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Published Version:
Chinni, G. (2023). On the sharp Gevrey regularity for a generalization of the Metivier operator. MATHEMATISCHE ANNALEN, Early Access, 1-47 [10.1007/s00208-022-02558-7].

Availability:
This version is available at: https://hdl.handle.net/11585/920131 since: 2023-03-08

Published:

DOI: http://doi.org/10.1007/s00208-022-02558-7

ON THE SHARP GEVREY REGULARITY FOR A GENERALIZATION OF THE MÉTIVIER OPERATOR

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ABSTRACT. We prove a sharp Gevrey hypoellipticity for the operator

$$D_x^2 + \left(x^{2n+1} D_y\right)^2 + \left(x^n y^m D_y\right)^2,$$

in Ω open neighborhood of the origin in $\mathbb{R}^2$, where n and m are positive integers. The operator is a non trivial generalization of the Métivier operator studied in [20]. However it has a symplectic characteristic manifold and a non symplectic stratum according to the Poisson-Treves stratification. According to Treves conjecture it turns out not to be analytic hypoelliptic.

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1. INTRODUCTION

In [20] G. Métivier studied the non-analytic hypoellipticity for a class of second order partial differential operators with analytic coefficients in Ω, open neighborhood of the origin in $\mathbb{R}^n$, whose principal symbol vanishes exactly of order two on a submanifold of $T^*\Omega$.

In the case of sum of squares of vector fields the most representative model of such class is the following

$$D_x^2 + x^2 D_y^2 + \left(y D_y\right)^2. \tag{1.1}$$

In [20] the author proves that the operator (1.1) is G^2-hypoelliptic and not better in a neighborhood of the origin.

Date: December 12, 2022.

2010 *Mathematics Subject Classification.* 35H10, 35H20, 35B65 .

Key words and phrases. Microlocal regularity, Gevrey Hypoellipticity, Sum of squares, Eigenfunctions.

This means that if u is a smooth function in the neighborhood of the origin U solving there the equation $Pu = f$, where P denotes the operator in (1.1) and f is analytic in U, then u belongs to the Gevrey class $G^2(U)$ and there are no solutions in $G^s(U)$ with $1 \leq s < 2$.
Here $G^s(U)$, $s \geq 1$, denotes the space of Gevrey functions in U, i.e. the set of all $f \in C^\infty(U)$ such that for every compact set $K \subset U$ there are two positive constants C_K and A such that for every $\alpha \in \mathbb{Z}_+^n$

$$|D^\alpha f(x)| \leq A C_K^{|\alpha|} |\alpha|^{s|\alpha|}, \qquad \forall x \in K.$$

The Gevrey index 2 corresponds to that obtained by Derridj and Zuily, [13] (see also [1]), when applied to this operator.
For this type of operators Hörmander, [18], stated the condition for C^∞ hypoellipticity:

(H) the Lie algebra generated by the vector fields and their commutators has dimension equal to the dimension of the ambient space.

We recall that a sum of squares operator satisfying the Hörmander condition is said to be G^s-hypoelliptic, $s \geq 1$, in Ω, open subset of $\mathbb{R}^n$, if for any U open subset of Ω the conditions $u \in \mathscr{D}'(U)$ and $Pu \in G^s(U')$ imply that $u \in G^s(U')$, U' open subset of U. Needless to say $G^1(U') = C^\omega(U')$, the class of real-analytic functions on U.

One of the main motivations to study the optimality of the Gevrey regularity of solutions to sums of squares in two variables is due to the fact that in dimension greater or equal than 4 the Treves conjecture has been proved to be false, see [2], [6]. In dimension 3 there are no results although in [7] a candidate has been produced that should violate the conjecture. On the contrary there are good reasons to surmise that the conjecture of Treves holds in two variables. We refer to [24], [10] for more details on the statement of the conjecture, as well as to [7] for a discussion of both the 3- and 2-dimensional cases.
This implies that it is of crucial importance to know if a certain Gevrey regularity, which may be relatively easy to obtain by using L^2 a priori estimates, is optimal or not.
In two variables this becomes particularly difficult because if the characteristic set is not a symplectic real analytic manifold, the Hamilton leaves corresponding to the kernel of the symplectic form have injective projection onto the fibers of the cotangent bundle, thus causing great technical complication in the construction of a singular solution, which is the method of proving optimality.
In order to exhibit a singular solution, meaning a solution which is not more regular than predicted, one constructs first a so called asymptotic formal solution. Once this is done, the formal solution gives a true solution solving the same equation with possibly a different right hand side and then one can finally achieve the proof. The construction of the formal solution in [20], e.g. for the operator in (1.1), uses the spectral theory of the harmonic oscillator, i.e. of the operator $D_x^2 + x^2$, the variable y being reduced to a parameter. For the eigenfunctions of the harmonic oscillator there are three terms recurrence formulas relating the derivative of an eigenfunction to those up and down one notch. As a result the k-th derivative with respect to x of an eigenfunction can be expressed as a linear combination of $2k$ eigenfunctions and this makes possible turning the formal solution into a true one.

In the case of an anharmonic oscillator, which occurs if one considers cases vanishing of order higher than 2, Gundersen has shown in [15] that such recurrence formulas do not exist, so that the optimality for the operators

$$D_x^2 + x^{2(q-1)} D_y^2 + (y^k D_y)^2,$$

as well as

$$D_x^2 + x^{2(q-1)} D_y^2 + x^{2(p-1)}(y^k D_y)^2, \quad 1 < p < q,$$

is not known.

In 2001 Bender and Wang, [3], studied the following class of eigenvalue problems

$$-u''(t) + t^{2N+2} u(t) = t^N E\, u(t), \qquad N = -1, 0, 1, 2, \dots, \tag{1.2}$$

on the interval $-\infty < t < +\infty$. Such a kind of problem arises in different contexts in physics, like the fluid-flow with a resonant internal boundary layer and in supersymmetric quantum mechanics. The eigenfunction u above is a confluent hypergeometric function and moreover can be written as a product of a polynomial and a function exponentially vanishing at infinity.

In this paper we study the optimality of the Gevrey regularity for the operator

$$M_{n,m}(x, y; D_x, D_y) = D_x^2 + \left(x^{2n+1} D_y\right)^2 + \left(x^n y^m D_y\right)^2, \tag{1.3}$$

in Ω open neighborhood of the origin in $\mathbb{R}^2$. Here n and m are positive integers. We point out that $M_{n,m}$ is a generalization of the Métivier operator, (1.1), which corresponds to the case $n = 0$ and $m = 1$.

This operator allows us to use the recurrence relations between the eigenfunctions of the operator in order to construct an asymptotic solution.

Here is the statement of the result.

Theorem 1.1. *The operator $M_{n,m}$, (1.3), is $G^{\frac{2m}{2m-1}}$-hypoelliptic and not better in any neighborhood of the origin.*

A few remarks are in order.

(a) The Gevrey regularity obtained for the operator $M_{n,m}$ is in accordance with that predicted in [9]. The Gevrey regularity obtained for the operator $M_{n,m}$ remains the same if we perturb the operator adding a pseudodifferential operator of order less then $(2n+2)^{-1}$ as showed in [4]. The strategy used to obtain the result could allowed, in same special cases, to show the optimality of the Gevrey regularity obtained for the models studied in [11].

(b) The characterization in terms of *Gel'fand-Shilov spaces*, see Theorem 5.1.12 below, of the eigenfunctions of the eigenvalue problem (5.1.1) allows us to precisely compute the partial Gevrey regularity for $M_{n,m}$. This means that we find the Gevrey regularity with respect to x and with respect to y of the solutions: the operator (1.3) is Gevrey hypoelliptc of order $s_0 = 1 + \frac{1}{(2m-1)(2n+2)}$ with respect to the variable x and of order $s_1 = \frac{2m}{2m-1}$ with respect to the variable y and not better in any neighborhood of the origin. We recall that a smooth function $u(x, y)$ belongs to the non-isotropic Gevrey space $G^{(s_0,s_1)}(U)$, U open subset of $\mathbb{R}^2$, if for every compact set $K \subset U$ there are two positive constants C_K and A such that for every α, $\beta \in \mathbb{Z}_+$

$$|D_x^\alpha D_y^\beta u(x, y)| \leq A C_K^{\alpha+\beta} \alpha^{s_0 \alpha} \beta^{s_1 \beta}, \qquad (x, y) \in K.$$

(c) As well as the Métivier operator (1.1), the operator $M_{n,m}$, in (1.3), is not globally analytic hypoelliptic on the two dimensional torus. This means that if for instance we consider in $\mathbb{T}^2$ the operator
$$D_x^2 + \left(\sin^{2n+1}(x) D_y\right)^2 + \left(\sin^n(x)\sin^m(y) D_y\right)^2,$$
then it is not globally analytic hypoelliptic in $\mathbb{T}^2$, it is G^s-globally hypoelliptic for every $s \geq \frac{2m}{2m-1}$. This can be obtained via Theorem 1.1 and or following the proof of the Theorem 2.1 in [25] (p. 325) or following the proof of the Proposition 1.1 in [5], concerning the subject see also [12]. We recall that a partial differential operator P is said to be G^s-globally hypoelliptic, $s \geq 1$, in $\mathbb{T}^n$ if the condition $u \in \mathscr{D}'(\mathbb{T}^n)$ and $Pu \in G^s(\mathbb{T}^n)$ imply that $u \in G^s(\mathbb{T}^n)$. Here $G^s(\mathbb{T}^n)$ denotes the space of the Gevrey functions on $\mathbb{T}^n$.

Here is the plan of the paper. In the first section we construct a formal solution to the problem $M_{n,m}u = 0$. More precisely first of all we construct a formal solution and subsequently, in order to justify the formal steps we went through, we turn the formal solution, u, into a true solution, $\tilde{u}$, with the aid of a family of smooth cutoff functions. Then $M_{n,m}\tilde{u}$ belongs to a suitable function space.
In the second section we deduce the proof of the Theorem 1.1. In order to focus on the construction of the formal solution, we shifted to the appendix a number of both known and unknown facts about the properties of the eigenfunctions of the eigenvalue problem (1.2), in the case $N = 2n$, as well as some auxiliary technical lemmas.

Aknowledgements. The preparation of this manuscript has been done partially while the author was supported by the Austrian Science Fund (FWF), Lise-Meitner position, project no. M2324-N35. The author would like to thank the people of the Department of Mathematics of Vienna University and in particular professor B. Lamel, for their hospitality and the numerous mathematical discussions.

2. Formal solution of the operator (1.3)

Following the ideas in [20], the purpose of the present section is to construct a formal and the associate "approximate" solution to the problem $M_{n,m}u = 0$ of the form
$$\mathscr{K}[u](x,y) = \int_0^{+\infty} e^{i\rho^\theta y}\rho^r u\left(\rho^\gamma x, \rho\right) d\rho = \int_0^{+\infty} e^{i\rho^\theta y}\rho^r \left[u\left(t,\rho\right)\right]_{|_{t=\rho^\gamma x}} d\rho, \tag{2.1}$$
where θ, γ and r are parameters that will be chosen later and the function $u(t,\rho)$ is an infinitely differentiable function in $\mathbb{R}^2_{t,\rho}$ with support in the region $\rho > 0$ and rapidly decreasing as ρ goes to infinity. The first step will be to establish the values of the parameters θ and γ.
We have

(1) $D_x^2\mathscr{K}[u] = \displaystyle\int_0^{+\infty} e^{i\rho^\theta y}\rho^r\rho^{2\gamma}\left[D_t^2 u(t,\rho)\right]_{|_{t=\rho^\gamma x}} d\rho;$

(2) $\left(x^{2n+1}D_y\right)^2\mathscr{K}[u] = \displaystyle\int_0^{+\infty} e^{i\rho^\theta y}\rho^r\rho^{2\theta-2(2n+1)\gamma}\left[t^{2(2n+1)}u(t,\rho)\right]_{|_{t=\rho^\gamma x}} d\rho;$

(3) and by applying the Lemma 5.2.3, see Appendix,
$$\left(x^n y^m D_y\right)^2\mathscr{K}[u] = \left(x^{2n}y^{2m}D_y^2 + \frac{m}{i}x^{2n}y^{2m-1}D_y\right)\mathscr{K}[u]$$

$$
= \left(\frac{1}{i\theta}\right)^{2m} \int_0^{+\infty} e^{i\rho^\theta y} \rho^{r+2\theta-2n\gamma+2m(1-\theta)} \left[t^{2n} \mathcal{P}(t,\rho,\partial_t,\partial_\rho) u(t,\rho)\right]_{|t=\rho^\gamma x} d\rho,
$$

where

$$
\begin{aligned}
\mathcal{P}(t,\rho,\partial_t,\partial_\rho) = \partial_\rho^{2m} &+ \frac{2m}{\rho}\left(\frac{2m+1}{2} - \theta(m-1) + r + \gamma t\partial_t\right)\partial_\rho^{2m-1} \\
&+ \sum_{i=2}^{2m} \frac{1}{\rho^i}\left(\mathscr{P}_i^1(t\partial_t) - m\theta \mathscr{P}_{i-1}^2(t\partial_t)\right)\partial_\rho^{2m-i},
\end{aligned}
$$

here

$$
\mathscr{P}_i^1(t\partial_t) = \sum_{j=0}^{i} \Upsilon_{i,j}^1 (t\partial_t)^j \text{ and } \mathscr{P}_{i-1}^2(t\partial_t) = \sum_{j=0}^{i-1} \Upsilon_{i-1,j}^2 (t\partial_t)^j.
$$

The coefficients $\Upsilon_{i,j}^1$ and $\Upsilon_{i,j}^2$ are obtained by the formulas (5.2.4) and (5.2.5), Lemma 5.2.3, setting $p = 2m$, $q = r + 2\theta - 2n\gamma$, $f = 2n$ and $p = 2m-1$, $q = r + \theta - 2n\gamma$, $f = 2n$ respectively.

Choosing $\theta = 2m(2m-1)^{-1}$ and $\gamma = m(n+1)^{-1}(2m-1)^{-1}$ we obtain

$$
\begin{aligned}
M_{n,m}\mathscr{K}[u] &= \left(D_x^2 + \left(x^{2n+1}D_y\right)^2 + \left(x^n y^m D_y\right)^2\right)\mathscr{K}[u](x,y) \\
&= \int_0^{+\infty} e^{iy\rho^{\frac{2m}{2m-1}}} \rho^{r+\frac{2m}{(n+1)(2m-1)}} \left[\sum_{i=0}^{2m} \frac{1}{\rho^i} \mathcal{P}_i(t,\partial_t,\partial_\rho) u(t,\rho)\right]_{|t=\rho^{\frac{m}{(n+1)(2m-1)}}x} d\rho,
\end{aligned} \tag{2.2}
$$

where

i) $\mathcal{P}_0(t,\partial_t,\partial_\rho) = -\partial_t^2 + t^{2(2n+1)} + \left(\frac{2m-1}{2im}\right)^{2m} t^{2n}\partial_\rho^{2m}$;

ii) $\mathcal{P}_1(t,\partial_t,\partial_\rho) = t^{2n}\left(\frac{2m-1}{2im}\right)^{2m}\left(\frac{(4m-1)2m}{2m-1} + 2mr + \frac{2m^2}{(n+1)(2m-1)}t\partial_t\right)\partial_\rho^{2m-1}$;

iii) $\mathcal{P}_i(t,\partial_t,\partial_\rho) = t^{2n}\left(\frac{2m-1}{2im}\right)^{2m}\mathscr{P}_i(t\partial_t)\partial_\rho^{2m-i}$, $i = 2,3,\dots,2m$, where $\mathscr{P}_i(t\partial_t) = \sum_{j=0}^{i} \Upsilon_{i,j}(t\partial_t)^j$; the coefficients $\Upsilon_{i,j}$ are obtained from the previous formulas replacing θ and γ with the above assigned values.

