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Capelli-Deruyts bitableaux and the classical Capelli generators of the center of the enveloping algebra $U(\mathfrak{gl}(n))$

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Capelli-Deruyts bitableaux and the classical Capelli generators of the center of the enveloping algebra $U(gl(n))$

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Abstract

In this paper, we consider a special class of Capelli bitableaux, namely the Capelli bitableaux of the form $\mathbf{K}^\lambda = [Der_\lambda^* | Der_\lambda] \in U(gl(n))$. The main results we prove are the hook coefficient lemma and the expansion theorem. Capelli-Deruyts bitableaux \mathbf{K}_n^p of rectangular shape are of particular interest since they are central elements in the enveloping algebra $U(gl(n))$. The expansion theorem implies that the central element \mathbf{K}_n^p is explicitly described as a polynomial in the classical Capelli central elements $\mathbf{H}_n^{(j)}$. The hook coefficient lemma implies that the Capelli-Deruyts bitableaux \mathbf{K}_n^p are (canonically) expressed as the products of column determinants.

Keyword: Capelli bitableaux; Capelli-Deruyts bitableaux; Capelli column determinants; central elements in $U(gl(n))$; Lie superalgebras.

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1 Introduction

The study of the center $\zeta(n)$ of the enveloping algebra $\mathbf{U}(gl(n))$ of the general linear Lie algebra $gl(n, \mathbb{C})$, and the study of the algebra $\Lambda^*(n)$ of shifted symmetric polynomials have noble and rather independent origins and motivations. The theme of central elements in $\mathbf{U}(gl(n))$ is a standard one in the general theory of Lie algebras, see e.g. [18]. It is an old and actual one, since it is an offspring of the celebrated Capelli identity (see e.g. [11], [14], [21], [22], [36], [41], [42]), relates to its modern generalizations and applications (see e.g. [1], [24], [25], [29], [30], [31], [32], [40]) as well as to the theory of *Yangians* (see, e.g. [27], [28]).

Capelli bitableaux $[S|T]$ and their variants (such as *Young-Capelli bitableaux* and *double Young-Capelli bitableaux*) have been proved to be relevant in the study of the enveloping algebra $\mathbf{U}(gl(n)) = \mathbf{U}(gl(n), \mathbb{C})$ of the general linear Lie algebra and of its center $\zeta(n)$.

To be more specific, the *superalgebraic method of virtual variables* (see, e.g. [4], [5], [6], [7], [8], [9], [10]) allowed us to express remarkable classes of elements in $\mathbf{U}(gl(n))$, namely,

- the class of *Capelli bitableaux* $[S|T] \in \mathbf{U}(gl(n))$
- the class of *Young-Capelli bitableaux* $[S|\boxed{T}] \in \mathbf{U}(gl(n))$
- the class of *double Young-Capelli bitableaux* $[\boxed{S | T}] \in \mathbf{U}(gl(n))$

as the images - with respect to the $Ad_{gl(n)}$ -adjoint equivariant Capelli *devirtualization epimorphism* - of simple expressions in an enveloping superalgebra $\mathbf{U}(gl(m_0|m_1 + n))$ (see, e.g [10]).

Capelli (determinantal) bitableaux are generalizations of the famous *column determinant element* in $\mathbf{U}(gl(n))$ introduced by Capelli in 1887 [11] (see, e.g. [9]). Young-Capelli bitableaux were introduced by the present authors several years ago [5], [6], [7] and might be regarded as generalizations of the Capelli column determinant elements in $\mathbf{U}(gl(n))$ as well as of the *Young symmetrizers* of the classical representation theory of symmetric groups (see, e.g. [42]). Double Young-Capelli bitableaux play a crucial role in the study of the center $\zeta(n)$ of the enveloping algebra ([8], [10]).

In plain words, the Young-Capelli bitableau $[S|\boxed{T}]$ is obtained by adding a *column symmetrization* to the Capelli bitableau $[S|T]$ and turn out to be a linear combination of

Capelli bitableaux (see, e.g [10], Proposition 2.13). The double Young-Capelli bitableau $[\boxed{S | T}]$ is obtained by adding a further *row skew-symmetrization* to the Young-Capelli bitableau $[S\boxed{T}]$ ([10], Proposition 5.1), turn out to be a linear combination of Young-Capelli bitableaux (see, e.g [10], Proposition 2.14) and, therefore, it is in turn a linear combination of Capelli bitableaux.

Capelli bitableaux are the preimages - with respect to the *Koszul linear* $\mathbf{U}(gl(n))$ -equivariant isomorphism \mathcal{K} from the enveloping algebra $\mathbf{U}(gl(n))$ to the polynomial algebra $\mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(gl(n))$ ([26], [7], [9]) - of the classical *determinant bitableaux* (see, e.g. [19], [17], [16], [20], [4]). Hence, they are ruled by the *straightening laws* and the set of standard Capelli bitableaux is a basis of $\mathbf{U}(gl(n))$.

The set of standard Young-Capelli bitableaux is another relevant basis of $\mathbf{U}(gl(n))$ whose elements act in a nondegenerate orthogonal way on the set of standard right symmetrized bitableaux (the *Gordan-Capelli basis* of $\mathbb{C}[M_{n,n}]$) and this fact leads to explicit complete decompositions of the semisimple $\mathbf{U}(gl(n))$ -module $\mathbb{C}[M_{n,n}]$ (see, e.g. [4], [5]).

The linear combinations of double Young-Capelli bitableaux

$$\mathbf{S}_\lambda(n) = \frac{1}{H(\tilde{\lambda})} \sum_S [\boxed{S | S}] \in \mathbf{U}(gl(n)), \quad (1)$$

where the sum is extended to all row (strictly) increasing tableaux S of shape $sh(S) = \tilde{\lambda} \vdash h$, $\tilde{\lambda}$ the conjugate shape/partition of λ ⁽¹⁾, are *central elements* of $\mathbf{U}(gl(n))$.

We called the elements $\mathbf{S}_\lambda(n)$ the *Schur elements*. The Schur elements $\mathbf{S}_\lambda(n)$ are the preimages - with respect to the Harish-Chandra isomorphism - of the elements of the basis of shifted Schur polynomials $s_{\lambda|n}^*$ of the algebra $\Lambda^*(n)$ of shifted symmetric polynomials [38], [33]. Hence, the Schur elements are the same [10] as the *quantum immanants* ([38], [31], [32], [33]), first presented by Okounkov as traces of *fusion matrices* ([31], [32]) and, recently, described by the present authors as linear combinations (with explicit coefficients) of “diagonal” *Capelli immanants* [8]. Presentation (1) of Schur elements/quantum immanants doesn’t involve the irreducible characters of symmetric groups. Furthermore, it is better suited to the study of the eigenvalues on irreducible $gl(n)$ -modules and of the duality in the algebra $\zeta(n)$, as well as to the study of the limit $n \rightarrow \infty$, via the *Olshanski decomposition* (see, Olshanski [34], [35] and Molev [27], pp. 928 ff.)

In this paper, we consider a special class of Capelli bitableaux, namely the class of *Capelli-Deruyts bitableaux*. These elements are Capelli bitableaux of the form

$$\mathbf{K}^\lambda = [Der_\lambda^* | Der_\lambda] \in \mathbf{U}(gl(n)),$$

¹Given a partition (shape) $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p) \vdash n$, let $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2 \geq \dots \geq \tilde{\lambda}_q) \vdash n$ denote its *conjugate* partition, where $\tilde{\lambda}_s = \#\{t; \lambda_t \geq s\}$.

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p)$ is a partition with $\lambda_1 \leq n$, and

- Der_λ is the *Deruyts tableaux* of shape λ , that is the Young tableau of shape λ :

$$Der_\lambda = \begin{bmatrix} 1 & 2 & \dots & \dots & \dots & \lambda_1 \\ 1 & 2 & \dots & \dots & \lambda_2 & \\ \dots & \dots & \dots & & & \\ 1 & 2 & \dots & \lambda_p & & \end{bmatrix}$$

- Der_λ^* is the *reverse Deruyts tableaux* of shape λ , that is the Young tableau of shape λ :

$$Der_\lambda^* = \begin{bmatrix} \lambda_1 & \dots & \dots & \dots & 2 & 1 \\ \lambda_2 & \dots & \dots & 2 & 1 & \\ \dots & \dots & \dots & & & \\ \lambda_p & \dots & 2 & 1 & & \end{bmatrix}.$$

Capelli-Deruyts bitableaux arise, in a natural way, as generalizations to arbitrary shapes $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p)$ of the well-known *Capelli column determinant*² elements:

$$\mathbf{H}_n^{(n)} = \mathbf{cdet} \begin{pmatrix} e_{1,1} + (n-1) & e_{1,2} & \dots & e_{1,n} \\ e_{2,1} & e_{2,2} + (n-2) & \dots & e_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n,1} & e_{n,2} & \dots & e_{n,n} \end{pmatrix} \in \mathbf{U}(gl(n)), \quad (2)$$

introduced by Alfredo Capelli [11] in the celebrated identities that bear his name (see, e.g. [11], [14], [21], [22], [36], [41], [42], [1], [24], [25], [29], [30], [31], [32], [40]).

The main results we prove are the following:

- **The hook coefficient lemma:** let v_μ be a $gl(n, \mathbb{C})$ -highest weight vector of weight $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_n)$, with $\mu_i \in \mathbb{N}$ for every $i = 1, 2, \dots, n$. Then, v_μ is an *eigenvector* of the action of the Capelli-Deruyts bitableau \mathbf{K}^λ with *eigenvalue* the (signed) product of *hook numbers* in the Ferrers diagram of the partition μ (Proposition 5).
- **The expansion theorem:** the Capelli-Deruyts bitableau $\mathbf{K}^\lambda \in \mathbf{U}(gl(n))$ expands as a polynomial, with explicit coefficients, in the *Capelli generators*

$$\mathbf{H}_k^{(j)} = \sum_{1 \leq i_1 < \dots < i_j \leq k} \mathbf{cdet} \begin{pmatrix} e_{i_1, i_1} + (j-1) & e_{i_1, i_2} & \dots & e_{i_1, i_j} \\ e_{i_2, i_1} & e_{i_2, i_2} + (j-2) & \dots & e_{i_2, i_j} \\ \vdots & \vdots & \ddots & \vdots \\ e_{i_k, i_1} & e_{i_k, i_2} & \dots & e_{i_k, i_j} \end{pmatrix}$$

²The symbol \mathbf{cdet} denotes the column determinat of a matrix $A = [a_{ij}]$ with noncommutative entries: $\mathbf{cdet}(A) = \sum_{\sigma} (-1)^{|\sigma|} a_{\sigma(1),1} a_{\sigma(2),2} \dots a_{\sigma(n),n}$.

of the centers of the enveloping algebras $\mathbf{U}(gl(k))$, $k = 1, 2, \dots, n$, $j = 1, 2, \dots, k$ (Theorem 3).

Capelli-Deruyts bitableaux \mathbf{K}_n^p of *rectangular shape* $\lambda = n^p = (n, \overset{p \text{ times}}{n}, \dots, n)$ are of particular interest since they are *central elements* in the enveloping algebra $\mathbf{U}(gl(n))$.

- The expansion theorem implies that the Capelli-Deruyts bitableau \mathbf{K}_n^p (with p rows) equals the product of the Capelli-Deruyts bitableau \mathbf{K}_n^{p-1} (with $p-1$ rows) and the central element

$$\mathbf{C}_n(p-1) = \sum_{j=0}^n (-1)^{n-j} (p-1)_{n-j} \mathbf{H}_n^{(j)}$$

(see Corollary 1). Hence, by iterating this procedure, the central element \mathbf{K}_n^p is explicitly described as a polynomial in the classical Capelli central elements $\mathbf{H}_n^{(j)}$ (see Corollary 3).

- The hook coefficient lemma implies -via the HarishChandra isomorphism- that the element $\mathbf{C}_n(p)$ also equals the column determinant element

$$\mathbf{H}_n(p) = \mathbf{cdet} [e_{h,k} + \delta_{hk}(-p + n - h)]_{h,k=1,\dots,n} \in \mathbf{U}(gl(n)).$$

Notice that

$$\mathbf{H}_n(0) = \mathbf{cdet} \begin{pmatrix} e_{1,1} + (n-1) & e_{1,2} & \dots & e_{1,n} \\ e_{2,1} & e_{2,2} + (n-2) & \dots & e_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n,1} & e_{n,2} & \dots & e_{n,n} \end{pmatrix} = \mathbf{H}_n^{(n)},$$

the classical Capelli column determinant element.

From these facts, the Capelli-Deruyts bitableaux \mathbf{K}_n^p are (canonically) expressed as the products of column determinants:

$$\mathbf{K}_n^p = (-1)^{n \binom{p}{2}} \mathbf{H}_n(p-1) \cdots \mathbf{H}_n(1) \mathbf{H}_n(0)$$

(see Corollary 7).

The method of *superalgebraic virtual variables* ([4], [5], [6], [7], [8], [9], [10]) plays a crucial role in the present paper; we provide a short presentation of the method in the Appendix.

2 The classical Capelli identities

The algebra of algebraic forms $\mathbf{f}(\underline{x}_1, \dots, \underline{x}_n)$ in n vector variables $\underline{x}_i = (x_{i1}, \dots, x_{id})$ of dimension d is the polynomial algebra in $n \times d$ (commutative) variables:

$$\mathbb{C}[M_{n,d}] = \mathbb{C}[x_{ij}]_{i=1,\dots,n;j=1,\dots,d},$$

and $M_{n,d}$ denotes the matrix with n rows and d columns with “generic” entries x_{ij} :

$$M_{n,d} = [x_{ij}]_{i=1,\dots,n;j=1,\dots,d} = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ x_{21} & \cdots & x_{2d} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nd} \end{bmatrix}. \quad (3)$$

The algebra $\mathbb{C}[M_{n,d}]$ is a $\mathbf{U}(gl(n))$ -module, with respect to the action:

$$e_{x_j, x_i} \cdot \mathbf{f} = D_{x_j, x_i}^l(\mathbf{f}),$$

for every $\mathbf{f} \in \mathbb{C}[M_{n,d}]$, where, for any $i, j = 1, 2, \dots, n$, where D_{x_j, x_i}^l is the unique derivation of the algebra $\mathbb{C}[M_{n,d}]$ such that

$$D_{x_j, x_i}^l(x_{hk}) = \delta_{ih} x_{jk},$$

for every $k = 1, 2, \dots, d$.

Proposition 1. (The Capelli identities, 1887)

$$\mathbf{H}_n^{(n)}(\mathbf{f}) = \begin{cases} 0 & \text{if } n > d \\ [\underline{x}_1, \dots, \underline{x}_n] \Omega_n(\mathbf{f}) & \text{if } n = d, \end{cases}$$

where $\mathbf{f}(\underline{x}_1, \dots, \underline{x}_n) \in \mathbb{C}[M_{n,d}]$ is an algebraic form (polynomial) in the n vector variables $\underline{x}_i = (x_{i1}, \dots, x_{id})$ of dimension d , and, if $d = n$, $[\underline{x}_1, \dots, \underline{x}_n]$ is the bracket

$$[\underline{x}_1, \dots, \underline{x}_n] = \det \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \vdots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix},$$

and Ω_n is the Cayley Ω -process

$$\Omega_n = \det \begin{bmatrix} \frac{\partial}{\partial x_{11}} & \cdots & \frac{\partial}{\partial x_{1n}} \\ \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_{n1}} & \cdots & \frac{\partial}{\partial x_{nn}} \end{bmatrix}.$$

□

From [9], we recall that the determinant element $\mathbf{H}_n^{(n)}$ can be written as the (one row) *Capelli-Deruyts bitableau* $[n \dots 21|12 \dots n]$ ([5], see also [8], [26]).

Proposition 2. *The element*

$$\mathbf{H}_n^{(n)} = \mathbf{cdet} \begin{pmatrix} e_{1,1} + (n-1) & e_{1,2} & \dots & e_{1,n} \\ e_{2,1} & e_{2,2} + (n-2) & \dots & e_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n,1} & e_{n,2} & \dots & e_{n,n} \end{pmatrix} \in \mathbf{U}(\mathfrak{gl}(n))$$

equals the one row Capelli-Deruyts bitableau (see, e.g. Subsection 9.6 below)

$$[n \dots 21|12 \dots n] = \mathbf{p}(e_{n,\alpha} \cdots e_{2,\alpha} e_{1,\alpha} \cdot e_{\alpha,1} e_{\alpha,2} \cdots e_{\alpha,n}),$$

where \mathbf{p} denotes the Capelli devirtualization epimorphism (see, e.g. Subsection 9.5 below).

