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ON THE PARTIAL AND MICROLOCAL REGULARITY FOR GENERALIZED MÉTIVIER OPERATORS

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ABSTRACT. The partial and the microlocal regularity are provided via L^2-estimate and via FBI-transform respectively, for the following generalization of the Métivier operator, [26],

$$D_x^2 + \left(x^{2n+1}D_y\right)^2 + \left(x^n y^m D_y\right)^2 ,$$

in Ω open neighborhood of the origin in $\mathbb{R}^2$, where n and m are positive integers.

1. INTRODUCTION

In [26] G. Métivier studied the hypoellipticity for a class of second order partial differential operators with analytic coefficients in Ω, open neighborhood of the origin in $\mathbb{R}^n$, whose principal symbol vanishes exactly of order two on a submanifold of $T^*\Omega$. In the case of sum of squares of vector fields the most representative model of such class is the following

$$P_M = D_x^2 + x^2 D_y^2 + \left(yD_y\right)^2 . \tag{1.1}$$

Métivier proves that the operator (1.1) is G^2-hypoelliptic and not better at the origin.
In [17] we studied the hypoellipticity of the following generalization of the Métivier operator

$$M_{n,m} = \sum_{j=1}^{3} X_j^2 = D_x^2 + \left(x^{2n+1}D_y\right)^2 + \left(x^n y^m D_y\right)^2 , \tag{1.2}$$

in Ω open neighborhood of the origin. Here n and m are positive integers. In [17] we show that the operator $M_{n,m}$ is $G^{\frac{2m}{2m-1}}$-hypoelliptic and not better in any neighborhood of the origin.
Throughout the paper we need the definition of Gevrey regularity as well as of partial Gevrey regularity:

Definition 1.1. *A smooth function f defined in U, open subset of $\mathbb{R}^n$, belongs to $G^s(U)$, $s \geq 1$, the class of Gevrey functions of order s in U, if for every compact set $K \Subset U$ there is a positive constant C_K such that*

$$|D^\alpha f(x)| \leq C_K^{|\alpha|+1} |\alpha|^{s|\alpha|}, \qquad \forall\, \alpha \in \mathbb{Z}_+^n \text{ and } \forall\, x \in K.$$

When $s = 1$ we shall say that u is analytic in U, $u \in C^\omega(U)$.

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The function $f(x_0, x_1, \dots, x_n)$ *belongs to the anisotropic Gevrey class* $G^{(\alpha_0,\alpha_1,\dots,\alpha_n)}(U)$ *at the point* x_0 *if there exists a neighborhood,* U*, of* x_0 *and a constant* C_f *such that for all multi-indices* β

$$|D^\beta f| \leq C_f^{|\beta|+1} \beta!^\alpha \quad \text{in } U,$$

where $\beta!^\alpha = \beta_0!^{\alpha_0} \beta_1!^{\alpha_1} \dots \beta_n!^{\alpha_n}$.

Definition 1.2. *A differential operator* P *is said to be* G^s*-hypoelliptic,* $s \geq 1$*, in* Ω*, open subset of* $\mathbb{R}^n$*, if for any* Ω_1 *open subset of* Ω*, the conditions* $u \in \mathscr{D}'(\Omega_1)$ *and* $Pu \in G^s(\Omega_1)$ *imply that* $u \in G^s(\Omega_1)$*. When* $s = 1$ *we shall say that* P *is analytic hypoelliptic.*

Definition 1.3. *A sum of squares operator*

$$P = \sum_{j=1}^{m} (X_j(x, D))^2, \tag{1.3}$$

where X_j *are vector fields in* Ω*, open subset of* $\mathbb{R}^n$*, with real-valued real analytic coefficients, satisfies the Hörmander condition if the Lie algebra generated by the vector fields* X_j *and their commutators has the dimension equal to the dimension of the ambient space, that is* n*.*

The operator $M_{n,m}$ satisfies the Hörmander condition.
Furthermore by the results in [23],[28] and [5] the following *subelliptic* a priori estimate holds:

$$\|u\|^2_{1/2(n+1)} + \sum_{j=1}^{3} \|X_j u\|^2 \leq C\big(|\langle M_{n,m}u, u\rangle| + \|u\|^2\big), \tag{1.4}$$

Here $u \in C_0^\infty(\Omega)$, $\|\cdot\|_0$ denotes the norm in $L^2(\Omega)$ and $\|\cdot\|_s$ the Sobolev norm of order s in Ω.
The study of these models is motivated by the problem of the analytic hypoellipticity sum of squares of vector fields with real-valued real analytic coefficients, satisfying the Hörmander condition.
A famous example, useful to better understand the complexity of the problem, is due to Baouendi and Goulaouic, [4]. They studied the operator $P_{BG} = D_1^2 + D_2^2 + x_1^2 D_3^2$, in $\mathbb{R}^3$, showing that it is G^2-hypoelliptic and no better in any neighborhood of the origin.
Both P_{BG} and P_M have non symplectic characteristic varieties, but in the case of the Baouendi-Goulaouic operator the Hamilton (bicharacteristic) leaf lies on the base of the cotangent bundle, whereas, in the case of the Métivier operator, the Hamilton leaf lies along the fiber of the cotangent bundle.
In 1996 [31] Treves suggested that the analytic hypoellipticity could depend on suitable geometrical properties of the characteristic variety, and he proposed to use the Poisson-Treves stratification (see [32] and [13]), formulating the following conjecture:

Conjecture: Let P be as in (1.3). Then P is analytic hypoelliptic if and only if every Poisson stratum of the characteristic variety is symplectic.

Recently in [3] and in [8] it has been shown that the sufficient part of the Treves conjecture does not hold neither locally nor microlocally in dimension greater or equal to 4. Moreover it is believed that it does not hold also in dimension 3.
The results in [3] and in [8] have de facto completely reopened the problem of the analytic hypoellipticity of operators of the form (1.3).

In dimension 2 no counterexamples are known, but it is believed that the Treves conjecture holds.
For completeness we recall that F. Treves, in [32], has also stated a conjecture in the global case: P is globally analytic hypoelliptic on Ω, compact analytic manifold without boundary and countable at infinity, if and only if every Poisson stratum of the characteristic variety has the following property: the closure in $T^*\Omega$ of every bicharacteristic leaf is compact.
We point out that the scenario can be very different moving from the local to the global case see [20], [7], [16], [18].
We emphasize that at present, contrary to what happens in the local case, the global Treves conjecture has not yet been proved or disproved.
In order to try to better understand what happens in dimension 2, we continue the study started in [17], and in this paper we analyze the partial and microlocal regularity of the operator (1.2).
In [12], Bove and Tartakoff studied the partial regularity of the Baouendi-Goulaouic operator and more in general of the Oleĭnik-Radkevič operator, [27], $P_{OR} = D_t^2 + t^{2(p-1)}D_x^2 + t^{2(q-1)}D_y^2$, where $1 < p \leq q$ and $p, q \in \mathbb{N}$. They prove that P_{OR} is $G^{q/p}$-hypoelliptic and moreover that if u solves the problem $P_{OR}u = f$, f analytic, then $u \in G^{(s_0,s_1,s_2)}$, where $s_0 \geq 1 - \frac{1}{q} + \frac{1}{p}$, $s_1 \geq 1$ and $s_2 \geq \frac{q}{p}$; their result is sharp (for generalization of this result in dimension greater then three see [14], [15].)
We state our result about the partial regularity of the operator (1.2).

Theorem 1.1. *Let u be a solution to the equation $M_{n,m}u = f$, $n \geq 1$, f analytic. Then $u \in G^{(s_0,s_1)}$ where $s_1 = \frac{2m}{2m-1}$ and $s_0 = 1 + \frac{1}{(n+1)(2m-1)}$.*

Remark 1.1. *We point out that in the above Theorem, we may assume that f belongs to $G^{(s_0,s_1)}$, moreover by Theorem 3.1 in* [25] *we deduce that $u \in G^{(s_0,s_1)}$ with $s_1 \geq \frac{2m}{2m-1}$ and $s_0 \geq 1 + \frac{1}{(n+1)(2m-1)}$.*

Remark 1.2. *The Gevrey regularity index, $\frac{2m}{2m-1}$, is in accordance with that obtained in* [11] *and in* [9].
In the case of the Métivier operator (1.1), using the same strategy adopted in the proof of the Theorem 1.1, we have that the solutions u to the problem $P_M u = f$, f analytic, belong to $G^{(s_0,s_1)}$, where $s_1 = 2$ and $s_0 = \frac{3}{2}$.
We notice that this partial regularity with respect to x is better if compared to that obtained in Theorem 1.1. This fact will be clarified at the end of the proof of the theorem, see Remark 2.2.

Concerning the microlocal regularity, the operator $M_{n,m}$ has the following characteristic set

$$\mathrm{Char}(M_{n,m}) = \{(x, y, \xi, \eta) \in T^*\mathbb{R}^2 \setminus \{0\} : x = 0 = \xi, \text{ and } \eta \neq 0\},$$

the related *Poisson-Treves* stratification is given by

$$\Sigma_0 = \{(x, y, \xi, \eta) \in T^*\mathbb{R}^2 \setminus \{0\} : x = 0 = \xi, y \neq 0, \eta \neq 0\},$$
$$\Sigma_1 = \{(x, y, \xi, \eta) \in T^*\mathbb{R}^2 \setminus \{0\} : x = 0 = \xi, y = 0, \eta \neq 0\}.$$

From the point of view of the microlocal regularity, we recall the definition of the Gevrey (analytic) wave front set of a distribution (function).
First of all we define the cutoff functions that are used for both the Gevrey and the analytic case defined e.g. in Ehrenpreis [21] (see also Hörmander [24]).

Definition 1.4. *For any N natural number, denote by $\phi_N = \phi_N(y)$ a function in $C_0^\infty(\mathbb{R}^m)$. We say that ϕ_N is an Ehrenpreis sequence of cutoff functions if there is a positive constant R such that for $|\alpha| \leq RN$ we have, for every N*

$$|\partial_y^\alpha \phi_N(y)| \leq C_\phi^{|\alpha|+1} N^{|\alpha|},$$

where $C_\phi > 0$ and independent of N.