We have to solve the following equation

$$
\sum_{i=0}^{2m} \frac{1}{\rho^i} \mathcal{P}_i(t,\partial_t,\partial_\rho) u(t,\rho) = 0. \tag{2.3}
$$

We do this formally. We set

$$
u(t,\rho) = \sum_{\ell \geq 0} u_\ell(t,\rho),
$$

with the purpose to obtain the functions $u_\ell(t,\rho)$ recursively taking advantage of the eigenvalue problem, (5.1.1), studied in the Appendix. More precisely we want to express the functions $u_\ell(t,\rho)$ in the following form:

$$
u_\ell(t,\rho) = \sum_{p=0}^{\ell} g_{\ell,p}(\rho) v_p(t),
$$

where $v_p(t)$ are the eigenfunctions given by (5.1.2) in the Appendix.
We remark that we allow the above sum for u_ℓ to be finite because of the relation (5.1.9), Lemma 5.1.3. We point out that the relation (5.1.9), and more generally

the relation (5.1.11), allow us to construct a suitable system in order to obtain recursively the functions $u_\ell(t,\rho)$. We stress that $\mathcal{P}_i u_{\ell-i}$ can be expressed by a linear combination of $v_0(t),\dots,v_\ell(t)$. This gives us the possibility, at any step of the process, to reduce the problem to solving a system of ordinary differential equations.

We choose

$$u_0(t,\rho) = g_{0,0}(\rho)v_0(t) = e^{-c_1\rho}v_0(t), \tag{2.4}$$

where $c_1 = \frac{2m}{2m-1}(2n+1)^{1/2m}\left(\sin\left(\frac{\pi}{2m}\right) - i\cos\left(\frac{\pi}{2m}\right)\right)$ and $v_0(t)$ is the even-parity eigenfunction to the problem (5.1.1), see (5.1.2) for its explicit form, corresponding to the eigenvalue $E_0 = 2n+1$. We have

$$\mathcal{P}_0(t,\partial_t,\partial_\rho)u_0(t,\rho) = 0 \tag{2.5}$$

We set $c_0 = \Re(c_1)$.

Before constructing our formal solution we point out that, in order to have the desired growth of the functions $g_{\ell,p}(\rho)$, since they have exponential nature, the derivative with respect to the parameter ρ does not help to push down the order with respect to negative power of ρ. The only instrument that allow to do it, it is the multiplication by negative powers of ρ. In particular the growth of $g_{\ell,0}(\rho)$ will be the most thorny; the precise choice of the parameter r plays a fundamental role in this situation.

Let us start with the action of the operators $\mathcal{P}_i(t,\partial_t,\partial_\rho)$ on $u_\ell(t,\rho)$:

i) case of $\mathcal{P}_0$; by (5.1.1), for every $\ell \geq 1$ we have

$$\begin{aligned}
\mathcal{P}_0 u_\ell &= \left(-\partial_t^2 + t^{2(2n+2)} + \left(\frac{2m-1}{2mi}\right)^{2m} t^{2n}\partial_\rho^{2m}\right)\left(\sum_{p=0}^{\ell} g_{\ell,p}(\rho)v_p(t)\right)\\
&= t^{2n}\left(\frac{2m-1}{2mi}\right)^{2m}\sum_{p=0}^{\ell}\left(\partial_\rho^{2m} + \left(\frac{2mi}{2m-1}\right)^{2m}E_p\right)g_{\ell,p}(\rho)v_p(t)\\
&\doteq t^{2n}\left(\frac{2m-1}{2mi}\right)^{2m}\sum_{p=0}^{\ell}\Theta_p g_{\ell,p}(\rho)v_p(t);
\end{aligned}$$

where $\Theta_p = \partial_\rho^{2m} + \left(\frac{2mi}{2m-1}\right)^{2m}E_p$, see Lemma 5.2.4 and (5.2.7) in the Appendix, here $E_p = 4p(n+1)+2n+1$ is the eigenvalue corresponding to the even-parity eigenfunction $v_p(t)$ to the problem (5.1.1), see (5.1.2).

ii) case of $\mathcal{P}_1$; for every $\ell \geq 2$ we have

$$\begin{aligned}
\mathcal{P}_1 u_{\ell-1} &= t^{2n}\left(\frac{2m-1}{2mi}\right)^{2m}\left(\upharpoonright_{1,0} + \upharpoonright_{1,1} t\partial_t\right)\partial_\rho^{2m-1}\left(\sum_{p=0}^{\ell-1} g_{\ell-1,p}(\rho)v_p(t)\right)\\
&= t^{2n}\left(\frac{2m-1}{2mi}\right)^{2m}\sum_{p=0}^{\ell} g_{\ell,p,1}(\rho)v_p(t),
\end{aligned}$$

where $\upharpoonright_{1,0} = \frac{(4m-1)2m}{2m-1} + 2mr$ and $\upharpoonright_{1,1} = \frac{2m^2}{(n+1)(2m-1)}$, see ii) after the equation (2.2). By the Lemma 5.1.3, in the Appendix, we have

$$g_{\ell,0,1}(\rho) = (\upharpoonright_{1,0} + \upharpoonright_{1,1}\delta_0^{0,1})g_{\ell-1,0}^{(2m-1)}(\rho) + \upharpoonright_{1,1}\delta_0^{1,1}g_{\ell-1,1}^{(2m-1)}(\rho), \tag{2.6}$$

$$g_{\ell,p,1}(\rho) = \pitchfork_{1,1}\delta_p^{p-1,1} g^{(2m-1)}_{\ell-1,p-1}(\rho) + g^{(2m-1)}_{\ell-1,p}(\rho)(\pitchfork_{1,0} + \pitchfork_{1,1}\delta_p^{p,1}) g^{(2m-1)}_{\ell-1,p}(\rho) + \pitchfork_{1,1}\delta_p^{p+1,1} g^{(2m-1)}_{\ell-1,p-1}(\rho), \qquad p = 1, 2, \dots \ell-2, \tag{2.7}$$

$$g_{\ell,\ell-1,1}(\rho) = \pitchfork_{1,1}\delta_{\ell-1}^{\ell-1,1} g^{(2m-1)}_{\ell-1,\ell-2}(\rho) + (\pitchfork_{1,0} + \pitchfork_{1,1}\delta_{\ell-1}^{\ell-1,1}) g^{(2m-1)}_{\ell-1,\ell-1}(\rho), \tag{2.8}$$

$$g_{\ell,\ell,1}(\rho) = \pitchfork_{1,1}\delta_\ell^{\ell-1,1} g^{(2m-1)}_{\ell-1,\ell-1}(\rho); \tag{2.9}$$

the symbols $\delta_\nu^{j,p}$ are defined in (5.1.11), Lemma 5.1.3, in the Appendix. In the case $\ell = 1$ we have

$$\mathcal{P}_1 u_0 = t^{2n}\left(\frac{2m-1}{2mi}\right)^{2m}\Big[\left(\pitchfork_{1,0} + \pitchfork_{1,1}\delta_0^{0,1}\right) g^{(2m-1)}_{0,0}(\rho) v_0(t) + \pitchfork_{1,1}\delta_1^{0,1} g^{(2m-1)}_{0,0}(\rho) v_1(t)\Big].$$

Since the term $\pitchfork_{1,0} = \frac{(4m-1)2m}{2m-1} + 2mr$, see ii) after the equation (2.2), is linear with respect to r with a non zero coefficient, we may make a suitable choice of the parameter r cancelling the coefficient of v_0 in the above expression.

iii) case of $\mathcal{P}_i$, $i = 2, 3, \dots, 2m$; we have

$$\mathcal{P}_i u_{\ell-i} = t^{2n}\left(\frac{2m-1}{2mi}\right)^{2m} \mathscr{P}_i(t\partial_t)\partial_\rho^{2m-i}\left(\sum_{p=0}^{\ell-i} g_{\ell-i,p}(\rho) v_p(t)\right) = t^{2n}\left(\frac{2m-1}{2mi}\right)^{2m}\sum_{p=0}^{\ell} g_{\ell,p,i}(\rho) v_p,$$

where

$$g_{\ell,p,i}(\rho) = \sum_{\nu=\max\{p-i,0\}}^{\min\{p+i,\ell-i\}} g^{(2m-i)}_{\ell-i,\nu}(\rho)\left(\sum_{j=|p-\nu|}^{i} \pitchfork_{i,j}\delta_{p-\nu}^{\nu,j}\right). \tag{2.10}$$

The symbols $\delta_\nu^{j,p}$ are defined in (5.1.11), Lemma 5.1.3 in the Appendix, and we have set $\delta_p^{p,0} = 1$.

We introduce the operator Π_0 and its "orthogonal" $(1-\Pi_0)$ acting on the functions u_ℓ in the following way

$$\Pi_0 u_\ell = g_{0,\ell}(\rho) v_0(t) \quad \text{and} \quad (1-\Pi_0)u_\ell = \sum_{p=1}^{\ell} g_{\ell,p}(\rho) v_p.$$

As consequence of the choice of the parameter r we have that

$$\Pi_0\mathcal{P}_1\Pi_0 u_0 = \Pi_0\mathcal{P}_1 u_0 = 0.$$

Moreover

$$\Pi_0\mathcal{P}_1\Pi_0 u_\ell = 0 \quad \text{for every } \ell.$$

This is crucial in order to obtain the right growth of the functions $g_{\ell,0}(\rho)$ with respect to (negative) powers of ρ (see Lemma 5.2.7 in the Appendix).

Following the idea in [20], to obtain the u_ℓ, we consider the system

$$
\begin{cases}
(1-\Pi_0)\,\mathcal{P}_0 u_\ell = -\,(1-\Pi_0) \displaystyle\sum_{i=1}^{\min\{\ell,2m\}} \frac{1}{\rho^i}\mathcal{P}_i u_{\ell-i}; \\
\Pi_0\mathcal{P}_0 u_\ell = -\dfrac{1}{\rho}\Pi_0\mathcal{P}_1\,(1-\Pi_0)\,u_\ell - \Pi_0 \displaystyle\sum_{i=2}^{\min\{\ell+1,2m\}} \frac{1}{\rho^i}\mathcal{P}_i u_{\ell+1-i}.
\end{cases}
\tag{2.11}
$$

This allows us, at any step of the process, to reduce the problem of computing the u_ℓ to that of solving a system of $\ell+1$ ordinary differential equations yielding the functions $g_{\ell,p}(\rho)$, $p=0,\dots\ell$.
With the purpose of understanding how to construct and subsequently estimate the functions $g_{\ell,p}(\rho)$, we begin to analyze the case $\ell=1$.
For $\ell=1$ system (2.11) in terms of the $g_{1,p}$, $p=0,1$, becomes

$$
\begin{cases}
\Theta_1 g_{1,1}(\rho) = -\dfrac{1}{\rho}\Lsh_{1,1}\delta_1^{0,1} g_{0,0}^{(2m-1)}(\rho); \\
\Theta_0 g_{1,0}(\rho) = -\dfrac{1}{\rho}\Lsh_{1,1}\delta_0^{1,1} g_{1,1}^{(2m-1)}(\rho) - \dfrac{1}{\rho^2}\left(\Lsh_{2,0}\delta_0^{0,0} + \Lsh_{2,1}\delta_0^{0,1} + \Lsh_{2,2}\delta_0^{0,2}\right) g_{0,0}^{(2m-2)}(\rho),
\end{cases}
$$

where $g_{0,0}(\rho)=e^{-c_1\rho}$ and $\Theta_k = \partial_\rho^{2m} + \left(2mi(2m-1)^{-1}\right)^{2m} E_k$, $k=0,1$.
We stress the fact that the choice of the parameter r has allowed us to make the right hand side of the second equation of order -2 with respect to the variable ρ.
We denote by $f_{1,j}(\rho)$, $j=1,0$, the functions on the right hand side of the above system.
In order to be able to apply Lemmas 5.2.5, 5.2.7, we need to make sure that the variable σ in those Lemmas belongs to the half line $[C_0(j+1),+\infty[$. To accomplish this, we use cutoff functions χ_ℓ so that the hypotheses of those Lemmas are satisfied.
We define a family of smooth functions $\{\chi_\ell(\rho)\}_{\ell\geq 0}$ such that

i) $\chi_\ell(\rho)$ is identically zero for $\rho < 2R_1(\ell+1)$ and identically one for $\rho > 4R_1(\ell+1)$, where R_1 denotes a suitable positive constant;
ii) there is a constant C_χ, independent of ρ and ℓ, such that

$$
|\chi_\ell^{(k)}(\rho)| \leq C_\chi^k \qquad \forall k \leq 2m. \tag{2.12}
$$

The construction of the functions χ_ℓ can be done using the same strategy used to construct the so called Ehrenpreis-Hörmander cut-off functions. More precisely the function χ_ℓ can be optained in the following way. Let $\widetilde{\Omega} = \{x\in\mathbb{R} : |x| \leq 2R_1(\ell+1)\}$. We choose a function $\varphi\in\mathscr{D}(\mathbb{R})$ with support in $\mathscr{B}_1(0) \doteq \{x\in\mathbb{R} : |x|\leq 1\}$ such that $\varphi\geq 0$ and $\int\varphi\,dx = 1$. For every $\delta>0$ we write $\psi_\delta(x)=\delta^{-1}\psi\left(\frac{x}{\delta}\right)$. Let $\tilde{\Theta}$ be the characteristic function of the set $\{x\in\mathbb{R} : \operatorname{dist}(x;\widetilde{\Omega}) \leq 3R_1(\ell+1)\}$. We set

$$
\widetilde{\chi}_\ell = \underbrace{\varphi*\varphi*\cdots*\varphi}_{2m-\text{times}} * \underbrace{\varphi_{c_1}*\cdots*\varphi_{c_1}}_{m-\text{times}} * \tilde{\Theta},
$$

where $c_1 = (R_1(\ell+1)-2m)m^{-1}$, we assume that $R_1 > 2m$. Since the support of a convolution is contained in the vector sum of the supports of the factors in the convolution the function $\widetilde{\chi}_\ell$ is supported in $\{x\in\mathbb{R} : |x|\leq 4R_1(\ell+1)\}$ and it is identically one on $\widetilde{\Omega}$. Let $k \in \mathbb{Z}_+$ with $k \leq 2m$, we have

$$D_\rho^k \widetilde{\chi}_\ell = \underbrace{(D_\rho \varphi) * \cdots * (D_\rho \varphi)}_{k-\text{times}} * \underbrace{\varphi * \cdots * \varphi}_{(2m-k)-\text{times}} * \varphi_{c_1} * \cdots * \varphi_{c_1} * \tilde{\Theta}.$$

Via the Hölder inequality we obtain

$$\|D_\rho^k \widetilde{\chi}_\ell\|_\infty \leq \prod_{i=1}^{k} \|D_\rho \varphi\|_{L^1} \prod_{i=k+1}^{2m} \|\varphi\|_{L^1} \prod_{j=1}^{m} \|\varphi_{c_1}\|_{L^1} \|\tilde{\Theta}\|_\infty \leq C_\chi^k,$$

where $C_\chi = \|D_\rho \varphi\|_{L^1}$.
Setting $\chi_\ell = 1 - \widetilde{\chi}_\ell$, the function χ_ℓ satisfies the properties in i) and ii) above.

