result by Kellerer; see Theorem (2.21) of [16].

Having noted these facts, fix $P \in M$. Suppose $P(c) = \inf_{Q \in M} Q(c)$ and take a function f satisfying (6). Then,

$$P(c) = \inf_{Q \in M} Q(c) = \sup_{g \in L,\, g \leq c} P(g) = P(f)$$

where the second equality is due to Theorem 3. Hence, condition (5) follows from $f \leq c$ and $P(c - f) = 0$. Conversely, if f is any function satisfying (5), one obtains

$$P(c) = P(f) = Q(f) \leq Q(c) \qquad \text{for each } Q \in M.$$

□

We recall that a probability measure μ on a measurable space $(S, \mathcal{E})$ is said to be *perfect* if, for each $\mathcal{E}$-measurable function $f : S \to \mathbb{R}$, there is a set $B \in \mathcal{B}(\mathbb{R})$ such that $B \subset f(S)$ and $\mu(f \in B) = 1$. As an example, if S is a separable metric space and $\mathcal{E} = \mathcal{B}(S)$, then μ is perfect if and only if it is tight. We refer to [20] for more on perfect measures.

In the standard framework (i.e., with Γ in the place of M), to settle problems (i)-(ii)-(iii), one needs some conditions. To fix ideas, suppose the Ω_i are separable metric spaces and $\mathcal{A}_i = \mathcal{B}(\Omega_i)$. Then, for the inf in problem (i) to be attained, c should be lower semicontinuous and the μ_i should be perfect. As regards (iii), duality holds if $0 \leq c < \infty$ and the Ω_i are Polish spaces. Moreover, in case the $(\Omega_i, \mathcal{A}_i)$ are arbitrary measurable spaces, duality holds if $c \leq f^*$ for some $f^* \in L$ and all but one the μ_i are perfect. Similarly, as regards problem (ii). An analogous of Theorem 4 holds, with Γ in the place of M, provided all but one the μ_i are perfect. See e.g. [1,6,14,16,19,22,23,25,27].

Thus, in a sense, this Subsection can be summarized by stating that all the above conditions are superfluous if the couplings are allowed to be finitely additive.

3.2. Examples

To point out some features of finitely additive couplings, we give three examples. In each of them, we let $n = 2$.

Example 5. Let $(\Omega_1, \mathcal{A}_1, \mu_1) = (\Omega_2, \mathcal{A}_2, \mu_2) = ([0,1], \mathcal{B}([0,1]), m)$, where m is the Lebesgue measure on $\mathcal{B}([0,1])$, and

$$T(A) = m\{x \in [0,1] : (x,x) \in A\} \qquad \text{for each } A \in \mathcal{B}([0,1]^2).$$

Note that $T \in \Gamma$ and T is supported by the diagonal of $[0,1]^2$.

We first note that there are two f.a.p.'s P and $\widetilde{P}$ on $\mathcal{A}$ such that

$$P, \widetilde{P} \in M \quad \text{and} \quad P(\pi_1 > \pi_2) = \widetilde{P}(\pi_1 < \pi_2) = 1.$$

To prove this fact, denote by $\mathcal{R}$ the field generated by the measurable rectangles $A_1 \times A_2$ with $A_1, A_2 \in \mathcal{B}([0,1])$. It is not hard to see that

$$T(A) = 1 \quad \text{whenever} \quad A \in \mathcal{R} \text{ and } A \supset \{\pi_1 > \pi_2\}. \tag{7}$$

Because of (7), the restriction $T|\mathcal{R}$ can be extended to a f.a.p. P on $\mathcal{A}$ such that $P(\pi_1 > \pi_2) = 1$; see e.g. [11, Theo. 3.3.3]. Moreover, since $T \in \Gamma$ and $P = T$ on $\mathcal{R}$, one obtains $P \in M$. The existence of $\widetilde{P}$ can be proved by the same argument.