From eq. (2) and Proposition 2, it follows:

Proposition 3. *We have:*

1. *Let v_μ be a $\mathfrak{gl}(n, \mathbb{C})$ -highest weight vector of weight $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_n)$, with $\mu_i \in \mathbb{N}$ for every $i = 1, 2, \dots, n$. Then v_μ is an eigenvector of the action of $\mathbf{H}_n^{(n)}$ with eigenvalue:*

$$(\mu_1 + n - 1)(\mu_2 + n - 2) \cdots \mu_n.$$

In symbols,

$$\mathbf{H}_n^{(n)} \cdot v_\mu = ((\mu_1 + n - 1)(\mu_2 + n - 2) \cdots \mu_n) v_\mu.$$

2. *The element $\mathbf{H}_n^{(n)}$ is central in the enveloping algebra $\mathbf{U}(\mathfrak{gl}(n))$.*

3 The Capelli-Deruyts bitableaux in $\mathbf{U}(\mathfrak{gl}(n))$

We generalize the *one row* Capelli bitableau $\mathbf{H}_n^{(n)} = [n \dots 21|12 \dots n]$ to arbitrary shapes (partitions)

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p), \quad \lambda_i \in \mathbb{Z}^+.$$

3.1 Capelli-Deruyts bitableaux \mathbf{K}^λ of shape λ .

Given a partition(shape) $\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p$, we recall that the *Deruyts tableaux* of shape λ is the Young tableau

$$Der_\lambda = (\underline{\lambda}_1, \underline{\lambda}_2, \dots, \underline{\lambda}_p) \quad (4)$$

and the *reverse Deruyts tableaux* of shape λ is the Young tableau

$$Der_\lambda^* = (\underline{\lambda}_1^*, \underline{\lambda}_2^*, \dots, \underline{\lambda}_p^*),$$

where

$$\underline{\lambda}_i = 1 \ 2 \ \dots \ \lambda_i$$

and

$$\underline{\lambda}_i^* = \lambda_i \ \dots \ 2 \ 1,$$

for every $i = 1, 2, \dots, p$.

The *Capelli-Deruyts bitableau* \mathbf{K}^λ is the Capelli bitableau in $\mathbf{U}(gl(n))$, $n \geq \lambda_1$:

$$\mathbf{K}^\lambda = [Der_\lambda^* | Der_\lambda] = \mathfrak{p}(e_{Der_\lambda^* C_\lambda} \cdot e_{C_\lambda Der_\lambda}),$$

where \mathfrak{p} denotes the Capelli devirtualization epimorphism and $e_{Der_\lambda^* C_\lambda}$, $e_{C_\lambda Der_\lambda}$ are *bitableaux monomials* (see., e.g. Subsection 9.6, eq. (9.6)).

Example 1. Let $\lambda = (3, 2, 2)$. Then

$$\begin{aligned} \mathbf{K}^{(3,2,2)} &= \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \\ 2 & 1 & & 1 & 2 & \end{array} \right] = \\ &= \mathfrak{p}(e_{3\alpha_1} e_{2\alpha_1} e_{1\alpha_1} e_{2\alpha_2} e_{1\alpha_2} e_{2\alpha_3} e_{1\alpha_3} \cdot e_{\alpha_1 1} e_{\alpha_1 2} e_{\alpha_1 3} e_{\alpha_2 1} e_{\alpha_2 2} e_{\alpha_3 1} e_{\alpha_3 2}) \in \mathbf{U}(gl(n)), \quad n \geq 3, \end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3$ are (arbitrary, distinct) positive virtual symbols.

□

Remark 1. Given a Young tableau

$$T = \left[\begin{array}{cccccc} x_{11} & x_{12} & \cdots & \cdots & \cdots & x_{1\lambda_1} \\ x_{21} & x_{22} & \cdots & \cdots & \cdots & x_{2\lambda_2} \\ \vdots & & & & & \\ x_{i1} & x_{i2} & \cdots & \cdots & \cdots & x_{i\lambda_i} \\ \vdots & & & & & \\ x_{p1} & x_{p2} & \cdots & \cdots & \cdots & x_{p\lambda_p} \end{array} \right], \quad x_{ij} \in X, \quad (5)$$

of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p)$ over the set X is said to be of Deruyts type whenever

$$\{x_{i1}, x_{i2}, \dots, x_{i\lambda_i}\} \subseteq \{x_{i-1\ 1}, x_{i-1\ 2}, \dots, x_{i-1\ \lambda_{i-1}}\},$$

for $i = 2, \dots, p$.

Clearly, any tableau of Deruyts type (5) can be regarded as a Deruyts tableau (4), by suitably renaming and reordering the entries.

3.2 The Capelli-Deruyts bitableaux \mathbf{K}_n^p of rectangular shape $\lambda = n^p$

Given any positive integer p , we define the *rectangular Capelli/Deruyts bitableau*, with p rows of length $\lambda_1 = \lambda_2 = \dots = \lambda_p = n$:

$$\mathbf{K}_n^p = \left[\begin{array}{cccccc|cccc} n & n-1 & \dots & 3 & 2 & 1 & 1 & 2 & 3 & \dots & n-1 & n \\ n & n-1 & \dots & 3 & 2 & 1 & 1 & 2 & 3 & \dots & n-1 & n \\ \dots & & & & & & \dots & & & & & \\ \dots & & & & & & \dots & & & & & \\ n & n-1 & \dots & 3 & 2 & 1 & 1 & 2 & 3 & \dots & n-1 & n \end{array} \right] \in \mathbf{U}(\mathfrak{gl}(n)).$$

From Proposition 26, we infer:

Proposition 4. *The elements \mathbf{K}_n^p are central in $\mathbf{U}(\mathfrak{gl}(n))$.*

Set, by definition, $\mathbf{K}_n^0 = \mathbf{1}$.

4 The hook eigenvalue Theorem for Capelli-Deruyts bitableaux

Any rectangular Capelli-Deruyts bitableau \mathbf{K}_n^p well behaves on $\mathfrak{gl}(n, \mathbb{C})$ -highest weight vectors (compare with Proposition 3, item 1)).

Theorem 1. *(The hook coefficient lemma)*

Let v_μ be a highest weight vector of weight $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_n)$, with $\mu_i \in \mathbb{N}$ for every $i = 1, 2, \dots, n$. Then v_μ is an eigenvector of the action of \mathbf{K}_n^p with eigenvalue the (signed) product of hook numbers in the Ferrers diagram of the partition μ :

$$(-1)^{\binom{p}{2}n} \left(\prod_{j=1}^p (\mu_1 - j + n)(\mu_2 - j + n - 1) \dots (\mu_n - j + 1) \right).$$

In symbols,

$$\mathbf{K}_n^p \cdot v_\mu = (-1)^{\binom{p}{2}n} \left(\prod_{j=1}^p (\mu_1 - j + n)(\mu_2 - j + n - 1) \cdots (\mu_n - j + 1) \right) v_\mu.$$

Theorem 1 generalizes to arbitrary Capelli-Deruyts bitableaux \mathbf{K}_λ of shape λ as follows:

Proposition 5. *Let v_μ be a highest weight vector of weight $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_n)$, with $\mu_i \in \mathbb{N}$ for every $i = 1, 2, \dots, n$. Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_p)$ be a partition(shape). Then*

$$\begin{aligned} \mathbf{K}^\lambda \cdot v_\mu = & (-1)^{\lambda_p(\lambda_{p-1} + \dots + \lambda_1) + \lambda_{p-1}(\lambda_{p-2} + \dots + \lambda_1) + \dots + \lambda_2 \lambda_1} \times \\ & \times \left(\prod_{i=1}^p (\mu_1 - i + \lambda_i)(\mu_2 - i + \lambda_i - 1) \cdots (\mu_{\lambda_i} - i + 1) \right) v_\mu. \end{aligned}$$

5 The factorization Theorem for Capelli-Deruyts bitableaux

Let $J = \{j_1 < j_2 < \dots < j_k\} \subseteq \underline{n} = \{1, 2, \dots, n\}$. With a slight abuse of notation, we write \underline{J} for the increasing word $\underline{J} = j_1 j_2 \cdots j_k$ and \underline{J}^* for the decreasing word $\underline{J}^* = j_k \cdots j_2 j_1$.

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p)$, set $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_p$.

We have

$$\mathbf{K}^\lambda = \left[\begin{array}{c|c} \underline{\lambda_1^*} & \underline{\lambda_1} \\ \underline{\lambda_2^*} & \underline{\lambda_2} \\ \vdots & \vdots \\ \underline{\lambda_p^*} & \underline{\lambda_p} \end{array} \right]$$

and, consistently, we write, for $J \subseteq M$,

$$\left[\begin{array}{c} \mathbf{K}^\lambda \\ J \end{array} \right] = \left[\begin{array}{c|c} \underline{\lambda_1^*} & \underline{\lambda_1} \\ \underline{\lambda_2^*} & \underline{\lambda_2} \\ \vdots & \vdots \\ \underline{\lambda_p^*} & \underline{\lambda_p} \\ \underline{J^*} & \underline{J} \end{array} \right], \quad [J] = [\underline{J^*} | \underline{J}].$$

Theorem 2. (The row insertion theorem) *Let $m \leq \lambda_p$. Given $M \subseteq \underline{\lambda_p}$, $|M| = m$, we have*

$$[M^* | M] \mathbf{K}^\lambda = \sum_{k=0}^m \langle p \rangle_{m-k} \sum_{J: J \subseteq M; |J|=k} (-1)^{|\lambda|k} \left[\begin{array}{c} \mathbf{K}^\lambda \\ J \end{array} \right],$$

where $\langle p \rangle_j$ denotes the raising factorial

$$\langle p \rangle_j = p(p+1) \cdots (p+j-1).$$

□

Theorem 3. (The expansion theorem) Let $m \leq \lambda_p$. Given $M \subseteq \underline{\lambda}_p$, $|M| = m$, we have

$$(-1)^{|\lambda|} \begin{bmatrix} \mathbf{K}^\lambda \\ M \end{bmatrix} = \sum_{k=0}^m (-1)^{m-k} (p)_{m-k} \sum_{J; J \subseteq M; |J|=k} [J^* | J] \mathbf{K}^\lambda,$$

where $(p)_j$ denotes the falling factorial

$$(p)_j = p(p-1) \cdots (p-j+1).$$

Proof. By Theorem 2,

$$\begin{aligned} & \sum_{k=0}^m (-1)^{m-k} (p)_{m-k} \sum_{J; J \subseteq M; |J|=k} [J] \mathbf{K}^\lambda = \\ &= \sum_{k=0}^m (-1)^{m-k} (p)_{m-k} \sum_{J; J \subseteq M; |J|=k} \sum_{i=0}^k \langle p \rangle_{k-i} \sum_{I; I \subseteq J; |I|=i} (-1)^{|\lambda|} \begin{bmatrix} \mathbf{K}^\lambda \\ I \end{bmatrix} = \\ &= \sum_{i=0}^m \sum_{k=i}^m \sum_{I; I \subseteq M; |I|=i} \left(\sum_{J; M \supseteq J \supseteq I; |J|=k} (-1)^{m-k} (p)_{m-k} \langle p \rangle_{k-i} \right) (-1)^{|\lambda|} \begin{bmatrix} \mathbf{K}^\lambda \\ I \end{bmatrix} = \\ &= \sum_{i=0}^m \sum_{I; I \subseteq M; |I|=i} \left(\sum_{k=i}^m (-1)^{m-k} (p)_{m-k} \langle p \rangle_{k-i} \binom{m-i}{k-i} \right) (-1)^{|\lambda|} \begin{bmatrix} \mathbf{K}^\lambda \\ I \end{bmatrix} = \\ &= \sum_{i=0}^m \sum_{I; I \subseteq M; |I|=i} \left((m-i)! \sum_{k=i}^m (-1)^{m-k} \binom{p}{m-k} \left\langle \frac{p}{k-i} \right\rangle \right) (-1)^{|\lambda|} \begin{bmatrix} \mathbf{K}^\lambda \\ I \end{bmatrix} = \\ &= \sum_{i=0}^m \sum_{I; I \subseteq M; |I|=i} \left((m-i)! \delta_{m-i,0} \right) (-1)^{|\lambda|} \begin{bmatrix} \mathbf{K}^\lambda \\ I \end{bmatrix} = \\ &= \sum_{i=0}^m \sum_{I; I \subseteq M; |I|=i} \left((m-i)! \delta_{m,i} \right) (-1)^{|\lambda|} \begin{bmatrix} \mathbf{K}^\lambda \\ I \end{bmatrix} = (-1)^{|\lambda|} \begin{bmatrix} \mathbf{K}^\lambda \\ M \end{bmatrix}. \end{aligned}$$

□

Example 2.

1. We have

$$\begin{aligned}
[21|12] \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \end{array} \right] &= 6 \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \end{array} \right] + 2 \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \\ 1 & & & & & 1 \end{array} \right] \\
&+ 2 \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \\ 2 & & & 2 & & \end{array} \right] + \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \\ 2 & 1 & & 1 & 2 & \end{array} \right].
\end{aligned}$$

2. We have

$$\begin{aligned}
\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \\ 2 & 1 & & 1 & 2 & \end{array} \right] &= 2 \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \end{array} \right] - 2[1|1] \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \end{array} \right] \\
&- 2[2|2] \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \end{array} \right] + [2\ 1|1\ 2] \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \end{array} \right].
\end{aligned}$$

6 The center $\zeta(n)$ of $\mathbf{U}(gl(n))$

6.1 The Capelli generators of the center $\zeta(n)$ of $\mathbf{U}(gl(n))$

In the enveloping algebra $\mathbf{U}(gl(n))$, given any increasing k -tuple integers $1 \leq i_1 < \dots < i_k \leq n$.

We recall that the column determinant

$$\mathbf{cdet} \begin{pmatrix} e_{i_1, i_1} + (k-1) & e_{i_1, i_2} & \dots & e_{i_1, i_k} \\ e_{i_2, i_1} & e_{i_2, i_2} + (k-2) & \dots & e_{i_2, i_k} \\ \vdots & \vdots & \vdots & \vdots \\ e_{i_k, i_1} & e_{i_k, i_2} & \dots & e_{i_k, i_k} \end{pmatrix} \in \mathbf{U}(gl(n))$$

equals the *one-row* Capelli-Deruyts bitableau

$$[i_k i_{k-1} \dots i_1 | i_1 \dots i_{k-1} i_k] = \mathfrak{p} (e_{i_k \alpha} e_{i_{k-1} \alpha} \dots e_{i_1 \alpha} e_{\alpha i_1} \dots e_{\alpha i_{k-1}} e_{\alpha i_k}) \in \mathbf{U}(gl(n))$$

(see, e.g. [9]).

Consider the k -th *Capelli element*

$$\mathbf{H}_n^{(k)} = \sum_{1 \leq i_1 < \dots < i_k \leq n} \mathbf{cdet} \begin{pmatrix} e_{i_1, i_1} + (k-1) & e_{i_1, i_2} & \dots & e_{i_1, i_k} \\ e_{i_2, i_1} & e_{i_2, i_2} + (k-2) & \dots & e_{i_2, i_k} \\ \vdots & \vdots & \vdots & \vdots \\ e_{i_k, i_1} & e_{i_k, i_2} & \dots & e_{i_k, i_k} \end{pmatrix}$$

Clearly, we have

$$\mathbf{H}_n^{(k)} = \sum_{1 \leq i_1 < \dots < i_k \leq n} [i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k]. \quad (6)$$

We recall the following fundamental result, proved by Capelli in two papers ([12], [13]) with deceiving titles.

Proposition 6. (Capelli, 1893) *Let $\zeta(n)$ denote be center of $\mathbf{U}(\mathfrak{gl}(n))$. We have:*

- *The elements $\mathbf{H}_n^{(k)}$, $k = 1, 2, \dots, n$ belong to the center $\zeta(n)$.*
- *The subalgebra $\zeta(n)$ of $\mathbf{U}(\mathfrak{gl}(n))$ is the polynomial algebra*

$$\zeta(n) = \mathbb{C}[\mathbf{H}_n^{(1)}, \mathbf{H}_n^{(2)}, \dots, \mathbf{H}_n^{(n)}],$$

where

$$\mathbf{H}_n^{(1)}, \mathbf{H}_n^{(2)}, \dots, \mathbf{H}_n^{(n)}$$

is a set of algebraically independent generators of $\zeta(n)$.

6.2 The factorization Theorem for rectangular Capelli-Deruyts bitableaux $\mathbf{K}_n^{\mathbf{p}}$

The crucial result in this section is that Capelli-Deruyts bitableaux $\mathbf{K}_n^{\mathbf{p}}$ of *rectangular* shape $\lambda = n^{\mathbf{p}}$ expand into *commutative* polynomials in the Capelli elements $\mathbf{H}_n^{(j)}$, with explicit coefficients.