We define the functions $g_{1,p}(\rho)$, $p = 0, 1$, as

$$g_{1,1}(\rho) = \left(G_1 * (\chi_1 f_1)\right)(\rho),$$

and

$$g_{1,0}(\rho) = \left(G_0 * (\chi_0 f_0)\right)(\rho) - h_0(\rho) - h_{m-1}(\rho),$$

where G_0 and G_1 are the fundamental solutions of Θ_0 and Θ_1, see Lemma 5.2.4, and the functions $h_0(\rho)$ and $h_{m-1}(\rho)$ are the solutions of the linear homogeneous equation $\Theta_0 h = 0$ defined in Lemma 5.2.7.
By Lemma 5.2.5, see also Remark 5.2.6, and Lemma 5.2.7, see also (5.2.10), (5.2.11) in Lemma 5.2.4, we obtain

$$|g_{1,p}(\rho)| \leq C^2 \frac{1}{\rho} e^{-c_0 \rho}, \qquad p = 0, 1,$$

for $\rho \geq R_1$, where C is a positive constant depending on m and n.

Let us now consider the case $\ell > 1$. To construct the functions $g_{\ell,p}(\rho)$ and consequently the functions $u_\ell(t, \rho)$, we proceed recursively adopting the technique described above.
We have to solve the following system of $\ell + 1$ equations

$$\begin{cases} \Theta_0 g_{\ell,0}(\rho) = f_{\ell,0}(\rho); \\ \Theta_1 g_{\ell,1}(\rho) = f_{\ell,1}(\rho); \\ \vdots \\ \Theta_\ell g_{\ell,\ell}(\rho) = f_{\ell,\ell}(\rho). \end{cases} \tag{2.13}$$

Where due to (2.6), (2.7), (2.8), (2.9) and (2.10) we have

$$f_{\ell,p}(\rho) = -\sum_{i=1}^{2m} \frac{1}{\rho^i} g_{\ell,p,i}(\rho), \quad p = 1, \ldots, \ell, \tag{2.14}$$

or, more explicitly, for $p = 1, \ldots, \ell$,

$$f_{\ell,p}(\rho) = \sum_{i=1}^{\min\{\ell,2m\}} \frac{1}{\rho^i} \left(\mathcal{P}_i u_{\ell-i}\right)_p = \sum_{i=1}^{\min\{\ell,2m\}} \frac{1}{\rho^i} \sum_{j=0}^{i} \sum_{p_1=0}^{\ell-i} \Upsilon_{i,j} \delta_{p-p_1}^{p,j} g_{\ell-i,p_1}^{(2m-i)},$$

where $\left(\mathcal{P}_i u_{\ell-i}\right)_p$ denotes the coefficient of v_p in the expression of $\mathcal{P}_i u_{\ell-i}$.

Moreover

(2.15)
$$f_{\ell,0}(\rho) = -\frac{1}{\rho} g_{\ell,1}^{(2m-1)}(\rho) \upharpoonright_{1,1} \delta_0^{1,1} - \sum_{i=2}^{\min\{\ell,2m\}} \sum_{\nu=0}^{\min\{i,\ell-i+1\}} \frac{1}{\rho^i} g_{\ell-i+1,\nu}^{(2m-i)}(\rho) \left(\sum_{j=\nu}^{i} \upharpoonright_{i,j} \delta_{-\nu}^{\nu,j} \right).$$

To comply with the hypotheses of Lemmas 5.2.5, 5.2.7, we solve instead the system

$$\begin{cases} \Theta_0 g_{\ell,0}(\rho) = f_{\ell,0}(\rho)\chi_\ell; \\ \Theta_1 g_{\ell,1}(\rho) = f_{\ell,1}(\rho)\chi_\ell; \\ \vdots \\ \Theta_\ell g_{\ell,\ell}(\rho) = f_{\ell,\ell}(\rho)\chi_\ell. \end{cases} \tag{2.16}$$

We remark that all the sums involved in the above formulas, see (2.10), are finite and they have at most $4m$ terms. We also point out that, due to the recurrence relation described in Lemma 5.1.3 in the Appendix, we have that $|\delta_p^{\nu,j}| \leq C^j \frac{(\nu+j)!}{\nu!}$.
The function $f_{\ell,0}(\rho)$ has order $-\ell-1$ with respect to ρ.
The system (2.16) is then solved by

$$\begin{cases} g_{\ell,p}(\rho) = G_p * (\chi_\ell f_{\ell,p})(\rho), \quad p = 1, \dots, \ell \\ g_{\ell,0}(\rho) = G_0 * (\chi_\ell f_{\ell,0})(\rho) - h_0(\rho) - h_{m-1}(\rho), \end{cases} \tag{2.17}$$

where the functions $h_0(\rho)$ and $h_{m-1}(\rho)$ are the solutions ofthe linear homogeneous equation $\Theta_0 h = 0$ defined in Lemma 5.2.7
The next lemma gives the estimates of the derivatives of the coefficients $g_{\ell,p}$.

Lemma 2.1. *With the above notations we have that*

$$|g_{\ell,p}(\rho)| \leq C^{\ell+1} \frac{(\ell+1)^{\ell\left(1-\frac{1}{2m}\right)}}{\rho^\ell} e^{-c_0\rho}, \qquad 0 \leq p \leq \ell, \tag{2.18}$$

and more generally,

$$|g_{\ell,p}^{(k)}(\rho)| \leq C^{\ell+1+(\frac{k}{2m}-1)_+} \frac{(\ell+1)^{\ell\left(1-\frac{1}{2m}\right)+\frac{k}{2m}}}{\rho^\ell} e^{-c_0\rho}, \qquad 0 \leq p \leq \ell, \tag{2.19}$$

with $k \in \mathbb{Z}_+$. Here $x_+ = x$ if $x \geq 0$ and $x_+ = 0$ if $x < 0$.

Proof. We apply Lemmas 5.2.5, 5.2.7 as well as Remark 5.2.6. We have

$$g_{\ell,p}^{(k)}(\rho) = G_p^{(k)} * \left(-\chi_\ell(\sigma) \sum_{i=1}^{\min\{\ell,2m\}} \frac{1}{\sigma^i} g_{\ell,p,i}(\sigma) \right)(\rho), \quad \text{for } p \geq 1, \tag{2.20}$$

where

$$g_{\ell,p,i}(\sigma) = \sum_{\nu=\max\{p-i,0\}}^{\min\{p+i,\ell-i\}} g_{\ell-i,\nu}^{(2m-i)}(\sigma) \left(\sum_{j=|p-\nu|}^{i} \upharpoonright_{i,j} \delta_{p-\nu}^{\nu,j} \right). \tag{2.21}$$

We proceed by induction on ℓ. Assume that for every $\ell' < \ell$, for every $k \in \mathbb{Z}_+$ and for every $0 \leq p \leq \ell'$, we have

$$|g_{\ell',p}^{(k)}(\rho)| \leq C^{\ell'+1+(\frac{k}{2m}-1)_+} (\ell'+1)^{\ell'(1-1/2m)+\frac{k}{2m}} \rho^{-\ell'} e^{-c_0\rho}.$$

Let us estimate first the quantity in (2.21). We have

$$|g_{\ell,p,i}(\sigma)| \leq \sum_{\nu=\max\{p-i,0\}}^{\min\{p+i,\ell-i\}} C^{\ell-i+1}\,(\ell-i+1)^{(\ell-i)(1-1/2m)+\frac{2m-i}{2m}}\,\sigma^{-(\ell-i)}e^{-c_0\sigma} \cdot C_{\Upsilon} \sum_{j=|p-\nu|}^{i} C_1^j\nu^j. \tag{2.22}$$

Hence

$$\begin{aligned}
|g^{(k)}_{\ell,p}(\rho)| &\leq C_{\Upsilon} \sum_{i=1}^{\min\{\ell,2m\}} \sum_{\nu=\max\{p-i,0\}}^{\min\{p+i,\ell-i\}} \sum_{j=|p-\nu|}^{i} C_1^j\nu^j \\
&\quad C^{\ell-i+1}\,(\ell-i+1)^{(\ell-i)(1-1/2m)+\frac{2m-i}{2m}} \int_R^{+\infty} |G^{(k)}_p(\rho-\sigma)|\frac{1}{\sigma^\ell}e^{-c_0\sigma}d\sigma \\
&\leq C_{\Upsilon}C_G \sum_{i=1}^{\min\{\ell,2m\}} \sum_{\nu=\max\{p-i,0\}}^{\min\{p+i,\ell-i\}} \sum_{j=|p-\nu|}^{i} C_1^j\nu^j C^{\ell-i+1}\,(\ell-i+1)^{(\ell-i)(1-1/2m)+\frac{2m-i}{2m}} \\
&\quad \cdot \left(\frac{2m-1}{2m}\right)^{2m-1-k} E_p^{\frac{k+1}{2m}-1} \int_R^{+\infty} e^{-c_p|\rho-\sigma|}\frac{1}{\sigma^\ell}e^{-c_0\sigma}d\sigma.
\end{aligned}$$

Let us now use Lemma 5.2.5 as well as the fact that the indices j, ν, i run over a finite number of values.
We point out that there are two positive constants $1 < C_- < C_+$ such that

$$C_-(p+1) \leq E_p \leq C_+(p+1).$$

As a consequence $E_p^\theta \leq C_+^\theta(p+1)^\theta$ if $\theta \geq 0$, whereas $E_p^\theta \leq C_-^\theta(p+1)^\theta \leq (p+1)^\theta$ if $\theta < 0$. Setting $C_E = C_+$ we have that $E_p^\theta \leq C_E^{\theta_+}(p+1)^\theta$, where $\theta_+ = \theta$ for $\theta \geq 0$, $\theta_+ = 0$ for $\theta < 0$. We note that C_E depends only on the problem data.
We then obtain

$$\begin{aligned}
|g^{(k)}_{\ell,p}(\rho)| &\leq C_2\frac{e^{-c_0\rho}}{\rho^\ell} \sum_{i=1}^{\min\{\ell,2m\}} \sum_{\nu=\max\{p-i,0\}}^{\min\{p+i,\ell-i\}} \left(\sum_{j=0}^{i} C_1^j\right) C^{\ell-i+1} \\
&\quad \cdot (\ell-i+1)^{(\ell-i)(1-\frac{1}{2m})+\frac{2m-i}{2m}+i}\; C_E^{(\frac{k}{2m}-1)_+}(p+1)^{\frac{k}{2m}-1} \\
&\leq C^{\ell+1}(\ell+1)^{\ell(1-\frac{1}{2m})+\frac{k}{2m}}\frac{e^{-c_0\rho}}{\rho^\ell}C_2C_3C_E^{(\frac{k}{2m}-1)_+} \sum_{i=1}^{\min\{\ell,2m\}} C^{-i} \\
&\leq C^{\ell+1+(\frac{k}{2m}-1)_+}(\ell+1)^{\ell(1-\frac{1}{2m})+\frac{k}{2m}}\frac{e^{-c_0\rho}}{\rho^\ell},
\end{aligned}$$

provided C is chosen suitably large.

$\square$

So far we proved the following

Proposition 2.2. *There are functions u_ℓ, $\ell = 0, 1, \dots,$ such that*

$$u_\ell(t,\rho) = \sum_{p=0}^{\ell} g_{\ell,p}(\rho)v_p(t), \tag{2.23}$$

where the v_p *are the eigenfunctions of the operator in* (5.1.1) *defined in* (5.1.2). $g_{0,0}$ *is defined in* (2.4) *and the* $g_{\ell,p}(\rho)$ *are defined by* (2.17), (2.14), (2.15) *and satisfy the estimates* (2.19). *Finally the function*

$$u(t,\rho)=\sum_{\ell\geq 0}u_\ell(t,\rho)$$

is a formal solution of (2.3) *in the sense that system* (2.11) *is verified and formally equivalent to* (2.3).
We say that $\mathscr{K}[u]$ *is a formal solution of equation* (1.3).

Next we need a lemma allowing us to estimate the derivatives with respect to t of the functions u_ℓ, $\ell \geq 0$.

Lemma 2.3. *For any* $\ell\in\mathbb{N}$, *any* α, β, γ, *we have the estimate*

$$|t^\alpha\partial_t^\beta\partial_\rho^\gamma u_\ell(t,\rho)|\leq C_u^{\ell+\alpha+\beta+\gamma+1}(\ell+1)^{\ell\frac{2m-1}{2m}+\frac{\gamma}{2m}}\alpha!^{\frac{1}{2n+2}}\beta!^{\frac{2n+1}{2n+2}}\frac{1}{\rho^\ell}e^{-c_0\rho}, \tag{2.24}$$

where C_u *denotes a positive constant independent of* ℓ, α, β, γ.

Proof. We apply Lemma 2.1 and Theorem 5.1.12 to

$$u_\ell(t,\rho)=\sum_{p=0}^{\ell}g_{\ell,p}(\rho)v_p(t).$$

Then

$$\begin{aligned}|t^\alpha\partial_t^\beta\partial_\rho^\gamma u_\ell(t,\rho)| &\leq \sum_{p=0}^{\ell}|\partial_\rho^\gamma g_{\ell,p}(\rho)||t^\alpha\partial_t^\beta v_p(t)|\\ &\leq \sum_{p=0}^{\ell}C_v^{p+\alpha+\beta+1}\alpha!^{\frac{1}{2n+2}}\beta!^{\frac{2n+1}{2n+2}}C_g^{\ell+1+(\frac{k}{2m}-1)_+}\frac{(\ell+1)^{\ell\left(1-\frac{1}{2m}\right)+\frac{k}{2m}}}{\rho^\ell}e^{-c_0\rho}\end{aligned}$$

Then we reach the conclusion choosing C_u large enough. □

3. Turning a formal solution into a true solution

Our next task is to turn the formal solution just constructed into a true solution. The obtained function is a solution of an equation of the form $M_{n,m}u=f$, with f, more regular than $G^{\frac{2m}{2m-1}}$.
In order to define the "approximate" solution to the problem $M_{n,m}u=0$, we need another family of cutoff functions.