Next we define the Gevrey and analytic wave front set (see [24].) Definition 3.1 gives an equivalent definition more suitable when working with the FBI transform.

Definition 1.5. *Let $x_0 \in U$ and $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ and $u \in \mathscr{D}'(U)$. Let $s \geq 1$. We say that the point $(x_0, \xi_0) \notin WF_s(u)$ if and only if there is an open neighborhood U' of x_0 and a conic neighborhood Γ of ξ_0 and a sequence of Ehrenpreis cutoff functions identically equal to 1 on U' such that*

$$|\widehat{\phi_N u}(\xi)| \leq C^{N+1} N^{Ns} |\xi|^{-N}, \quad N = 1, 2, \dots \tag{1.5}$$

for every $\xi \in \Gamma$ and for a suitable positive constant C independent of N.

Theorem 1.2. *Let u be a solution of the problem $M_{n,m}u = f$, then If $\rho_1 \in \Sigma_1$ and $\rho_1 \notin WF_{s_0}(f)$ then $\rho_1 \notin WF_{s_0}(u)$, where $s_0 = \frac{2m}{2m-1}$, if $\rho_0 \in \Sigma_0$ and $\rho_0 \notin WF_a(f)$ then $\rho_0 \notin WF_a(u)$.*

We remark that the microlocal regularity obtained for the operator $M_{n,m}$ remains the same if we perturb the operator adding a pseudodifferential operator of order less then $(2n+2)^{-1}$, this is a consequence of the result in [6].
The partial regularity of the solutions of the problem $M_{n,m}u = f$, $f \in C^\omega(\Omega)$, will be obtained using the same technique as in [12], more precisely we will estimate a suitable localization of a high derivative in the direction D_y and D_x iteratively using the subelliptic estimate (1.4); the microlocal regularity of the points in Σ_0 will be obtained via FBI-technique (FBI=Fourier-Bros-Iagolnitzer), taking advantage of a result in [2].

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2. Partial regularity, proof of the Theorem 1.1

2.1. **Regularity with respect y-direction.** Let $\psi_N(x, y)$ be an Ehrenpreis cutoff sequence. Since in the region $x \neq 0$ the operator $M_{n,m}$ is elliptic we may assume that ψ_N is independent of the variable x, since in that region analytic regularity is expected.
In view of [9] we have the estimate

$$\|\psi_k D_y^k u\|_{\frac{1}{2(n+1)}} \leq C^{k+1} k^{s_1 k}, \tag{2.1}$$

where $s_1 = \frac{2m}{2m-1}$ and C is a suitable positive constant independent of k.

2.2. **Regularity with respect x-direction.** Let ψ_N be an Ehrenpreis-Hörmander cut-off sequence as in the previous paragraph. We replace u by $\psi_N(y) D_x^k u$ in (1.4). We have

$$\|\psi_N D_x^k u\|_\varepsilon^2 + \sum_{j=1}^{3} \|X_j \psi_N D_x^k u\|_0^2 \leq C \left(|\langle M \psi_N D_x^k u, \psi_N D_x^k u \rangle| + \|\psi_N D_x^k u\|_0^2 \right), \tag{2.2}$$

where we set $\frac{1}{2(n+1)} = \varepsilon$ and $M_{n,m} = M$.
We start by treating the last term on the right hand side. Since it can be rewritten as $\|X_1 \psi_N D_x^{k-1} u\|_0^2$, we can use the subelliptic estimate and restart the process. More precisely, we have

$$\begin{aligned} C\|\psi_N D_x^k u\|_0^2 = C\|X_1 \psi_N D_x^{k-1} u\|_0^2 \leq C \sum_{j=1}^{3} \|X_j \psi_N D_x^{k-1} u\|_0^2 \\ \leq C^2 \left(|\langle M \psi_N D_x^{k-1} u, \psi_N D_x^{k-1} u\rangle| + \|\psi_N D_x^{k-1} u\|_0^2 \right). \end{aligned}$$

Now, focusing on the last term on the right hand side and using the same strategy adopted above, it can be estimate by

$$\begin{aligned} C^2\|\psi_N D_x^{k-1} u\|_0^2 = C^2\|X_1 \psi_N D_x^{k-2} u\|_0^2 \\ \leq C^3 \left(|\langle M \psi_N D_x^{k-2} u, \psi_N D_x^{k-2} u\rangle| + \|\psi_N D_x^{k-2} u\|_0^2 \right). \end{aligned}$$

So, after k steps we obtain

$$C\|\psi_N D_x^k u\|_0^2 \leq \sum_{j=1}^{k} C^j |\langle M \psi_N D_x^{k-j} u, \psi_N D_x^{k-j} u\rangle + C^k \|\psi_N u\|_0^2.$$

The last term in the sum gives analytic growth, the other terms in the sum, $|\langle M \psi_N D_x^{k-j} u, \psi_N D_x^{k-j} u\rangle$, $j = 1, \ldots, k-1$, can be handled as the first term on the right hand side of (2.2), which we are about to analyze. The last term can be bounded by C_1^{k+1}.

Let us now focus on the scalar product:

$$\begin{aligned} (2.3) \quad |\langle M \psi_N D_x^k u, \psi_N D_x^k u\rangle| \leq |\langle \psi_N D_x^k M u, \psi_N D_x^k u\rangle| \\ + 2|\langle \left[x^{2n+1} D_y, \psi_N D_x^k\right] u, X_2 \psi_N D_x^k u\rangle| + 2|\langle \left[x^n y^m D_y, \psi_N D_x^k\right] u, X_3^* \psi_N D_x^k u\rangle| \\ + |\langle \left[x^{2n+1} D_y, \left[x^{2n+1} D_y, \psi_N D_x^k\right]\right] u, \psi_N D_x^k u\rangle| \\ + |\langle \left[x^n y^m D_y, \left[x^n y^m D_y, \psi_N D_x^k\right]\right] u, \psi_N D_x^k u\rangle| \\ = |\langle \psi_N D_x^k M u, D_x^k u\rangle| + \sum_{q=1}^{4} I_q. \end{aligned}$$

Let us estimate the terms in the sum above.
$\mathbf{I_1}$. Using the following formula

$$(2.4) \qquad \left[D_x^k, x^p\right] u = \sum_{\ell=1}^{\inf\{p,k\}} \frac{k!}{(k-\ell)!} \binom{p}{\ell} \left(\frac{1}{i}\right)^{\ell} x^{p-\ell} D_x^{k-\ell},$$

where k and p are positive integers, we have

$$\begin{aligned} (2.5) \quad I_1 \leq 2|\langle x^{2n+1} \psi_N^{(1)} D_x^k u, X_2 \psi_N D_x^k u\rangle| \\ + 2 \sum_{\ell=1}^{\inf\{2n+1,k\}} \frac{k!}{(k-\ell)!} \binom{2n+1}{\ell} |\langle x^{2n+1-\ell} \psi_N D_x^{k-\ell} D_y u, X_2 \psi_N D_x^k u\rangle| \\ \leq 2C_0 \delta^{-1} \|\psi_N^{(1)} D_x^k u\| + \frac{\delta}{2} \|X_2 \psi_N D_x^k u\| \end{aligned}$$

$$+4^{2n+1}\sum_{\ell=1}^{\inf\{2n+1,k\}} C_\ell k^{2\ell}\|x^{2n+1-\ell}\psi_N D_x^{k-\ell}D_y u\|^2+\left(\sum_{\ell=1}^{\inf\{2n+1,k\}}\frac{1}{C_\ell}\right)\|X_2\psi_N D_x^k u\|^2,$$

where C_j, $j=1,\dots k+1$, are arbitrary constants. We choose $C_j=\delta^{-1}2^{j+1}$, where δ is a fixed small positive constant. Since $\left(\sum_{\ell=1}^{\inf\{2n+1,k\}}\frac{1}{C_\ell}\right)\le\delta/2$, the sum of the second and the last term on the right hand side of (2.5) can be estimated by $\delta\|X_2\psi_N D_x^k u\|^2$; it can be absorbed on the left hand side of (2.2). Concerning the first term on the right hand side, setting $2C_0\delta^{-1}=\tilde{C}_0$, we can use the subelliptic estimate and restart the process:

$$\begin{aligned}\tilde{C}_0\|\psi_N^{(1)}D_x^k u\|=\tilde{C}_0\|X_1\psi_N^{(1)}D_x^{k-1}u\|&\le\tilde{C}_0\sum_{j=1}^{3}\|X_j\psi_N^{(1)}D_x^{k-1}u\|_0^2\\&\le\tilde{C}_0 C\left(|\langle M\psi_N^{(1)}D_x^{k-1}u,\psi_N^{(1)}D_x^{k-1}u\rangle|+\|\psi_N^{(1)}D_x^{k-1}u\|_0^2\right).\end{aligned}$$

The last term on the right hand side can be submitted to the same treatment used to handle the last term on the right of (2.2). The scalar product can be handled as done in (2.3), producing, modulo terms which give analytic growth, terms of the form $|\langle X_i[X_i,\psi_N^{(1)}D_x^{k-1}]u,\psi_N^{(1)}D_x^{k-1}u\rangle|$ and $|\langle[X_i,[X_i,\psi_N^{(1)}D_x^{k-1}]]u,\psi_N^{(1)}D_x^{k-1}u\rangle|$, $i=2,3$. On all these terms we can restart the processes used to treat the terms I_q, $q=1,2,3,4$, on the right hand side of (2.3). We point out that one x-derivative on u is turned in one derivative on ψ_N. Now, the term $|\langle X_2[X_2,\psi_N^{(1)}D_x^{k-1}]u,\psi_N^{(1)}D_x^{k-1}u\rangle|$ can be estimated as done in (2.5), obtaining a term of the form $\|\psi_N^{(2)}D_x^{k-1}u\|$. Such term has the some form of the first term on the right hand side of (2.5), we can restart the process above described. So, after k steps, modulo terms which give analytic growth or terms of the form $|\langle X_i[X_i,\psi_N^{(j)}D_x^{k-j}]u,\psi_N D_x^k u\rangle|$ and $|\langle[X_i,[X_i,\psi_N^{(j)}D_x^{k-j}]]u,\psi_N D_x^k u\rangle|$, $i=2,3$ and $1\le j\le k-1$, we will obtain a term of the form $\tilde{C}_1^k\|\psi_N^{(k)}u\|$, which gives analytic growth.