The next result was announced, without proof, in [3]. By eq. (6), it is a special case of Theorem 3.

Corollary 1. (Expansion Theorem)

Let $p \in \mathbb{N}$ and set $\mathbf{H}_n^{(0)} = \mathbf{1}$, by definition. The following identity in $\zeta(n)$ holds:

$$\mathbf{K}_n^{\mathbf{p}} = (-1)^{n(p-1)} \mathbf{C}_n(p-1) \mathbf{K}_n^{\mathbf{p}-1},$$

where, given $p \in \mathbb{N}$,

$$\mathbf{C}_n(p-1) = \sum_{j=0}^n (-1)^{n-j} (p-1)_{n-j} \mathbf{H}_n^{(j)}. \quad (7)$$

where

$$(m)_k = m(m-1) \cdots (m-k+1), \quad m, k \in \mathbb{N}$$

denotes the falling factorial coefficient.

If $p = 0$, eq. (7) collapses to

$$\mathbf{K}_n^1 = \mathbf{H}_n^{(n)} = \mathbf{C}_n(0).$$

Notice that the linear relations (7), for $p = 0, \dots, n-1$, yield a nonsingular triangular coefficients matrix.

Corollary 2. *The subalgebra $\zeta(n)$ of $\mathbf{U}(\mathfrak{gl}(n))$ is the polynomial algebra*

$$\zeta(n) = \mathbb{C}[\mathbf{C}_n(0), \mathbf{C}_n(1), \dots, \mathbf{C}_n(n-1)],$$

where

$$\mathbf{C}_n(0), \mathbf{C}_n(1), \dots, \mathbf{C}_n(n-1)$$

is a set of algebraically independent generators of $\zeta(n)$.

Corollary 3. *The rectangular Capelli-Deruyts bitableau \mathbf{K}_n^p equals the commutative polynomial in the Capelli generators:*

$$\mathbf{K}_n^p = (-1)^{n \binom{p}{2}} \mathbf{C}_n(p-1) \cdots \mathbf{C}_n(1) \mathbf{C}_n(0).$$

Example 3. *Let $n = 3$, $p = 2$. Then*

$$\mathbf{K}_3^2 = \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 3 & 2 & 1 & 1 & 2 & 3 \end{array} \right] = -\mathbf{C}_3(1) \mathbf{C}_3(0) = \left(\mathbf{H}_3^{(2)} - \mathbf{H}_3^{(3)} \right) \mathbf{H}_3^{(3)}.$$

□

6.3 The Harish-Chandra isomorphism and the algebra $\Lambda^*(n)$ of shifted symmetric polynomials

In this subsection we follow A. Okounkov and G. Olshanski [33].

As in the classical context of the algebra $\Lambda(n)$ of symmetric polynomials in n variables x_1, x_2, \dots, x_n , the algebra $\Lambda^*(n)$ of *shifted symmetric polynomials* is an algebra of polynomials $p(x_1, x_2, \dots, x_n)$ but the ordinary symmetry is replaced by the *shifted symmetry*:

$$f(x_1, \dots, x_i, x_{i+1}, \dots, x_n) = f(x_1, \dots, x_{i+1} - 1, x_i + 1, \dots, x_n),$$

for $i = 1, 2, \dots, n-1$.

The *shifted elementary symmetric polynomials* are the elements of $\Lambda^*(n)$

– for every $r \in \mathbb{Z}^+$,

$$\mathbf{e}_k^*(x_1, x_2, \dots, x_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} (x_{i_1} + k - 1)(x_{i_2} + k - 2) \cdots (x_{i_k}),$$

$$- \mathbf{e}_0^*(x_1, x_2, \dots, x_n) = \mathbf{1}.$$

The *Harish-Chandra isomorphism* is the algebra isomorphism

$$\chi_n : \zeta(n) \longrightarrow \Lambda^*(n), \quad A \mapsto \chi_n(A),$$

$\chi_n(A)$ being the shifted symmetric polynomial such that, for every highest weight module V_μ , the evaluation $\chi_n(A)(\mu_1, \mu_2, \dots, \mu_n)$ equals the eigenvalue of $A \in \zeta(n)$ in V_μ ([33], Proposition 2.1).

6.4 The Harish-Chandra isomorphism interpretation of Theorem 1 and Theorem 3

Notice that

$$\chi_n(\mathbf{H}_n^{(r)}) = \mathbf{e}_r^*(x_1, x_2, \dots, x_n) \in \Lambda^*(n),$$

for every $r = 1, 2, \dots, n$.

Furthermore, from Theorem 1 it follows

Corollary 4.

$$\chi_n(\mathbf{K}_n^p) = (-1)^{\binom{p}{2}n} \left(\prod_{j=1}^p (x_1 - j + n)(x_2 - j + n - 1) \cdots (x_n - j - 1) \right).$$

By Corollary 1, we have

$$\chi_n(\mathbf{K}_n^{p+1}) = \chi_n(\mathbf{C}_n(p)) \chi_n(\mathbf{K}_n^p).$$

and Corollary 4 implies

Proposition 7. *For every $p \in \mathbb{N}$,*

$$\chi_n(\mathbf{C}_n(p)) = (x_1 - p + n - 1)(x_2 - p + n - 2) \cdots (x_n - p).$$

Proposition 8. *The set*

$$\chi_n(\mathbf{C}_n(0)), \chi_n(\mathbf{C}_n(1)), \dots, \chi_n(\mathbf{C}_n(n-1))$$

is a system of algebraically independent generators of the ring $\Lambda^(n)$ of shifted symmetric polynomials in the variables x_1, x_2, \dots, x_n .*

Given $p \in \mathbb{N}$, consider the column determinant

$$\mathbf{H}_n(p) = \mathbf{cdet} \begin{pmatrix} e_{1,1} - p + (n-1) & e_{1,2} & \cdots & e_{1,n} \\ e_{2,1} & e_{2,2} - p + (n-2) & \cdots & e_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n,1} & e_{n,2} & \cdots & e_{n,n} - p \end{pmatrix}. \quad (8)$$

We recall a standard result (for an elementary proof see e.g. [41]):

Proposition 9. *For every $p \in \mathbb{N}$, the element*

$$\mathbf{H}_n(p) = \mathbf{cdet}[e_{h,k} + \delta_{hk}(-p + n - h)]_{h,k=1,\dots,n} \in \mathbf{U}(\mathfrak{gl}(n)).$$

is central. In symbols, $\mathbf{H}_n(p) \in \zeta(n)$.

Equation (8), Proposition 9 and Proposition 7 imply

$$\chi_n(\mathbf{H}_n(p)) = (x_1 - p + n - 1)(x_2 - p + n - 2) \cdots (x_n - p) = \chi_n(\mathbf{C}_n(p)).$$

Hence, we get the well-known identity (see, e.g. [27]):

Corollary 5. *For every $p \in \mathbb{N}$, we have*

$$\begin{aligned} \mathbf{H}_n(p) &= \mathbf{cdet}[e_{h,k} + \delta_{hk}(-p + n - h)]_{h,k=1,\dots,n} \\ &= \sum_{j=0}^n (-1)^{n-j} (p)_{n-j} \mathbf{H}_n^{(j)} = \mathbf{C}_n(p). \end{aligned}$$

Corollary 6. *The subalgebra $\zeta(n)$ of $\mathbf{U}(\mathfrak{gl}(n))$ is the polynomial algebra*

$$\zeta(n) = \mathbb{C}[\mathbf{H}_n(0), \mathbf{H}_n(1), \dots, \mathbf{H}_n(n-1)],$$

where

$$\mathbf{H}_n(0), \mathbf{H}_n(1), \dots, \mathbf{H}_n(n-1)$$

is a set of algebraically independent generators of $\zeta(n)$.

Corollary 7. *The rectangular Capelli-Deruyts bitableau \mathbf{K}_n^p equals the product of column determinants:*

$$\mathbf{K}_n^p = (-1)^n \binom{p}{2} \mathbf{H}_n(p-1) \cdots \mathbf{H}_n(1) \mathbf{H}_n(0).$$

Example 4. Let $n = 3$, $p = 2$. Then

$$\begin{aligned} \mathbf{K}_3^2 &= \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 3 & 2 & 1 & 1 & 2 & 3 \end{array} \right] = -\mathbf{H}_3(1) \mathbf{H}_3(0) = \\ &= -\mathbf{cdet} \begin{pmatrix} e_{1,1} + 1 & e_{1,2} & e_{1,3} \\ e_{2,1} & e_{2,2} & e_{2,3} \\ e_{3,1} & e_{3,2} & e_{3,3} - 1 \end{pmatrix} \mathbf{cdet} \begin{pmatrix} e_{1,1} + 2 & e_{1,2} & e_{1,3} \\ e_{2,1} & e_{2,2} + 1 & e_{2,3} \\ e_{3,1} & e_{3,2} & e_{3,3} \end{pmatrix}. \end{aligned}$$

□

Corollaries 3 and 7 generalize to Capelli-Deruyts bitableaux \mathbf{K}^λ of arbitrary shape λ . Theorem 3 implies:

Proposition 10. Let $n \in \mathbb{Z}$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$, $\lambda_1 \leq n$. Set $\lambda' = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{p-1})$. Then

$$\mathbf{K}^\lambda = (-1)^{\lambda_p(\lambda_{p-1} + \cdots + \lambda_1)} \mathbf{C}_{\lambda_p}(p-1) \mathbf{K}^{\lambda'},$$

where

$$\mathbf{C}_{\lambda_p}(p-1) = \sum_{j=0}^{\lambda_p} (-1)^{\lambda_p-j} (p-1)_{\lambda_p-j} \mathbf{H}_{\lambda_p}^{(j)}.$$

□

Corollary 8. Let $n \in \mathbb{Z}$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$, $\lambda_1 \leq n$. For $i = 1, 2, \dots, p$, set

$$\mathbf{C}_{\lambda_i}(i-1) = \sum_{j=0}^{\lambda_i} (-1)^{\lambda_i-j} (i-1)_{\lambda_i-j} \mathbf{H}_{\lambda_i}^{(j)}.$$

Then,

1. The element $\mathbf{C}_{\lambda_i}(i-1)$ is central in the enveloping algebra $\mathbf{U}(\mathfrak{gl}(\lambda_i))$, for $i = 1, 2, \dots, p$.
2. The Capelli-Deruyts bitableau \mathbf{K}^λ equals the polynomial in the Capelli elements $\mathbf{H}_{\lambda_i}^{(j)}$:

$$\mathbf{K}^\lambda = (-1)^{\lambda_p(\lambda_{p-1} + \cdots + \lambda_1) + \cdots + \lambda_2 \lambda_1} \mathbf{C}_{\lambda_p}(p-1) \cdots \mathbf{C}_{\lambda_2}(1) \mathbf{C}_{\lambda_1}(0).$$

□

Example 5. Let $n = 3$, $\lambda = (3, 2)$ and let

$$\mathbf{K}^{(3,2)} = \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \end{array} \right].$$

Then,

$$\mathbf{K}^{(3,2)} = \mathbf{C}_2(1) \mathbf{C}_3(0) = \left(\mathbf{H}_2^{(2)} - \mathbf{H}_2^{(1)} \right) \mathbf{H}_3^{(3)}.$$

□

For $i = 1, 2, \dots, p$, consider the center $\zeta(\lambda_i)$ of $\mathbf{U}(\mathfrak{gl}(\lambda_i))$ and the Harish-Chandra isomorphisms

$$\chi_{\lambda_i} : \zeta(\lambda_i) \longrightarrow \Lambda^*(\lambda_i).$$

Proposition 5 and Proposition 10 imply:

$$\chi_{\lambda_i}(\mathbf{C}_{\lambda_i}(i-1)) = (x_1 - i + \lambda_i)(x_2 - i + \lambda_i - 1) \cdots (x_{\lambda_i} - i + 1). \quad (9)$$

Proposition 9 implies that the element

$$\mathbf{H}_{\lambda_i}(i-1) = \mathbf{cdet} [e_{h,k} + \delta_{hk}(\lambda_i - i - h + 1)]_{h,k=1,\dots,\lambda_i} \in \mathbf{U}(\mathfrak{gl}(\lambda_i)).$$

is central in the enveloping algebra $\mathbf{U}(\mathfrak{gl}(\lambda_i))$. In symbols, $\mathbf{H}_n(p) \in \zeta(\lambda_i)$.

Clearly,

$$\chi_{\lambda_i}(\mathbf{H}_{\lambda_i}(i-1)) = (x_1 - i + \lambda_i)(x_2 - i + \lambda_i - 1) \cdots (x_{\lambda_i} - i + 1),$$

and, therefore, from eq. (9), we have

Corollary 9. $\mathbf{H}_{\lambda_i}(i-1) = \mathbf{C}_{\lambda_i}(i-1)$.

From Corollary 8, we have

Corollary 10. The Capelli-Deruyts bitableau \mathbf{K}_λ equals the product of column determinants:

$$\mathbf{K}^\lambda = (-1)^{\lambda_p(\lambda_{p-1} + \cdots + \lambda_1) + \cdots + \lambda_2 \lambda_1} \mathbf{H}_{\lambda_p}(p-1) \cdots \mathbf{H}_{\lambda_2}(1) \mathbf{H}_{\lambda_1}(0).$$

Example 6. We have

$$\begin{aligned} \mathbf{K}^{(3,2)} &= \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \end{array} \right] = \mathbf{H}_2(1) \mathbf{H}_3(0) = \\ &= \mathbf{cdet} \begin{pmatrix} e_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} - 1 \end{pmatrix} \mathbf{cdet} \begin{pmatrix} e_{1,1} + 2 & e_{1,2} & e_{1,3} \\ e_{2,1} & e_{2,2} + 1 & e_{2,3} \\ e_{3,1} & e_{3,2} & e_{3,3} \end{pmatrix}. \end{aligned}$$

6.5 Polynomial identities

Let t be a variable and consider the polynomial

$$\begin{aligned} \mathbf{H}_n(t) &= \mathbf{cdet} \begin{pmatrix} e_{1,1} - t + (n-1) & e_{1,2} & \cdots & e_{1,n} \\ e_{2,1} & e_{2,2} - t + (n-2) & \cdots & e_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n,1} & e_{n,2} & \cdots & e_{n,n} - t \end{pmatrix} = \\ &= \mathbf{cdet} [e_{i,j} + \delta_{ij}(-t + n - i)]_{i,j=1,\dots,n} \end{aligned}$$

with coefficients in $\mathbf{U}(\mathfrak{gl}(n))$.

Corollary 11. (see, e.g. [41]) *In the polynomial algebra $\zeta(n)[t]$, the following identity holds:*

$$\mathbf{H}_n(t) = \sum_{j=0}^n (-1)^{n-j} \mathbf{H}_n^{(j)}(t)_{n-j},$$

where, for every $k \in \mathbb{N}$, $(t)_k = t(t-1)\cdots(t-k+1)$ denotes the k -th falling factorial polynomial.

Corollary 12. *In the polynomial algebra $\Lambda^*(n)[t]$, the following identity holds:*

$$(x_1 - t + n - 1)(x_2 - t + n - 2) \cdots (x_n - t) = \sum_{j=0}^n (-1)^{n-j} \mathbf{e}_j^*(x_1, x_2, \dots, x_n) (t)_{n-j}.$$

Following Molev [28] Chapt. 7 (see also Howe and Umeda [22]), consider the ‘‘Capelli determinant’’

$$\begin{aligned} \mathcal{C}_n(s) &= \mathbf{cdet} \begin{pmatrix} e_{1,1} + s & e_{1,2} & \cdots & e_{1,n} \\ e_{2,1} & e_{2,2} + s - 1 & \cdots & e_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n,1} & e_{n,2} & \cdots & e_{n,n} + s - (n-1) \end{pmatrix} = \\ &= \mathbf{cdet} [e_{i,j} + \delta_{ij}(s - i + 1)]_{i,j=1,\dots,n}, \end{aligned}$$

regarded as a polynomial in the variable s .

By the formal (column) Laplace rule, the coefficients $\mathcal{C}_n^{(h)} \in \mathbf{U}(\mathfrak{gl}(n))$ in the expansion

$$\mathcal{C}_n(s) = s^n + \mathcal{C}_n^{(1)} s^{n-1} + \mathcal{C}_n^{(2)} s^{n-2} + \cdots + \mathcal{C}_n^{(n)},$$

are the sums of the minors:

$$\mathcal{C}_n^{(h)} = \sum_{1 \leq i_1 < i_2 < \dots < i_h \leq n} \mathcal{M}_{i_1, i_2, \dots, i_h},$$

where $\mathcal{M}_{i_1, i_2, \dots, i_h}$ denotes the column determinant of the submatrix of the matrix $\mathcal{C}_n(0)$ obtained by selecting the rows and the columns with indices $i_1 < i_2 < \dots < i_h$.