Lemma 3.1 ([8])**.** *Let* $\sigma>1$. *There exists a family of cutoff functions* $\omega_j\in G^\sigma(\mathbb{R}^n_x)$, $0\leq\omega_j(x)\leq 1$, $j=0,1,2,\dots$, *such that*

1- $\omega_j\equiv 0$ *if* $|x|\leq 2R(j+1)$, $\omega_j\equiv 1$ *if* $|x|\geq 4R(j+1)$, *with* R *an arbitrary positive constant.*

2- *There is a suitable constant* C_ω, *independent of* j, α, R, *such that*

$$|D^\alpha\omega_j(x)|\leq C_\omega^{|\alpha|+1}R^{-|\alpha|},\qquad \text{if } |\alpha|\leq 3j. \tag{3.1}$$

and

$$|D^\alpha\omega_j(x)|\leq (RC_\omega)^{|\alpha|+1}\frac{\alpha!^\sigma}{|x|^{|\alpha|}},\qquad \text{for every } \alpha. \tag{3.2}$$

Let ω_ℓ denote cutoff functions like those in the above lemma, with $\sigma > 1$ sufficiently close to 1 and R large to be chosen later. We shall be more precise further on. Define

$$\tilde{u}(t,\rho) = \sum_{\ell\geq 0} u_\ell(t,\rho)\omega_\ell(\rho), \tag{3.3}$$

where u_ℓ is given by Proposition 2.2. We point out that the above series is actually a locally finite sum, so that we do not have any convergence problem.
We consider

$$\mathscr{K}[\tilde{u}](x,y) = \int_0^{+\infty} e^{iy\rho^{\frac{2m}{2m-1}}} \rho^{r+\frac{2m}{(n+1)(2m-1)}} \tilde{u}(t,\rho)_{\big| t=\rho^{\frac{m}{(n+1)(2m-1)}}x} d\rho, \tag{3.4}$$

where $\tilde{u}$ is given by (3.3).
Applying $M_{n,m}$ to $\mathscr{K}[\tilde{u}](x,y)$, we obtain

$$\mathscr{K}\left[\sum_{i=0}^{2m} \rho^{-i}\mathcal{P}_i(t,\partial_t,\partial_\rho)\sum_{\ell\geq 0} u_\ell\omega_\ell\right](x,y).$$

Let us look at the quantity in square brackets. We have

$$\sum_{i=0}^{2m} \rho^{-i}\mathcal{P}_i(t,\partial_t,\partial_\rho)\sum_{\ell\geq 0} u_\ell\omega_\ell = \sum_{i=0}^{2m}\sum_{\ell\geq 0}\sum_{\gamma=0}^{2m} \frac{1}{\rho^i}\partial_\rho^\gamma\omega_\ell\frac{1}{\gamma!}\mathcal{P}_i^{(\gamma)}(t,\partial_t,\partial_\rho)u_\ell,$$

where $\mathcal{P}_i^{(\gamma)}$denotes the differential operator whose symbol is $\partial_\sigma^\gamma\mathcal{P}_i(t,\tau,\sigma)$.
We are going to consider separately the cases where the cutoff function ω_ℓ takes derivatives and those where it does not take any derivative. The above quantity becomes

$$\sum_{i=0}^{2m}\sum_{\ell\geq 0}\frac{1}{\rho^i}\omega_\ell\mathcal{P}_i(t,\partial_t,\partial_\rho)u_\ell + \sum_{i=0}^{2m}\sum_{\ell\geq 0}\sum_{\gamma=1}^{2m}\frac{1}{\rho^i}\partial_\rho^\gamma\omega_\ell\frac{1}{\gamma!}\mathcal{P}_i^{(\gamma)}(t,\partial_t,\partial_\rho)u_\ell = A_1 + A_2.$$

Let us consider A_1 first. We have

$$\begin{aligned} A_1 &= \sum_{\ell\geq 0}\omega_\ell\mathcal{P}_0u_\ell + \sum_{\ell\geq 0}\sum_{i=1}^{2m}\frac{1}{\rho^i}\omega_\ell\mathcal{P}_i(t,\partial_t,\partial_\rho)u_\ell \\ &= \sum_{\ell\geq 0}\omega_\ell\mathcal{P}_0u_\ell + \sum_{\ell\geq 1}\sum_{i=1}^{\min\{\ell,2m\}}\frac{1}{\rho^i}\omega_{\ell-i}\mathcal{P}_i(t,\partial_t,\partial_\rho)u_{\ell-i} \\ &= \sum_{\ell\geq 1}\omega_\ell(1-\Pi_0)\mathcal{P}_0u_\ell + \sum_{\ell\geq 1}\sum_{i=1}^{\min\{\ell,2m\}}\frac{1}{\rho^i}\omega_{\ell-i}(1-\Pi_0)\mathcal{P}_i(t,\partial_t,\partial_\rho)u_{\ell-i} \\ &\quad + \sum_{\ell\geq 1}\omega_\ell\Pi_0\mathcal{P}_0u_\ell + \sum_{\ell\geq 1}\sum_{i=1}^{\min\{\ell,2m\}}\frac{1}{\rho^i}\omega_{\ell-i}\Pi_0\mathcal{P}_i(t,\partial_t,\partial_\rho)u_{\ell-i} \\ &= \sum_{\ell\geq 1}\left[\omega_\ell(1-\Pi_0)\mathcal{P}_0u_\ell + \sum_{i=1}^{\min\{\ell,2m\}}\frac{1}{\rho^i}\omega_{\ell-i}(1-\Pi_0)\mathcal{P}_i(t,\partial_t,\partial_\rho)u_{\ell-i}\right] \end{aligned} \tag{3.5}$$

$$+\sum_{\ell\geq 1}\omega_\ell\Pi_0\mathcal{P}_0u_\ell+\sum_{\ell\geq 2}\frac{1}{\rho}\omega_{\ell-1}\Pi_0\mathcal{P}_1(1-\Pi_0)u_{\ell-1}+\sum_{\ell\geq 2}\sum_{i=2}^{\min\{\ell,2m\}}\frac{1}{\rho^i}\omega_{\ell-i}\Pi_0\mathcal{P}_i(t,\partial_t,\partial_\rho)u_{\ell-i}$$

$$= \sum_{\ell\geq 1}\left[\omega_\ell(1-\Pi_0)\mathcal{P}_0 u_\ell + \sum_{i=1}^{\min\{\ell,2m\}} \frac{1}{\rho^i}\omega_{\ell-i}(1-\Pi_0)\mathcal{P}_i(t,\partial_t,\partial_\rho)u_{\ell-i}\right]$$

$$+\sum_{\ell\geq 1}\left[\omega_\ell\Pi_0\mathcal{P}_0 u_\ell + \frac{1}{\rho}\omega_\ell\Pi_0\mathcal{P}_1(1-\Pi_0)u_\ell + \sum_{i=2}^{\min\{\ell+1,2m\}} \frac{1}{\rho^i}\omega_{\ell+1-i}\Pi_0\mathcal{P}_i(t,\partial_t,\partial_\rho)u_{\ell+1-i}\right],$$

where we used the fact that $\mathcal{P}_0 u_0 = 0$.
We immediately see that if the functions ω_ℓ are identically 1 or 0 the quantities in brackets above are zero because of (2.11). As a consequence the quantities in square brackets above have ρ–support for $2R(\ell+1-2m) \leq \rho \leq 4R(\ell+1)$.
Consider now A_2. Since γ runs over a finite interval, the derivatives of the functions ω_ℓ are uniformly bounded and as a consequence

$$\sum_{\gamma=1}^{2m} \frac{1}{\rho^i}\partial_\rho^\gamma \omega_\ell \frac{1}{\gamma!}\mathcal{P}_i^{(\gamma)}(t,\partial_t,\partial_\rho)u_\ell$$

has ρ–support in the interval $[2R(\ell+1), 4R(\ell+1)]$. Hence

$$M_{n,m}\mathscr{K}[\tilde{u}](x,y) = \mathscr{K}\Big[\sum_{\ell\geq 1}\tilde{w}_\ell\Big](x,y),$$

where the ρ–support of $\tilde{w}_\ell$ is contained in the interval $[2R(\ell+1-2m), 4R(\ell+1)]$.
We now show that $\mathscr{K}[\tilde{w}](x,y)$, where $\tilde{w} = \sum_{\ell\geq 1}\tilde{w}_\ell$ belongs to the following class of functions.

Definition 3.2. *Let $s > 1$, we denote by $\gamma^s(\Omega)$, Ω open subset of $\mathbb{R}^n$, the set of all $\varphi \in C^\infty(\Omega)$, for which, to every compact set K in Ω and every $\varepsilon > 0$, there is a constant C_ε such that, for every $\alpha \in \mathbb{Z}_+^n$,*

$$|D^\alpha \varphi(x)| \leq C_\varepsilon \varepsilon^{|\alpha|}|\alpha|^{s|\alpha|}, \quad x \in K. \tag{3.6}$$

The classes $\gamma^s(\Omega)$ are the (local) Beurling classes of order s and, for further reference on the subject, we refer to [17]. We may also define the global version of the classes γ^s, by using uniform constants. These classes are denoted as $\gamma_g^s(\Omega)$.

Proposition 3.3. *With the above notation the function $M_{n,m}\mathscr{K}[\tilde{u}](x,y)$ belongs to $\gamma_g^{2m/(2m-1)}(\mathbb{R}^2)$.*

Proof. We start off by estimating the derivatives of $\mathscr{K}\Big[\sum_{\ell\geq 1}\tilde{w}_\ell\Big](x,y)$. Set $s_0 = \frac{2m}{2m-1}$ and $r' = r + \frac{s_0}{n+1}$. Then

$$|\partial_y^\alpha\partial_x^\beta \mathscr{K}\Big[\sum_{\ell\geq 1}\tilde{w}_\ell\Big](x,y)| = \left|\int_0^{+\infty} e^{iy\rho^{s_0}}\rho^{r'+\alpha s_0+\frac{\beta s_0}{2(n+1)}}\left(\partial_t^\beta \tilde{w}(t,\rho)\right)\Big|_{t=\rho^{\frac{s_0}{2(n+1)}}x} d\rho\right|. \tag{3.7}$$

As we pointed out above the quantity $\partial_t^\beta\tilde{w}(t,\rho)$ is the sum of two functions: $\partial_t^\beta\tilde{w}(t,\rho) = \partial_t^\beta f_1(t,\rho) + \partial_t^\beta f_2(t,\rho)$, where f_1 contains the terms where the cutoffs ω_ℓ are not getting derived, while f_2 has the terms where the ω_ℓ takes derivatives.

$$f_1(t,\rho) = \sum_{i=0}^{2m}\sum_{\ell\geq 0}\frac{1}{\rho^i}\omega_\ell\mathcal{P}_i(t,\partial_t,\partial_\rho)u_\ell,$$

$$f_2(t,\rho) = \sum_{i=0}^{2m} \sum_{\ell \geq 0} \sum_{\gamma=1}^{2m} \frac{1}{\rho^i} \partial_\rho^\gamma \omega_\ell \frac{1}{\gamma!} \mathcal{P}_i^{(\gamma)}(t, \partial_t, \partial_\rho) u_\ell.$$

Consider the above integral where $\partial_t^\beta \tilde{w}$ has been replaced by $\partial_t^\beta f_2$. We have to estimate

$$\left| \int_0^{+\infty} e^{iy\rho^{s_0}} \rho^{r'+\alpha s_0 + \frac{\beta s_0}{2(n+1)}} \left(\partial_t^\beta \sum_{i=0}^{2m} \sum_{\ell \geq 0} \sum_{\gamma=1}^{2m} \frac{1}{\rho^i} \partial_\rho^\gamma \omega_\ell \frac{1}{\gamma!} \mathcal{P}_i^{(\gamma)}(t, \partial_t, \partial_\rho) u_\ell \right)_{\big| t = \rho^{\frac{s_0}{2(n+1)}} x} d\rho \right|$$

$$\leq \sum_{i=0}^{2m} \sum_{\ell \geq 0} \sum_{\gamma=1}^{2m} \int_0^{+\infty} \rho^{r'+\alpha s_0 + \frac{\beta s_0}{2(n+1)}} \left| \partial_t^\beta \frac{1}{\rho^i} \partial_\rho^\gamma \omega_\ell \frac{1}{\gamma!} \mathcal{P}_i^{(\gamma)}(t, \partial_t, \partial_\rho) u_\ell \right|_{\big| t = \rho^{\frac{s_0}{2(n+1)}} x} d\rho$$

$$\leq \sum_{i=0}^{2m} \sum_{\ell \geq 0} \sum_{\gamma=1}^{2m} \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} \int_0^{+\infty} \rho^{r'+\alpha s_0 + \frac{\beta s_0}{2(n+1)}} \left| \frac{1}{\rho^i} \partial_\rho^\gamma \omega_\ell \frac{1}{\gamma!} \mathcal{P}_{i,(\beta')}^{(\gamma)}(t, \partial_t, \partial_\rho) \partial_t^{\beta-\beta'} u_\ell(t,\rho) \right|_{\big| t = \rho^{\frac{s_0}{2(n+1)}} x} d\rho,$$

Where $\mathcal{P}_{i,(\beta')}^{(\gamma)}(t, \partial_t, \partial_\rho)$ denotes the differential operator whose symbol is given by $\partial_\sigma^\gamma \partial_t^{\beta'} \mathcal{P}_i(t, \tau, \sigma)$.

Next we need to bound $|\partial_t^\beta f_2|$. This derivative is a sum of terms of the form

$$\binom{\beta}{\beta'} \frac{1}{\rho^i} |\partial_\rho^\gamma \omega_\ell| |t^{2n+j-\beta'} \partial_t^{j+\beta-\beta'} \partial_\rho^{2m-i-\gamma} u_\ell(t,\rho)|, \ j \leq i,$$

modulo a constant independent of α, β. We apply Lemma 2.3 to obtain the bound for the above quantity

$$\binom{\beta}{\beta'} \frac{1}{\rho^i} |\partial_\rho^\gamma \omega_\ell| C_u^{\ell+j-\beta'+j+\beta-\beta'+2m-i-\gamma+1}$$
$$\cdot (\ell+1)^{\ell \frac{2m-1}{2m} + \frac{2m-i-\gamma}{2m}} (j-\beta')!^{\frac{1}{2n+2}} (j+\beta-\beta')!^{\frac{2n+1}{2n+2}} \frac{e^{-c_0\rho}}{\rho^\ell}.$$

Since $0 \leq i \leq 2m$, $1 \leq \gamma \leq 2m - i$, $j \leq i$, we may bound j by i and replace β' by zero. We also point out that the base of each factorial in the second line above is bounded by $\ell + \beta + 2m$. As a consequence we get the estimate

$$\binom{\beta}{\beta'} \frac{e^{-c_0\rho}}{\rho^{i+\ell}} |\partial_\rho^\gamma \omega_\ell| C_1^{\ell+\beta+1} (\ell+\beta+2m)^{\ell \frac{2m-1}{2m} + i \frac{2m-1}{2m} + \frac{2m-\gamma}{2m} + \beta \frac{2n+1}{2n+2}}$$
$$\leq \frac{e^{-c_0\rho}}{\rho^{i+\ell}} |\partial_\rho^\gamma \omega_\ell| C_2^{\ell+\beta+1} (\ell+\beta+2m)^{\ell \frac{2m-1}{2m} + \beta \frac{2n+1}{2n+2}}$$
$$\leq \frac{e^{-c_0\rho}}{\rho^{i+\ell}} |\partial_\rho^\gamma \omega_\ell| C_3^{\ell+\beta+1} \ell^{\ell \frac{2m-1}{2m}} \beta^{\beta \frac{2n+1}{2n+2}}.$$

Now the support of $\partial_\rho^\gamma \omega_\ell$ is contained in the interval $[2R(\ell+1), 4R(\ell+1)]$, so that $\ell \leq \frac{\rho}{2R} - 1$. As a consequence

$$\frac{1}{\rho^\ell} \ell^{\ell \frac{2m-1}{2m}} \leq \left(\frac{1}{2R} \right)^{\ell \frac{2m-1}{2m}} \rho^{-\frac{\ell}{2m}} \leq \left(\frac{1}{2R} \right)^{\ell \frac{2m-1}{2m}} \rho^{-\rho \frac{1}{4mR} + \frac{1}{2m}}.$$

This allows us to conclude that

$$|\partial_t^\beta f_2(t,\rho)| \leq \sum_{\ell \geq 0} \left(\frac{C_4}{2R} \right)^{\ell \frac{2m-1}{2m}} C_4^{\beta+1} \beta^{\beta \frac{2n+1}{2n+2}} \rho^{\frac{1}{2m}} e^{-\frac{1}{4mR} \rho \log \rho - c_0 \rho}.$$