Next, define

$$c = 1(\pi_1 \leq \pi_2).$$

In this case, duality holds for Γ. Therefore,

$$\inf_{Q\in\Gamma} Q(c) = \sup_{f_1,f_2} \sum_{i=1}^{2} m(f_i) \le P(c) = P(\pi_1 \le \pi_2) = 0,$$

where sup is over the pairs (f_1, f_2) where f_1 and f_2 are m-integrable functions such that $f_1(x) + f_2(y) \le c(x,y)$ for all $(x,y) \in [0,1]^2$. However, for each $Q \in \Gamma$,

$$Q(\pi_1 - \pi_2) = Q(\pi_1) - Q(\pi_2) = 0 \quad \Rightarrow \quad Q(c) = Q(\pi_1 \le \pi_2) > 0.$$

Hence, the inf in problem (i) is not attained. On the contrary, by Theorem 2, the inf is attained if Γ is replaced by M. In fact,

$$\inf_{Q\in M} Q(c) = 0 = P(c).$$

Next, for all (x,y), define

$$c(x,y) = \infty \text{ if } x < y, \quad c(x,y) = 1 \text{ if } x = y, \quad c(x,y) = 0 \text{ if } x > y.$$

If $Q \in \Gamma$ and $Q(c) < \infty$, then $Q(\pi_1 = \pi_2) = 1$. Therefore, $\inf_{Q\in\Gamma} Q(c) = 1$. However, as shown in [6, Ex. 4.1], duality fails for Γ. In fact, if f_1 and f_2 are m-integrable functions such that $f_1(x) + f_2(y) \le c(x,y)$ for all (x,y), then

$$m(f_1) + m(f_2) = \lim_{\epsilon\to 0} \left\{ \int_{\epsilon}^{1} f_1(x)\,dx + \int_0^{1-\epsilon} f_2(x)\,dx \right\}$$
$$= \lim_{\epsilon\to 0} \int_0^{1-\epsilon} \left\{ f_1(x+\epsilon) + f_2(x) \right\} dx \le \lim_{\epsilon\to 0} \int_0^{1-\epsilon} c(x+\epsilon, x)\,dx = 0.$$

Once again, because of Theorem 3, duality holds if Γ is replaced by M. In fact, since $\{c = 0\} = \{\pi_1 > \pi_2\}$, one obtains

$$\sup_{f_1,f_2} \sum_{i=1}^{2} m(f_i) = 0 = P(c) = \inf_{Q\in M} Q(c).$$

Example 6. Let again $(\Omega_1, \mathcal{A}_1, \mu_1) = (\Omega_2, \mathcal{A}_2, \mu_2) = ([0,1], \mathcal{B}([0,1]), m)$. Define T, P and $\widetilde{P}$ as in Example 5 and

$$K = \left\{ Q \in M : (\pi_1, \pi_2) \text{ is a } Q\text{-martingale} \right\}.$$

Then, P and $\widetilde{P}$ belong to K. In fact, if $f(x,y) = g(x)(y-x)$ for all (x,y) and some bounded Borel function $g : [0,1] \to \mathbb{R}$, then

$$|P(f)| \le P(|f|) \le \sup|g|\, P(|\pi_2 - \pi_1|)$$
$$= \sup|g|\, P(\pi_1 - \pi_2) = \sup|g| \left\{ P(\pi_1) - P(\pi_2) \right\} = 0.$$

Hence $P(f) = 0$, and similarly $\widetilde{P}(f) = 0$.

The fact that $P, \widetilde{P} \in K$ suggests two (related) remarks.

- K contains various elements in addition to T. For instance,

$$a\,P + b\,\widetilde{P} + (1 - a - b)T \in K$$

whenever $a, b \geq 0$ and $a + b \leq 1$. On the contrary, the only member of $K \cap \Gamma$ is T.

- Let $c = 1(\pi_1 = \pi_2)$. Arguing as in [5, Ex. 8.1], it can be shown that

$$m(f_1) + m(f_2) \leq 0$$

whenever f_1 and f_2 are m-integrable and satisfy

$$f_1(x) + f_2(y) \leq c(x, y) - g(x)(y - x) \tag{8}$$

for all (x, y) and some bounded Borel function g. Since $K \cap \Gamma = \{T\}$, one obtains

$$\inf_{Q \in K \cap \Gamma} Q(c) = T(c) = 1 > 0 = \sup_{f_1, f_2} \big\{m(f_1) + m(f_2)\big\}$$

where sup is over the pairs (f_1, f_2) of m-integrable functions satisfying condition (8). Hence, in the standard framework, there is a duality gap in the martingale problem. To avoid this gap, a suitable almost sure formulation of the dual problem is to be adopted; see [5] again. On the contrary, the gap does not arise if Γ is replaced by M (so that $K \cap \Gamma$ is replaced by K). In fact,

$$0 \leq \inf_{Q \in K} Q(c) \leq P(c) = P(\pi_1 = \pi_2) = 0.$$

A conjecture is that, with M in the place of Γ, the absence of duality gap in the martingale problem is a general fact, and not a lucky feature of this example. Another conjecture is that some of the stability results, which fail in the standard framework when $\Omega_i = \mathbb{R}^d$, are valid in a finitely additive setting; see [2,4,12].