Since $\mathcal{C}_n(s) = \mathbf{H}_n(-s + (n - 1))$, from Proposition 11 it follows:

Corollary 13.

$$\mathcal{C}_n(s) = \sum_{j=0}^n (-1)^{n-j} (-s + (n - 1))_{n-j} \mathbf{H}_n^{(j)}.$$

Corollary 14. *We have:*

- *The elements $\mathcal{C}_n^{(h)}$, $h = 1, 2, \dots, n$ are central and provide a system of algebraically independent generators of $\zeta(n)$.*
- *$\chi_n(\mathcal{C}_n^{(h)}) = \bar{\mathbf{e}}_h(x_1, x_2, \dots, x_n) = \mathbf{e}_h(x_1, x_2 - 1, \dots, x_n - (n - 1))$, where \mathbf{e}_h denotes the h -th elementary symmetric polynomial.*

6.6 The shaped Capelli central elements $\mathbf{K}_\lambda(n)$

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$, $\lambda_1 \leq n$, consider the *shaped Capelli elements* (see [9])

$$\mathbf{K}_\lambda(n) = \sum_S \mathfrak{p}(e_{S, C_\lambda^*} \cdot e_{C_\lambda^*, S}) = \sum_S [S|S] \in \mathbf{U}(\mathfrak{gl}(n)),$$

where the sum is extended to all row-increasing tableaux S , $sh(S) = \lambda$.

Notice that the elements $\mathbf{K}_\lambda(n)$ are *radically different* from the elements $\mathbf{H}_\lambda(n) = \mathbf{H}_{\lambda_1}(n) \cdots \mathbf{H}_{\lambda_p}(n)$ and are *radically different* from the elements \mathbf{K}^λ .

Since the adjoint representation acts by derivation, we have

$$ad(e_{ij}) \left(\sum_S e_{S, C_\lambda^*} \cdot e_{C_\lambda^*, S} \right) = 0,$$

for every $e_{ij} \in \mathfrak{gl}(n)$ and, then, from Proposition 26, it follows

Proposition 11. *The elements $\mathbf{K}_\lambda(n)$ are central in $\mathbf{U}(\mathfrak{gl}(n))$.*

Let $\zeta(n)^{(m)}$ be the m -th filtration element of the center $\zeta(n)$ of $\mathbf{U}(\mathfrak{gl}(n))$.

Clearly, $\mathbf{K}_\lambda(n), \mathbf{H}_\lambda(n) \in \zeta(n)^{(m)}$ if and only if $m \geq |\lambda|$.

Proposition 12.

$$\mathbf{K}_\lambda(n) = \pm \mathbf{H}_\lambda(n) + \sum c_{\lambda,\mu} \mathbf{F}_\mu(n),$$

where $\mathbf{F}_\mu(n) \in \zeta(n)^{(m)}$ for some $m < |\lambda|$.

Proof. Immediate from Corollary 16. □

Therefore, the central elements $\mathbf{K}_\lambda(n)$, $|\lambda| \leq m$ are linearly independent in $\zeta(n)^{(m)}$, and the next result follows at once.

Proposition 13. *The set*

$$\{ \mathbf{K}_\lambda(n); \lambda_1 \leq n \}$$

is a linear basis of the center $\zeta(n)$.

Let \mathcal{K} be the Koszul equivariant isomorphism [9]

$$\begin{aligned} \mathcal{K} : \mathbf{U}(\mathfrak{gl}(n)) &\rightarrow \mathbb{C}[M_{n,n}], \\ \mathcal{K} : [S|S] &\mapsto (S|S). \end{aligned} \tag{10}$$

Clearly, the Koszul map \mathcal{K} induces, by restriction, an isomorphism from the center $\zeta(n)$ of $\mathbf{U}(\mathfrak{gl}(n))$ to the algebra $\mathbb{C}[M_{n,n}]^{ad_{\mathfrak{gl}(n)}}$ of $ad_{\mathfrak{gl}(n)}$ -invariants in $\mathbb{C}[M_{n,n}]$.

Consider to the polynomial

$$\begin{aligned} \mathbf{h}_k(n) &= \sum_{1 \leq i_1 < \dots < i_k \leq n} (i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k) \\ &= \sum_{1 \leq i_1 < \dots < i_k \leq n} \det \begin{pmatrix} (i_1 | i_1) & \dots & (i_1 | i_k) \\ \vdots & & \vdots \\ (i_k | i_1) & \dots & (i_k | i_k) \end{pmatrix} \in \mathbb{C}[M_{n,n}]. \end{aligned}$$

Clearly, $\mathbf{h}_k(n) \in \mathbb{C}[M_{n,n}]^{ad_{\mathfrak{gl}(n)}}$.

Notice that the polynomials $\mathbf{h}_k(n)$'s appear as coefficients (in $\mathbb{C}[M_{n,n}]$) of the characteristic polynomial:

$$P_{M_{n,n}}(t) = \det(tI - M_{n,n}) = t^n + \sum_{i=1}^n (-1)^i \mathbf{h}_i(n) t^{n-i}.$$

From (10), we have

Proposition 14.

$$\mathcal{K}(\mathbf{K}_\lambda(n)) = (-1)^{\binom{|\lambda|}{2}} \mathbf{h}_{\lambda_1}(n) \mathbf{h}_{\lambda_2}(n) \cdots \mathbf{h}_{\lambda_p}(n), \quad |\lambda| = \sum_i \lambda_i.$$

Proposition 13 implies (is actually equivalent to) the well-known theorem for the algebra of invariants $\mathbb{C}[M_{n,n}]^{ad_{gl(n)}}$:

Proposition 15.

$$\mathbb{C}[M_{n,n}]^{ad_{gl(n)}} = \mathbb{C}[\mathbf{h}_1(n), \mathbf{h}_2(n), \dots, \mathbf{h}_n(n)].$$

Moreover, the $\mathbf{h}_k(n)$'s are algebraically independent.

Proposition 15 is usually stated in terms of the algebra $\mathbb{C}[M_{n,n}]^{GL(n)} = \mathbb{C}[M_{n,n}]^{ad_{gl(n)}}$, where $\mathbb{C}[M_{n,n}]^{GL(n)}$ is the subalgebra of invariants with respect to the *conjugation action* of the general linear group $GL(n)$ on $\mathbb{C}[M_{n,n}]$ (see, e.g. [36]).

7 Proof of Theorem 2

7.1 A commutation identity for enveloping algebras of Lie superalgebras

Let $(L = L_0 \oplus L_1, [\ , \])$ be a *Lie superalgebra* over \mathbb{C} (see, e.g. [23], [39]), where $[\ , \]$ denotes the *superbracket* bilinear form.

Given $a \in L$, consider the linear operator T_a from $U(L)$ to itself defined by setting

$$T_a(\mathbf{N}) = a \mathbf{N} - (-1)^{|\mathbf{N}|} \mathbf{N} a,$$

for every $\mathbf{N} \in U(L)$, \mathbb{Z}_2 -homogeneous of degree $|\mathbf{N}|$.

We recall that T_a is the unique (left) superderivation of $U(L)$, \mathbb{Z}_2 -homogeneous of degree $|a|$, such that

$$T_a(b) = [a, b],$$

for every $b \in L$.

Furthermore, given $a, b \in L = L_0 \oplus L_1$, from (super) skew-symmetry and the (super) Jacobi identity, it follows:

$$T_a \circ T_b - (-1)^{|a||b|} T_b \circ T_a = T_{[a,b]}.$$

The Lie algebra representation

$$Ad_L : L = L_0 \oplus L_1 \rightarrow \text{End}_{\mathbb{C}}[\mathbf{U}(L)] = \text{End}_{\mathbb{C}}[\mathbf{U}(L)]_0 \oplus \text{End}_{\mathbb{C}}[\mathbf{U}(L)]_1$$

$$e_a \mapsto T_a$$

is the *adjoint representation* of $U(L)$ on itself.

Proposition 16.

$$\begin{aligned}
a_{i_1} a_{i_2} \cdots a_{i_m} \omega &= \omega a_{i_1} a_{i_2} \cdots a_{i_m} (-1)^{|\omega|(|a_{i_1}|+|a_{i_2}|+\cdots+|a_{i_m}|)} + \\
&+ \sum_{k=1}^m \sum_{\sigma(1)<\cdots<\sigma(k); \sigma(k+1)<\cdots<\sigma(m)} \left((T_{a_{i_{\sigma(1)}}} \cdots T_{a_{i_{\sigma(k)}}}(\omega)) a_{i_{\sigma(k+1)}} \cdots a_{i_{\sigma(m)}} \times \right. \\
&\quad \left. \times \operatorname{sgn}(a_{i_{\sigma(1)}} \cdots a_{i_{\sigma(k)}}; a_{i_{\sigma(k+1)}} \cdots a_{i_{\sigma(m)}}) (-1)^{|\omega|(|a_{i_{\sigma(k+1)}}|+\cdots+|a_{i_{\sigma(m)}}|)} \right).
\end{aligned}$$

Proof. By induction hypothesis,

$$\begin{aligned}
a_{i_1} (a_{i_2} \cdots a_{i_m}) \omega &= a_{i_1} \omega a_{i_2} \cdots a_{i_m} (-1)^{|\omega|(|a_{i_2}|+\cdots+|a_{i_m}|)} + \\
&+ a_{i_1} \sum_{h=2}^m \sum_{\tau(2)<\cdots<\tau(h); \tau(h+1)<\cdots<\tau(m)} \left(T_{a_{i_{\tau(2)}}} \cdots T_{a_{i_{\tau(h)}}}(\omega) a_{i_{\tau(h+1)}} \cdots a_{i_{\tau(m)}} \times \right. \\
&\quad \left. \times \operatorname{sgn}(a_{i_{\tau(2)}} \cdots a_{i_{\tau(h)}}; a_{i_{\tau(h+1)}} \cdots a_{i_{\tau(m)}}) (-1)^{|\omega|(|a_{i_{\tau(h+1)}}|+\cdots+|a_{i_{\tau(m)}}|)} \right) = \\
&= \omega a_{i_1} a_{i_2} \cdots a_{i_m} (-1)^{|\omega|(|a_{i_1}|+|a_{i_2}|+\cdots+|a_{i_m}|)} + T_{a_{i_1}}(\omega) a_{i_2} \cdots a_{i_m} (-1)^{|\omega|(|a_{i_2}|+\cdots+|a_{i_m}|)} + \\
&+ \sum_{h=2}^m \sum_{\tau(2)<\cdots<\tau(h); \tau(h+1)<\cdots<\tau(m)} \left(T_{a_{i_1}} T_{a_{i_{\tau(2)}}} \cdots T_{a_{i_{\tau(h)}}}(\omega) a_{i_{\tau(h+1)}} \cdots a_{i_{\tau(m)}} \times \right. \\
&\quad \left. \times \operatorname{sgn}(a_{i_{\tau(2)}} \cdots a_{i_{\tau(h)}}; a_{i_{\tau(h+1)}} \cdots a_{i_{\tau(m)}}) (-1)^{|\omega|(|a_{i_{\tau(h+1)}}|+\cdots+|a_{i_{\tau(m)}}|)} + \right. \\
&\quad \left. + T_{a_{i_{\tau(2)}}} \cdots T_{a_{i_{\tau(h)}}}(\omega) a_{i_1} a_{i_{\tau(h+1)}} \cdots a_{i_{\tau(m)}} \times \right. \\
&\quad \left. (-1)^{|a_{i_1}|(|\omega|+|a_{i_{\tau(2)}}|+\cdots+|a_{i_{\tau(m)}}|)} \times \operatorname{sgn}(a_{i_{\tau(2)}} \cdots a_{i_{\tau(h)}}; a_{i_{\tau(h+1)}} \cdots a_{i_{\tau(m)}}) (-1)^{|\omega|(|a_{i_{\tau(h+1)}}|+\cdots+|a_{i_{\tau(m)}}|)} \right),
\end{aligned}$$

where

$$\begin{aligned}
&(-1)^{|a_{i_1}|(|\omega|+|a_{i_{\tau(2)}}|+\cdots+|a_{i_{\tau(m)}}|)+|\omega|(|a_{i_{\tau(h+1)}}|+\cdots+|a_{i_{\tau(m)}}|)} \times \operatorname{sgn}(a_{i_{\tau(2)}} \cdots a_{i_{\tau(m)}}; a_{i_{\tau(h+1)}} \cdots a_{i_{\tau(m)}}) = \\
&= \operatorname{sgn}(a_{i_{\tau(2)}} \cdots a_{i_{\tau(h)}}; a_{i_1} a_{i_{\tau(h+1)}} \cdots a_{i_{\tau(m)}}) (-1)^{|\omega|(|a_{i_1}|+|a_{i_{\tau(h+1)}}|+\cdots+|a_{i_{\tau(m)}}|)}.
\end{aligned}$$

Then, the assertion follows. \square

In the Sweedler notation of the *supersymmetric* superbialgebra $Super(L)$, Theorem 16 can be stated in the following compact form:

Proposition 17. *Let*

$$\alpha = a_{i_1} a_{i_2} \cdots a_{i_m}.$$

Then

$$\alpha \omega = \sum_{(\alpha)} T_{\alpha(1)}(\omega) \alpha(2) (-1)^{|\omega||\alpha(2)|}.$$

Proof. Let

$$\alpha = a_{i_1} a_{i_2} \cdots a_{i_m}.$$

Then, the coproduct (in the Sweedler notation)

$$\Delta(\alpha) = \sum_{(\alpha)} \alpha_{(1)} \otimes \alpha_{(2)}$$

equals

$$\begin{aligned} & \sum_{k=0}^m \sum_{\sigma(1) < \cdots < \sigma(k); \sigma(k+1) < \cdots < \sigma(m)} \left(a_{i_{\sigma(1)}} \cdots a_{i_{\sigma(k)}} \otimes a_{i_{\sigma(k+1)}} \cdots a_{i_{\sigma(m)}} \times \right. \\ & \quad \left. \times \operatorname{sgn}(a_{i_{\sigma(1)}} \cdots a_{i_{\sigma(k)}}; a_{i_{\sigma(k+1)}} \cdots a_{i_{\sigma(m)}}) \right). \end{aligned}$$

□

Furthermore

Lemma 1. *Let $T_\alpha = T_{a_1} T_{a_2} \cdots T_{a_m}$. Then*

$$T_\alpha(\omega_1 \cdot \omega_2) = \sum_{(\alpha)} T_{\alpha_{(1)}}(\omega_1) T_{\alpha_{(2)}}(\omega_2) (-1)^{|\alpha_{(2)}||\omega_1|}.$$

7.2 Some preliminary remarks and definitions

7.2.1 The virtual algebra and the Capelli devirtualization epimorphism

Given a vector space V of dimension n , we will regard it as a subspace of a \mathbb{Z}_2 -graded vector space $V_0 \oplus V_1$, where $V_1 = V$. The vector spaces V_0 (we assume that $\dim(V_0) = m$ is “sufficiently large”) is called the *positive virtual (auxiliary) vector space* and V is called the *(negative) proper vector space*.

Let $\mathcal{A}_0 = \{\alpha_1, \dots, \alpha_{m_0}\}$, $\mathcal{L} = \{1, 2, \dots, n\}$ denote *fixed bases* of V_0 and $V = V_1$, respectively; therefore $|\alpha_s| = 0 \in \mathbb{Z}_2$, and $|i| = 1 \in \mathbb{Z}_2$.

Let

$$\{e_{a,b}; a, b \in \mathcal{A}_0 \cup \mathcal{L}\}, \quad |e_{a,b}| = |a| + |b| \in \mathbb{Z}_2$$

be the standard \mathbb{Z}_2 -homogeneous basis of the Lie superalgebra $gl(m|n)$ provided by the elementary matrices. The elements $e_{a,b} \in gl(m|n)$ are \mathbb{Z}_2 -homogeneous of \mathbb{Z}_2 -degree $|e_{a,b}| = |a| + |b|$.

The superbracket of the Lie superalgebra $gl(m|n)$ has the following explicit form:

$$[e_{a,b}, e_{c,d}] = \delta_{bc} e_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)} \delta_{ad} e_{c,b},$$

$a, b, c, d \in \mathcal{A}_0 \cup \mathcal{L}$.

In the following, the elements of the sets $\mathcal{A}_0, \mathcal{L}$ will be called *positive virtual symbols* and *negative proper symbols*, respectively.

The inclusion $V \subset V_0 \oplus V_1$ induces a natural embedding of the ordinary general linear Lie algebra $gl(n) = gl(0|n)$ of V into the *auxiliary* general linear Lie *superalgebra* $gl(m|n)$ of $V_0 \oplus V_1$ (see, e.g. [23], [39]) and, hence, a natural embedding $\mathbf{U}(gl(n)) \subset \mathbf{U}(gl(m|n))$.