Choosing R large enough allows us to bound the sum over ℓ and absorb it into the constant.
Going back to (3.7) where $\partial_t^\beta \tilde{w}$ has been replaced by $\partial_t^\beta f_2$ we obtain

$$\left| \int_0^{+\infty} e^{iy\rho^{s_0}} \rho^{r'+\alpha s_0 + \frac{\beta s_0}{2(n+1)}} (\partial_t^\beta f_2(t,\rho))_{\big|t=\rho^{\frac{s_0}{2(n+1)}}x} d\rho \right|$$
$$\leq C_4^{\beta+1} \beta^{\beta\frac{2n+1}{2n+2}} \int_{2R}^{+\infty} \rho^{s_0\left(\alpha+\frac{\beta}{2(n+1)}\right)} e^{-\frac{1}{4mR}\rho\log\rho} \rho^{r'+\frac{1}{2m}} e^{-c_0\rho} d\rho.$$

Since $\rho^{r'+\frac{1}{2m}} e^{-c_0\rho} \leq C_5$, C_5 independents by α, β and s_0, and applying Lemma 5.2.9, for any $\epsilon > 0$, the right hand side of the above relation is bounded by

$$C_5 C_4^{\beta+1} \beta^{\beta\frac{2n+1}{2n+2}} \epsilon^{\alpha+\frac{\beta}{2n+2}} C_\epsilon \left(\alpha + \frac{\beta}{2n+2}\right)^{\left(\alpha+\frac{\beta}{2n+2}\right)s_0}.$$

With a slight change of notation we see that the above quantity is estimated by

$$C_\epsilon \epsilon^{\alpha+\beta} \alpha^{\alpha s_0} \beta^{\beta s_0}, \tag{3.8}$$

since

$$\frac{s_0}{2n+2} + \frac{2n+1}{2n+2} < s_0.$$

Consider now

$$\int_0^{+\infty} e^{iy\rho^{s_0}} \rho^{r'+\alpha s_0 + \frac{\beta s_0}{2(n+1)}} (\partial_t^\beta f_1(t,\rho))_{\big|t=\rho^{\frac{s_0}{2(n+1)}}x} d\rho,$$

where f_1 has been defined after (3.7). Arguin as in (3.5) we may rewrite the integral above as

$$\int_0^{+\infty} e^{iy\rho^{s_0}} \rho^{r'+\alpha s_0 + \frac{\beta s_0}{2(n+1)}} \Bigg[\partial_t^\beta \sum_{\ell\geq 1} \Big(\omega_\ell (1-\Pi_0)\mathcal{P}_0 u_\ell$$
$$+ \sum_{i=1}^{\min\{\ell,2m\}} \frac{1}{\rho^i} \omega_{\ell-i}(1-\Pi_0)\mathcal{P}_i(t,\partial_t,\partial_\rho) u_{\ell-i} \Big) \Bigg]_{\big|t=\rho^{\frac{s_0}{2(n+1)}}x} d\rho$$
$$+ \int_0^{+\infty} e^{iy\rho^{s_0}} \rho^{r'+\alpha s_0 + \frac{\beta s_0}{2(n+1)}} \Bigg[\partial_t^\beta \sum_{\ell\geq 1} \Big(\omega_\ell \Pi_0 \mathcal{P}_0 u_\ell + \frac{1}{\rho}\omega_\ell \Pi_0 \mathcal{P}_1 (1-\Pi_0) u_\ell$$
$$+ \sum_{i=2}^{\min\{\ell+1,2m\}} \frac{1}{\rho^i} \omega_{\ell+1-i} \Pi_0 \mathcal{P}_i(t,\partial_t,\partial_\rho) u_{\ell+1-i} \Big) \Bigg]_{\big|t=\rho^{\frac{s_0}{2(n+1)}}x} d\rho.$$

The quantities in square brackets above have ρ–support for $2R(\ell+1-2m) \leq \rho \leq 4R(\ell+1)$.
The estimate of the derivatives of order β with respect to t of the above terms is made according to the same lines of the estimate above, since the order of ρ-derivative of ω_ℓ played no role in the above estimate.
Hence we get (3.8) also in this case. This ends the proof of the proposition. $\square$

4. Proof of Theorem 1.1

For the sake of completeness and to make the paper self contained, we recall, in the Gevrey setting, the following result due to Métivier.

Theorem 4.1 ([19]). *Let P be a differential operator with analytic coefficient. Assume that in Ω open neighborhood of $x_0 \in \mathbb{R}^n$ there exist a continuous operator R from $L^2(\Omega)$ in $L^2(\Omega)$ such that $PR = Id$. Then P is G^s-hypoelliptic in x_0 if and only if for any neighborhood $\Omega_2 \subset \Omega$ of x_0, there exists a neighborhood $\Omega_3 \Subset \Omega_2$ of x_0 and constants C and L such that for any $k \in \mathbb{Z}_+$ and any $u \in \mathscr{D}'(\Omega_2)$ we have:*

i) *$Pu \in H^k(\Omega_2) \Rightarrow u_{|\Omega_3} \in H^k(\Omega_3)$;*

ii) *the following estimate holds*

$$\|u\|_{k,\Omega_3} \leq CL^k \left(|||Pu|||_{k,\Omega_2} + k^{sk} \|u\|_{0,\Omega_2} \right), \tag{4.1}$$

where

$$|||Pu|||_{k,\Omega_2} = \sum_{|\alpha| \leq k} k^{s(k-|\alpha|)} \|D^\alpha Pu\|_{0,\Omega_2}.$$

We are going to use the above theorem to prove Theorem 1.1. Since the operator $M_{n,m}$ in (1.3), as well as its adjoint, satisfies the Hörmander condition in Ω, open neighborhood of the origin, both $M_{n,m}$ and its adjoint are subelliptic with a loss of $2(1-1/(2n+1))$ derivatives (see [18] and [21].) As a consequence there is $\Omega_1 \Subset \Omega$, neighborhood of the origin, where $M_{n,m}$ satisfies the assumption of Theorem 4.1.
Arguing by contradiction, assume that $M_{n,m}$ is G^s-hypoelliptic in a neighborhood of the origin for $s < \frac{2m}{2m-1}$.
We showed before, in Proposition 3.3, that $M_{n,m}\mathscr{K}[\tilde{u}] \in \gamma_g^{2m/(2m-1)}(\mathbb{R}^2)$; due to estimate (4.1) and to the fact that $G^{s_1}(\Omega_2) \subset \gamma^{s_2}(\Omega_2)$ if $s_1 < s_2$ (see Lemma 5.2.10,) we have that $\mathscr{K}[\tilde{u}] \in \gamma^{2m/(2m-1)}(\Omega_3)$, $\Omega_3 \Subset \Omega_2$ open neighborhood of the origin.
Next we need to prove the following

Proposition 4.2. *For any $\epsilon > 0$ there exists a $C_\epsilon > 0$ such that for any $\alpha \in \mathbb{N}$ we have*

$$\|\langle y \rangle^2 \partial_y^\alpha \mathscr{K}[\tilde{u}](0,y)\|_{L^\infty(\mathbb{R})} \leq C_\epsilon \epsilon^\alpha \alpha!^{s_0}, \tag{4.2}$$

where

$$s_0 = \frac{2m}{2m-1}.$$

Let us show first that Proposition 4.2 allows us to prove Theorem 1.1.

Lemma 4.3. *For any $\epsilon > 0$ there exists a $C_\epsilon > 0$, $B > 0$, B independents of ε, such that*

$$|\mathscr{F}\big(\mathscr{K}[\tilde{u}]\big)(0,\eta)| \leq C_\epsilon e^{-B\left(\frac{|\eta|}{\epsilon}\right)^{\frac{1}{s_0}}}. \tag{4.3}$$

Proof. Since

$$|\mathscr{F}\big(\mathscr{K}[\tilde{u}]\big)(0,\eta)| \leq \frac{1}{|\eta|^\alpha} \int_{\mathbb{R}} |\langle y \rangle^2 \partial_y^\alpha \mathscr{K}[\tilde{u}](0,y)| \frac{1}{\langle y \rangle^2} dy$$
$$\leq C_\epsilon \epsilon^\alpha \alpha!^{s_0} |\eta|^{-\alpha} \int_{\mathbb{R}} \frac{1}{\langle y \rangle^2} dy \leq C'_\epsilon \epsilon^\alpha \alpha!^{s_0} |\eta|^{-\alpha}.$$

Hence

$$|\mathscr{F}\big(\mathscr{K}[\tilde{u}]\big)(0,\eta)|^{\frac{1}{s_0}}\left(\frac{|\eta|}{2\epsilon}\right)^{\frac{\alpha}{s_0}}\frac{1}{\alpha!}\leq C''_\epsilon\left(\frac{1}{2}\right)^{\frac{\alpha}{s_0}},$$

so that, summing over α we finally obtain

$$|\mathscr{F}\big(\mathscr{K}[\tilde{u}]\big)(0,\eta)|^{\frac{1}{s_0}}e^{\left(\frac{|\eta|}{2\epsilon}\right)^{\frac{1}{s_0}}}\leq C'''_\epsilon,$$

from which we conclude the proof of the lemma. □

We need also a bound from below of the same quantity of Lemma 4.3.

Lemma 4.4. *There are constants $M>0$, $\mu\in\mathbb{R}$ such that for $\eta\geq 4R$, R large enough, we have*

$$|\mathscr{F}\big(\mathscr{K}[\tilde{u}]\big)(0,\eta)|\geq M|\eta|^{\mu}e^{-c_0|\eta|^{\frac{1}{s_0}}},\tag{4.4}$$

where $c_0=\Re(c_1)$, $c_1=\frac{2m}{2m-1}(2n+1)^{1/2m}\left(\sin\left(\frac{\pi}{2m}\right)-i\cos\left(\frac{\pi}{2m}\right)\right)$.

Proof. Since

$$\mathscr{K}[\tilde{u}](0,y)=\int_0^{+\infty}e^{iy\rho^{s_0}}\rho^r\tilde{u}(0,\rho)d\rho=\frac{1}{s_0}\int_{-\infty}^{+\infty}e^{iy\eta}\eta^{\frac{r+1}{s_0}-1}\tilde{u}(0,\eta^{\frac{1}{s_0}})d\eta,$$

we have that

$$\mathscr{F}\big(\mathscr{K}[\tilde{u}]\big)(0,\eta)=\frac{2\pi}{s_0}\eta^{\frac{r+1}{s_0}-1}\tilde{u}(0,\eta^{\frac{1}{s_0}}).$$

Then, using Lemma 2.3 we may write that

$$\begin{aligned}
|\mathscr{F}\big(\mathscr{K}[\tilde{u}]\big)(0,\eta)| &\geq \frac{2\pi}{s_0}|\eta|^{\frac{r+1}{s_0}-1}\Big(\omega_0(\eta^{\frac{1}{s_0}})v_0(0)e^{-c_0|\eta|^{\frac{1}{s_0}}}-\sum_{\ell\geq1}\omega_\ell(\eta^{\frac{1}{s_0}})|u_\ell(0,\eta^{\frac{1}{s_0}})|\Big)\\
&=\frac{2\pi}{s_0}|\eta|^{\frac{r+1}{s_0}-1}\Big(\omega_0(\eta^{\frac{1}{s_0}})v_0(0)e^{-c_0|\eta|^{\frac{1}{s_0}}}-\sum_{\ell\geq1}\omega_\ell(\eta^{\frac{1}{s_0}})e^{-c_0|\eta|^{\frac{1}{s_0}}}|u_\ell(0,\eta^{\frac{1}{s_0}})|e^{c_0|\eta|^{\frac{1}{s_0}}}\Big)\\
&\geq\frac{2\pi}{s_0}|\eta|^{\frac{r+1}{s_0}-1}\Big(\omega_0(\eta^{\frac{1}{s_0}})v_0(0)e^{-c_0|\eta|^{\frac{1}{s_0}}}-e^{-c_0|\eta|^{\frac{1}{s_0}}}\sum_{\ell\geq1}\omega_\ell(\eta^{\frac{1}{s_0}})C_u^{\ell+1}(\ell+1)^{\frac{\ell}{s_0}}|\eta|^{-\frac{\ell}{s_0}}\Big).
\end{aligned}$$

On the other hand, on $\operatorname{supp}\omega_\ell$, $\eta^{\frac{1}{s_0}}\geq 2R(\ell+1)$, so that the above inequality becomes, if

$$\begin{aligned}
|\mathscr{F}\big(\mathscr{K}[\tilde{u}]\big)(0,\eta)|\geq\frac{2\pi}{s_0}|\eta|^{\frac{r+1}{s_0}-1}\Big(&\omega_0(\eta^{\frac{1}{s_0}})v_0(0)e^{-c_0|\eta|^{\frac{1}{s_0}}}\\
&-e^{-c_0|\eta|^{\frac{1}{s_0}}}\sum_{\ell\geq1}\left(\frac{1}{2R}\right)^{\ell}C_u^{\ell+1}(\ell+1)^{\ell\left(\frac{1}{s_0}-1\right)}\Big).
\end{aligned}$$

Since $\frac{1}{s_0}-1=-\frac{1}{2m}$ and taking $R>C_u$, we have

$$\sum_{\ell\geq1}\left(\frac{1}{2R}\right)^{\ell}C_u^{\ell+1}(\ell+1)^{-\frac{\ell}{2m}}\leq C_u\sum_{\ell\geq1}\left(\frac{C_u}{2R}\right)^{\ell}=C_u\left(\frac{1}{1-\frac{C_u}{2R}}-1\right)=\frac{C_u^2}{2R-C_u}.$$

Then

$$|\mathscr{F}\big(\mathscr{K}[\tilde{u}]\big)(0,\eta)|\geq\frac{2\pi}{s_0}|\eta|^{\frac{r+1}{s_0}-1}e^{-c_0|\eta|^{\frac{1}{s_0}}}\left(\omega_0(\eta^{\frac{1}{s_0}})v_0(0)-\frac{C_u^2}{2R-C_u}\right)$$

$$\geq M|\eta|^{\frac{r+1}{s_0}-1} e^{-c_0|\eta|^{\frac{1}{s_0}}},$$

provided $|\eta|^{\frac{1}{s_0}} \geq 4R$ and R is large enough, so that $v_0(0) - C_u^2(2R - C_u)^{-1} > 0$. This ends the proof of the lemma. □

Lemma 4.3 and Lemma 4.4 for η large and ϵ small give a contradiction. Hence the operator $M_{n,m}$ is Gevrey $\frac{2m}{2m-1}$ hypoelliptic and this value is optimal.
To finish the proof we have to show that (4.2) holds, or, in other words, that $\mathscr{K}[\tilde{u}](0, y)$ belongs to the global Beurling class $\gamma_g^{\frac{2m}{2m-1}}(\mathbb{R})$.