Example 7. For each $i \in I$, let $\mathcal{F}_i$ be a field such that $\mathcal{A}_i = \sigma(\mathcal{F}_i)$. In this example, M is defined as

$$M = \big\{P \in \mathbb{P} : P \circ \pi_i^{-1} = \mu_i \text{ on } \mathcal{F}_i \text{ for all } i \in I\big\}. \tag{9}$$

If P is a f.a.p., $P \circ \pi_i^{-1} = \mu_i$ on $\mathcal{F}_i$ does not imply $P \circ \pi_i^{-1} = \mu_i$ on $\mathcal{A}_i$. Therefore, this example is not completely consistent with the rest of the paper.

Given a measurable space $(S, \mathcal{E})$, with $\mathcal{E}$ including the singletons, let $\mathcal{X} = S^\infty$ be the set of S-valued sequences $x = (x_1, x_2, \ldots)$. Let $X_n : \mathcal{X} \to S$ be the n-th canonical projection on $\mathcal{X}$ and

$$\mathcal{D} = \sigma(X_1, X_2, \ldots), \quad \mathcal{T} = \cap_n \sigma(X_n, X_{n+1}, \ldots), \quad G = \big\{(x, y) \in \mathcal{X}^2 : x = y \text{ eventually}\big\}.$$

Both $\mathcal{D}$ and $\mathcal{T}$ are σ-fields over $\mathcal{X}$, usually called the *product σ-field* and the *tail σ-field*, respectively. Moreover, G is the graph of the equivalence relation on $\mathcal{X}$ defined by

$$x \sim y \quad \Leftrightarrow \quad x \text{ and } y \text{ belong to the same atom of } \mathcal{T}.$$

Define $(\Omega_1, \mathcal{A}_1) = (\Omega_2, \mathcal{A}_2) = (\mathcal{X}, \mathcal{D})$ and take μ_1 and μ_2 such that

$$\mu_1(A) = \mu_2(A^c) = 1 \qquad \text{for some } A \in \mathcal{T}.$$

Since $A \in \mathcal{T}$, if $(x, y) \in G$ and $x \in A$, then $y \in A$. Moreover, each $Q \in \Gamma$ satisfies $Q\big(A \times \mathcal{X}\big) = 1$ and $Q\big(\mathcal{X} \times A\big) = 0$. Hence,

$$Q(G) = Q\big(G \cap (A \times \mathcal{X})\big) = Q\big(G \cap (A \times A)\big) = 0 \qquad \text{for each } Q \in \Gamma.$$

A finite dimensional cylinder is a subset $C \subset \mathcal{X}$ of the form

$$C = \big\{x \in \mathcal{X} : (x_1, \ldots, x_n) \in B\big\} \qquad \text{for some } n \geq 1 \text{ and some } B \in \mathcal{E}^n.$$

Let $\mathcal{F}$ be the field over $\mathcal{X}$ consisting of all finite dimensional cylinders and $\mathcal{S}$ the field over $\mathcal{X}^2$ generated by $C_1 \times C_2$ for all $C_1, C_2 \in \mathcal{F}$. It is straightforward to verify that

$$H \in \mathcal{S} \text{ and } H \supset G \quad \Rightarrow \quad H = \mathcal{X}^2. \tag{10}$$

Fix $Q \in \Gamma$ and denote by $Q_0 = Q|\mathcal{S}$ the restriction of Q on $\mathcal{S}$. Because of (10), the Q_0-outer measure of G is 1. Hence, Q_0 can be extended to a f.a.p. P on $\mathcal{A}$ such that $P(G) = 1$; see e.g. [11, Theo. 3.3.3]. Such a P has marginals μ_1 and μ_2 on $\mathcal{F}$ and satisfies $P(G) = 1$.