We focus on the terms in the sum on the right hand side of the above inequality. We use the Young inequality for products

$$\begin{aligned}(\sqrt{2}k)^\ell|x|^{2n+1-\ell}&=(\sqrt{2}k)^{(1-\theta_\ell)\ell}|x|^{2n+1-\ell}(\sqrt{2}k)^{\theta_\ell\ell}\\&\le\left((\sqrt{2}k)^{(1-\theta_\ell)\ell}|x|^{2n+1-\ell}\right)^q+\left((\sqrt{2}k)^{\theta_\ell\ell}\right)^p,\end{aligned}$$

where $q^{-1}+p^{-1}=1$ and θ_ℓ is a parameter that we will select later. Choosing $q=\frac{2n+1}{2n+1-\ell}$ and $p=\frac{2n+1}{\ell}$ we obtain

$$(\sqrt{2}k)^\ell|x|^{2n+1-\ell}\le(\sqrt{2}k)^{(1-\theta_\ell)\frac{\ell(2n+1)}{2n+1-\ell}}|x|^{2n+1}+(\sqrt{2}k)^{\theta_\ell(2n+1)}.$$

Thus we can estimate each term in the sum by

(2.6)

$$\begin{aligned}C2^\ell k^{2\ell}\|x^{2n+1-\ell}\psi_N D_x^{k-\ell}D_y u\|^2&\le C(\sqrt{2}k)^{2(1-\theta_\ell)\frac{\ell(2n+1)}{2n+1-\ell}}\|x^{2n+1}\psi_N D_x^{k-\ell}D_y u\|^2\\&\quad+C(\sqrt{2}k)^{2\theta_\ell(2n+1)}\|\psi_N D_x^{k-\ell}D_y u\|^2\\=C(\sqrt{2}k)^{2(1-\theta_\ell)\frac{\ell(2n+1)}{2n+1-\ell}}\|X_2\psi_N D_x^{k-\ell}u\|^2&+C(\sqrt{2}k)^{2\theta_\ell(2n+1)}\|X_1\psi_N D_x^{k-(\ell+1)}D_y u\|^2\end{aligned}$$

$$+ C(\sqrt{2}k)^{2(1-\theta_\ell)\frac{\ell(2n+1)}{2n+1-\ell}}\|x^{2n+1}\psi_N^{(1)}D_x^{k-\ell}u\|^2.$$

For the first term on the right hand side we use the subelliptic estimate (1.4), thus restarting the process with u replaced by $C^{1/2}(\sqrt{2}k)^{(1-\theta_\ell)\frac{\ell(2n+1)}{2n+1-\ell}}\psi_N D_x^{k-\ell}u$. This will produce a term of the form

$$C^2(\sqrt{2}k)^{2\left[2(1-\theta_\ell)\frac{\ell(2n+1)}{2n+1-\ell}\right]}\|X_2\psi_N D_x^{k-2\ell}u\|^2,$$

on which we can restart the process again. We point out that at any step ℓ x-derivatives on u are "turned into" a factor $k^{(1-\theta_\ell)\frac{\ell(2n+1)}{2n+1-\ell}}$. In the case $k/\ell \in \mathbb{N}$, iterating this procedure k/ℓ-times, the first term on the right hand side of (2.6) will give a contribution of the form

$$C^{2(k+1)}k^{2k(1-\theta_\ell)\frac{(2n+1)}{2n+1-\ell}}\|X_2\psi_N u\|^2, \tag{2.7}$$

where C is a suitable positive constant independent of k and N.
In the case $k/\ell \notin \mathbb{N}$, this term can be handled essentially in the same way with some more technical difficulties not significant for the purpose of the proof of the result. We point out that in this case we can apply the same strategy only $\lfloor k/\ell \rfloor$- times, where $\lfloor k/\ell \rfloor$ denotes the integer part of k/ℓ. Iterating this procedure $\lfloor k/\ell \rfloor$-times, we obtain a term of the form $C^{2(\lfloor k/\ell\rfloor+1)}k^{2\lfloor k/\ell\rfloor(1-\theta_\ell)\frac{(2n+1)}{2n+1-\ell}}|\langle M\psi_N D_x^{k'}u, \psi_N D_x^{k'}u\rangle|$, where $k' = k - \lfloor k/\ell \rfloor \ell$; on this term we can restart the procedure described in (2.3) up to removing all the x-derivatives acting on u.

Concerning the second term on the right hand side of (2.6), we can once again use the subelliptic estimate restarting the process. We point out that, in this case, $\ell+1$ x-derivatives on u are "turned into" a factor $k^{\theta_\ell(2n+1)}$ and in one y-derivative on u.
Iterating this procedure $k/(\ell+1)$ times, this term gives a contribution of the form

$$C^{2\frac{k}{\ell+1}}2^{k\theta_\ell\frac{(2n+1)}{\ell+1}}k^{2k\theta_\ell\frac{(2n+1)}{\ell+1}}\|\psi_N D_y^{\frac{k}{\ell+1}}u\|^2,$$

where C is a suitable positive constant independent of k and N. Since in the direction y we have a Gevrey growth of order $\frac{2m}{2m-1}$, we can estimate the this term by

$$C^{2(k+1)}k^{2k\left(\theta_\ell\frac{(2n+1)}{\ell+1}+\frac{2m}{2m-1}\frac{1}{\ell+1}\right)}. \tag{2.8}$$

Now, we choose θ_ℓ so that the powers of k in (2.7) and (2.8) are equal. We set

$$\theta_\ell\frac{2n+1}{\ell+1}+\frac{2m}{2m-1}\frac{1}{\ell+1} = (1-\theta_\ell)\frac{2n+1}{2n+1-\ell}.$$

We obtain

$$\theta_\ell = \frac{2m\ell(2n+2)-(\ell+1)(2n+1)}{(2n+2)(2m-1)(2n+1)}. \tag{2.9}$$

Replacing the value of θ_ℓ obtained in (2.7) and (2.8) we have

$$C^{2(k+1)}k^{2k(1+\frac{1}{2(n+1)(2m-1)})}, \tag{2.10}$$

that is a Gevrey growth of order $1+\frac{1}{2(n+1)(2m-1)}$.

Concerning the third term on the right hand side of (2.6), we have

$$C(\sqrt{2}k)^{2(1-\theta_\ell)\frac{\ell(2n+1)}{2n+1-\ell}}\|x^{2n+1}\psi_N^{(1)}D_x^{k-\ell}u\|^2 \le C_1(\sqrt{2}k)^{2(1-\theta_\ell)\frac{\ell(2n+1)}{2n+1-\ell}}\|X_1\psi_N^{(1)}D_x^{k-(\ell+1)}u\|^2.$$

Using the subelliptic estimate we restart the process. We point out that in this case $\ell+1$ x-derivatives on u are turned into a factor $k^{(1-\theta_\ell)\frac{\ell(2n+1)}{2n+1-\ell}}$ and in one y-derivative on ψ_N.
Iterating this procedure $k/(\ell+1)$ times, gives a contribution of the form

$$C_2^{2(k+1)}k^{2(1-\theta_\ell)\frac{\ell}{\ell+1}\frac{(2n+1)}{2n+1-\ell}}N^{2\frac{k}{\ell+1}}.$$

Substituting the value of θ_ℓ obtained in (2.9) and taking $N\sim 2k$ we conclude that this term has a Gevrey growth of order $1+\frac{1}{2(n+1)(2m-1)}$.

$\mathbf{I_2}$. Consider I_2 on the right hand side of (2.3). Using formula (2.4) and since $X_3^*=X_3-imx^ny^{m-1}$, we have

$$\begin{aligned}(2.11)\quad I_2 &\le 2|\langle x^ny^m\psi_N^{(1)}D_x^ku, X_3\psi_ND_x^ku\rangle| + 2m|\langle x^ny^m\psi_N^{(1)}D_x^ku, x^ny^{m-1}\psi_ND_x^ku\rangle|\\ &+2\sum_{\ell=1}^{\inf\{n,k\}}\frac{k!}{(k-\ell)!}\binom{n}{\ell}|\langle x^{n-\ell}y^m\psi_ND_x^{k-\ell}D_yu, X_3\psi_ND_x^ku\rangle|\\ &+2m\sum_{\ell=1}^{\inf\{n,k\}}\frac{k!}{(k-\ell)!}\binom{n}{\ell}\langle x^{n-\ell}y^m\psi_ND_x^{k-\ell}D_yu, x^ny^{m-1}\psi_ND_x^ku\rangle|\\ &= I_{2,1}+I_{2,2}+I_{2,3}+I_{2,4}.\end{aligned}$$

We analyze the terms obtained separately.
$\mathbf{I_{2,1}}$. We have

$$I_{2,1}\le C\frac{1}{\delta}\|\psi_N^{(1)}D_x^ku\|^2+\delta\|X_3\psi_ND_x^ku\|^2 = C\frac{1}{\delta}\|X_1\psi_N^{(1)}D_x^{k-1}u\|^2+\delta\|X_3\psi_ND_x^ku\|^2,$$

where δ is a small positive number. The second term can be absorbed on the right hand side of (2.2). Concerning the first term we can restart the process, we remark that one x-derivative on u is turned into one derivative on ψ_N, so that, iterating the process, yields analytic growth.

$\mathbf{I_{2,2}}$. We have

$$I_{2,2}\le C\|X_1\psi_N^{(1)}D_x^{k-1}u\|^2+C\|\psi_ND_x^ku\|^2.$$

The first term can be handled as done a few lines above. The second term can be handled as the second term on the right hand side of (2.2).