Proof of Proposition 4.2. Since $\mathscr{K}[\tilde{u}] \in \gamma^{\frac{2m}{2m-1}}(\Omega)$, where Ω is a neighborhood of the origin in $\mathbb{R}^2$, we need only to show that $\mathscr{K}[\tilde{u}](0, y) \in \gamma_g^{\frac{2m}{2m-1}}(\{|y| \geq \delta\})$, for a suitable positive δ.
Actually we are going to prove (4.2) for $|y| \geq \delta$.
For $\alpha \in \mathbb{N}$, consider

$$y^2 \partial_y^\alpha \mathscr{K}[\tilde{u}](0, y).$$

If the factor y^2 is missing the argument is the same, hence we skip it.
Let us compute the above quantity:

$$y^2 D_y^\alpha \int_0^{+\infty} e^{iy\rho^{s_0}} \rho^r \tilde{u}(0, \rho) d\rho = y^2 \int_0^{+\infty} e^{iy\rho^{s_0}} \rho^{r+\alpha s_0} \tilde{u}(0, \rho) d\rho.$$

Now

$$e^{iy\rho^{s_0}} = \frac{1}{y s_0 \rho^{s_0 - 1}} D_\rho e^{iy\rho^{s_0}},$$

so that

$$y^2 D_y^\alpha \int_0^{+\infty} e^{iy\rho^{s_0}} \rho^r \tilde{u}(0, \rho) d\rho = y^2 \int_0^{+\infty} e^{iy\rho^{s_0}} \left(-D_\rho \frac{1}{y s_0 \rho^{s_0-1}}\right)^\beta \rho^{r+\alpha s_0} \tilde{u}(0, \rho) d\rho,$$

where we made β integrations by parts using the fact that $\tilde{u}(0, \rho)$ is rapidly vanishing at infinity.
As for the factor y^2 we transform it into derivatives with respect to ρ, finally obtaining

$$\begin{aligned} y^2 D_y^\alpha &\int_0^{+\infty} e^{iy\rho^{s_0}} \rho^r \tilde{u}(0, \rho) d\rho \\ &= \int_0^{+\infty} e^{iy\rho^{s_0}} \left(-D_\rho \frac{1}{s_0 \rho^{s_0-1}}\right)^2 \left(-D_\rho \frac{1}{y s_0 \rho^{s_0-1}}\right)^\beta \rho^{r+\alpha s_0} \tilde{u}(0, \rho) d\rho \\ &\qquad = \frac{1}{y^\beta} \int_0^{+\infty} e^{iy\rho^{s_0}} \left(-\partial_\rho \frac{1}{i s_0 \rho^{s_0-1}}\right)^{\beta+2} \rho^{r+\alpha s_0} \tilde{u}(0, \rho) d\rho. \end{aligned}$$

We use the formula

$$\left(-\partial_\rho \frac{1}{i s_0 \rho^{s_0-1}}\right)^{\beta+2} = \sum_{h=0}^{\beta+2} \gamma_{\beta+2,h} \frac{1}{\rho^{s_0(\beta+2)-h}} \partial_\rho^h, \tag{4.5}$$

where

$$|\gamma_{\beta+2,h}| \leq C_\gamma'^{\beta+2+h} \frac{(\beta+2)!}{h!} \leq C_\gamma^{\beta+2+h} (\beta + 2 - h)!. \tag{4.6}$$

Here C'_γ, C_γ are positive constants independent of β, h. In particular we have $\gamma_{\beta+2,\beta+2} = \left(\frac{i}{s_0}\right)^{\beta+2}$, and for convenience we set $\gamma_{00} = 1$. Thus the above integral becomes

$$\frac{1}{y^\beta}\sum_{h=0}^{\beta+2}\gamma_{\beta+2,h}\int_0^{+\infty} e^{iy\rho^{s_0}}\frac{1}{\rho^{s_0(\beta+2)-h}}\partial_\rho^h\left(\rho^{r+\alpha s_0}\tilde{u}(0,\rho)\right)d\rho.$$

We use the formula

$$\partial_\rho^p\left(\rho^\lambda u\right) = \sum_{k=0}^{p}\binom{p}{k}(\lambda)_{p-k}\rho^{\lambda-p+k}\partial_\rho^k u, \tag{4.7}$$

where $(\lambda)_\beta$ is the Pochhammer symbol defined by

$$(\lambda)_\beta = \lambda(\lambda-1)\cdots(\lambda-\beta+1), \quad (\lambda)_0 = 1, \quad \lambda\in\mathbb{C}. \tag{4.8}$$

Then the above integral becomes

$$\frac{1}{y^\beta}\sum_{h=0}^{\beta+2}\sum_{k=0}^{h}\binom{h}{k}(r+\alpha s_0)_{h-k}\gamma_{\beta+2,h}\int_0^{+\infty} e^{iy\rho^{s_0}}\rho^{r+\alpha s_0+k-s_0(\beta+2)}\partial_\rho^k\tilde{u}(0,\rho)d\rho. \tag{4.9}$$

Consider now

$$\partial_\rho^k\tilde{u}(0,\rho) = \sum_{\ell\ge0}\partial_\rho^k\left(u_\ell(0,\rho)\omega_\ell(\rho)\right) = \sum_{\ell\ge0}\sum_{i=0}^{k}\binom{k}{i}\partial_\rho^i u_\ell(0,\rho)\partial_\rho^{k-i}\omega_\ell(\rho).$$

by (3.3). The absolute value of the quantity in (4.9) for $|y|\ge\delta$ can be estimated, applying Lemma 3.1 and Lemma 2.3, by

$$\begin{aligned}
&\delta^{-\beta}\sum_{h=0}^{\beta+2}\sum_{k=0}^{h}\sum_{\ell\ge0}\sum_{i=0}^{k}\binom{k}{i}\binom{h}{k}|(r+\alpha s_0)_{h-k}||\gamma_{\beta+2,h}|\\
&\qquad\qquad\cdot\int_0^{+\infty}\rho^{r+\alpha s_0+k-s_0(\beta+2)}|\partial_\rho^i u_\ell(0,\rho)||\partial_\rho^{k-i}\omega_\ell(\rho)|d\rho\\
&\qquad\le\delta^{-\beta}\sum_{h=0}^{\beta+2}\sum_{k=0}^{h}\sum_{\ell\ge0}\sum_{i=0}^{k}C_\gamma'^{\beta+2-h}\binom{k}{i}\binom{h}{k}|(r+\alpha s_0)_{h-k}|\frac{(\beta+2)!}{h!}\\
&\cdot(RC_\omega)^{k-i+1}(k-i)!^\sigma C_u^{\ell+i+1}(\ell+1)^{\frac{\ell}{s_0}+\frac{i}{2m}}\int_{2R(\ell+1)}^{+\infty}\rho^{r+\alpha s_0+i-s_0(\beta+2)-\ell}e^{-c_0\rho}d\rho
\end{aligned}$$

Choose $\beta+2=\alpha$. There is no problem in assuming that $\alpha>2$, since we are interested in large values of α. Furthermore on the domain of integration we have that

$$\frac{1}{\rho}\le\frac{1}{2R(\ell+1)},$$

so that the quantity on the right hand side of the above inequality is bounded by

$$\begin{aligned}
&\delta^{-\alpha+2}C^\alpha\sum_{h=0}^{\alpha}\sum_{k=0}^{h}\sum_{\ell\ge0}\sum_{i=0}^{k}C^{-h+k+1}\left(\frac{C}{R}\right)^\ell\frac{\alpha!}{i!}\frac{|(r+\alpha s_0)_{h-k}|}{(h-k)!}(k-i)!^{\sigma-1}\\
&\qquad\qquad\cdot R^{k-i+1-\frac{i}{2m}}(\ell+1)^{\ell\left(\frac{1}{s_0}-1\right)}\int_{2R(\ell+1)}^{+\infty}\rho^{r+i\left(1+\frac{1}{2m}\right)}e^{-c_0\rho}d\rho.
\end{aligned}$$

The sum over $\ell \geq 0$ yields a constant independent of α:

$$\delta^{-\alpha+2}C^{\alpha+1}\sum_{h=0}^{\alpha}\sum_{k=0}^{h}\sum_{i=0}^{k}C^{-h+k+1}\frac{\alpha!}{i!}\frac{|(r+\alpha s_0)_{h-k}|}{(h-k)!}(k-i)!^{\sigma-1}$$
$$\cdot R^{k-i+1-\frac{i}{2m}}\int_0^{+\infty}\rho^{r+i\left(1+\frac{1}{2m}\right)}e^{-c_0\rho}d\rho.$$

Now

$$\begin{aligned}\frac{|(r+\alpha s_0)_{h-k}|}{(h-k)!} &= \frac{|r+\alpha s_0|\ |r+\alpha s_0-1|\dots|r+\alpha s_0-h+k+1|}{(h-k)!}\\ &\leq \frac{(|r|+\alpha s_0)(|r|+\alpha s_0-1)\dots(|r|+\alpha s_0-(h-k)+1)}{(h-k)!}\\ &= s_0^{h-k}\frac{\left(\frac{|r|}{s_0}+\alpha\right)\left(\frac{|r|}{s_0}+\alpha-\frac{1}{s_0}\right)\dots\left(\frac{|r|}{s_0}+\alpha-\frac{h-k+1}{s_0}\right)}{(h-k)!} \leq s_0^{h-k}\frac{\left(\frac{|r|}{s_0}+\alpha\right)^{h-k}}{(h-k)!}\\ &\leq C_0^{h-k}\frac{\alpha^{h-k}}{(h-k)!} = C_0^{h-k}\frac{\alpha!}{(\alpha-(h-k))!}\frac{\alpha^{h-k}}{\alpha!}\frac{(\alpha-(h-k))!}{(h-k)!}\\ &= C_0^{h-k}\binom{\alpha}{h-k}(\alpha-(h-k))!\frac{\alpha^{h-k}}{\alpha!} \leq C_1^{\alpha}\frac{(\alpha-(h-k))!}{\alpha^{\alpha-(h-k)}} \leq C_2^{\alpha}.\end{aligned}$$

Here we assumed, without loss of generality, that $\alpha \geq \frac{|r|}{s_0}$ and we also used the estimate $n! \geq C^n n^n$. Plugging this into the above expression we get

$$\delta^{-\alpha+2}C^{\alpha+1}\sum_{h=0}^{\alpha}\sum_{k=0}^{h}\sum_{i=0}^{k}C^{-h+k+1}\frac{\alpha!}{i!}(k-i)!^{\sigma-1}$$
$$\cdot R^{k-i+1-\frac{i}{2m}}\int_0^{+\infty}\rho^{r+i\left(1+\frac{1}{2m}\right)}e^{-c_0\rho}d\rho,$$

with a different meaning of the constant involved. As a consequence the above quantity can be further estimated as

$$\delta^{-\alpha+2}C^{\alpha+1}\sum_{h=0}^{\alpha}\sum_{k=0}^{h}\sum_{i=0}^{k}C^{-h+k+1}\frac{\alpha!}{i!}(k-i)!^{\sigma-1}R^{k-i+1-\frac{i}{2m}}M^i i!^{1+\frac{1}{2m}}.$$

If we choose σ such that $\sigma-1=\frac{1}{2m}$, we have

$$\delta^{-\alpha+2}C^{\alpha+1}\sum_{h=0}^{\alpha}\sum_{k=0}^{h}\sum_{i=0}^{k}C^{-h+k+1}\alpha!R^{k-i+1-\frac{i}{2m}}M^i k!^{\frac{1}{2m}} \leq \delta^{-\alpha+2}C_1^{\alpha+1}\alpha!^{1+\frac{1}{2m}}.$$

Hence for $|y| \geq \delta$ we have that $\mathscr{K}[\tilde{u}](0,y)$ belongs to $G^{1+\frac{1}{2m}}(\{|y| \geq \delta\})$ and, since

$$1+\frac{1}{2m} < \frac{2m}{2m-1} = s_0,$$

we proved that $\mathscr{K}[\tilde{u}](0,y)$ belongs to $\gamma_g^{\frac{2m}{2m-1}}(\mathbb{R})$ and, moreover, each derivative is L^1 summable in the variable y.

$\square$

5. Appendix

In this appendix we collect some results on the eigenfunctions of the Bender and Wang operator. We also include here some estimates we use in the preceding sections.

5.1. **On a special eigenvalue problem.** In [3], Bender and Wang study the eigenvalue problem

$$\left(-\partial_t^2 + t^{2N+2}\right) u(t) = E t^N u(t), \quad N = -1, 0, 1, 2, \dots,$$

on the interval $-\infty \leq t \leq +\infty$. The eigenfunction $u(t)$ is required to obey the boundary conditions that $u(t)$ vanish exponentially rapidly as $t \to \pm\infty$.
In this paragraph we focus on the case $N = 2n$, $n \in \mathbb{Z}_+$,

$$\left(-\partial_t^2 + t^{2(2n+1)}\right) u(t) = E t^{2n} u(t). \tag{5.1.1}$$

In [3] the authors show that the even-parity eigenfunctions have the form

$$v_k(t) = e^{-\frac{t^{2n+2}}{2n+2}} L_k^{(-1/(2n+2))}\left(\frac{t^{2n+2}}{n+1}\right), \quad k \in \mathbb{Z}_+, \tag{5.1.2}$$

with corresponding eigenvalue

$$E_k = 4k(n+1) + 2n + 1;$$

and the odd-parity eigenfunctions have the form

$$w_k(t) = e^{-\frac{t^{2n+2}}{2n+2}}\, t\, L_k^{(1/(2n+2))}\left(\frac{t^{2n+2}}{n+1}\right), \quad k \in \mathbb{Z}_+ \tag{5.1.3}$$

with corresponding eigenvalue

$$\tilde{E}_k = 4k(n+1) + 2n + 3.$$

$L_k^{(\alpha)}(\cdot)$, $\alpha = \pm 1/(2n+2)$, are the generalized *Laguerre polynomials* given by

$$L_k^{(\alpha)}\left(\frac{t^{2n+2}}{n+1}\right) = \sum_{i=0}^{k} (-1)^i \binom{k+\alpha}{k-i} \frac{t^{2i(n+1)}}{(n+1)^i i!}\,.$$

In the next Proposition we list some properties of the *Laguerre polynomials* that we use in what follows:

Proposition 5.1.1. *The following properties concerning the generalized Laguerre polynomials hold:*

(L-1) *let $k \in \mathbb{Z}_+$ and $\alpha \geq -1/2$ then*

$$(k+1)L_{k+1}^{\alpha}(s) - (2k+1-\alpha-s)\, L_k^{\alpha}(s) + (k+\alpha)L_{k-1}^{\alpha}(s) = 0, \tag{5.1.4}$$

and

$$s\frac{d}{ds}L_k^{\alpha}(s) = kL_k^{\alpha}(s) - (k+\alpha)L_{k-1}^{\alpha}(s); \tag{5.1.5}$$

(L-2) *let $k, q \in \mathbb{Z}_+$ and $\alpha > -1$ then*

$$\int_0^{+\infty} e^{-s} s^{\alpha} L_k^{\alpha}(s) L_q^{\alpha}(s) ds = \Gamma(\alpha+1)\binom{k+\alpha}{k}\delta_{k,q}, \tag{5.1.6}$$

where $\delta_{k,q}$ is the Kronecker symbol;

(L-3) *due to the Theorem 6.23,* [22]*, if $\alpha > -1$ and $k \in \mathbb{Z}_+$ then $L_k^{\alpha}(s)$ has k positive zeros.*

Remark 5.1.2. *Due to the property (L-3), Proposition 5.1.1, we have that $v_k(t)$, (5.1.2), has $2k$ zeros each one of multiplicity $2n+2$ and $w_k(t)$, (5.1.3), has $2k+1$ zeros, $2k$ of them with multiplicity $2n+2$, 0 is a zero.*
From the property (L-2), Proposition 5.1.1, we obtain

$$\|t^n v_k(t)\|_0^2 = \frac{1}{2}\left(\frac{1}{n+1}\right)^{\frac{1}{2n+2}} \Gamma(1-\alpha)\binom{k-\alpha}{k} \tag{5.1.7}$$

and

$$\|t^n w_k(t)\|_0^2 = \frac{1}{2}\left(\frac{1}{n+1}\right)^{\frac{1}{2n+2}} \Gamma(1+\alpha)\binom{k+\alpha}{k} \tag{5.1.8}$$

where $\alpha = (2n+2)^{-1}$.
Moreover we have $\langle t^n v_k, t^n v_m\rangle = 0$ and $\langle t^n w_k, t^n w_m\rangle = 0$ for any $k, m \in \mathbb{Z}_+$ with $k \neq m$. From now on we shall consider the eigenfunctions, v_k, w_k, as normalized with respect the norms in (5.1.7), (5.1.8).