In the following, we will systematically refer to the *Capelli devirtualization epimorphism*

$$\mathfrak{p} : Virt(m, n) \twoheadrightarrow \mathbf{U}(gl(0|n)) = \mathbf{U}(gl(n)),$$

where $Virt(m, n)$ is the *virtual subalgebra* of $\mathbf{U}(gl(m|n))$.

For definitions and details, we refer the reader to Subsection 9.5.

7.2.2 A more readable notation

In the following, we will adopt the more readable notation:

- We write $\{a|b\}$ for the elements $e_{a,b}$ of the standard basis of $gl(m|n)$.
- Given two words $I = i_1 i_2 \cdots i_p, J = j_1 j_2 \cdots j_p$, with $i_h, j_h \in \mathcal{L}$ and a virtual symbol α , we write

$$\{J|\alpha\} = \{j_1 j_2 \cdots j_p|\alpha\}, \quad \{\alpha|I\} = \{\alpha|i_1 i_2 \cdots i_p\}$$

in place of

$$e_{j_1, \alpha} e_{j_2, \alpha} \cdots e_{j_p, \alpha}, \quad e_{\alpha, i_1} e_{\alpha, i_2} \cdots e_{\alpha, i_p},$$

respectively.

In this notation, given a pair of Young tableaux

$$S = (w_1, w_2, \dots, w_p), \quad T = (\bar{w}_1, \bar{w}_2, \dots, \bar{w}_p), \quad sh(S) = sh(T) = \lambda,$$

the *Capelli bitableau*

$$[S|T] = \mathfrak{p}(e_{S C_\lambda} \cdot e_{C_\lambda T}) \in \mathbf{U}(gl(n))$$

is

$$[S|T] = \mathfrak{p}(\mathbf{P}_S \cdot \mathbf{P}_T),$$

where

$$\mathbf{P}_S = \{w_1|\beta_1\}\{w_2|\beta_2\} \cdots \{w_p|\beta_p\}, \quad \mathbf{P}_T = \{\beta_1|\bar{w}_1\}\{\beta_2|\bar{w}_2\} \cdots \{\beta_p|\bar{w}_p\}.$$

Furthermore, for the adjoint representation

$$Ad_{gl(m|n)} : gl(m|n) \rightarrow End_{\mathbb{C}}[\mathbf{U}(gl(m|n))]$$

we write

- $T_{i\alpha}, T_{\alpha i}$ in place of $Te_{i\alpha}, Te_{\alpha i}$.
- $T_{I\alpha}, T_{\alpha I}$ in place of $T_{i_1,\alpha}T_{i_2,\alpha} \cdots T_{i_p,\alpha}, T_{\alpha,i_1}T_{\alpha,i_2} \cdots T_{\alpha,i_p}$, respectively.

7.2.3 The coproduct in $\Lambda(V) = \Lambda(\mathcal{L})$, Sweedler notation and *split notation*

Given a word $I = i_1 i_2 \cdots i_m$, $i_t \in \mathcal{L}$ in $\Lambda(V) = \Lambda(\mathcal{L})$, and a natural integer k , $k = 0, 1, \dots, m$, consider the homogeneous component

$$\Delta_{k,m-k} : \Lambda(\mathcal{L}) \rightarrow \Lambda(\mathcal{L})_k \otimes \Lambda(\mathcal{L})_{m-k}$$

of the coproduct

$$\Delta : \Lambda(\mathcal{L}) \rightarrow \Lambda(\mathcal{L}) \otimes \Lambda(\mathcal{L}).$$

Given a permutation σ with

$$\sigma(1) < \cdots < \sigma(k), \quad \sigma(k+1) < \cdots < \sigma(m),$$

and the two subwords

$$I_{(1)} = i_{\sigma(1)} \cdots i_{\sigma(k)}, \quad I_{(2)} = i_{\sigma(k+1)} \cdots i_{\sigma(m)}$$

we call the pair $(I_{(1)}, I_{(2)})$ a *split* of I of step $(k, m-k)$ of signature $sgn(I; I_{(1)}, I_{(2)}) = sgn(\sigma)$. Clearly, $I = sgn(I; I_{(1)}, I_{(2)}) I_{(1)}I_{(2)}$.

We denote by $\mathbf{S}(I; k, m-k)$ the set of all splits of I of step $(k, m-k)$.

Then, the coproduct component

$$\Delta_{k,m-k}(I) = \sum_{(I)_{k,m-k}} I_{(1)} \otimes I_{(2)}$$

can be explicitly written as

$$\Delta_{k,m-k}(I) = \sum_{(I_{(1)}, I_{(2)}) \in \mathbf{S}(I; k, m-k)} sgn(I; I_{(1)}, I_{(2)}) I_{(1)} \otimes I_{(2)}.$$

7.3 Some lemmas

Consider the Capelli bitableau

$$[S|T] = \mathfrak{p}(\mathbf{P}_S \cdot \mathbf{P}_T)$$

as in Eq. (7.2.2).

From Proposition 17, we derive the following pair of Lemmas.

Lemma 2. *Let $I = i_1 i_2 \cdots i_m$, $J = j_1 j_2 \cdots j_m$, $m \leq \lambda_p$.*

Then

$$\{J|\alpha\}\{\alpha|I\} \mathbf{P}_S$$

equals

$$\{J|\alpha\} \sum_{k=0}^m \sum_{(I)_{k,m-k}} T_{\alpha I_{(1)}}(\mathbf{P}_S) \{\alpha|I_{(2)}\} (-1)^{|\mathbf{P}_S|(m-k)}.$$

Since

$$\mathfrak{p}(\{J|\alpha\}\{\alpha|I\} \mathbf{P}_S \cdot \mathbf{P}_T) = [J|I] [S|T],$$

Lemma 3. *We have*

$$\begin{aligned} [J|I] [S|T] &= (-1)^{(|\mathbf{P}_T|+k)(m-k)} \times \\ &\times \mathfrak{p}\left(\sum_{k=0}^m \sum_{(I)_{k,m-k}} \sum_{(J)_{k,m-k}} T_{J_{(1)}\alpha} T_{\alpha I_{(1)}}(\mathbf{P}_S) \{J_{(2)}|\alpha\} \mathbf{P}_T \{\alpha|I_{(2)}\}\right). \end{aligned} \quad (11)$$

Proof. We have

$$\begin{aligned} \{J|\alpha\}\{\alpha|I\} \mathbf{P}_S \mathbf{P}_T &= \\ &= \{J|\alpha\} \sum_{k=0}^m \sum_{(I)_{k,m-k}} T_{\alpha I_{(1)}}(\mathbf{P}_S) \{\alpha|I_{(2)}\} \mathbf{P}_T (-1)^{|\mathbf{P}_S|(m-k)} = \\ &= \sum_{k=0}^m \sum_{(I)_{k,m-k}} \{J|\alpha\} T_{\alpha I_{(1)}}(\mathbf{P}_S) \{\alpha|I_{(2)}\} \mathbf{P}_T (-1)^{|\mathbf{P}_S|(m-k)} = \\ &= \sum_{k=0}^m \sum_{(I)_{k,m-k}} \left(\sum_{h=0}^m \sum_{(J)_{h,m-h}} T_{J_{(1)}\alpha} (T_{\alpha I_{(1)}}(\mathbf{P}_S)) \{J_{(2)}|\alpha\} (-1)^{(|\mathbf{P}_S|+h)(m-h)} \right) \times \\ &\quad \times \{\alpha|I_{(2)}\} \mathbf{P}_T (-1)^{|\mathbf{P}_S|(m-k)}. \end{aligned}$$

Now, if $h < k$, then $m - h > m - k$ and, hence,

$$\sum_{(I)_{k,m-k}} \left(\sum_{(J)_{h,m-h}} T_{J_{(1)}\alpha}(T_{\alpha I_{(1)}}(\mathbf{P}_S)) \{J_{(2)}|\alpha\} (-1)^{(|\mathbf{P}_S|+h)(m-h)} \right) \times \\ \times \{\alpha|I_{(2)}\} \mathbf{P}_T (-1)^{|\mathbf{P}_S|(m-k)}$$

is an *irregular element*, since the $\{J_{(2)}|\alpha\}\{\alpha|I_{(2)}\}$ are irregular monomials; so, its image with respect to the Capelli epimorphism \mathfrak{p} equals zero.

If $h > k$, then,

$$T_{J_{(1)}\alpha}(T_{\alpha I_{(1)}}(\mathbf{P}_S)) = 0.$$

and, hence,

$$\sum_{(I)_{k,m-k}} \left(\sum_{(J)_{h,m-h}} T_{J_{(1)}\alpha}(T_{\alpha I_{(1)}}(\mathbf{P}_S)) \{J_{(2)}|\alpha\} (-1)^{(|\mathbf{P}_S|+h)(m-h)} \right) \times \\ \times \{\alpha|I_{(2)}\} \mathbf{P}_T (-1)^{|\mathbf{P}_S|(m-k)} = 0.$$

Then,

$$\begin{aligned} [J|I] [S|T] &= (-1)^{(|\mathbf{P}_S|+k)(m-k)} (-1)^{|\mathbf{P}_S|(m-k)} \times \\ &\quad \times \mathfrak{p} \left(\sum_{k=0}^m \sum_{(I)_{k,m-k}} \sum_{(J)_{k,m-k}} T_{J_{(1)}\alpha} T_{\alpha I_{(1)}}(\mathbf{P}_S) \{J_{(2)}|\alpha\} \{\alpha|I_{(2)}\} \mathbf{P}_T \right) \\ &= (-1)^{(|\mathbf{P}_T|+k)(m-k)} \times \\ &\quad \times \mathfrak{p} \left(\sum_{k=0}^m \sum_{(I)_{k,m-k}} \sum_{(J)_{k,m-k}} T_{J_{(1)}\alpha} T_{\alpha I_{(1)}}(\mathbf{P}_S) \{J_{(2)}|\alpha\} \mathbf{P}_T \{\alpha|I_{(2)}\} \right). \end{aligned}$$

□

Corollary 15. *Let $m \leq \lambda_p$. Then*

$$[J|I] [S|T] = \pm \left[\begin{array}{c|c} S & T \\ \hline J & I \end{array} \right] + \sum c_{m,\lambda} \mathbf{G}_{m,\lambda},$$

where

$$[J|I] [S|T], \quad \left[\begin{array}{c|c} S & T \\ \hline J & I \end{array} \right] \notin \mathbf{U}(\mathfrak{gl}(n))^{(n)} \quad \text{whenever } n < m + |\lambda|,$$

and

$$\mathbf{G}_{m,\lambda} \in \mathbf{U}(\mathfrak{gl}(n))^{(n)} \quad \text{for some } n < m + |\lambda|.$$

Corollary 16. *Let $m \leq \lambda_p$. Then*

$$[S|T] = \pm [\omega_1|\bar{\omega}_1] [\omega_2|\bar{\omega}_2] \cdots [\omega_p|\bar{\omega}_p] + \sum d_\lambda \mathbf{F}_\lambda,$$

where

$$[S|T], [\omega_1|\bar{\omega}_1] [\omega_2|\bar{\omega}_2] \cdots [\omega_p|\bar{\omega}_p] \notin \mathbf{U}(\mathfrak{gl}(n))^{(n)} \quad \text{whenever } n < |\lambda|,$$

and

$$\mathbf{F}_\lambda \in \mathbf{U}(\mathfrak{gl}(n))^{(n)} \quad \text{for some } n < |\lambda|.$$

We specialize the previous results to Capelli-Deruyts bitableaux \mathbf{K}^λ .

Let

$$\mathbf{M}^* = \{\underline{\lambda}_1^*|\beta_1\} \cdots \{\underline{\lambda}_p^*|\beta_p\}, \quad \mathbf{M} = \{\beta_1|\underline{\lambda}_1\} \cdots \{\beta_p|\underline{\lambda}_p\},$$

where $\lambda = (\lambda_1 \geq \cdots \geq \lambda_p)$ and $|\mathbf{M}^*| = |\mathbf{M}| = |\lambda| = \lambda_1 + \cdots + \lambda_p \in \mathbb{Z}_2$.

Given an increasing word $W = h_1 h_2 \cdots h_p$ on $\mathcal{L} = \{1, 2, \dots, n\}$, denote by W^* its reverse word, that is:

$$W^* = h_p \cdots h_2 h_1.$$

Let $I = 1 2 \cdots m$, $I^* = m m-1 \cdots 1$, $m \leq \lambda_p$.

In this notation

$$\mathbf{K}^\lambda = \mathfrak{p}(\mathbf{M}^* \cdot \mathbf{M})$$

and

$$[I^*|I] \mathbf{K}^\lambda = \mathfrak{p}(\{I^*|\alpha\}\{\alpha|I\} \mathbf{M}^* \cdot \mathbf{M}).$$

We apply Lemma 3 to the element $[I^*|I] \mathbf{K}^\lambda$. As we shall see, the double sum

$$\sum_{(I^*)_{k,m-k}} \sum_{(I)_{k,m-k}}$$

in eq. (11) reduces to a single sum

$$\sum_{(I)_{k,m-k}}$$

since the only splits $I_{(1)}^*$, $I_{(2)}^*$ in $(I^*)_{k,m-k}$ that give rise to nonzero summands are those for

$$I_{(1)}^* = (I_{(1)})^* \quad \text{and} \quad I_{(2)}^* = (I_{(2)})^*,$$

where $(I_{(1)})^*$, $(I_{(2)})^*$ are the reverse words of $I_{(1)}$ and $I_{(2)}$, respectively.

Lemma 4. *The element*

$$[I^*|I] \mathbf{K}^\lambda = \mathfrak{p}(\{I^*|\alpha\}\{\alpha|I\} \mathbf{M}^* \cdot \mathbf{M})$$

equals

$$\sum_{k=0}^m (-1)^{(|\mathbf{M}|+k)(m-k)} \sum_{(I)_{k,m-k}} \mathfrak{p}(T_{(I_{(1)})^* \alpha}(T_{\alpha I_{(1)}}(\mathbf{M}^*))\{(I_{(2)})^*|\alpha\} \mathbf{M}\{\alpha|I_{(2)}\}).$$

Proof. From Lemma 3, we have

$$\begin{aligned} \mathfrak{p}(\{I^*|\alpha\}\{\alpha|I\} \mathbf{M}^* \cdot \mathbf{M}) &= \sum_{k=0}^m (-1)^{(|\mathbf{M}|+k)(m-k)} \\ &\quad \left(\sum_{(I)_{k,m-k}} \sum_{(I^*)_{k,m-k}} \mathfrak{p}(T_{I_{(1)}^* \alpha}(T_{\alpha I_{(1)}}(\mathbf{M}^*)) \{(I_{(2)}^*|\alpha\} \mathbf{M} \{\alpha|I_{(2)}\}) \right). \end{aligned}$$

Let $k = 0, 1, \dots, m$ and examine the element

$$\begin{aligned} &\sum_{(I)_{k,m-k}} \sum_{(I^*)_{k,m-k}} T_{I_{(1)}^* \alpha}(T_{\alpha I_{(1)}}(\mathbf{M}^*))\{I_{(2)}^*|\alpha\} \mathbf{M}\{\alpha|I_{(2)}\} = \\ &= \sum_{(I)_{k,m-k}} \sum_{(I^*)_{k,m-k}} T_{I_{(1)}^* \alpha}(T_{\alpha I_{(1)}}(\{\underline{\lambda}_1^*|\beta_1\} \cdots \{\underline{\lambda}_p^*|\beta_p\}))\{(I_{(2)}^*|\alpha\} \mathbf{M}\{\alpha|I_{(2)}\}). \end{aligned}$$

If $i \in I_{(2)}$, then $i \notin I_{(1)}$. Hence, all the variables

$$\{i|\beta_q\} \quad q = 1, 2, \dots, p$$

appear in

$$T_{\alpha, I_{(1)}}(\{\underline{\lambda}_1^*|\beta_1\} \cdots \{\underline{\lambda}_p^*|\beta_p\}),$$

for every $q = 1, 2, \dots, p$.