$\mathbf{I_{2,3}}$. We have

(2.12)

$$I_{2,3}\le 4^{2n+1}\sum_{\ell=1}^{\inf\{n,k\}}C_\ell k^{2\ell}\|x^{n-\ell}y^m\psi_ND_x^{k-\ell}D_yu\|^2+\left(\sum_{\ell=1}^{\inf\{n,k\}}\frac{1}{C_\ell}\right)\|X_3\psi_ND_x^ku\|^2,$$

where we set $C_\ell=\delta^{-1}2^\ell$, δ a fixed small positive constant. Since $\left(\sum_{\ell=1}^{\inf\{n,k\}}\frac{1}{C_\ell}\right)\le\delta$, the last term on the right hand side can be absorbed on the left hand side of

(2.2). In order to handle the terms in the sum we use the Young inequality for products

$$(\sqrt{2}k)^{\ell}|x|^{n-\ell} \;=\; (\sqrt{2}k)^{\ell-\theta_\ell}|x|^{n-\ell}(\sqrt{2}k)^{\theta_\ell} \;\le\; \left((\sqrt{2}k)^{\ell-\theta_\ell}|x|^{n-\ell}\right)^{q} + (\sqrt{2}k)^{\theta_\ell p},$$

where $q^{-1} + p^{-1} = 1$ and θ_ℓ is a parameter that we will choose later. Taking $q = \frac{n}{n-\ell}$ and $p = \frac{n}{\ell}$ we obtain

$$(\sqrt{2}k)^{\ell}|x|^{n-\ell} \le (\sqrt{2}k)^{(\ell-\theta_\ell)\frac{n}{n-\ell}}|x|^{n} + (\sqrt{2}k)^{\theta_\ell \frac{n}{\ell}}.$$

Thus we can estimate each term in the sum by

$$\begin{aligned}(2.13)\quad 2^{\ell}k^{2\ell}\|x^{n-\ell}y^{m}\psi_N D_x^{k-\ell}D_y u\|^2 &\le (\sqrt{2}k)^{2(\ell-\theta_\ell)\frac{n}{n-\ell}}\|x^{n}y^{m}\psi_N D_x^{k-\ell}D_y u\|^2\\ &\quad + C(\sqrt{2}k)^{2\theta_\ell \frac{n}{\ell}}\|\psi_N D_x^{k-\ell}D_y u\|^2.\end{aligned}$$

The first term on the right hand side can be estimated by

$$(2.14)\quad (\sqrt{2}k)^{2(\ell-\theta_\ell)\frac{n}{n-\ell}}\|X_3\psi_N D_x^{k-\ell}u\|^2 + C(\sqrt{2}k)^{2(\ell-\theta_\ell)\frac{n}{n-\ell}}\|X_1\psi_N^{(1)} D_x^{k-(\ell+1)}u\|^2.$$

Let us examine the first term on the right. Using the subelliptic estimate we restart the process. We point out that in this case ℓ x-derivatives on u are turned into a factor $k^{(\ell-\theta_\ell)\frac{n}{n-\ell}}$. Iterating this procedure k/ℓ times gives a contribution of the form

$$(2.15)\qquad C^{2(k+1)}k^{2k(\ell-\theta_\ell)\frac{n}{\ell(n-\ell)}}\|X_3\psi_N u\|^2.$$

Concerning the second term on the right hand side of (2.13), it can be handled using the same strategy adopted to treat the term in the second line of (2.6), more precisely we have

$$C(\sqrt{2}k)^{2\theta_\ell \frac{n}{\ell}}\|\psi_N D_x^{k-\ell}D_y u\|^2 = C(\sqrt{2}k)^{2\theta_\ell \frac{n}{\ell}}\|X_1\psi_N D_x^{k-(\ell+1)}D_y u\|^2.$$

Now, we can take advantage from the subelliptic estimate, (1.4), with u replaced by $C^{1/2}(\sqrt{2}k)^{\theta_\ell \frac{n}{\ell}}\psi_N D_x^{k-(\ell+1)}D_y u$ and restart the process. We point out that at any time that we restart the process $\ell+1$ x-derivatives on u are turned into a factor $k^{2\theta_\ell \frac{n}{\ell}}$ and in one y-derivative on u.
Iterating the process $k/(\ell+1)$ times gives a contribution of the form

$$(2.16)\qquad C^{2\frac{k}{\ell+1}}2^{k\theta_\ell \frac{n}{\ell(\ell+1)}}k^{2k\theta_\ell \frac{n}{\ell(\ell+1)}}\|\psi_N D_y^{\frac{k}{\ell+1}}u\|^2,$$

where C is a suitable positive constant independent of k and N.
Since in the direction y we have a Gevrey growth of order $\frac{2m}{2m-1}$, (2.1), we have

$$\|\psi_N D_y^{\frac{k}{\ell+1}}u\|^2 \le C_1^{2(k+1)}k^{2\frac{2m}{2m-1}\frac{k}{\ell+1}}.$$

We can estimate (2.16) by

$$(2.17)\qquad C_2^{2(k+1)}k^{2k\left(\theta_\ell\frac{n}{\ell(\ell+1)}+\frac{2m}{2m-1}\frac{1}{\ell+1}\right)}.$$

where C_2 is a suitable positive constant independent of k and N.
Now, we choose θ_ℓ so that the powers of k in (2.15) and (2.17) are equal

$$\theta_\ell\frac{n}{\ell(\ell+1)} + \frac{2m}{2m-1}\frac{1}{\ell+1} = (\ell-\theta_\ell)\frac{n}{\ell(n-\ell)}.$$

Hence

$$(2.18)\qquad \theta_\ell = \frac{\ell[2m\ell(n+1)-n(\ell+1)]}{n(n+1)(2m-1)}.$$

Substituting the value of θ_ℓ in (2.15) and (2.17) we have a contribution of the form

$$C^{2(k+1)} k^{2k(1+\frac{1}{(n+1)(2m-1)})}, \tag{2.19}$$

that is a Gevrey growth of order $1 + \frac{1}{(n+1)(2m-1)}$.

We come back to (2.14) and we focus us on the second term on the right hand side. We can take advantage of the subelliptic estimate restarting the process. We stress that in this case $\ell + 1$ x-derivatives on u are turned into a factor $k^{(\ell-\theta_\ell)\frac{n}{n-\ell}}$ and in one derivative on ψ_N. Iterating this procedure $k/(\ell+1)$ times gives a contribution of the form

$$C^{2(k+1)} k^{2k\frac{(\ell-\theta_\ell)n}{(\ell+1)(n-\ell)}} N^{2\frac{k}{\ell+1}}.$$

Replacing θ_ℓ by the value obtained in (2.18), and taking $N \sim 2k$ we have a contribution of the form

$$C_1^{2(k+1)} k^{2k\left(1+\frac{\ell}{(\ell+1)}\frac{1}{(n+1)(2m-1)}\right)},$$

that is a Gevrey growth of order $1 + \frac{\ell}{(\ell+1)}\frac{1}{(n+1)(2m-1)}$. We remark that this index is smaller than $1 + \frac{1}{(n+1)(2m-1)}$.

$\mathbf{I_{2,4}}$. It can be handled as $I_{2,3}$. The only difference is the presence of the norm $\|\psi_N D_x^k u\|^2$, which can be bound as done in the analysis of $I_{2,2}$.

$\mathbf{I_3}$. Consider I_3 on the right hand side of (2.3). Using two times formula (2.4), we have

$$\begin{aligned}
&[x^{2n+1}D_y, [x^{2n+1}D_y, \psi_N D_x^k]] \\
&= \frac{1}{i}[x^{2n+1}D_y, x^{2n+1}\psi_N^{(1)} D_x^k] + \sum_{\ell=1}^{\inf\{2n+1,k-1\}} C_{k,2n+1,\ell}[x^{2n+1}D_y, x^{2n+1-\ell}\psi_N D_x^{k-\ell} D_y] \\
&\qquad = -x^{2(2n+1)}\psi_N^{(2)} D_x^k + \frac{2}{i}\sum_{\ell=1}^{\inf\{m,k\}} C_{k,2n+1,\ell} x^{2(2n+1)-\ell}\psi_N^{(1)} D_x^{k-\ell} D_y \\
&\quad + \sum_{\ell=1}^{\inf\{2n+1,k\}} \sum_{\ell_1=1}^{\inf\{2n+1,k-\ell\}} C_{k,2n+1,\ell}\, C_{k-\ell,2n+1,\ell_1}\, x^{2(2n+1)-\ell-\ell_1}\psi_N D_x^{k-\ell-\ell_1} D_y^2,
\end{aligned}$$

where

$$C_{k,2n+1,\ell} = \frac{k!}{(k-\ell)!}\binom{2n+1}{\ell} \quad \text{and} \quad C_{k-\ell,2n+1,\ell_1} = \frac{(k-\ell)!}{(k-\ell-\ell_1)!}\binom{2n+1}{\ell_1}.$$

We remark that the last sum above can be rewritten in the following form

$$\begin{aligned}
&\sum_{h=2}^{\inf\{k,2(2n+1)\}} \left(\sum_{\substack{\ell+\ell_1=h \\ 1\le \ell,\ell_1 \le \inf\{k-1,2n+1\}}} C_{k,2n+1,\ell} C_{k-\ell,2n+1,\ell_1} \right) x^{2(2n+1)-h}\psi_N D_x^{k-h} D_y^2 \\
&\qquad\qquad = \sum_{h=2}^{\inf\{k,2(2n+1)\}} D_h\, x^{2(2n+1)-h}\psi_N D_x^{k-h} D_y^2,
\end{aligned}$$

where

$$D_h = \begin{cases} \displaystyle\sum_{\ell=1}^{h-1} C_{k,2n+1,\ell} C_{k-\ell,2n+1,h-\ell} & \text{if } \; h = 2,3,\dots,\inf\{k,2n+2\}, \\ \\ \displaystyle\sum_{\ell=h-(2n+1)}^{2n+1} C_{k,2n+1,\ell} C_{k-\ell,2n+1,h-\ell} & \text{if } \; h = 2n+1+2,\dots,\inf\{k,2(2n+1)\}. \end{cases}$$

We point out that D_h is given by the sum of at most $2n+1$ summands, moreover since $C_{k,2n+1,\ell} \leq 2^{2n+1}k^{\ell}$ and $C_{k-\ell,2n+1,h-\ell} \leq 2^{2n+1}k^{h-\ell}$ we have that $D_h \leq (2n+1)2^{2(2n+1)}k^h$.
We obtain

$$\begin{aligned} |\langle \left[x^{2n+1}D_y, \left[x^{2n+1}D_y, \psi_N D_x^k\right]\right] u, \psi_N D_x^k u\rangle| &\leq |\langle x^{2(2n+1)}\psi_N^{(2)} D_x^k u, \psi_N D_x^k u\rangle| \\ &+ 2^{2n+2} \sum_{\ell=1}^{\inf\{m,k\}} k^{\ell} |\langle x^{2(2n+1)-\ell}\psi_N^{(1)} D_x^{k-\ell} D_y u, \psi_N D_x^k u\rangle| \\ &+ 2^{3(2n+1)} \sum_{h=2}^{\inf\{k,2(2n+1)\}} k^h |\langle x^{2(2n+1)-h}\psi_N D_x^{k-h} D_y^2 u, \psi_N D_x^k u\rangle| \\ &= I_{3,1} + I_{3,2} + I_{3,3}. \end{aligned}$$

We analyze the terms obtained separately.