The next lemma gives us a three terms recurrence relation for some derivative of the eigenfunctions. This relation is crucial for our computation of the asymptotic solution.

Lemma 5.1.3. *The following recurrence relation holds*

$$t\frac{d}{dt}v_k(t) = (n+1)\Big((k+1)v_{k+1}(t) - (1+\alpha)v_k(t) - (k+\alpha)v_{k-1}(t)\Big),\ k \geq 1, \tag{5.1.9}$$

and

$$t\frac{d}{dt}v_0(t) = (n+1)\left(v_1(t) - (1+\alpha)v_0(t)\right), \tag{5.1.10}$$

where $\alpha = -\frac{1}{2n+2}$. More in general for every $i, k \in \mathbb{Z}_+$ we have

$$\left(t\frac{d}{dt}\right)^i v_k(t) = \sum_{j=-i}^{i} \delta_j^{k,i} v_{k+j}(t) = \sum_{j=\max\{k-i,0\}}^{k+i} \delta_{j-k}^{k,i} v_j(t). \tag{5.1.11}$$

where:

(i) $\delta_i^{k,i} = (n+1)^i \frac{(k+i)!}{k!}$, *and, more generally,*
(ii) $|\delta_j^{k,i}| \leq C^i \frac{(k+i)!}{k!}$, $j = -i, \dots, i$, *where C is a suitable positive constant.*

In the first sum in (5.1.11) *we used the convention that $v_\ell(t)$ is identically zero if ℓ is negative.*

Proof. Taking advantage of the relations (5.1.4) and (5.1.5) in (L-1), Proposition 5.1.1, it easy to obtain the recurrence relation (5.1.9) and (5.1.10).
Setting $\delta_j^{k,i} = (n+1)^i \tilde{\delta}_j^{k,i}$ a direct computation allows us to obtain recursively the coefficients in the sum (5.1.11):

- $\tilde{\delta}_i^{k,i} = (k+i)\tilde{\delta}_{i-1}^{k,i-1}$;
- $\tilde{\delta}_{i-1}^{k,i} = (k+i-1)\tilde{\delta}_{i-2}^{k,i-1} - (1+\alpha)\tilde{\delta}_{i-1}^{k,i-1}$;
- $\tilde{\delta}_j^{k,i} = (k+j)\tilde{\delta}_{j-1}^{k,i-1} - (1+\alpha)\tilde{\delta}_j^{k,i-1} - (k+j+1+\alpha)\tilde{\delta}_{j+1}^{k,i-1}$, $j = -i+2, \dots, i-2$;
- $\tilde{\delta}_{-i+1}^{k,i} = (k+\alpha-i+2)\tilde{\delta}_{-i+2}^{k,i-1} - (1+\alpha)\tilde{\delta}_{-i+1}^{k,i-1}$;
- $\tilde{\delta}_{-i}^{k,i} = (k+\alpha-i+1)\tilde{\delta}_{-i+1}^{k,i-1}$;

where $\tilde{\delta}_{\ell}^{k,i-1}$, $\ell \in \{i-1, -i+1\}$, are the coefficients of $(n+1)^{-i+1}\left(t\frac{d}{dt}\right)^{i-1} v_k(t)$. From the above relations and arguing by induction we obtain $\tilde{\delta}_i^{k,i} = \frac{(k+i)!}{k!}$, $|\tilde{\delta}_{i-1}^{k,i}| \leq i(1+\alpha)\frac{(k+i-1)!}{k!}$, $\tilde{\delta}_{-i}^{k,i} = \prod_{\ell=0}^{i-1}(k+\alpha-\ell)$ and $|\tilde{\delta}_j^{k,i}| \leq C^i \frac{(k+i)!}{k!}$, $j = -i+1, \dots, i-2$, where C is a suitable positive constant. $\square$

Remark 5.1.4. *From the above Lemma and taking advantage of relation (5.1.7), Remark 5.1.2, we can describe the action of the operators $\mathscr{P}_i(t\partial_t)$, (5.2.3), on the even-parity eigenfunctions. We have*

(1) *for $p \leq i$*

$$\mathscr{P}_i(t\partial_t) v_p(t) = \sum_{\nu=0}^{i+p} \left(\sum_{j=|\nu-p|}^{i} \mathscr{r}_{i,j}\, \delta_{\nu-p}^{p,j} \right) v_\nu(t); \tag{5.1.12}$$

(2) *for $p \geq i$*

$$\mathscr{P}_i(t\partial_t) v_p(t) = \sum_{\nu=p-i}^{p+i} \left(\sum_{j=|\nu-p|}^{i} \mathscr{r}_{i,j}\, \delta_{\nu-p}^{p,j} \right) v_\nu(t). \tag{5.1.13}$$

Next we show that the eigenfunctions $v_k(t)$ belong to the *Gel'fand-Shilov space* $S_{1/(2n+2)}^{(2n+1)/(2n+2)}(\mathbb{R})$; the same approach can be used in the case of the odd-parity eigenfunctions $w_k(t)$.

Definition 5.1.5. *Let α and β be real positive numbers. By $S_\beta^\alpha(\mathbb{R})$ we denote the set of infinitely differentiable functions $f(t)$, in $\mathbb{R}$, satisfying the inequality*

$$|t^k f^{(q)}(t)| \leq C A^k B^q k^{\alpha k} q^{\beta q} \tag{5.1.14}$$

where the positive constants C, A and B depend only on $f(t)$.

Fore more details on the subject we refer to [14].
In order to prove the following proposition we follow the ideas of Gundersen, [16], and of Titchmarsh, [23].

Proposition 5.1.6. *There exists a positive constant C_0, such that the following estimates hold*

$$\|v_k(t)\|_\infty \leq C_0 E_k^{\frac{3}{2}+\frac{1}{4n+4}} \quad \text{and} \quad \|w_k(t)\|_\infty \leq C_0 \tilde{E}_k^{\frac{3}{2}+\frac{1}{4n+4}}. \tag{5.1.15}$$

Proof. We set $Q_k(t) \doteq t^{2n}\left(t^{2n+2} - E_k\right)$. Even though Q_k is not a polynomial of the form considered in [16], [23], we observe that it differs from a polynomial of that form by the factor t^{2n}. Hence if we work on the complement of a small interval centered at the origin, we may argue along the same lines of [16], [23], Sections 5.4 and 8.4.1.
In what follows we work for $t > 0$. A symmetric argument holds for $t < 0$.
Set $T_k \doteq E_k^{1/(2n+2)}$, so that $Q_k(T_k) = 0$ and $Q_k(t) \neq 0$ if $t > T_k$.
Since $v_k(t)$ and $v_k''(t)$ have the same sign for $t > T_k$, all the zeros of $v_k(t)$ are in the interval $(0, T_k)$.
Let t_1 be a zero of $v_k(t)$, so that $v_k''(t_1) = 0$, and let $t_2 > t_1$ be the next critical point of $v_k(t)$. To be definite we assume that t_2 is a maximum, the case of a minimum being treated in the same way.

Let s be a point in the interval $[t_1, t_2]$. Since $v_k''(t) = -Q_k(t)v_k(t)$ we have

$$\begin{aligned}|v_k(t_2)| &= \left|\int_{t_1}^{t_2} v_k'(s)\,ds\right| = \left|\int_{t_1}^{t_2}\int_s^{t_2} t^{2n}\left(E_k - t^{2n+2}\right)v_k(t)\,dt\,ds\right| \\ &\le E_k\left|\int_{t_1}^{t_2}\int_s^{t_2} t^{2n}v_k(t)\,dt\,ds\right| \le E_k|T_k|^n\int_{t_1}^{t_2}(t_2-s)^{1/2}\,\|t^n v_k(t)\|_{L^2(\mathbb{R})}\,ds \\ &\le \frac{2}{3}E_k^{1+\frac{n}{2n+2}}(t_2-t_1)^{3/2} \le \frac{2}{3}E_k^{1+\frac{n}{2n+2}}T_k^{3/2}.\end{aligned}$$

Same argument if t_2 is a minimum. As a consequence the first of the estimates in (5.1.15) follows from the fact that both v_k is rapidly decreasing at infinity.
The same argument gives the second estimate in (5.1.15). □

Proposition 5.1.6 allows us to estimate how the L^∞ norm of v_k depends on k.

Corollary 5.1.7. *There is a positive constant, C_v, such that*

$$\|v_k\|_\infty \le C_v(k+1)^{\frac{3}{2}+\frac{1}{4n+4}}. \tag{5.1.16}$$

Proposition 5.1.8. *There exists a positive constant C_1, such that the following estimates hold*

$$\|v_k'(t)\|_\infty \le C_1E_k^{\frac{7}{2}-\frac{1}{4n+4}} \quad \text{and} \quad \|w_k'(t)\|_\infty \le C_1\tilde{E}_k^{\frac{7}{2}-\frac{1}{4n+4}}. \tag{5.1.17}$$

Proof. Let t_0 be such that $|t_0| < T_k$ and let t_1 denote the closest critical point for $v_k(t)$. Then

$$\begin{aligned}|v_k'(t)| &\le \left|\int_{t_1}^{t_0} v_k''(s)\,ds\right| = \left|\int_{t_1}^{t_0} s^{2n}\left(s^{2n+2}-E_k\right)v_k(s)\,ds\right| \\ &\le E_kT_k^{2n}\|v_k\|_\infty|t_0-t_1| \le E_kT_k^{2n+1}\|v_k\|_\infty.\end{aligned}$$

On the other hand, since $|v_k(t)| \to 0$ and $|v_k'(t)| \to 0$ for $t \to \pm\infty$ we obtain the first of the estimates in (5.1.17).
Same argument for the second estimate in (5.1.17). □

Proposition 5.1.9. *The following estimate holds*

$$|v_k(t)| \le C_0^{k+1}e^{-Bt^{2n+2}}, \tag{5.1.18}$$

where

$$B = \frac{1-\delta^{2n+1}}{2n+2}$$

where $\delta \le 1/2$ and C_0 is a suitable constant independent of k.

Proof. Let t_0 be such that $Q_k(t_0)$ and $v_k(t_0)$ are both positive. (In the odd-parity case, $w_k(t)$, we have to consider $|w_k(t_0)|$ since for $t_0 < -T_k$, $Q_k(t_0) > 0$ and $w_k(t_0) < 0$.)
We assume that $t_0 > T_k$. We have that for $t > t_0$, $Q_k(t) > 0$, $v_k(t) > 0$ and $v_k'(t) < 0$, $v_k(t) \to 0$ as $t \to +\infty$ (in the odd parity case $w_k(t) < 0$ and $w_k'(t) > 0$ as $t \to -\infty$.) We have

$$-v_k'(t)v_k''(t) = Q_k(t)v_k(t)(-v_k'(t)) \ge Q_k(t_0)v_k(t)(-v_k'(t)).$$

We have

$$\int_{t_0}^{t_1} v_k'(t)v_k''(t)dt = \left.\frac{(v_k'(t))^2}{2}\right|_{t_0}^{t_1} \quad \text{and} \quad \int_{t_0}^{t_1} v_k(t)v_k'(t)dt = \left.\frac{(v_k(t))^2}{2}\right|_{t_0}^{t_1}.$$

We obtain

$$-\frac{(v_k'(t_1))^2}{2}+\frac{(v_k'(t_0))^2}{2}\geq Q_k(t_0)\left[-\frac{(v_k(t_1))^2}{2}+\frac{(v_k(t_0))^2}{2}\right].$$

Taking $t_1\to+\infty$, since both $v_k(t_1)\to 0$ and $v_k'(t_1)\to 0$, we have

$$(v_k'(t_0))^2\geq Q_k(t_0)\,(v_k(t_0))^2 \text{ or equivalently } \left(-\frac{v_k'(t_0)}{v_k(t_0)}\right)^2\geq Q_k(t_0).$$

Without loss of generality we replace t_0 by t, $t>T_k$. Integrating both side on the interval (t_0,t) we have

$$-\int_{t_0}^{t}\frac{v_k'(s)}{v_k(s)}\,ds=\log v_k(t_0)-\log v_k(t)>\int_{t_0}^{t}(Q_k(s))^{1/2}\,ds.$$

We obtain

$$v_k(t)\leq v_k(t_0)e^{-\int_{t_0}^{t}(Q_k(s))^{1/2}\,ds}.$$

In the region $s\geq E_k^{1/(2n+2)}=T_k$ we have $\left(s^{2n+1}-E_k^{1/2}s^n\right)^2\leq s^{4n+2}-E_ks^{2n}$. We set $Z_0=\delta^{-2}E_k^{1/(2n+2)}(\geq T_k)$, $\delta\leq 1/2$. Taking $t_0=Z_0$ in the above formula we have

$$\begin{aligned}v_k(t)&\leq v_k(Z_0)e^{-\int_{Z_0}^{t}(Q_k(s))^{1/2}\,ds}\leq v_k(Z_0)e^{-\int_{Z_0}^{t}(s^{2n+1}-E_k^{1/2}s^n)\,ds}\\&\leq v_k(Z_0)e^{\frac{Z_0^{2n+2}}{2n+2}-E_k^{1/2}\frac{Z_0^{n+1}}{n+1}}e^{-\frac{t^{2n+2}}{2n+2}+E_k^{1/2}\frac{t^{n+1}}{n+1}}=C(Z_0)e^{-\frac{t^{2n+2}}{2n+2}+E_k^{1/2}\frac{t^{n+1}}{n+1}}.\end{aligned}$$

Let $B=\frac{1-2\delta^{2(n+1)}}{2n+2}$, we have

$$v_k(t)\leq C(Z_0)e^{-Bt^{2n+2}},\qquad \text{for } t\geq Z_0.$$

We remark that

$$C(Z_0)=v_k(Z_0)e^{\frac{\delta^{-4(n+1)}}{2n+2}E_k-E_k\frac{\delta^{-2(n+1)}}{n+1}}\leq\|v_k\|_\infty e^{\delta^{-4(n+1)}\frac{E_k}{2n+2}}.$$

On the other side if $t\in[0,Z_0]$ we have

$$v_k(t)\leq\|v_k\|_\infty e^{B\delta^{-4(n+1)}E_k}.$$

Hence we have obtained the estimate

$$|v_k(t)|\leq C_0^{k+1}e^{-Bt^{2(n+1)}}.$$

where C_0, B are suitable positive constants independent of k. □

Proposition 5.1.10. *The following estimate holds*

$$|v_k'(t)|\leq C_1C_0^ke^{-B't^{2n+2}}, \tag{5.1.19}$$

where $0<B'<B$, B, C_0 *are given in Proposition 5.1.9 and* C_1 *suitable independent of* k *and such that* $C_1=\mathscr{O}((B-B')^{-2})$.