Finally, let $c = 1 - 1_G$ and define M by (9) with $n = 2$ and $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$. As proved above, $Q(G) = 0$ for each $Q \in \Gamma$ while there is $P \in M$ such that $P(G) = 1$. Therefore,

$$\inf_{Q \in \Gamma} Q(c) = \inf_{Q \in \Gamma} (1 - Q(G)) = 1 \quad \text{and} \quad \inf_{P \in M} P(c) = \inf_{P \in M} (1 - P(G)) = 0.$$

Once again, we come across a striking difference between σ-additive and finitely additive couplings.

4. Existence of f.a.p.'s and probability measures satisfying certain conditions

For Theorem 2 to apply, certain collections K_j of f.a.p.'s should be non-empty. We now give conditions for $K_j \neq \emptyset$. By the same argument, we also obtain conditions for the existence of certain probability measures. This section is based on ideas from [9, Sect. 3] which in turn was inspired by Strassen [26].

Let $\mathcal{U} \subset \mathbb{P}$ be any collection of f.a.p.'s on $\mathcal{A}$. If $P \in M \cap \mathcal{U}$ and $f_i \in B(\Omega_i, \mathcal{A}_i)$ for each $i \in I$, one trivially obtains

$$\sum_{i=1}^{n} \mu_i(f_i) = P\Big(\oplus_{i=1}^n f_i\Big) \geq \inf_{Q \in \mathcal{U}} Q\Big(\oplus_{i=1}^n f_i\Big)$$

where the equality is due to $P \in M$ and the inequality to $P \in \mathcal{U}$. Hence,

$$\sum_{i=1}^{n} \mu_i(f_i) \geq \inf_{Q \in \mathcal{U}} Q\Big(\oplus_{i=1}^n f_i\Big) \qquad \text{whenever } f_i \in B(\Omega_i, \mathcal{A}_i) \text{ for all } i \in I. \tag{11}$$

Our next goal is proving that, for some choices of $\mathcal{U}$, condition (11) is actually equivalent to $M \cap \mathcal{U} \neq \emptyset$. In what follows, we adopt the usual convention

$$\inf \emptyset = \infty.$$

Therefore, condition (11) is trivially false if $\mathcal{U} = \emptyset$.

Theorem 8. *Let $P^* \in \mathbb{P}$ and F a collection of $\mathcal{A}$-measurable functions $f : \Omega \to \mathbb{R}$. If $\mathcal{U} = \{Q \in \mathbb{P} : Q \ll P^*\}$, condition (11) is equivalent to $M \cap \mathcal{U} \neq \emptyset$. Moreover, if*

$$\mathcal{U} = \{Q \in \mathbb{P} : Q \ll P^*, Q(|f|) < \infty \text{ and } Q(f) = 0 \text{ for each } f \in F\},$$

then (11) is equivalent to $M \cap \mathcal{U} \neq \emptyset$ provided

$$\lim_k \sup_{Q \in \mathcal{U}} Q\{|f|\, 1(|f| > k)\} = 0 \quad \text{for all } f \in F. \tag{12}$$

Proof. Let $\mathcal{U} \subset \mathbb{P}$. As proved above, condition (11) holds if $M \cap \mathcal{U} \neq \emptyset$. Hence, suppose condition (11) holds. As in the proof of Theorem 3, define D to be the collection of functions f of the form $f = \oplus_{i=1}^n f_i$ where $f_i \in B(\Omega_i, \mathcal{A}_i)$ for each $i \in I$. For $f = \oplus_{i=1}^n f_i \in D$, let

$$X_f(Q) = \sum_{i=1}^n \mu_i(f_i) - Q(f) \qquad \text{for all } Q \in \mathcal{U}.$$

Then, X_f is a bounded function on $\mathcal{U}$ and $\{X_f : f \in D\}$ is a linear space. Moreover, for fixed $f \in D$, condition (11) yields

$$\sup_{Q \in \mathcal{U}} X_f(Q) = \sum_{i=1}^n \mu_i(f_i) - \inf_{Q \in \mathcal{U}} Q(f) \geq 0.$$