$\mathbf{I_{3,1}}$. We have

$$\begin{aligned} |\langle x^{2(2n+1)}\psi_N^{(2)} D_x^k u, \psi_N D_x^k u\rangle| &\leq C|\langle (C_\psi N)^{-1}\psi_N^{(2)} D_x^k u, (C_\psi N)\psi_N D_x^k u\rangle| \\ &\leq C(C_\psi N)^{-2}\|X_1\psi_N^{(2)} D_x^{k-1}\|^2 + C(C_\psi N)^2\|X_1\psi_N D_x^{k-1}u\|^2. \end{aligned}$$

The weight $C_\psi N$ introduced above helps to balance the number of x-derivatives on u with the number of derivatives on ψ_N. We consider the factor $C_\psi N$ as a derivative on ψ_N, $C_\psi N \psi_N$ and $(C_\psi N)^{-1}\psi_N^{(2)}$ behave as $\psi_N^{(1)}$. On each term we can restart the process.
We remark that both terms have the same behavior as the term $\|X_1\psi_N^{(1)} D_y^{k-1}u\|^2$, that is we have a shift of one derivative from u to ψ_N; iterating the process we will obtain analytic growth.

$\mathbf{I_{3,2}}$. Since each term in the sum can be estimated by

$$2^{\ell}k^{2\ell}\|x^{2n+1-\ell}\psi_N D_x^{k-\ell} D_y u\|^2 + \frac{1}{2^{\ell}}\|X_1\psi_N^{(1)} D_x^{k-1}u\|^2,$$

we have

$$I_{3,2} \leq C \sum_{\ell=1}^{\inf\{2n+1,k\}} 2^{\ell}k^{2\ell}\|x^{2n+1-\ell}\psi_N D_x^{k-\ell} D_y u\|^2 + 2C\|X_1\psi_N^{(1)} D_x^{k-1}u\|^2.$$

The norms in the sum have the same form of the norms in the first sum on the right hand side of (2.5), so that they give a Gevrey growth of order $1 + \frac{1}{2(n+1)(2m-1)}$.
To handle the last term on the right hand side we apply the subelliptic estimate and remark that we have shifted one derivative from u to ψ_N, so that, iterating the

process, yields analytic growth.

$\mathbf{I_{3,3}}$. We distinguish two cases. First: $h \le 2n+1$. We have

$$
\begin{aligned}
& k^h |\langle x^{2n+1} x^{2n+1-h} \psi_N D_x^{k-h} D_y^2 u, \psi_N D_x^k u\rangle| \\
\le\ & k^h |x^{2n+1-h} \psi_N D_x^{k-h} D_y u, X_2 \psi_N D_x^k u\rangle| + k^h |\langle x^{2n+1} x^{2n+1-h} \psi_N D_x^{k-h} D_y u, \psi_N^{(1)} D_x^k u\rangle| \\
\le\ & (\delta^{-1}+1) k^{2h} \|x^{2n+1-h} \psi_N D_x^{k-h} D_y u\|^2 + \delta \|X_2 \psi_N D_x^k u\|^2 + C \|X_1 \psi_N^{(1)} D_x^{k-1} u\|^2,
\end{aligned}
$$

where δ is a small suitable positive number. The second term on the right can be absorbed on the left hand side of (2.4). The first term can be handled as the terms in the sum on the right hand side of (2.5); thus this term will give a Gevrey growth of order $1 + \frac{1}{2(n+1)(2m-1)}$. Concerning the last term, we point out that we shifted one derivative from u to ψ_N, yielding analytic growth.

Case $2n+1 < h \le 2(2n+1)$. We set $h = 2n+1+h_1$. Using (2.4), we have

$$
\begin{aligned}
& k^{2n+1} k^{h_1} |\langle x^{2n+1-h_1} \psi_N D_x^{k-(2n+1)-h_1} D_y^2 u, \psi_N D_x^k u\rangle| \\
&\qquad \le k^{2n+1} k^{h_1} \Big(|\langle x^{2n+1-h_1} \psi_N D_x^{k-(2n+1)-h_1} D_y u, \psi_N D_x^k D_y u\rangle| \\
&\qquad\qquad + 2 |\langle x^{2n+1-h_1} \psi_N D_x^{k-(2n+1)-h_1} D_y u, \psi_N^{(1)} D_x^k u\rangle| \Big) \\
&\qquad \le |\langle k^{h_1} x^{2n+1-h_1} \psi_N D_x^{k-h_1} D_y u, k^{2n+1} \psi_N D_x^{k-(2n+1)} D_y u\rangle| \\
&\qquad\quad + 2 |\langle k^{h_1} x^{2n+1-h_1} \psi_N D_x^{k-(2n+1)-h_1} D_y u, k^{2n+1} \psi_N^{(1)} D_x^k u\rangle| \\
& + \sum_{\beta=1}^{2n+1-h_1} \frac{1}{\beta!} \frac{(2n+1-h_1)!}{(2n+1-h_1-\beta)!} \frac{(2n+1)!}{(2n+1-\beta)!} \\
&\qquad \times \Big(|\langle k^{h_1} x^{2n+1-(h_1+\beta)} \psi_N D_x^{k-(h_1+\beta)} D_y u, k^{2n+1} \psi_N D_x^{k-(2n+1)} D_y u\rangle| \\
&\qquad + |\langle k^{h_1} x^{2n+1-(h_1+\beta)} \psi_N D_x^{k-(h_1+\beta)} D_y u, k^{2n+1} \psi_N^{(1)} D_x^{k-(2n+1)} u\rangle| \Big) \\
&\qquad \le C \sum_{\beta=0}^{2n+1-h_1} k^{2(h_1+\beta)} \|x^{2n+1-(h_1+\beta)} \psi_N D_x^{k-(h_1+\beta)} D_y u\|^2 \\
& + C^{2n+1} k^{2(2n+1)} \|X_1 \psi_N D_x^{k-(2n+2)} D_y u\|^2 + C^{2n+1} k^{2(2n+1)} \|X_1 \psi_N^{(1)} D_x^{k-(2n+2)} u\|^2.
\end{aligned}
$$

The second term and the terms in the sum can be handled as the terms in the sum on the right hand side of (2.5); thus they give a Gevrey growth $1 + \frac{1}{2(n+1)(2m-1)}$. Concerning the last term we can restart the process. We point out that $2n+1$ derivatives on u are turned into a factor $k^{2n+1} \psi_N^{(1)}$ and, since $N \sim 2k$, it behaves as $\psi_N^{(2n+2)}$. Iterating gives analytic growth.

$\mathbf{I_4}$. Consider I_4 on the right hand side of (2.3). Using two times formula (2.4), we have

$$
\begin{aligned}
(2.20)\quad & [x^n y^m D_y, [x^n y^m D_y, \psi_N D_x^k]] \\
& = \frac{1}{i} [x^n y^m D_y, x^n y^m \psi_N^{(1)} D_x^k] + \sum_{\ell=1}^{\inf\{n,k\}} C_{k,n,\ell} [x^n y^m D_y, x^{n-\ell} y^m \psi_N D_x^{k-\ell} D_y]
\end{aligned}
$$

$$= -x^{2n}y^{2m}\psi_N^{(2)}D_x^k - mx^{2n}y^{2m-1}\psi_N^{(1)}D_x^k + \frac{2}{i}\sum_{\ell=1}^{\inf\{n,k\}} C_{k,n,\ell}\, x^{2n-\ell}y^{2m}\psi_N^{(1)}D_x^{k-\ell}D_y$$

$$+ \frac{2m}{i}\sum_{\ell=1}^{\inf\{n,k\}} C_{k,n,\ell}x^{2n-\ell}y^{2m-1}\psi_N D_x^{k-\ell}D_y$$

$$+ \frac{m}{i}\sum_{\ell=1}^{\inf\{n,k\}}\sum_{\ell_1=1}^{\inf\{n,k-\ell\}} C_{k,n,\ell}\,C_{k-\ell,n,\ell_1}\,x^{2n-\ell-\ell_1}y^{2m-1}\psi_N D_x^{k-\ell-\ell_1}D_y,$$

$$+ \sum_{\ell=1}^{\inf\{n,k\}}\sum_{\ell_1=1}^{\inf\{n,k-\ell\}} C_{k,n,\ell}\,C_{k-\ell,n,\ell_1}\,x^{2n-\ell-\ell_1}y^{2m}\psi_N D_x^{k-\ell-\ell_1}D_y^2,$$

where

$$C_{k,n,\ell} = \frac{k!}{(k-\ell)!}\binom{n}{\ell} \quad \text{and} \quad C_{k-\ell,n,\ell_1} = \frac{(k-\ell)!}{(k-\ell-\ell_1)!}\binom{n}{\ell_1}.$$