Assume that $i \notin I_{(2)}^*$, then $i \in I_{(1)}^*$. Hence, $\exists \underline{q} \in \{1, 2, \dots, p\}$ such that the variable

$$\{i|\beta_{\underline{q}}\}$$

is *created* by the action of

$$T_{I_{(1)}^* \alpha}$$

on

$$T_{\alpha, I_{(1)}}(\{\underline{\lambda}_1^*|\beta_1\} \cdots \{\underline{\lambda}_p^*|\beta_p\}) \quad (*).$$

Then (*) contains two occurrences of $\{i|\beta_q\}$ and, hence, *equals zero*. Therefore

$$T_{I_{(1)}^* \alpha} T_{\alpha, I_{(1)}}(\{\underline{\lambda}_1^* | \beta_1\} \cdots \{\underline{\lambda}_p^* | \beta_p\}) \neq 0$$

implies

$$i \in I_{(2)} \implies i \in I_{(2)}^*.$$

Since $I_{(2)}$ and $I_{(2)}^*$ are words of the same length $m - k$, this implies that the only *not zero* summands - *with respect to the action of the Capelli epimorphism \mathfrak{p}* - in

$$\sum_{(I)_{k, m-k}} \sum_{(I^*)_{k, m-k}} \mathfrak{p}(T_{I_{(1)}^* \alpha}(T_{\alpha, I_{(1)}}(\mathbf{M}^*)) \{I_{(2)}^* | \alpha\} \mathbf{M}\{\alpha | I_{(2)}\})$$

are for $I_{(1)}^* = (I_{(1)})^*$ and $I_{(2)}^* = (I_{(2)})^*$, that is

$$\mathfrak{p}(T_{(I_{(1)})^* \alpha}(T_{\alpha, I_{(1)}}(\mathbf{M}^*)) \{(I_{(2)})^* | \alpha\} \mathbf{M}\{\alpha | I_{(2)}\}).$$

□

Let us examine the expression

$$\sum_{(I)_{k, m-k}} (-1)^{k(m-k)} T_{(I_{(1)})^* \alpha}(T_{\alpha, I_{(1)}}(\mathbf{M}^*)) \{(I_{(2)})^* | \alpha\} \{\alpha | I_{(2)}\}. \quad (12)$$

in the notation of *splits*.

Corollary 17. *The expression (12) equals*

$$\sum_{(A, B) \in S(I; k, m-k)} T_{A^* \alpha}(T_{\alpha, A}(\mathbf{M}^*)) \{B^* | \alpha\} \{\alpha | B\}.$$

Proof. In the notation of *splits*, the expression (12) equals

$$\begin{aligned} (-1)^{k(m-k)} \sum_{(A, B) \in S(I; k, m-k)} T_{A^* \alpha}(T_{\alpha, A}(\mathbf{M}^*)) \{B^* | \alpha\} \{\alpha | B\} \times \\ \text{sgn}(I; A, B) \text{sgn}(I^*; A^*, B^*). \end{aligned}$$

We have

$$\begin{aligned} (-1)^{k(m-k)} \text{sgn}(I; A, B) \text{sgn}(I^*; A^*, B^*) = \\ = (-1)^{k(m-k)} (-1)^{k(m-k)} \text{sgn}(I; A, B) \text{sgn}(I^*; B^*, A^*). \end{aligned}$$

But $\text{sgn}(I; A, B) \text{sgn}(I^*; B^*, A^*) = 1$. □

Given $(A, B) \in S(I; k, m-k)$, let $A = a_1 a_2 \cdots a_k$, $\{a_1 < a_2 < \cdots < a_k\} \subseteq \{1, 2, \dots, m\}$ and recall

$$\mathbf{M}^* = \{\underline{\lambda}_1 | \beta_1\} \cdots \{\underline{\lambda}_p^* | \beta_p\};$$

we examine the element

$$T_{A^* \alpha} T_{\alpha A}(\mathbf{M}^*). \quad (13)$$

Lemma 5. *We have*

$$T_{A^* \alpha} T_{\alpha A}(\mathbf{M}^*) = \langle p \rangle_k \{\underline{\lambda}_1^* | \beta_1\} \cdots \{\underline{\lambda}_p^* | \beta_p\} = \langle p \rangle_k \mathbf{M}^*,$$

where

$$\langle p \rangle_k = p(p+1) \cdots (p+k-1)$$

is the raising factorial coefficient.

Proof. By skew-symmetry, a simple computation shows that (13) equals

$$\sum_{h_1 + \cdots + h_p = k} \sum_{(A_1, \dots, A_p) \in S(A; h_1, \dots, h_p)} T_{(A_1)^* \alpha} T_{\alpha A_1}(\{\underline{\lambda}_1^* | \beta_1\}) \cdots T_{(A_p)^* \alpha} T_{\alpha A_p}(\{\underline{\lambda}_p^* | \beta_p\}). \quad (14)$$

We examine the value of

$$T_{C^* \alpha} T_{\alpha C}(\{\underline{q}^* | \beta\})$$

for $C = c_1 c_2 \cdots c_h$, $\{c_1 < c_2 < \cdots < c_h\} \subseteq \{1, 2, \dots, q\}$.

Clearly

$$\{\underline{q}^* | \beta\} = \{\underline{q} | \beta\} (-1)^{\binom{q}{2}},$$

and a simple computation shows that

$$T_{C^* \alpha} T_{\alpha C}(\{\underline{q} | \beta\}) = h! \{\underline{q} | \beta\}.$$

Indeed, we have

$$\begin{aligned} T_{\alpha C}(\{\underline{q} | \beta\}) &= T_{c_1 \alpha} \cdots T_{c_h \alpha}(\{1 | \beta\} \cdots \{q | \beta\}) \\ &= \{1 | \beta\} \cdots \widehat{\{c_1 | \beta\}} \{\alpha | \beta\} \cdots \widehat{\{c_h | \beta\}} \{\alpha | \beta\} \cdots \{q | \beta\} (-1)^{c_h - 1 + \cdots + c_1 - 1} \\ &= \{\alpha | \beta\}^h \{1 | \beta\} \cdots \widehat{\{c_1 | \beta\}} \cdots \widehat{\{c_h | \beta\}} \cdots \{q | \beta\} (-1)^{c_h - 1 + \cdots + c_1 - 1}; \end{aligned}$$

now,

$$\begin{aligned} T_{C \alpha} T_{\alpha C}(\{\underline{q} | \beta\}) &= T_{c_h \alpha} \cdots T_{c_1 \alpha}(\{\alpha | \beta\}^h \{1 | \beta\} \cdots \widehat{\{c_1 | \beta\}} \cdots) (-1)^{c_h - 1 + \cdots + c_1 - 1} \\ &= h! \{c_h | \beta\} \cdots \{c_1 | \beta\} \cdots \widehat{\{c_1 | \beta\}} \cdots \widehat{\{c_h | \beta\}} (-1)^{c_h - 1 + \cdots + c_1 - 1} \\ &= h! \{1 | \beta\} \cdots \{q | \beta\} = h! \{\underline{q} | \beta\}. \end{aligned}$$

Then,

$$T_{C^* \alpha} T_{\alpha C}(\{\underline{q}^* | \beta\}) = (-1)^{\binom{q}{2}} T_{C^* \alpha} T_{\alpha C}(\{\underline{q} | \beta\}) = (-1)^{\binom{q}{2}} h! \{\underline{q} | \beta\} = h! \{\underline{q}^* | \beta\}.$$

Hence, (14) equals

$$\begin{aligned} & \sum_{(h_1, \dots, h_p); h_1 + \dots + h_p = k} \sum_{(A_1, \dots, A_p) \in S(A; h_1, \dots, h_p)} h_1! \cdots h_p! (\{\underline{\lambda}_1^* | \beta_1\} \cdots \{\underline{\lambda}_p^* | \beta_p\}) = \\ & = \sum_{h_1 + \dots + h_p = k} \frac{k!}{h_1! \cdots h_p!} h_1! \cdots h_p! (\{\underline{\lambda}_1^* | \beta_1\} \cdots \{\underline{\lambda}_p^* | \beta_p\}) \end{aligned}$$

that equals

$$\left\langle \begin{matrix} p \\ k \end{matrix} \right\rangle k! (\{\underline{\lambda}_1^* | \beta_1\} \cdots \{\underline{\lambda}_p^* | \beta_p\}) = \langle p \rangle_k \{\underline{\lambda}_1^* | \beta_1\} \cdots \{\underline{\lambda}_p^* | \beta_p\}.$$

□

Hence, from Lemma 4 and Lemma 5, we infer:

Proposition 18. *Let $I = 12 \cdots m$, $I^* = m \cdots 21$. Then*

$$\begin{aligned} [I^* | I] \mathbf{K}^\lambda &= \mathfrak{p}(\{I^* | \alpha\} \{\alpha | I\} \mathbf{M}^* \cdot \mathbf{M}) \\ &= \mathfrak{p}(\{I^* | \alpha\} \{\alpha | I\} \{\underline{\lambda}_1^* | \beta_1\} \cdots \{\underline{\lambda}_p^* | \beta_p\} \{\beta_1 | \underline{\lambda}_1\} \cdots \{\beta_p | \underline{\lambda}_p\}) \end{aligned}$$

equals

$$\sum_{k=0}^m (-1)^{|\mathbf{M}|(m-k)} \sum_{(A, B) \in S(I; k, m-k)} \langle p \rangle_k \mathfrak{p}(\mathbf{M}^* \{B^* | \alpha\} \mathbf{M} \{\alpha | B\}).$$

7.4 Proof of Theorem 2

Let $m \leq \lambda_p$ and $M \subseteq \underline{\lambda}_p$, $|M| = m$, as in Theorem 2.

Recall that $|\mathbf{M}| = |\mathbf{M}^*| = |\lambda| \in \mathbb{Z}_2$, where $|\lambda| = \lambda_1 + \cdots + \lambda_p$.

From Remark 1 and Proposition 18, we have:

$$\begin{aligned} [M^* | M] \mathbf{K}^\lambda &= \mathfrak{p}(\{M^* | \alpha\} \{\alpha | M\} \mathbf{M}^* \cdot \mathbf{M}) \\ &= \sum_{k=0}^m \langle p \rangle_{m-k} (-1)^{|\lambda|k} \sum_{J; J \subseteq M; |J|=k} \mathfrak{p}(\mathbf{M}^* \{J^* | \alpha\} \mathbf{M} \{\alpha | J\}) \\ &\stackrel{\text{def}}{=} \sum_{k=0}^m \langle p \rangle_{m-k} (-1)^{|\lambda|k} \sum_{J; J \subseteq M; |J|=k} \left[\begin{matrix} \mathbf{K}^\lambda \\ J \end{matrix} \right]. \end{aligned}$$

8 Proof of Theorem 1

Proof. Recall that

$$v_\mu = (Der_{\tilde{\mu}} | Der_{\tilde{\mu}}^P),$$

where $(Der_{\tilde{\mu}} | Der_{\tilde{\mu}}^P)$ is the Young bitableau (see, e.g. Subsection 9.7 below)

$$\left(\begin{array}{cccc|cccc} 1 & 2 & \cdots & \cdots & \tilde{\mu}_1 & 1 & 2 & \cdots & \cdots & \tilde{\mu}_1 \\ 1 & 2 & \cdots & \tilde{\mu}_2 & & 1 & 2 & \cdots & \tilde{\mu}_2 & \\ \cdots & \cdots & \cdots & & & \cdots & \cdots & \cdots & & \\ \cdots & \cdots & & & & \cdots & \cdots & & & \\ 1 & 2 & \tilde{\mu}_q & & & 1 & 2 & \tilde{\mu}_q & & \end{array} \right)$$

in the polynomial algebra $\mathbb{C}[M_{n,d}]$.

Set

$$e_{Der_{n^p}^*, Coder_{n^p}} = e_{n\alpha_1} \cdots e_{1\alpha_1} \cdots e_{n\alpha_{p-1}} \cdots e_{1\alpha_{p-1}} e_{n\alpha_p} \cdots e_{1\alpha_p}.$$

Set

$$e_{Coder_{n^p}, Der_{n^p}} = e_{\alpha_1 1} \cdots e_{\alpha_1 n} \cdots e_{\alpha_{p-1} 1} \cdots e_{\alpha_{p-1} n} e_{\alpha_p 1} \cdots e_{\alpha_p n}.$$

Since

$$\mathbf{K}_n^P = \mathfrak{p}(e_{Der_{n^p}^*, Coder_{n^p}} e_{Coder_{n^p}, Der_{n^p}}),$$

the action of \mathbf{K}_n^P on $v_\mu = (Der_{\tilde{\mu}} | Der_{\tilde{\mu}}^P)$ is the same as the action of

$$e_{Der_{n^p}^*, Coder_{n^p}} e_{Coder_{n^p}, Der_{n^p}}.$$

We follow [37] (see Proposition 5).

Now, if $\mu_n = 0$, then

$$e_{\alpha_p n} \cdot (Der_{\tilde{\mu}} | Der_{\tilde{\mu}}^P)$$

is zero.

In the following, we limit ourselves to write the left parts of the Young bitableaux involved.

If $\mu_n \geq 1$, then

$$e_{\alpha_p n} \cdot (Der_{\tilde{\mu}} | Der_{\tilde{\mu}}^P)$$

equals

$$(-1)^{n-1} \begin{vmatrix} 1 & 2 & \cdots & n-1 & \alpha_p \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} + \cdots + (-1)^{n(\mu_n-1)+n-1} \begin{vmatrix} 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & n-1 & \alpha_p \\ 1 & 2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}, \quad (15)$$

by Proposition 30.

A simple sign computation shows that (15) equals

$$(-1)^{n-1} \mu_n (-1)^{n-1} \begin{vmatrix} 1 & 2 & \cdots & n-1 & \alpha_p \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

Now, again by Proposition 30 and simple computation, we have:

$$\begin{aligned}
e_{\alpha_p n-1} \cdot \left(\begin{array}{cccc|c} 1 & 2 & \cdots & n-1 & \alpha_p \\ 1 & 2 & \cdots & \cdots & n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right) &= \\
&= (-1)^{n-2} \left(\begin{array}{cccc|c} 1 & 2 & \cdots & \alpha_p & \alpha_p \\ 1 & 2 & \cdots & \cdots & n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & \cdots & n \\ 1 & 2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right) + \\
&+ \sum_{i=2}^{\mu_n} (-1)^{(n-1)+(i-2)n+(n-2)} \left(\begin{array}{cccc|c} 1 & 2 & \cdots & n-1 & \alpha_p \\ 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & \alpha_p & n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & \cdots & n \\ 1 & 2 & \cdots & n-1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & n-1 & \cdots \\ 1 & 2 & \cdots & \cdots & \cdots \end{array} \right) + \\
&+ \sum_{i=\mu_n+1}^{\mu_{n-1}} (-1)^{(n-1)+(\mu_n-1)n+(i-\mu_n-1)(n-1)+(n-2)} \left(\begin{array}{cccc|c} 1 & 2 & \cdots & n-1 & \alpha_p \\ 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & \cdots & n \\ 1 & 2 & \cdots & n-1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & \alpha_p & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & n-1 & \cdots \\ 1 & 2 & \cdots & \cdots & \cdots \end{array} \right),
\end{aligned}$$

where the tableaux in the two sums are the tableaux with the second occurrence of α_p in the i th row.