In turn, the above condition implies the existence of a f.a.p. $P_{\mathcal{U}}$, defined on the collection of all subsets of $\mathcal{U}$, such that

$$\int X_f(Q)\, P_{\mathcal{U}}(dQ) = 0 \qquad \text{for each } f \in D$$

or equivalently

$$\int Q\left(\oplus_{i=1}^n f_i\right) P_{\mathcal{U}}(dQ) = \sum_{i=1}^n \mu_i(f_i) \quad \text{if } f_i \in B(\Omega_i, \mathcal{A}_i) \text{ for all } i \in I; \tag{13}$$

see [10, Sect. 2]. Using $P_{\mathcal{U}}$, define

$$P(A) = \int Q(A)\, P_{\mathcal{U}}(dQ) \qquad \text{for all } A \in \mathcal{A}.$$

Then, P is a f.a.p. on $\mathcal{A}$. For each $i \in I$ and each $B \in \mathcal{A}_i$, equation (13) yields

$$P(\pi_i \in B) = \int Q(\pi_i \in B)\, P_{\mathcal{U}}(dQ) = \mu_i(B).$$

Therefore, $P \in M$. In addition, $P \in \mathcal{U}$ if $\mathcal{U} = \{Q \in \mathbb{P} : Q \ll P^*\}$. In this case, in fact, $Q(A) = 0$ for each $Q \in \mathcal{U}$ whenever $A \in \mathcal{A}$ and $P^*(A) = 0$. Hence, one trivially obtains $P \ll P^*$.

It remains to see that $P \in \mathcal{U}$ if condition (12) holds and

$$\mathcal{U} = \{Q \in \mathbb{P} : Q \ll P^*,\ Q(|f|) < \infty \text{ and } Q(f) = 0 \text{ for each } f \in F\}.$$

Arguing as above, it is obvious that $P \ll P^*$. Hence, we have to see that $P(|f|) < \infty$ and $P(f) = 0$ for each $f \in F$. If $g \in B(\Omega, \mathcal{A})$, since g is the uniform limit of a sequence g_k of simple functions, one obtains

$$P(g) = \lim_k P(g_k) = \lim_k \int Q(g_k)\, P_{\mathcal{U}}(dQ) = \int Q(g)\, P_{\mathcal{U}}(dQ).$$

Having noted this fact, fix $f \in F$. For each integer $j \geq 1$,

$$P(|f|) \leq j + P\big\{|f|\,1(|f| > j)\big\} = j + \sup_k P\big\{|f| \wedge k\,1(|f| > j)\big\}$$

$$= j + \sup_k \int Q\big\{|f| \wedge k\,1(|f| > j)\big\}\, P_{\mathcal{U}}(dQ) \leq j + \sup_{Q \in \mathcal{U}} Q\big\{|f|\,1(|f| > j)\big\}.$$

Hence, $P(|f|) < \infty$ follows from condition (12). Similarly,

$$P(f) = \lim_k P\big\{f\,1(|f| \leq k)\big\} = \lim_k \int Q\big\{f\,1(|f| \leq k)\big\}\, P_{\mathcal{U}}(dQ).$$

For every $Q \in \mathcal{U}$, since $Q(f) = 0$, one obtains

$$Q\big\{f\,1(|f| \leq k)\big\} = -Q\big\{f\,1(|f| > k)\big\}.$$

Therefore, condition (12) implies again

$$P(f) = \lim_k \int Q\big\{-f\,1(|f| > k)\big\}\, P_{\mathcal{U}}(dQ) \leq \lim_k \sup_{Q \in \mathcal{U}} Q\big\{|f|\,1(|f| > k)\big\} = 0.$$

This proves that $P(f) \leq 0$. Replacing f with $-f$, one also obtains $P(f) \geq 0$. Thus, $P(f) = 0$ and this concludes the proof. □

Theorem 8 applies to martingale transport.

Corollary 9. *Let $P^* \in \mathbb{P}$ and $(\Omega_i, \mathcal{A}_i) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for all $i \in I$. Suppose that $P^*(A) = 1$ for some compact set $A \subset \mathbb{R}^n$. Then, condition (11) is equivalent to $M \cap \mathcal{U} \neq \emptyset$ where*

$$\mathcal{U} = \big\{Q \in \mathbb{P} : Q \ll P^* \text{ and } (\pi_1, \ldots, \pi_n) \text{ is a } Q\text{-martingale}\big\}.$$