We remark that the last two sums above can be rewritten in the following form

$$\sum_{h=2}^{\inf\{k,2n\}}\left(\sum_{\substack{\ell+\ell_1=h\\ 1\le\ell,\ell_1\le\inf\{k-1,n\}}} C_{k,n,\ell}\,C_{k-\ell,n,\ell_1}\right)x^{2n-\ell-\ell_1}y^{2m-\alpha}\psi_N D_x^{k-\ell-\ell_1}D_y^{2-\alpha}$$

$$= \sum_{h=2}^{\inf\{k,2n\}} D_h\,x^{2(2n+1)}x^{2n-h}y^{2m-\alpha}\psi_N D_x^{k-h}D_y^{2-\alpha}, \qquad \alpha=0,1,$$

where

$$D_h = \begin{cases} \displaystyle\sum_{\ell=1}^{h-1} C_{k,n,\ell}C_{k-\ell,2n,h-\ell} & \text{if } \; h=2,3,\dots,\inf\{k,n+1\},\\[2ex] \displaystyle\sum_{\ell=h-n}^{n} C_{k,n,\ell}C_{k-\ell,n,h-\ell} & \text{if } \; h=n+2,\dots,\inf\{k,2n\}.\end{cases}$$

We point out that D_h is given by the sum of at most n terms, moreover since $C_{k,n,\ell}\le 2^n k^\ell$ and $C_{k-\ell,n,h-\ell}\le 2^n k^{h-\ell}$ we have that $D_h\le n2^{2n}k^h$.
Hence

$$|\langle\left[x^n y^m D_y,\left[x^n y^m D_y,\psi_N D_x^k\right]\right]u,\psi_N D_x^k u\rangle|$$

$$\le |\langle x^{2n}y^{2m}\psi_N^{(2)}D_x^k,\psi_N D_x^k u\rangle| + m|\langle x^{2n}y^{2m-1}\psi_N^{(1)}D_x^k u,\psi_N D_x^k u\rangle|$$

$$+ 2^{n+1}\sum_{\ell=1}^{\inf\{n,k\}} k^\ell|\langle x^{2n-\ell}y^{2m}\psi_N^{(1)}D_x^{k-\ell}D_y u,\psi_N D_x^k u\rangle|$$

$$+ m2^{2n+1}\sum_{\ell=1}^{\inf\{n,k\}} k^\ell|\langle x^{2n-\ell}y^{2m-1}\psi_N D_x^{k-\ell}D_y u,\psi_N D_x^k u\rangle|$$

$$+ m2^{2n}\sum_{h=2}^{\inf\{k,2n\}} k^h|\langle x^{2n-h}y^{2m-1}\psi_N D_x^{k-h}D_y u,\psi_N D_x^k u\rangle|$$

$$+ 2^{2n} \sum_{h=2}^{\inf\{k,2n\}} k^h |\langle x^{2n-h} y^{2m} \psi_N D_x^{k-h} D_y^2 u, \psi_N D_x^k u\rangle| = \sum_{i=1}^{6} I_{4,i}.$$

First we remark that $I_{4,1}$ can be handled as $I_{3,1}$. $I_{4,2}$ is analogous to $I_{4,1}$.

$\mathbf{I_{4,3}}$. Since each term in the sum can be estimated by

$$2^\ell k^{2\ell} \|x^{n-\ell} y^m \psi_N D_x^{k-\ell} D_y u\|^2 + \frac{1}{2^\ell} \|X_1 \psi_N^{(1)} D_x^{k-1} u\|^2.$$

We have

$$I_{4,3} \leq C \sum_{\ell=1}^{\inf\{n,k\}} 2^\ell k^{2\ell} \|x^{n-\ell} y^m \psi_N D_x^{k-\ell} D_y u\|^2 + 2C \|X_1 \psi_N^{(1)} D_x^{k-1} u\|^2.$$

The summands have the same form of the those in the first sum on the right hand side of (2.12); they give a Gevrey growth of order $1 + \frac{1}{(n+1)(2m-1)}$. To handle the last term on the right hand side we can use the subelliptic estimate. We remark that we have shifted one derivative from u to ψ_N, this term has analytic growth.

$\mathbf{I_{4,4}}$. Since each term in the sum can be estimated by

$$2^\ell k^{2\ell} \|x^{n-\ell} y^m \psi_N D_x^{k-\ell} D_y u\|^2 + \frac{1}{2^\ell} \|\psi_N D_x^k u\|^2,$$

we have

$$I_{4,4} \leq C \sum_{\ell=1}^{\inf\{n,k\}} 2^\ell k^{2\ell} \|x^{n-\ell} y^m \psi_N D_x^{k-\ell} D_y u\|^2 + 2C \|\psi_N D_x^k u\|^2.$$

The terms in the sum are analogous to the term in the first sum on the right hand side of (2.12), and they have a Gevrey growth of order $1 + \frac{1}{(n+1)(2m-1)}$. The second term can be handled as the second term on the right hand side of (2.2).

$\mathbf{I_{4,5}}$. We focus on a single term in the sum. We distinguish two cases. Case $h \leq n$. We have

$$\begin{aligned} &k^h |\langle x^{2n-h} y^{2m-1} \psi_N D_x^{k-h} D_y u, \psi_N D_x^k u\rangle| \\ &\qquad\qquad \leq C k^{2h} \|x^{n-h} y^m \psi_N D_x^{k-h} D_y u\|^2 + \|\psi_N D_x^k u\|^2. \end{aligned}$$

The first term can be handled as the terms in the sum on the right hand side of (2.12). The second term can be estimated as the second term on the right hand side of (2.2).

Case $h > n$. We set $h = n + h_1$. We rewrite the summands in $I_{4,5}$ as

$$\begin{aligned} &k^h |\langle x^{2n-h} y^{2m-1} \psi_N D_x^{k-h} D_y u, \psi_N D_x^k u\rangle| \\ &\qquad\qquad = k^n k^{h_1} |\langle D^n x^{n-h_1} y^{2m-1} \psi_N D_x^{k-n-h_1} D_y u, \psi_N D_x^{k-n} u\rangle| \end{aligned}$$

Using (2.4), we have

$$\begin{aligned} &k^n k^{h_1} |\langle D^n x^{n-h_1} y^{2m-1} \psi_N D_x^{k-n-h_1} D_y u, \psi_N D_x^{k-n} u\rangle| \\ &\leq k^n k^{h_1} \sum_{\beta=0}^{n-h_1} \frac{1}{\beta!} \frac{(n-h_1)!}{(n-h_1-\beta)!} \frac{n!}{(n-\beta)!} |\langle x^{n-(h_1+\beta)} y^m \psi_N D_x^{k-(h_1+\beta)} D_y u, \psi_N D_x^{k-n} u\rangle| \end{aligned}$$

$$\leq 2^{n-h_1}\left(\sum_{\beta=0}^{n-h_1} n^{2j}k^{2(h_1+\beta)}\|x^{n-(h_1+\beta)}y^m\psi_N D_x^{k-(h_1+\beta)}D_y u\|^2 + k^{2n}\|\psi_N D_x^{k-n}u\|^2\right).$$

All these terms are analogous to those in the sum on the right hand side of (2.12).

$\mathbf{I_{4,6}}$. In order to handle this term we use the same strategy used to analyze $I_{4,5}$. We distinguish tow cases.
Case $h \leq n$. We have

$$\begin{aligned} k^h|\langle x^{2n-h}y^{2m}\psi_N D_x^{k-h}D_y^2 u, \psi_N D_x^k u\rangle| &\\ \leq k^{2h}\|x^{n-h}y^m\psi_N D_x^{k-h}D_y u\|^2 &+ \|X_3\psi_N D_x^k u\|^2 \\ &+ 2mk^h|\langle x^{2n-h}y^{2m-1}\psi_N D_x^{k-h}D_y u, \psi_N D_x^k u\rangle| \\ &+ 2k^h|\langle x^{2n-h}y^{2m}\psi_N D_x^{k-h}D_y^2 u, \psi_N^{(1)}D_x^k u\rangle|. \end{aligned}$$

The first term has the same form of the terms in the sum on the right hand side of (2.12). The second term can be absorbed on the left hand side of (2.2). The last two terms have the same form as $I_{4,4}$ and $I_{4,3}$.

Case $h > n$. We set $h = n + h_1$. We have

$$\begin{aligned} k^{n+h_1}|\langle x^{n-h_1}y^{2m}\psi_N D_x^{k-n-h_1}D_y^2 u, \psi_N D_x^k u\rangle| &\\ \leq k^{n+h_1}|\langle x^{n-h_1}y^{2m}\psi_N D_x^{k-n-h_1}D_y u, \psi_N D_x^k D_y u\rangle| &\\ + 2k^{n+h_1}|\langle x^{n-h_1}y^{2m}\psi_N D_x^{k-n-h_1}D_y u, \psi_N^{(1)}D_x^k u\rangle| &\\ + 2mk^{n+h_1}|\langle x^{n-h_1}y^{2m-1}\psi_N D_x^{k-n-h_1}D_y u, \psi_N D_x^k u\rangle|. & \end{aligned}$$

Using (2.4), the right hand side of the above inequality can be estimated by

$$\begin{aligned} &\sum_{\beta=0}^{n-h_1}\frac{1}{\beta!}\frac{(n-h_1)!}{(n-h_1-\beta)!}\frac{n!}{(n-\beta)!} \\ &\quad\times\Big(|\langle k^{h_1}x^{n-(h_1+\beta)}y^m\psi_N D_x^{k-(h_1+\beta)}D_y u, y^m k^n\psi_N D_x^{k-n}D_y u\rangle| \\ &\quad\quad + 2|\langle k^{h_1}x^{n-(h_1+\beta)}y^{2m}\psi_N D_x^{k-(h_1-\beta)}D_y u, k^n\psi_N^{(1)}D_x^{k-n}u\rangle| \\ &\quad\quad +2m|\langle k^{h_1}x^{n-(h_1+\beta)}y^{2m-1}\psi_N D_x^{k-(h_1+\beta)}D_y u, k^n\psi_N D_x^{k-n}u\rangle|\Big) \\ &\leq (2m+3)^2 2^{n-h_1}\sum_{\beta=0}^{n-h_1} n^{2j}k^{2(h_1+\beta)}\|x^{n-(h_1+\beta)}y^m\psi_N D_x^{k-(h_1+\beta)}D_y u\|^2 \\ &\quad + 2^{n-h_1}\Big(k^{2n}\|X_1\psi_N D_x^{k-(n+1)}D_y u\|^2 + k^{2n}\|X_1\psi_N^{(1)}D_x^{k-(n+1)}u\|^2 \\ &\quad\quad + k^{2(n+1)}\|X_1\psi_N D_x^{k-(n+1)}u\|\Big). \end{aligned}$$

The terms in the sum can be handled as those in the sum on the right hand side of (2.12). Iteration gives a Gevrey growth of order $1+\frac{1}{(n+1)(2m-1)}$. The last two terms have analytic growth.