By the *Straightening Law* of Grosshans, Rota and Stein ([20], Proposition 10, see also

[2], Thm. 8.1), each summand in the two sums equals

$$(-1)^{n-2} \frac{1}{2} \left(\begin{array}{cccc|c} 1 & 2 & \cdots & \alpha_p & \alpha_p \\ 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & \cdots & n \\ 1 & 2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right)$$

and, hence,

$$e_{\alpha_p n-1} \cdot \left(\begin{array}{cccc|c} 1 & 2 & \cdots & n-1 & \alpha_p \\ 1 & 2 & \cdots & \cdots & n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right) = (-1)^{n-2} \frac{(\mu_{n-1} + 1)}{2} \left(\begin{array}{cccc|c} 1 & 2 & \cdots & \alpha_p & \alpha_p \\ 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & \cdots & n \\ 1 & 2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right).$$

By iterating this argument, we obtain:

$$e_{\alpha_p j} \cdot \left(\frac{1}{(n-j)!} \left(\begin{array}{cccc|c} 1 & 2 & \cdots & j & \alpha_p^{n-j} \\ 1 & 2 & \cdots & j & \cdots & n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & j & \cdots & n \\ 1 & 2 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right) \right) =$$

$$= (-1)^{j-1} \frac{\mu_j + n - j}{(n-j+1)!} \left(\begin{array}{cccc|c} 1 & 2 & \cdots & j-1 & \alpha_p^{n-j+1} \\ 1 & 2 & \cdots & j-1 & j & \cdots & n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & j-1 & j & \cdots & n \\ 1 & 2 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right).$$

By iterating this procedure,

$$e_{\alpha_p 1} \cdots e_{\alpha_p n} \cdot (Der_{\vec{\mu}} | Der_{\vec{\mu}}^P) =$$

$$= \frac{(-1)^{\binom{n}{2}}}{n!} (\mu_1 + n - 1)(\mu_2 + n - 2) \cdots \mu_n \left(\begin{array}{cccc|c} \alpha_p & \alpha_p & \cdots & \alpha_p \\ 1 & 2 & \cdots & n \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array} \right)$$

and

$$\begin{aligned}
& e_{\text{Coder}_{np}, \text{Der}_{np}} \cdot (\text{Der}_{\bar{\mu}} | \text{Der}_{\bar{\mu}}^P) = \\
& = \left(\prod_{i=0}^{p-1} (\mu_1 - i + n - 1) \cdots (\mu_n - i) \right) \frac{(-1)^{\binom{n}{2}p}}{(n!)^p} \left(\begin{array}{cccc} \alpha_p & \alpha_p & \cdots & \alpha_p \\ \alpha_{p-1} & \alpha_{p-1} & \cdots & \alpha_{p-1} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1 & \alpha_1 & \cdots & \alpha_1 \\ 1 & 2 & \cdots & \\ \cdots & \cdots & & \end{array} \right) = \\
& = \left(\prod_{i=0}^{p-1} (\mu_1 - i + n - 1) \cdots (\mu_n - i) \right) \frac{(-1)^{\binom{n}{2}p + \binom{p}{2}n}}{(n!)^p} \left(\begin{array}{cccc} \alpha_1 & \alpha_1 & \cdots & \alpha_1 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{p-1} & \alpha_{p-1} & \cdots & \alpha_{p-1} \\ \alpha_p & \alpha_p & \cdots & \alpha_p \\ 1 & 2 & \cdots & \\ \cdots & & & \end{array} \right).
\end{aligned}$$

Since

$$\begin{aligned}
& e_{\text{Der}_{np}^*, \text{Coder}_{np}} \cdot \left(\begin{array}{cccc} \alpha_1 & \alpha_1 & \cdots & \alpha_1 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{p-1} & \alpha_{p-1} & \cdots & \alpha_{p-1} \\ \alpha_p & \alpha_p & \cdots & \alpha_p \\ 1 & 2 & \cdots & \\ \cdots & & & \end{array} \right) = (-1)^{\binom{n}{2}p} (n!)^p (\text{Der}_{\bar{\mu}} | \text{Der}_{\bar{\mu}}^P) = \\
& = \mathbf{K}_n^P(v_\mu) = \mathbf{K}_n^P \cdot (\text{Der}_{\bar{\mu}} | \text{Der}_{\bar{\mu}}^P) = e_{\text{Der}_{np}^*, \text{Coder}_{np}} e_{\text{Coder}_{np}, \text{Der}_{np}} \cdot (\text{Der}_{\bar{\mu}} | \text{Der}_{\bar{\mu}}^P) = \\
& = \left(\prod_{i=0}^{p-1} (\mu_1 - i + n - 1) \cdots (\mu_n - i) \right) \frac{(-1)^{\binom{n}{2}p}}{(n!)^p} (-1)^{\binom{p}{2}n} \times \\
& \quad \times e_{\text{Der}_{np}^*, \text{Coder}_{np}} \cdot \left(\begin{array}{cccc} \alpha_1 & \alpha_1 & \cdots & \alpha_1 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{p-1} & \alpha_{p-1} & \cdots & \alpha_{p-1} \\ \alpha_p & \alpha_p & \cdots & \alpha_p \\ 1 & 2 & \cdots & \\ \cdots & & & \end{array} \right) = \\
& = \left(\prod_{i=0}^{p-1} (\mu_1 - i + n - 1) \cdots (\mu_n - i) \right) (-1)^{\binom{p}{2}n} (\text{Der}_{\bar{\mu}} | \text{Der}_{\bar{\mu}}^P).
\end{aligned}$$

Notice that, if $\mu_n < p$, then $\mathbf{K}_n^P(v_\mu) = 0$.

□

9 Appendix. A glimpse on the superalgebraic method of virtual variables

In this section, we summarize the main features of the superalgebraic method of virtual variables. We follow [8] and [9].

9.1 The general linear Lie super algebra $gl(m|n)$

Given a vector space V of dimension n , we will regard it as a subspace of a \mathbb{Z}_2 -graded vector space $V_0 \oplus V_1$, where $V_1 = V$. The vector spaces V_0 (we assume that $\dim(V_0) = m$ is “sufficiently large”) is called the *positive virtual (auxiliary) vector space* and V is called the *(negative) proper vector space*.

The inclusion $V \subset V_0 \oplus V_1$ induces a natural embedding of the ordinary general linear Lie algebra $gl(n)$ of V_n into the *auxiliary* general linear Lie *superalgebra* $gl(m|n)$ of $V_0 \oplus V_1$ (see, e.g. [23], [39]).

Let $\mathcal{A}_0 = \{\alpha_1, \dots, \alpha_{m_0}\}$, $\mathcal{L} = \{x_1, x_2, \dots, x_n\}$ denote *fixed bases* of V_0 and $V = V_1$, respectively; therefore $|\alpha_s| = 0 \in \mathbb{Z}_2$, and $|i| = 1 \in \mathbb{Z}_2$.

Let

$$\{e_{a,b}; a, b \in \mathcal{A}_0 \cup \mathcal{L}\}, \quad |e_{a,b}| = |a| + |b| \in \mathbb{Z}_2$$

be the standard \mathbb{Z}_2 -homogeneous basis of the Lie superalgebra $gl(m|n)$ provided by the elementary matrices. The elements $e_{a,b} \in gl(m|n)$ are \mathbb{Z}_2 -homogeneous of \mathbb{Z}_2 -degree $|e_{a,b}| = |a| + |b|$.

The superbracket of the Lie superalgebra $gl(m_0|m_1+n)$ has the following explicit form:

$$[e_{a,b}, e_{c,d}] = \delta_{bc} e_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)} \delta_{ad} e_{c,b},$$

$a, b, c, d \in \mathcal{A}_0 \cup \mathcal{L}$.

For the sake of readability, we will frequently write $\mathcal{L} = \{1, 2, \dots, n\}$ in place of $\mathcal{L} = \{x_1, x_2, \dots, x_n\}$.

The elements of the sets $\mathcal{A}_0, \mathcal{L}$ are called *positive virtual symbols* and *negative proper symbols*, respectively.

9.2 The supersymmetric algebra $\mathbb{C}[M_{m|n,d}]$

For the sake of readability, given $n, d \in \mathbb{Z}^+$, $n \leq d$, we write

$$M_{n,d} = [(i|j)]_{i=1, \dots, n, j=1, \dots, d} = \begin{pmatrix} (1|1) & \dots & (1|d) \\ \vdots & & \vdots \\ (n|1) & \dots & (n|d) \end{pmatrix}$$

in place of

$$M_{n,d} = [x_{ij}]_{i=1,\dots,n;j=1,\dots,d} = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ x_{21} & \cdots & x_{2d} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nd} \end{bmatrix}.$$

(compare with eq. (3)) and, consistently,

$$\mathbb{C}[M_{n,d}] = \mathbb{C}[(i|j)]_{i=1,\dots,n;j=1,\dots,d}$$

in place of

$$\mathbb{C}[M_{n,d}] = \mathbb{C}[x_{ij}]_{i=1,\dots,n;j=1,\dots,d}$$

for the polynomial algebra in the (commutative) entries $(i|j)$ of the matrix $M_{n,d}$.

We regard the commutative algebra $\mathbb{C}[M_{n,d}]$ as a subalgebra of the “auxiliary” supersymmetric algebra

$$\mathbb{C}[M_{m|n,d}]$$

generated by the (\mathbb{Z}_2 -graded) variables

$$(a|j), \quad a \in \mathcal{A}_0 \cup \mathcal{L}, \quad j \in \mathcal{P} = \{j = 1, \dots, d; |j| = 1 \in \mathbb{Z}_2\},$$

with $|(a|j)| = |a| + |j| \in \mathbb{Z}_2$, subject to the commutation relations:

$$(a|h)(b|k) = (-1)^{|(a|h)|| (b|k)|} (b|k)(a|h).$$

In plain words, $\mathbb{C}[M_{m|n,d}]$ is the free supersymmetric algebra

$$\mathbb{C}[(\alpha_s|j), (i|j)]$$

generated by the (\mathbb{Z}_2 -graded) variables $(\alpha_s|j), (i|j)$, $j = 1, 2, \dots, d$, where all the variables commute each other, with the exception of pairs of variables $(\alpha_s|j), (\alpha_t|j)$ that skew-commute:

$$(\alpha_s|j)(\alpha_t|j) = -(\alpha_t|j)(\alpha_s|j).$$

In the standard notation of multilinear algebra, we have:

$$\mathbb{C}[M_{m|n,d}] \cong \Lambda[V_0 \otimes P_d] \otimes \text{Sym}[V_1 \otimes P_d]$$

where $P_d = (P_d)_1$ denotes the trivially \mathbb{Z}_2 -graded vector space with distinguished basis $\mathcal{P} = \{j = 1, \dots, d; |j| = 1 \in \mathbb{Z}_2\}$.

9.3 Left superderivations and left superpolarizations

A *left superderivation* D^l (\mathbb{Z}_2 -homogeneous of degree $|D^l|$) (see, e.g. [39], [23]) on $\mathbb{C}[M_{m|n,d}]$ is an element of the superalgebra $End_{\mathbb{C}}[\mathbb{C}[M_{m|n,d}]]$ that satisfies "Leibniz rule"

$$D^l(\mathbf{p} \cdot \mathbf{q}) = D^l(\mathbf{p}) \cdot \mathbf{q} + (-1)^{|D^l||\mathbf{p}|} \mathbf{p} \cdot D^l(\mathbf{q}),$$

for every \mathbb{Z}_2 -homogeneous of degree $|\mathbf{p}|$ element $\mathbf{p} \in \mathbb{C}[M_{m|n,d}]$.

Given two symbols $a, b \in \mathcal{A}_0 \cup \mathcal{L}$, the *left superpolarization* $D_{a,b}^l$ of b to a is the unique *left superderivation* of $\mathbb{C}[M_{m|n,d}]$ of \mathbb{Z}_2 -degree $|D_{a,b}^l| = |a| + |b| \in \mathbb{Z}_2$ such that

$$D_{a,b}^l((c|j)) = \delta_{bc} (a|j), \quad c \in \mathcal{A}_0 \cup \mathcal{L}, \quad j = 1, \dots, n.$$

Informally, we say that the operator $D_{a,b}^l$ *annihilates* the symbol b and *creates* the symbol a .

9.4 The superalgebra $\mathbb{C}[M_{m|n,d}]$ as a $\mathbf{U}(gl(m|n))$ -module

Since

$$D_{a,b}^l D_{c,d}^l - (-1)^{(|a|+|b|)(|c|+|d|)} D_{c,d}^l D_{a,b}^l = \delta_{b,c} D_{a,d}^l - (-1)^{(|a|+|b|)(|c|+|d|)} \delta_{a,d} D_{c,b}^l,$$

the map

$$e_{a,b} \mapsto D_{a,b}^l, \quad a, b \in \mathcal{A}_0 \cup \mathcal{L}$$

is a Lie superalgebra morphism from $gl(m|n)$ to $End_{\mathbb{C}}[\mathbb{C}[M_{m|n,d}]]$ and, hence, it uniquely defines a representation:

$$\varrho : \mathbf{U}(gl(m|n)) \rightarrow End_{\mathbb{C}}[\mathbb{C}[M_{m|n,d}]],$$

where $\mathbf{U}(gl(m|n))$ is the enveloping superalgebra of $gl(m|n)$.

In the following, we always regard the superalgebra $\mathbb{C}[M_{m|n,d}]$ as a $\mathbf{U}(gl(m|n))$ -supermodule, with respect to the action induced by the representation ϱ :

$$e_{a,b} \cdot \mathbf{p} = D_{a,b}^l(\mathbf{p}),$$

for every $\mathbf{p} \in \mathbb{C}[M_{m|n,d}]$.

We recall that $\mathbf{U}(gl(m|n))$ -module $\mathbb{C}[M_{m|n,d}]$ is a semisimple module, whose simple submodules are - up to isomorphism - *Schur supermodules* (see, e.g. [4], [5], [2]). For a more traditional presentation, see also [15]).

Clearly, $\mathbf{U}(gl(0|n)) = \mathbf{U}(gl(n))$ is a subalgebra of $\mathbf{U}(gl(m|n))$ and the subalgebra $\mathbb{C}[M_{n,d}]$ is a $\mathbf{U}(gl(n))$ -submodule of $\mathbb{C}[M_{m|n,d}]$.

9.5 The virtual algebra $Virt(m, n)$ and the virtual presentations of elements in $\mathbf{U}(gl(n))$

We say that a product

$$e_{a_m, b_m} \cdots e_{a_1, b_1} \in \mathbf{U}(gl(m|n)), \quad a_i, b_i \in \mathcal{A}_0 \cup \mathcal{L}, \quad i = 1, \dots, m$$

is an *irregular expression* whenever there exists a right subword

$$e_{a_i, b_i} \cdots e_{a_2, b_2} e_{a_1, b_1},$$

$i \leq m$ and a virtual symbol $\gamma \in \mathcal{A}_0$ such that

$$\#\{j; b_j = \gamma, j \leq i\} > \#\{j; a_j = \gamma, j < i\}.$$

The meaning of an irregular expression in terms of the action of $\mathbf{U}(gl(m|n))$ by left superpolarization on the algebra $\mathbb{C}[M_{m|n, d}]$ is that there exists a virtual symbol γ and a right subsequence in which the symbol γ is *annihilated* more times than it was already *created* and, therefore, the action of an irregular expression on the algebra $\mathbb{C}[M_{n, d}]$ is *zero*.

Example 7. Let $\gamma \in \mathcal{A}_0$ and $x_i, x_j \in \mathcal{L}$. The product

$$e_{\gamma, x_j} e_{x_i, \gamma} e_{x_j, \gamma} e_{\gamma, x_i}$$

is an irregular expression. □

Let \mathbf{Irr} be the *left ideal* of $\mathbf{U}(gl(m|n))$ generated by the set of irregular expressions.

Proposition 19. The superpolarization action of any element of \mathbf{Irr} on the subalgebra $\mathbb{C}[M_{n, d}] \subset \mathbb{C}[M_{m|n, d}]$ - via the representation ϱ - is identically zero.

Proposition 20. The sum $\mathbf{U}(gl(0|n)) + \mathbf{Irr}$ is a direct sum of vector subspaces of $\mathbf{U}(gl(m|n))$.

Proposition 21. The direct sum vector subspace $\mathbf{U}(gl(0|n)) \oplus \mathbf{Irr}$ is a subalgebra of $\mathbf{U}(gl(m|n))$.

The subalgebra

$$Virt(m, n) = \mathbf{U}(gl(0|n)) \oplus \mathbf{Irr} \subset \mathbf{U}(gl(m|n)).$$

is called the *virtual algebra*.

Proposition 22. *The left ideal \mathbf{Irr} of $\mathbf{U}(gl(m|n))$ is a two sided ideal of $Virt(m, n)$.*

The *Capelli devirtualization epimorphism* is the surjection

$$\mathfrak{p} : Virt(m, n) = \mathbf{U}(gl(0|n)) \oplus \mathbf{Irr} \twoheadrightarrow \mathbf{U}(gl(0|n)) = \mathbf{U}(gl(n))$$

with $Ker(\mathfrak{p}) = \mathbf{Irr}$.

Any element in $\mathbf{M} \in Virt(m, n)$ defines an element in $\mathfrak{m} \in \mathbf{U}(gl(n))$ - via the map \mathfrak{p} - and \mathbf{M} is called a *virtual presentation* of \mathfrak{m} .

Furthermore,

Proposition 23. *The subalgebra $\mathbb{C}[M_{n,d}] \subset \mathbb{C}[M_{m|n,d}]$ is invariant with respect to the action of the subalgebra $Virt(m, n)$.*

Proposition 24. *For every element $\mathfrak{m} \in \mathbf{U}(gl(n))$, the action of \mathfrak{m} on the subalgebra $\mathbb{C}[M_{n,d}]$ is the same of the action of any of its virtual presentation $\mathbf{M} \in Virt(m, n)$. In symbols,*

$$\text{if } \mathfrak{p}(\mathbf{M}) = \mathfrak{m} \text{ then } \mathfrak{m} \cdot \mathbf{P} = \mathbf{M} \cdot \mathbf{P}, \text{ for every } \mathbf{P} \in \mathbb{C}[M_{n,d}].$$

Since the map \mathfrak{p} a surjection, any element $\mathfrak{m} \in \mathbf{U}(gl(n))$ admits several virtual presentations. In the sequel, we even take virtual presentations as the *definition* of special elements in $\mathbf{U}(gl(n))$, and this method will turn out to be quite effective.

The superalgebra $\mathbf{U}(gl(m|n))$ is a Lie module with respect to the adjoint representation $Ad_{gl(m|n)}$. Since $gl(n) = gl(0|n)$ is a Lie subalgebra of $gl(m|n)$, then $\mathbf{U}(gl(m|n))$ is a $gl(n)$ -module with respect to the adjoint action $Ad_{gl(n)}$ of $gl(n)$.