Proof. The collection $\mathcal{U}$ can be written as in the second part of Theorem 8 with $F = H$, where H has been introduced in Subsection 2.2. Hence, it suffices to check condition (12) with $F = H$. Let

$$B = \bigcup_{i=1}^n \pi_i(A) = \big\{\pi_i(x) : i \in I,\, x \in A\big\}.$$

Then, B is a compact subset of $\mathbb{R}$ and

$$P^*\big(\pi_i \in B \text{ for each } i \in I\big) \geq P^*(A) = 1.$$

Take k such that $P^*\big(\max_i |\pi_i| \leq k\big) = 1$. Since $Q \ll P^*$ for each $Q \in \mathcal{U}$,

$$Q(|\pi_1| > k) = Q(|\pi_{i+1} - \pi_i| > 2k) = 0 \qquad \text{for all } Q \in \mathcal{U} \text{ and } 1 \leq i < n.$$

Fix now $f \in H$. If $f = \pi_1$, then $Q\{|\pi_1|\,1(|\pi_1| > k)\} = 0$ for each $Q \in \mathcal{U}$. If

$$f = (\pi_{i+1} - \pi_i)\, g(\pi_1,\dots,\pi_i),$$

for some i and $g \in B(\mathbb{R}^i, \mathcal{B}(\mathbb{R}^i))$, take an integer $j \geq 2\,k\ \sup|g|$ and note that

$$Q\{|f|\,1(|f| > j)\} \leq \sup|g|\, Q\{|\pi_{i+1} - \pi_i|\,1(|\pi_{i+1} - \pi_i| > 2k)\} = 0$$

for each $Q \in \mathcal{U}$. Hence, condition (12) holds. □

The same argument used for Theorem 8 and Corollary 9 can be applied to prove the existence of an absolutely continuous, martingale, *probability measure* with given marginals. More precisely, our last result provides conditions for the existence of $P \in \Gamma$ (so that P is a probability measure on $\mathcal{A}$ with marginals $\mu_1,\dots,\mu_n$) such that $P \ll P^*$ and $(\pi_1,\dots,\pi_n)$ is a P-martingale.

Theorem 10. *Let P^* be a probability measure on $\mathcal{A}$ and $(\Omega_i, \mathcal{A}_i) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for all $i \in I$. Suppose P^* has discrete marginals, except possibly one, and $P^*(A) = 1$ for some compact $A \subset \mathbb{R}^n$. Then, condition* (11) *is equivalent to $\Gamma \cap \mathcal{U} \neq \emptyset$ where*

$$\mathcal{U} = \{Q \in \mathbb{P} : Q \ll P^* \text{ and } (\pi_1,\dots,\pi_n) \text{ is a } Q\text{-martingale}\}.$$

Proof. If $P \in \Gamma \cap \mathcal{U}$, then $P \in M \cap \mathcal{U}$, and condition (11) follows exactly as above. Conversely, assume condition (11). Let us adopt the same notation as in the proof of Theorem 8. Arguing as in such a proof, because of (11), there is a f.a.p. $P_{\mathcal{U}}$, defined on the collection of all subsets of $\mathcal{U}$, satisfying equation (13).

Let $\mathcal{R}$ be the field on Ω generated by the measurable rectangles $A_1 \times \dots \times A_n$, where $A_i \in \mathcal{A}_i$ for each $i \in I$. Define

$$P_0(A) = \int Q(A)\, P_{\mathcal{U}}(dQ) \qquad \text{for all } A \in \mathcal{R}.$$

(Note that P_0 has been defined only on $\mathcal{R}$, not on all of $\mathcal{A}$). Because of (13),

$$P_0(\pi_i \in B) = \mu_i(B) \qquad \text{for all } i \in I \text{ and } B \in \mathcal{A}_i.$$

Hence, $\mu_1,\dots,\mu_n$ are the marginals of P_0. Since $\mu_1,\dots,\mu_n$ are all σ-additive and perfect, it follows that P_0 is σ-additive as well; see e.g. [21]. Let P be the (only) σ-additive extension of P_0 to $\mathcal{A} = \sigma(\mathcal{R})$. Then, P is a probability measure on $\mathcal{A}$ with marginals $\mu_1,\dots,\mu_n$, namely, $P \in \Gamma$. Furthermore,

$$P(A) = P_0(A) = 0 \quad \text{whenever} \quad A \in \mathcal{R} \text{ and } P^*(A) = 0. \tag{14}$$

By Lemma 4 of [9], since P^* is a probability measure and all but one its marginals are discrete, condition (14) implies $P \ll P^*$.