Combining and iterating the above processes, modulo terms which give analytic growth, we obtain terms of the form

$$N^{2(i-j)}k^{2\frac{2m}{2m-1}\left(\sum_{\beta=1}^{p}\ell_\beta+\sum_{\alpha=1}(l_\alpha-1-\varepsilon)\right)}\|X_j\psi_N^{(j)}D_y^{k-\left(\sum_{\beta=1}^{p}\ell_\beta+\sum_{\alpha=1}^{q}(l_\alpha-1-\varepsilon)\right)-i}u\|^2$$

or

$$N^{2(i-j)}k^{2\frac{2m}{2m-1}\left(\sum_{\beta=1}^{p}\ell_\beta+\sum_{\alpha=1}(l_\alpha-1-\varepsilon)\right)}\|\psi_N^{(j)}D_y^{k-\left(\sum_{\beta=1}^{p}\ell_\beta+\sum_{\alpha=1}^{q}(l_\alpha-1-\varepsilon)\right)-i}u\|_\varepsilon^2.$$

Due to the result in the previous sub-section we can conclude that

$$\|\psi_N D_y^k u\|^2_{\frac{1}{2(n+1)}}+\sum_{j=1}^{3}\|X_j\psi_N D_y^k u\|_0^2\leq C^{2(k+1)}k^{2\left(1+\frac{1}{(n+1)(2m-1)}\right)k},$$

where C is a suitable positive constant. Thus we have a Gevrey growth of order $1+\frac{1}{(n+1)(2m-1)}$ in the x-direction.

This concludes the proof of Theorem 1.1.

Remark 2.2. *In the case of the Métivier operator (1.1) we obtain that* $u\in G^{(2,\frac{3}{2})}$. *This is due to the fact that the vector field* yD_y *commutes with* D_x. *More precisely, following the strategy used to estimate* $\psi_N D_x^k u$, *we remark that we have to handle only the terms involving the commutators with the vector field* xD_y, *i.e.* $|\langle[xD_y,\psi_N D_x^k]\,u,(xD_y)\psi_N D_x^k u\rangle|$ *and* $|\langle[xD_y,[xD_y,\psi_N D_x^k]]\,u,\psi_N D_x^k u\rangle|$.
These terms give a Gevrey growth of order $\frac{3}{2}$.

3. Short background on the FBI-transform

In order to make the paper self-consistent and more readable we recall below some of the main definitions and properties concerning the Fourier-Bros-Iagolnitzer (FBI) transform. For further details on the FBI-transform we refer to the papers [29], [22] and to the lecture notes [30].

3.1. **FBI Transform.** Let $u\in\mathscr{E}'(\Omega)$; here $\mathscr{E}'(\Omega)$ denotes the space of distributions with compact support in Ω, open subset of $\mathbb{R}^n$, i.e. the dual space of the space of smooth functions in Ω equipped with its natural topology.
In what follows Ω is a neighborhood of a point x_0. A phase function $\varphi(z,w)$, $z\in W\subset\mathbb{C}^n$, $w\in W'\subset\mathbb{C}^n$, is a holomorphic function such that $\partial_w\varphi(z_0,w_0)=-\eta_0\neq 0$, $\det\partial_z\partial_w\varphi(z_0,w_0)\neq 0$ and $\Im\partial_w^2\varphi(z_0,w_0)>0$.
We define the *FBI transform* of u as

$$Tu(z,\lambda)=\int_{\mathbb{R}^n}e^{i\lambda\varphi(z,y)}u(y)dy,$$

where $z\in W$, $\lambda\geq 1$ is a large parameter. Moreover we assume that $W'\cap\mathbb{R}^n\supset\Omega$ and we take $w_0=x_0$.
We remark that if φ is a complex quadratic form, then the localization becomes trivial.
To the phase φ we associate a weight function $\phi(z)$, defined as

$$\phi(z)=\sup_{y\in\Omega}-\Im\varphi(z,y),\qquad z\in W.$$

Example 1. *A typical phase function is* $\varphi(z,y)=\frac{i}{2}(z-y)^2$, *called the classical phase. The corresponding weight function is given by* $\phi(z)=\frac{1}{2}(\Im z)^2$.

We recall that T is associated to the following complex canonical transformation:

$$\mathscr{H}_T\colon\ (w, -\partial_w\varphi(z,w)) \mapsto (z, \partial_z\varphi(z,w))\,, \tag{3.1}$$

with φ as a generating function, from a complex neighborhood of (x_0, η_0) to a complex neighborhood of (z_0, ξ_0), where $\xi_0 = \frac{2}{i}\partial_z\phi(z_0)$.
We denote by $\Lambda_\phi = \{(z, -2i\partial_z\phi(z))\,;\, z \in W\}$. In the case of the classical phase function, we write

$$\mathscr{H}_0(x,\xi) = (x - i\xi, \xi), \qquad (x,\xi) \in \mathbb{R}^{2n},$$

and set $\mathscr{H}_0(\mathbb{R}^{2n}) = \Lambda_{\phi_0}$.
We recall the definition of s–Gevrey wave front set of a distribution via classical FBI transform (see Example 1.)

Definition 3.1. *Let u be a compactly supported distribution on $\mathbb{R}^n$. Let $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$. We say that $(x_0, \xi_0) \notin WF_s(u)$, $s \geq 1$, if there exist a neighborhood Ω of $x_0 - i\xi_0 \in \mathbb{C}^n$ and positive constants C, ε such that*

$$|e^{-\lambda\phi_0(z)} Tu(z,\lambda)| \leq C e^{-\varepsilon\lambda^{1/s}},$$

for every $z \in \Omega$ and $\lambda > 1$.

3.2. **Pseudodifferential Operators.** Let $\lambda \geq 1$ be a large positive parameter. We write

$$\tilde{D} = \frac{1}{\lambda} D, \qquad D = \frac{1}{i}\partial.$$

Denote by $q(x,\xi,\lambda)$ an analytic classical symbol (see [29]) and by $Q(x,\tilde{D},\lambda)$ the formal classical pseudodifferential operator associated to q:

$$Q(x,\tilde{D},\lambda)u(x,\lambda) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x-y)\xi} q(x,\xi,\lambda) u(y,\lambda)\, dy d\xi.$$

The above expression is formal and we realize it by choosing an integration path in the (complex) variable ξ of the form

$$\xi = \frac{2}{i}\frac{\partial\Phi}{\partial x}(x) + iR(\overline{x-y}),$$

where R is a sufficiently large positive constant and $|x-y| \leq r$, with r a sufficiently small positive constant and Φ denotes a weight function associated to a phase function.
Using "Kuranishi's trick" and applying Stokes theorem we have the following Ω-realization of Q

$$Q^\Omega u(x,\lambda) = \left(\frac{\lambda}{\pi}\right)^n \int_\Omega e^{2\lambda\psi(x,\bar{y})} \tilde{q}(x,\bar{y},\lambda) u(y) e^{-2\lambda\Phi(y)} L(dy), \tag{3.2}$$

where $L(dy) = (2i)^{-n} dy \wedge d\bar{y}$, Ω is a small neighborhood of $(x_0, \bar{x}_0) \in \mathbb{C}^{2n}$ and the integration path is $\theta = \bar{y}$. Here $\psi(x,y)$ is the holomorphic function defined near $(x_0, \bar{x}_0)$ by $\psi(x,\bar{x}) = \Phi(x)$. We point out that since Φ is plurisubharmonic, we have $\det \partial_x\partial_y\psi \neq 0$.
For more details on the Ω-realization we refer to [29], [22] as well as the appendix in [10].

The advantages of such a definition are:

1- if the principal symbol is real, Q^Ω is formally self adjoint in $L^2(\Omega, e^{-2\lambda\Phi})$ where $L^2(\Omega, e^{-2\lambda\Phi}) = \{u \mid \|u\|_\Phi < +\infty\}$ and

$$\|u\|_\Phi^2 = \int_\Omega e^{-2\lambda\Phi(x)}|u(x)|^2 L(dx).$$

2- If $\tilde{q}$ is a classical symbol of order zero, Q^Ω is uniformly bounded as $\lambda \to +\infty$, from $H_\Phi(\Omega)$ into itself. Here $H_\Phi(\Omega)$ is the space of all holomorphic functions $u(x,\lambda)$ such that for every $\epsilon > 0$ we have

$$|u(x,\lambda)| \le Ce^{\lambda(\Phi(x)+\epsilon)}, \qquad C > 0, \qquad x \in \Omega$$

with C independent of x and λ.

We recall the basic proposition on the composition of two pseudodifferential operators.

Proposition 3.1 ([22], Proposition 1.3). *Let Q_1 and Q_2 be pseudodifferential operators of order zero. Then they can be composed and*

$$Q_1^\Omega \circ Q_2^\Omega = (Q_1 \circ Q_2)^\Omega + R^\Omega,$$

where R^Ω is an error term whose norm is $\mathscr{O}(1)$ as an operator from $H_{\Phi+(1/C)d^2}$ to $H_{\Phi-(1/C)d^2}$. Here $d(x) = dist(x, \complement\Omega)$ denotes the distance of x to the boundary of Ω.