Proposition 25. *The virtual algebra $Virt(m, n)$ is a submodule of $\mathbf{U}(gl(m|n))$ with respect to the adjoint action $Ad_{gl(n)}$ of $gl(n)$.*

Proposition 26. *The Capelli epimorphism*

$$\mathfrak{p} : Virt(m, n) \twoheadrightarrow \mathbf{U}(gl(n))$$

is an $Ad_{gl(n)}$ -equivariant map.

Corollary 18. *The isomorphism \mathfrak{p} maps any $Ad_{gl(n)}$ -invariant element $\mathfrak{m} \in Virt(m, n)$ to a central element of $\mathbf{U}(gl(n))$.*

Balanced monomials are elements of the algebra $\mathbf{U}(gl(m|n))$ of the form:

$$- e_{i_1, \gamma_{p_1}} \cdots e_{i_k, \gamma_{p_k}} \cdot e_{\gamma_{p_1}, j_1} \cdots e_{\gamma_{p_k}, j_k},$$

$$- e_{i_1, \theta_{q_1}} \cdots e_{i_k, \theta_{q_k}} \cdot e_{\theta_{q_1}, \gamma_{p_1}} \cdots e_{\theta_{q_k}, \gamma_{p_k}} \cdot e_{\gamma_{p_1}, j_1} \cdots e_{\gamma_{p_k}, j_k},$$

– and so on,

where $i_1, \dots, i_k, j_1, \dots, j_k \in L$, i.e., the $i_1, \dots, i_k, j_1, \dots, j_k$ are k proper (negative) symbols, and the $\gamma_{p_1}, \dots, \gamma_{p_k}, \theta_{q_1}, \dots, \theta_{q_k}, \dots$ are virtual symbols. In plain words, a balanced monomial is product of two or more factors where the rightmost one *annihilates* (by superpolarization) the k proper symbols j_1, \dots, j_k and *creates* (by superpolarization) some virtual symbols; the leftmost one *annihilates* all the virtual symbols and *creates* the k proper symbols i_1, \dots, i_k ; between these two factors, there might be further factors that annihilate and create virtual symbols only.

Proposition 27. *Every balanced monomial belongs to $\text{Virt}(m, n)$. Hence, the Capelli epimorphism \mathfrak{p} maps balanced monomials to elements of $\mathbf{U}(\mathfrak{gl}(n))$.*

9.6 Bitableaux monomials and Capelli bitableaux in $\mathbf{U}(\mathfrak{gl}(n))$

We will introduce two classes of remarkable elements of the enveloping algebra $\mathbf{U}(\mathfrak{gl}(n))$, that we call *bitableaux monomials*, *Capelli bitableaux*, respectively.

Let $\lambda \vdash h$ be a partition, and label the boxes of its Ferrers diagram with the numbers $1, 2, \dots, h$ in the following way:

$$\begin{array}{ccccccc} 1 & 2 & \cdots & \cdots & \lambda_1 & & \\ \lambda_1 + 1 & \lambda_1 + 2 & \cdots & \lambda_1 + \lambda_2 & & & \\ \cdots & \cdots & \cdots & & & & \\ \cdots & \cdots & h & & & & \end{array}.$$

A *Young tableau* T of shape λ over the alphabet $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{L}$ is a map $T : \underline{h} = \{1, 2, \dots, h\} \rightarrow \mathcal{A}$; the element $T(i)$ is the symbol in the cell i of the tableau T .

The sequences

$$\begin{aligned} & T(1)T(2) \cdots T(\lambda_1), \\ & T(\lambda_1 + 1)T(\lambda_1 + 2) \cdots T(\lambda_1 + \lambda_2), \\ & \cdots \end{aligned}$$

are called the *row words* of the Young tableau T .

We will also denote a Young tableau by its sequence of rows words, that is $T = (\omega_1, \omega_2, \dots, \omega_p)$. Furthermore, the *word of the tableau* T is the concatenation

$$w(T) = \omega_1 \omega_2 \cdots \omega_p.$$

The *content* of a tableau T is the function $c_T : \mathcal{A} \rightarrow \mathbb{N}$,

$$c_T(a) = \#\{i \in \underline{h}; T(i) = a\}.$$

Given a shape/partition λ , we assume that $|\mathcal{A}_0| = m \geq \tilde{\lambda}_1$, where $\tilde{\lambda}$ denotes the conjugate shape/partition of λ . Let us denote by $\alpha_1, \dots, \alpha_p \in \mathcal{A}_0$ an *arbitrary* family of *distinct positive symbols*. Set

$$C_\lambda^* = \begin{pmatrix} \alpha_1 \dots \dots \alpha_1 \\ \alpha_2 \dots \dots \alpha_2 \\ \dots \dots \\ \alpha_p \dots \dots \alpha_p \end{pmatrix}. \quad (16)$$

The tableaux of kind (16) are called *virtual Coderuyts tableaux* of shape λ .

Let S and T be two Young tableaux of same shape $\lambda \vdash h$ on the alphabet $\mathcal{A}_0 \cup \mathcal{L}$:

$$S = \begin{pmatrix} z_{i_1} \dots \dots z_{i_{\lambda_1}} \\ z_{j_1} \dots \dots z_{j_{\lambda_2}} \\ \dots \dots \\ z_{s_1} \dots z_{s_{\lambda_p}} \end{pmatrix}, \quad T = \begin{pmatrix} z_{h_1} \dots \dots z_{h_{\lambda_1}} \\ z_{k_1} \dots \dots z_{k_{\lambda_2}} \\ \dots \dots \\ z_{t_1} \dots z_{t_{\lambda_p}} \end{pmatrix}.$$

To the pair (S, T) , we associate the *bitableau monomial*:

$$e_{S,T} = e_{z_{i_1}, z_{h_1}} \cdots e_{z_{i_{\lambda_1}}, z_{h_{\lambda_1}}} e_{z_{j_1}, z_{k_1}} \cdots e_{z_{j_{\lambda_2}}, z_{k_{\lambda_2}}} \cdots \cdots e_{z_{s_1}, z_{t_1}} \cdots e_{z_{s_{\lambda_p}}, z_{t_{\lambda_p}}}$$

in $\mathbf{U}(gl(m|n))$.

Given a pair of Young tableaux S, T of the same shape λ on the proper alphabet L , consider the elements

$$e_{S, C_\lambda^*} e_{C_\lambda^*, T} \in \mathbf{U}(gl(m|n)).$$

Since these elements are *balanced monomials* in $\mathbf{U}(gl(m|n))$, then they belong to the *virtual subalgebra* $Virt(m, n)$.

Hence, we can consider their images in $\mathbf{U}(gl(n))$ with respect to the Capelli epimorphism \mathfrak{p} .

We set

$$\mathfrak{p}\left(e_{S, C_\lambda^*} e_{C_\lambda^*, T}\right) = [S|T] \in \mathbf{U}(gl(n)), \quad (17)$$

and call the element $[S|T]$ a *Capelli bitableau*.

The elements defined in (17) do not depend on the choice of the virtual Coderuyts tableau C_λ^* .

9.7 Biproducts and bitableaux in $\mathbb{C}[M_{m|n,d}]$

Embed the algebra

$$\mathbb{C}[M_{m|n,d}] = \mathbb{C}[(\alpha_s|j), (i|j)]$$

into the (supersymmetric) algebra $\mathbb{C}[(\alpha_s|j), (i|j), (\gamma|j)]$ generated by the (\mathbb{Z}_2 -graded) variables $(\alpha_s|j), (i|j), (\gamma|j)$, $j = 1, 2, \dots, d$, where

$$|(\gamma|j)| = 1 \in \mathbb{Z}_2 \text{ for every } j = 1, 2, \dots, d,$$

and denote by $D_{z_i, \gamma}^l$ the superpolarization of γ to z_i .

Let $\omega = z_1 z_2 \cdots z_p$ be a word on $\mathcal{A}_0 \cup \mathcal{L}$, and $\varpi = j_{t_1} j_{t_2} \cdots j_{t_q}$ a word on the alphabet $P = \{1, 2, \dots, d\}$. The *biproduct*

$$(\omega|\varpi) = (z_1 z_2 \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_q})$$

is the element

$$D_{z_1, \gamma}^l D_{z_2, \gamma}^l \cdots D_{z_p, \gamma}^l \left((\gamma|j_{t_1}) (\gamma|j_{t_2}) \cdots (\gamma|j_{t_q}) \right) \in \mathbb{C}[M_{m|n,d}]$$

if $p = q$ and is set to be zero otherwise.

Claim 1. *The biproduct $(\omega|\varpi) = (z_1 z_2 \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_q})$ is supersymmetric in the z 's and skew-symmetric in the j 's. In symbols*

1. $(z_1 z_2 \cdots z_i z_{i+1} \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_q}) = (-1)^{|z_i||z_{i+1}|} (z_1 z_2 \cdots z_{i+1} z_i \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_q})$
2. $(z_1 z_2 \cdots z_i z_{i+1} \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_i} j_{t_{i+1}} \cdots j_{t_q}) = - (z_1 z_2 \cdots z_i z_{i+1} \cdots z_p | j_{t_1} \cdots j_{t_{i+1}} j_{t_i} \cdots j_{t_q})$.

Proposition 28. (Laplace expansions) *We have*

1. $(\omega_1 \omega_2 | \varpi) = \Sigma_{(\varpi)} (-1)^{|\varpi_{(1)}||\omega_2|} (\omega_1 | \varpi_{(1)}) (\omega_2 | \varpi_{(2)})$.
2. $(\omega | \varpi_1 \varpi_2) = \Sigma_{(\omega)} (-1)^{|\varpi_1||\omega_{(2)}|} (\omega_{(1)} | \varpi_1) (\omega_{(2)} | \varpi_2)$.

where

$$\Delta(\varpi) = \Sigma_{(\varpi)} \varpi_{(1)} \otimes \varpi_{(2)}, \quad \Delta(\omega) = \Sigma_{(\omega)} \omega_{(1)} \otimes \omega_{(2)}$$

denote the coproducts in the Sweedler notation of the elements ϖ and ω in the supersymmetric Hopf algebra of W (see, e.g. [2]) and in the free exterior Hopf algebra generated by $j = 1, 2, \dots, d$, respectively.

Let $\omega = i_1 i_2 \cdots i_p$, $\varpi = j_1 j_2 \cdots j_p$ be words on the negative alphabet $\mathcal{L} = \{1, 2, \dots, n\}$ and on the positive alphabet $\mathcal{P} = \{1, 2, \dots, d\}$.

From Proposition 28, we infer

Corollary 19. *The biproduct of the two words ω and ϖ*

$$(\omega|\varpi) = (i_1 i_2 \cdots i_p | j_1 j_2 \cdots j_p)$$

is the signed minor:

$$(\omega|\varpi) = (-1)^{\binom{p}{2}} \det \left((i_r | j_s) \right)_{r,s=1,2,\dots,p} \in \mathbb{C}[M_{n,d}].$$

Following the notation introduced in the previous sections, let

$$\text{Super}[V_0 \oplus V_1] = \text{Sym}[V_0] \otimes \Lambda[V_1]$$

denote the (*super*)*symmetric* algebra of the space

$$V_0 \oplus V_1$$

(see, e.g. [39]).

By multilinearity, the algebra $\text{Super}[V_0 \oplus V_1]$ is the same as the superalgebra $\text{Super}[\mathcal{A}_0 \cup \mathcal{L}]$ generated by the "variables"

$$\alpha_1, \dots, \alpha_{m_0} \in \mathcal{A}_0, \quad 1, \dots, n \in L,$$

modulo the congruences

$$z z' = (-1)^{|z||z'|} z' z, \quad z, z' \in \mathcal{A}_0 \cup \mathcal{L}.$$

Let $d_{z,z'}^l$ denote the (left)polarization operator of z' to z on

$$\text{Super}[W] = \text{Super}[\mathcal{A}_0 \cup \mathcal{L}],$$

that is the unique superderivation of \mathbb{Z}_2 -degree

$$|z| + |z'| \in \mathbb{Z}_2$$

such that

$$d_{z,z'}^l(z'') = \delta_{z',z''} \cdot z,$$

for every $z, z', z'' \in \mathcal{A}_0 \cup \mathcal{L}$.

Clearly, the map

$$e_{z,z'} \rightarrow d_{z,z'}^l$$

is a Lie superalgebra map and, therefore, induces a structure of

$$gl(m|n) - \text{module}$$

on $\text{Super}[\mathcal{A}_0 \cup \mathcal{L}] = \text{Super}[V_0 \oplus V_1]$.

Proposition 29. Let $\varpi = j_{t_1} j_{t_2} \cdots j_{t_q}$ be a word on $P = \{1, 2, \dots, d\}$. The map

$$\Phi_\varpi : \omega \mapsto (\omega|\varpi),$$

ω any word on $\mathcal{A}_0 \cup \mathcal{L}$, uniquely defines $gl(m|n)$ -equivariant linear operator

$$\Phi_\varpi : Super[\mathcal{A}_0 \cup \mathcal{L}] \rightarrow \mathbb{C}[M_{m|n,d}],$$

that is

$$\Phi_\varpi(e_{z,z'} \cdot \omega) = \Phi_\varpi(d_{z,z'}^l(\omega)) = D_{z,z'}^l((\omega|\varpi)) = e_{z,z'} \cdot (\omega|\varpi),$$

for every $z, z' \in \mathcal{A}_0 \cup \mathcal{L}$.

With a slight abuse of notation, we will write (29) in the form

$$D_{z,z'}^l((\omega|\varpi)) = (d_{z,z'}^l(\omega)|\varpi). \quad (18)$$

Let $S = (\omega_1, \omega_2, \dots, \omega_p)$ and $T = (\varpi_1, \varpi_2, \dots, \varpi_p)$ be Young tableaux on $\mathcal{A}_0 \cup \mathcal{L}$ and $P = \{1, 2, \dots, d\}$ of shapes λ and μ , respectively.

If $\lambda = \mu$, the *Young bitableau* $(S|T)$ is the element of $\mathbb{C}[M_{m|n,d}]$ defined as follows:

$$(S|T) = \left(\begin{array}{c|c} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{array} \right) = \pm (\omega_1|\varpi_1)(\omega_2|\varpi_2) \cdots (\omega_p|\varpi_p),$$

where

$$\pm = (-1)^{|\omega_2||\varpi_1| + |\omega_3|(|\varpi_1| + |\varpi_2|) + \cdots + |\omega_p|(|\varpi_1| + |\varpi_2| + \cdots + |\varpi_{p-1}|)}.$$

If $\lambda \neq \mu$, the *Young bitableau* $(S|T)$ is set to be zero.

By naturally extending the slight abuse of notation (18), the action of any polarization on bitableaux can be explicitly described:

Proposition 30. Let $z, z' \in \mathcal{A}_0 \cup \mathcal{L}$, and let $S = (\omega_1, \dots, \omega_p)$, $T = (\varpi_1, \dots, \varpi_p)$. We have the following identity:

$$\begin{aligned} e_{z,z'} \cdot (S|T) &= D_{z,z'}^l \left(\begin{array}{c|c} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{array} \right) \\ &= \sum_{s=1}^p (-1)^{(|z| + |z'|)\epsilon_s} \left(\begin{array}{c|c} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ d_{z,z'}^l(\omega_s) & \vdots \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{array} \right), \end{aligned}$$

where

$$\epsilon_1 = 1, \quad \epsilon_s = |\omega_1| + \cdots + |\omega_{s-1}|, \quad s = 2, \dots, p.$$

Example 8. Let $\alpha_i \in \mathcal{A}_0$, $1, 2, 3, 4 \in L$, $|D_{\alpha_i, 2}| = 1$. Then

$$\begin{aligned} e_{\alpha_i, 2} \cdot \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 2 & 3 \\ 2 & 3 & & 2 & 3 & \\ 4 & 2 & & 3 & 1 & \end{array} \right) &= D_{\alpha_i, 2}^l \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 2 & 3 \\ 2 & 3 & & 2 & 3 & \\ 4 & 2 & & 3 & 1 & \end{array} \right) = \\ &= \left(\begin{array}{ccc|ccc} 1 & 3 & \alpha_i & 1 & 2 & 3 \\ 2 & 3 & & 2 & 3 & \\ 4 & 2 & & 3 & 1 & \end{array} \right) - \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 2 & 3 \\ \alpha_i & 3 & & 2 & 3 & \\ 4 & 2 & & 3 & 1 & \end{array} \right) + \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 2 & 3 \\ 2 & 3 & & 2 & 3 & \\ 4 & \alpha_i & & 3 & 1 & \end{array} \right). \end{aligned}$$

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