It remains to see that $(\pi_1,\dots,\pi_n)$ is a P-martingale, namely, $P(|f|) < \infty$ and $P(f) = 0$ for all $f \in H$ where H has been introduced in Subsection 2.2. Since P^* has compact support, there is k such that $P^*(\max_i|\pi_i| \leq k) = 1$. Since $P \ll P^*$ and $Q \ll P^*$ for each $Q \in \mathcal{U}$,

$$P(\max_i|\pi_i| \leq k) = Q(\max_i|\pi_i| \leq k) = 1 \qquad \text{for each } Q \in \mathcal{U}.$$

In particular, $P(|f|) < \infty$ for all $f \in H$. Let $C(\mathcal{R})$ be the class of those functions $f : \Omega \to \mathbb{R}$ such that $f_k \to f$ *uniformly* for some sequence f_k of $\mathcal{R}$-simple functions. (Such functions are called $\mathcal{R}$-continuous in [11]). If $f \in C(\mathcal{R})$,

$$P(f) = \lim_k P(f_k) = \lim_k P_0(f_k) = \lim_k \int Q(f_k)\, P_{\mathcal{U}}(dQ) = \int Q(f)\, P_{\mathcal{U}}(dQ)$$

where f_k is any sequence of $\mathcal{R}$-simple functions such that $f_k \to f$ uniformly. Since $\pi_1\, 1(|\pi_1| \le k) \in C(\mathcal{R})$ and $Q(\pi_1) = 0$ for all $Q \in \mathcal{U}$, it follows that

$$P(\pi_1) = P\{\pi_1\, 1(|\pi_1| \le k)\} = \int Q\{\pi_1\, 1(|\pi_1| \le k)\}\, P_{\mathcal{U}}(dQ) = \int Q(\pi_1)\, P_{\mathcal{U}}(dQ) = 0.$$

Suppose now that f is of the form

$$f = (\pi_{i+1} - \pi_i)\, 1(\pi_1 \in A_1) \ldots 1(\pi_i \in A_i), \tag{15}$$

where $1 \le i < n$ and $A_j \in \mathcal{A}_j$ for $j = 1, \ldots, i$. On noting that

$$\tilde{f} := f\, 1(|\pi_i| \le k)\, 1(|\pi_{i+1}| \le k) \in C(\mathcal{R}),$$

one obtains

$$P(f) = P(\tilde{f}) = \int Q(\tilde{f})\, P_{\mathcal{U}}(dQ) = \int Q(f)\, P_{\mathcal{U}}(dQ) = 0.$$

To sum up, $P(\pi_1) = 0$ and $P(f) = 0$ for all f as in (15). Since P is a probability measure, this implies $P(f) = 0$ for all $f \in H$. □

We conclude this paper with two remarks.

Firstly, it would be nice to have an analogous of Theorem 10, as well as of the other results of this paper, with $P \sim P^*$ in the place of $P \ll P^*$, where $P \sim P^*$ means $P \ll P^*$ and $P^* \ll P$. To this end, however, the techniques of this paper seem not to work. A few partial results are in [7–9].

Secondly, the class $\mathcal{U}$ involved in Corollary 9 and Theorem 10 may be empty. Let us consider the case where P^* is a probability measure. Then, by [13], one obtains $\mathcal{U} \neq \emptyset$ if P^* satisfies the *non-arbitrage* condition

$$P^*(f \ge 0) = 1 \quad \Rightarrow \quad P^*(f = 0) = 1 \qquad \text{for each } f \text{ in the linear span of } H.$$

The non-arbitrage condition, however, is stronger than $\mathcal{U} \neq \emptyset$. In fact, it implies the existence of a probability measure P on $\mathcal{A}$ such that $P \sim P^*$ and $(\pi_1, \ldots, \pi_n)$ is a P-martingale. Assuming P^* has compact support, a characterization of $\mathcal{U} \neq \emptyset$ is provided by [7, Theo. 3]. According to the latter, $\mathcal{U} \neq \emptyset$ if and only if

$$P^*(f > a) > 0 \qquad \text{for all } a < 0 \text{ and all } f \text{ in the linear span of } H.$$

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