3.3. **A subelliptic estimate for sums of squares on the FBI side.** Let $X_1(x,\xi), \dots, X_\nu(x,\xi)$ be classical analytic symbols of the first order defined in an open neighborhood Ω of $(x_0,\xi_0) \in \Lambda_\Phi$. We assume also that the $X_{j|_{\Lambda_\Phi}}$ are real valued, so that we may think of the corresponding pseudodifferential operators as formally self-adjoint in H_Φ. Let

$$L(x,\tilde{D}) = \sum_{j=1}^{\nu} X_j^2(x,\tilde{D}). \tag{3.3}$$

Arguing as in [22], using Proposition 3.1 and (3.2), we see that the Ω-realization of L can be written as

$$L^\Omega = \sum_{j=1}^{\nu} (X_j^\Omega)^2 + \mathscr{O}(\lambda^2), \tag{3.4}$$

where $\mathscr{O}(\lambda^2)$ is continuous from $H_{\tilde{\Phi}}$ to $H_{\Phi-(1/C)d^2}$ with norm bounded by $C'\lambda^2$ (see Proposition (3.1).)

We assume also that there is a Poisson bracket of the symbols of the vector fields of length $\nu(x_0,\xi_0)$ which is elliptic at $(x_0,\xi_0) \in \Lambda_\Phi$ and it is shortest iterated Poisson bracket non zero at (x_0,ξ_0).

Next we state the sub-elliptic estimate on the FBI side.

Theorem 3.1 ([1]). *Let L^Ω be as in (3.4). We write $r = \nu(x_0,\xi_0)$. Let Ω_1 be a neighborhood of (x_0,ξ_0) such that $\Omega_1 \Subset \Omega$, then*

$$\lambda^{\frac{2}{r}}\|u\|_\Phi^2 + \sum_{j=1}^{\nu} \|X_j^\Omega u\|_\Phi^2 \le C\left(\langle L^\Omega u, u\rangle_\Phi + \lambda^\alpha \|u\|^2_{\Phi,\Omega\setminus\Omega_1}\right), \tag{3.5}$$

where C is a suitable positive constant and α is a suitable positive integer.

3.4. **A criterion of analytic hypoellipticity.** We consider an operator of the form

$$P(x,D) = \sum_{i,j=1}^{N} X_i(x,D)a_{ij}(x,D)X_j(x,D) + \sum_{j=1}^{N} b_j(x,D)X_j(x,D) + c(x,D), \tag{3.6}$$

where $D_j = D_{x_j} = i^{-1}\partial_{x_j}$ and the $a_{ij}(x,\xi)$, $b_j(x,\xi)$, $c(x,\xi)$ are analytic symbols of order zero such that

$$[a_{ij}]_{i,j=1,\dots,N} + [\bar{a}_{ji}]_{i,j=1,\dots,N} \geq c, \tag{3.7}$$

where $c > 0$ is a positive constant.
Let $(x_0,\xi_0) \in \mathit{Char}(P)$. Denote by U a neighborhood of the point (x_0,ξ_0) in $\mathbb{R}^{2n}$. We assume that $\mathit{Char}(P) \cap U$ is a symplectic real analytic manifold.
We assume that there exists a function $r\colon U \to [0,+\infty[$, real analytic in U, such that

(1) $r(x_0,\xi_0) = 0$ and $r(x,\xi) > 0$ in $U \setminus \{(x_0,\xi_0)\}$.
(2) There exist real analytic functions, $\alpha_{j,k}(x,\xi)$, defined in U, such that

$$\{r(x,\xi), X_j(x,\xi)\} = \sum_{\ell=1}^{N} \alpha_{j,\ell}(x,\xi)X_\ell(x,\xi), \tag{3.8}$$

where $j = 1,\dots,N$, $(x,\xi) \in U$.

Then the following result holds.

Theorem 3.2 ([2]). *Let (x_0,ξ_0) be a point in* $\mathrm{Char}(P)$. *Assume that* $\mathrm{Char}(P)$ *is symplectic near (x_0,ξ_0) and, as above, that there exists a function $r \in C^\omega(U)$, satisfying conditions (1), (2). Then we have that if $(x_0,\xi_0) \notin WF_a(Pu)$, then $(x_0,\xi_0) \notin WF_a(u)$.*

Our purpose will be to use Theorem 3.2 in order to characterize the microlocal hypoellipticity of the operator $M_{n,m}$ at the stratum Σ_0.

4. Microlocal regularity, proof of Theorem 1.2

We recall that the characteristic set of $M_{n,m}$ is

$$\mathrm{Char}(M_{n,m}) = \{(x,y,\xi,\eta) \in T^*\mathbb{R}^2 \setminus \{0\} \,:\, x = 0 = \xi, \text{ and } \eta \neq 0\},$$

and that the related *Poisson-Treves* stratification is given by

$$\Sigma_0 = \{(x,y,\xi,\eta) \in T^*\mathbb{R}^2 \setminus \{0\} \,:\, x = 0 = \xi,\, y \neq 0,\, \eta \neq 0\},$$
$$\Sigma_1 = \{(x,y,\xi,\eta) \in T^*\mathbb{R}^2 \setminus \{0\} \,:\, x = 0 = \xi,\, y = 0,\, \eta \neq 0\}.$$

We point out that Σ_0 is symplectic manifold of codimension two.
Let $\rho_1 \in \Sigma_1$, $\rho_1 = (0,0,0,1)$ with $\eta \neq 0$. In view of [9] we have the estimate

$$\|\psi_k D_y^k u\|_{\frac{1}{2(n+1)}} \leq C^{k+1} k^{s_1 k},$$

where $s_1 = \frac{2m}{2m-1}$, C is a suitable positive constant independent of k. Here u is a solution of the problem $M_{n,m}u = f$, $\rho_1 \notin WF_{s_1}(f)$, and ψ_k is an Ehrenpreis cut-off sequence associated to the couple (Ω_0, Ω_1), both open neighborhoods of the origin with $\overline{\Omega}_0 \Subset \Omega_1$. We conclude that

$$|\widehat{\phi_k u}(\zeta)| \leq C^{k+1} k^{ks_1} |\zeta|^{-k}, \quad k = 1, 2, \dots,$$

$\zeta = (\xi, \eta)$, for every $\zeta \in \Gamma$ conic neighborhood of $(0, 1)$ and for a suitable positive constant C independent of N. By the Definition 3.1, we have that $\rho_1 \notin WF_{s_1}(u)$.

Let $\rho_0 \in \Sigma_0$, without loss of generality we choose $\rho_0 = (0, y_0, 0, 1)$, $y_0 \neq 0$. The depth of ρ_0 is $n+1$, i.e. the shortest iterated Poisson bracket of the symbols of the vector fields non zero at ρ_0 has length $n+1$. Let $\tilde{u}$, $\widetilde{M}_{n,m}$ and $\widetilde{X}_j$, $j = 1, 2$, the FBI transform of u, $M_{n,m}$ and the vector fields X_j respectively. By (3.2), the Proposition 3.1 and the Theorem (3.1) we have

$$\lambda^{2/n+1} \|\tilde{u}\|^2_{\phi_0} + \sum_{j=0}^{2} \|\widetilde{X}^{\Omega}_j \tilde{u}\|^2_{\phi_0} \leq C \left(\langle \widetilde{M}^{\Omega}_{n,m} \tilde{u}, \tilde{u} \rangle_{\phi_0} + \lambda^{\alpha} \|\tilde{u}\|^2_{\phi_0, \Omega \setminus \Omega_1} \right), \tag{4.1}$$

where Ω and Ω_1 are neighborhoods of the point $(0, y_0 - i) \in \mathbb{C}^2$, $\Omega_1 \Subset \Omega$ and $\widetilde{M}^{\Omega}_{n,m}$ is the $\Omega-$realization of the operator $\widetilde{M}_{n.m}$.
In order to use the Theorem 3.2, we have to construct a real analytic function r microlocally defined in a conic neighborhood of the point $\rho_0 \in \Sigma_0$, satisfying conditions (1), (2), see the previous subsection. Let U be a small neighborhood of the point ρ_0 in $\mathbb{R}^4 \setminus \{0\}$. Let $r \colon U \to [0, +\infty[$, the real analytic function of the form

$$r(x, y, \xi, \eta) = \xi^2 + x^{2n+2} + (y - y_0)^2 + (\eta - 1)^2. \tag{4.2}$$

We have that $r(\rho_0) = 0$, $r(x, y, \xi, \eta) > 0$ in $U \setminus \{\rho_0\}$ and

$$\{X_1, r\} = \{\xi, r\} = (2n+2) x^{2n+1} = \frac{2n+2}{\eta} x^{2n+1} \eta = \alpha_1(\eta) X_2,$$

$$\begin{aligned}\{X_2, r\} &= \{x^{2n+1}\eta, r\} = -2(2n+1) x^{2n} \eta \xi + 2(y - y_0) x^{2n+1} \\ &= \left(-2(2n+1) x^{2n} \eta\right) \xi + \left(\frac{2(y - y_0)}{\eta} \right) x^{2n+1} \eta = \alpha_{2,1}(x, \eta) X_1 + a_{2,2}(y, \eta) X_2,\end{aligned}$$

$$\begin{aligned}\{X_3, r\} &= \{x^n y^m \eta, r\} = -2n x^{n-1} y^m \eta \xi + 2(y - y_0) x^n y^m - 2m x^n y^{m-1} (\eta - 1) \eta \\ &= \left(-2n x^{n-1} y^m \eta\right) \xi + \left(\frac{2(y - y_0)}{\eta} \right) x^n y^m \eta - \left(2m \frac{\eta - 1}{y} \right) x^n y^m \eta \\ &= \alpha_{3,1}(x, y, \eta) X_1 + \alpha_{2,2}(y, \eta) X_3.\end{aligned}$$

r satisfies the properties (1) and (2), see (3.8). Then by Theorem 3.2 we have that if $\rho_0 \notin WF_a(M_{n,m} u)$, then $\rho_0 \notin WF_a(u)$.

This concludes the proof of Theorem 1.2.

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