Proof. Let $t>T_k$. By the Mean-Value Theorem we have

$$|v_k(t+1)-v_k(t)|=|v_k'(t_1)|,\qquad t<t_1<t+1.$$

Since both $|v_k'(t)|$ and $|v_k(t)|$ tend to zero as $t\to\infty$, we have

$$|v_k'(t+1)|\leq|v_k'(t_1)|\leq|v_k(t)|+|v_k(t+1)|\leq 2|v_k(t)|.$$

Arguing as above we have

$$|v_k'(t+1) - v_k'(t)| = |v_k''(t_2)| = |t_2^{2n}(E_k - t_2^{2n+2})v_k(t_2)|, \qquad t < t_2 < t+1.$$

For $t \geq T_k$ we obtain

$$\begin{aligned}|v_k'(t)| &\leq |v_k'(t+1)| + |t_2^{2n}(E_k - t_2^{2n+2})v_k(t_2)| \\ &\leq |v_k(t)| \left(2 + t^{4n+2}(E_k t^{-(2n+2)} + 1)\right) \leq 2|v_k(t)| \left(1 + t^{4n+2}\right).\end{aligned}$$

We apply the result in Proposition 5.1.9:

$$|v_k'(t)| \leq 2\left(1 + t^{4n+2}\right) C_0^{k+1} e^{-Bt^{2n+2}}.$$

Then we may conclude that

$$|v_k'(t)| \leq C_1 C_0^k e^{-B' t^{2n+2}},$$

where $0 < B' < B$ and $C_1 = \mathscr{O}((B - B')^{-2})$. This proves the assertion. □

We consider $v_k(z)$, $z \in \mathbb{C}$, which is the entire continuation of $v_k(t)$. We are interested to the *order* of $v_k(z)$. We recall that an entire function $f(z)$ is of finite order if there exist C, $\alpha > 0$ such that

$$|f(z)| \leq e^{|z|^\alpha}, \qquad \forall\, |z| \geq C.$$

We have

Proposition 5.1.11. *$v_k(z)$ has order $2n+2$, is of finite type, and the following estimate holds*

$$|v_k(z)| \leq C^{k+1} e^{c_1 |z|^{2n+2}}, \tag{5.1.20}$$

for some c_1 and C positive, independent of k.

Proof. We set $f_0(z) = f_0 = v_k(0)$ and

$$f_\ell(z) = f_0(z) + \int_0^z Q_k(\zeta) f_{\ell-1}(\zeta)(z-\zeta)d\zeta \qquad \ell = 1, 2, \dots .$$

We have

$$\begin{aligned}|f_1(z) - f_0(z)| &= \left| \int_0^z Q_k(\zeta) f_0(\zeta)(z-\zeta)d\zeta \right| \\ &= \left| \int_0^z Q_k(\zeta)\left(v_k(0) + \zeta v_k'(0)\right)(z-\zeta)dz \right| \leq S\frac{\tilde{Q}_k(z)|z|^2}{2},\end{aligned}$$

where $S = |v_k(0)| = |f_0|$ and $\tilde{Q}_k(z) = |z|^{2n}\left(|z|^{2n+2} + E_k\right)$. Using the above estimate at the step two we have

$$|f_2(z) - f_1(z)| = \left| \int_0^z Q_k(\zeta)\left(f_1(\zeta) - f_0(\zeta)\right)(z-\zeta)d\zeta \right| \leq S\frac{\tilde{Q}^2(z)|z|^4}{4!}.$$

By induction argument we obtain

$$|f_\ell(z) - f_{\ell-1}(z)| \leq S\frac{\tilde{Q}^\ell(z)|z|^{2\ell}}{(2\ell)!}, \qquad \forall \ell \geq 2.$$

Due to the above estimate we have that the series $\sum_{\ell\geq 1}\left(f_\ell(z)-f_{\ell-1}(z)\right)$ converges uniformly on compact sets and the term by term differentiation is permitted. Hence we may take a term by term second derivative of it. We consider the function

$$f(z)=f_0(z)+\sum_{\ell=1}^{\infty}\left(f_\ell(z)-f_{\ell-1}(z)\right).$$

We have that

$$f''(z)=Q_k(z)f(z).$$

$f(z)$ satisfies our equation with the same initial conditions at $z=0$ as $v_k(z)$, so that

$$v_k(z)=f_0(z)+\sum_{\ell=1}^{\infty}\left(f_\ell(z)-f_{\ell-1}(z)\right).$$

If $|z|\geq cE_k^{1/(2n+2)}$ we have

$$|v_k(z)|\leq S\left(1+\sum_{\ell=1}^{\infty}\frac{\left(c_1|z|^{2n+2}\right)^{2\ell}}{(2\ell)!}\right)=S\cosh(c_1|z|^{2n+2})\leq Ce^{c_1|z|^{2n+2}},$$

where $c_1\geq 1+c^{-2(n+1)}$ and $C>0$ does not depend on k.
On the other side since $v_k(z)$ is a holomorphic function in the bounded domain $|z|\leq cE_k^{1/(2n+2)}$, its absolute value attains its maximum at some points on the boundary of the disc $|z|\leq cE_k^{1/(2n+2)}$:

$$|v_k(z)|\leq Ce^{c_2E_k}\leq C_2^{k+1}.$$

We conclude that

$$|v_k(z)|\leq C_2^{k+1}e^{c_1|z|^{2n+2}},$$

where C_2 is a suitable positive constant independent of k. This shows that the order of $v_k(z)$ is less or equal than $2n+2$; on the other hand Proposition 5.1.9 shows that the order is greater or equal than $2n+2$. □

Due to the Propositions 5.1.9 and 5.1.11 and by the Remark at page 220 of [14] we have

Theorem 5.1.12. *For any k the function $v_k(t)$ belongs to the Gel'fand-Shilov space $S^{\frac{1}{2n+2}}_{\frac{2n+1}{2n+2}}(\mathbb{R})$ and moreover satisfies the estimates*

$$|t^\alpha\partial_t^\beta v_k(t)|\leq C_v^{k+\alpha+\beta+1}\alpha!^{\frac{1}{2n+2}}\beta!^{\frac{2n+1}{2n+2}}. \tag{5.1.21}$$

Proof. The proof is contained in [14] and we sketch it here for convenience.
By estimates (5.1.18), (5.1.20) we may apply Theorem 1 of [14], p. 213, and conclude that on a domain of the complex plane of the form $|s|\leq K_1(1+|t|)$, K_1 a suitable positive constant independent of k, we have

$$|v_k(t+is)|\leq C_3^{k+1}e^{-B't^{2n+2}}, \tag{5.1.22}$$

where $C_3=\max\{C,C_0\}$ and B' differs as little as desired from B.
Applying Theorem 2 of [14], p. 216, because of the above estimate, we obtain that

$$|v_k(t+is)|\leq C_3^{k+1}e^{-B't^{2n+2}+c_2s^{2n+2}}, \tag{5.1.23}$$

for any $t+is=z\in\mathbb{C}$. Here c_2 is a constant depending on B', c_1 and K_1.

Finally we apply Theorem 3 of [14], p. 219, and obtain that

$$|\partial_t^\beta v_k(t)| \leq C_3^{k+1} B^\beta \beta^{\beta \frac{2n+1}{2n+2}} e^{-B'' t^{2n+2}}, \tag{5.1.24}$$

where B'' differs as little as desired from B' and B is a positive constant independent of k.
This concludes the proof of the theorem. □

5.2. **Technical Lemmas.** The following paragraph is devoted to list several technical results used in the first two Sections.

Lemma 5.2.1. *Let θ be a positive real number. Then for every positive integer p the following identity holds*

$$\left[\rho^{-\theta}\left(1-\theta+\rho\partial_\rho\right)\right]^p = \rho^{-\theta p} \sum_{\ell=1}^{p} (-\theta)^{l-1} b_{p,\ell} \left(1-\theta+\rho\partial_\rho\right)^{p+1-\ell}, \tag{5.2.1}$$

where the constants $b_{p,\ell}$ satisfy the estimate

$$b_{p,\ell} \leq 2^{p-1}(p-1)^\ell.$$

In particular $b_{p,1}=1$, $b_{p,2}=p(p-1)/2$, $b_{p,p}=(p-1)!$.

Proof. By induction we have that the coefficients $b_{p,\ell}$ satisfy the recurrence relations: $b_{p,1} = 1$, $b_{p,p} = b_{p-1,p-1}(p-1)$ and $b_{p,\ell} = b_{p-1,\ell} + (p-1)b_{p-1,\ell-1}$, $\ell = 3,\dots,p-1$, where $b_{p-1,\ell}$ are the coefficients of $(AB)^{p-1}$. By induction we have the estimate. □

Lemma 5.2.2. *Let ν be a positive integer number. Then the following identity holds*

$$(\rho\partial_\rho)^\nu = \sum_{i=1}^{\nu} d_{\nu,i}\, \rho^{\nu+1-i} \partial_\rho^{\nu+1-i},$$

where the constants $d_{\nu,i}$ satisfy the estimate

$$d_{\nu,i} \leq 2^{\nu+i}(\nu+1-i)^i.$$

In particular $d_{\nu,1} = 1 = d_{\nu,\nu}$, $d_{\nu,2} = \nu(\nu-1)/2$ and $d_{\nu,i}$, $i = 3,\dots,\nu-1$.

Proof. By induction we have that the coefficients $d_{\nu,i}$ satisfy the recurrence relations: $d_{\nu,1} = 1 = d_{\nu,\nu}$, $d_{\nu,2} = \nu(\nu-1)/2$ and $d_{\nu,i} = d_{\nu-1,i} + (\nu+1-i)d_{\nu-1,i-1}$, $i = 3,\dots,\nu-1$, where $d_{\nu-1,i}$ are the coefficients of $(\rho\partial_\rho)^{\nu-1}$. By induction we have the estimate. □

Lemma 5.2.3. *Let θ, f, q and γ real number, $\theta > 0$. Then for every integer p the following identity holds*

(5.2.2)

$$\begin{aligned}
\left[\rho^{-\theta}\left(1-\theta+\rho\partial_\rho\right)\right]^p \rho^q \big[t^f u(t,\rho)\big]_{|t=\rho^\gamma x} &= \rho^{q-\theta p}\left[t^f \sum_{i=0}^{p} \rho^{-i} \mathscr{P}_i(t\partial_t) \partial_\rho^{p-i} u(t,\rho)\right]_{|t=\rho^\gamma x} \\
&= \rho^{q-\theta p}\left[t^f\left(\partial_\rho^p + \frac{p}{\rho}\left(\frac{p+1}{2}(1-\theta) + q + \gamma f + \gamma t\partial_t\right)\partial_\rho^{p-1}\right.\right. \\
&\qquad \left.\left. + \sum_{i=2}^{p} \rho^{-i} \mathscr{P}_i(t\partial_t)\partial_\rho^{p-i}\right) u(t,\rho)\right]_{|t=\rho^\gamma x},
\end{aligned}$$

where

$$\mathscr{P}_i(t\partial_t) = \sum_{j=0}^{i} \mathscr{p}_{i,j}(t\partial_t)^j. \tag{5.2.3}$$

The coefficients $\mathscr{p}_{i,j}$, $i = 2, \ldots, p-1$, *are given by*

$$\gamma^j \sum_{\nu=j}^{i} \sum_{\mu=\nu}^{i} \frac{(p+\nu-\mu)!}{j!(\nu-j)!(p-\mu)!}(-\theta)^{\mu-\nu}(1-\theta+q+\gamma f)^{\nu-j} b_{p,\mu-\nu+1} d_{p-\mu,i-\mu+1}, \tag{5.2.4}$$

and

$$\mathscr{p}_{p,j} = \sum_{\nu=j}^{p} \binom{\nu}{j} (-\theta)^{\nu-j}(1-\theta+q+\gamma f)^{\nu-j} b_{p,p-\nu+1}. \tag{5.2.5}$$

The constants $b_{p,\mu-\nu+1}$ *and* $d_{p-\mu,i-\mu+1}$ *are the same of the previous Lemmas.*

Proof. We have

$$\begin{aligned}(1-\theta+\rho\partial_\rho)\, \rho^q \left[t^f u(t,\rho)\right]_{|t=\rho^\gamma x} &= \rho^q \left(1-\theta+q+\rho\partial_\rho\right) \left[t^f u(t,\rho)\right]_{|t=\rho^\gamma x} \\ &= \rho^q \left[t^f \left(1-\theta+q+\gamma f+\gamma t\partial_t+\rho\partial_\rho\right) u(t,\rho)\right]_{|t=\rho^\gamma x}.\end{aligned}$$

Applying the Lemma 5.2.1 we can rewrite the left hand side of (5.2.2) in the following form

$$\begin{aligned}&\rho^{-\theta p} \sum_{\ell=1}^{p} (-\theta)^{l-1} b_{p,\ell} \left(1-\theta+\rho\partial_\rho\right)^{p+1-\ell} \rho^q \left[t^f u(t,\rho)\right]_{|t=\rho^\gamma x} \\ &= \rho^{q-\theta p} \left[t^f \sum_{\ell=1}^{p} (-\theta)^{l-1} b_{p,\ell} \left(1-\theta+q+\gamma f+\gamma t\partial_t+\rho\partial_\rho\right)^{p+1-\ell} u(t,\rho)\right]_{|t=\rho^\gamma x}.\end{aligned} \tag{5.2.6}$$

We observe that if Q_1 and Q_2 be two operators such that $[Q_1, Q_2] = 0$ for every positive integer ν we have

$$(Q_1+Q_2)^\nu = \sum_{i=0}^{\nu} \binom{\nu}{i} Q_1^i Q_2^{\nu-i}.$$

Taking advantage from the above formula and from the Lemma 5.2.2 we obtain (5.2.2) from (5.2.6). □

Lemma 5.2.4. *Let* Θ_k *be the ordinary differential equation of order* $2m$, $m \in \mathbb{Z}_+$, *given by*

$$\Theta_k = \left(\frac{d}{d\rho}\right)^{2m} + \left(\frac{2mi}{2m-1}\right)^{2m} E_k, \tag{5.2.7}$$

where $E_k = 4k(n+1)+2n+1$ *is the eigenvalue of the eigenfunction* $v_k(t)$, *(see (5.1.2).) We shall use the following fundamental solution of* Θ_k

$$G_k(\rho) = (ic_k)^{-(2m-1)} \frac{1}{m} \sum_{j=0}^{[\frac{m-1}{2}]} 2^{-[\sin\theta_j]} e^{-c_k \sin\theta_j |\rho|} \sin\left(c_k|\rho|\cos\theta_j + \theta_j\right), \tag{5.2.8}$$

where $c_k = 2m(2m-1)^{-1} E_k^{1/2m}$, $\theta_j = \frac{\pi}{2m}(1+2j)$, $j = 0, 1, \ldots, 2m-1$ *and* $[x] = \max\{n \in \mathbb{Z}_+ \mid n \leq x\}$, $x \geq 0$. *We point out that when* m *is odd the*

summand in (5.2.8) corresponding to $j = \frac{m-1}{2}$ *is* $e^{-c_k|\rho|}$ *and* $[\sin\theta_j] = 0$ *if* $j < \frac{m-1}{2}$, $[\sin\theta_j] = 1$ *when* $j = \frac{m-1}{2}$.
Moreover we have

$$|G_k(\rho)| \le c_k^{-(2m-1)} e^{-c_k \sin(\pi/2m)|\rho|}. \tag{5.2.9}$$

Moreover for every ℓ, $0 \le \ell \le 2m-1$, *we have*
(5.2.10)

$$G_k^{(\ell)}(\rho) = i^{-(2m-1)} c_k^{-(2m-1-\ell)} \frac{1}{m} \sum_{j=0}^{[\frac{m-1}{2}]} 2^{-[\sin\theta_j]} e^{-c_k \sin\theta_j |\rho|} \cos\left(c_k|\rho|\cos\theta_j + (\ell+1)\theta_j\right) \operatorname{sign}(\rho),$$

if ℓ *is odd and*
(5.2.11)

$$G_k^{(\ell)}(\rho) = i^{-(2m-1)} c_k^{-(2m-1-\ell)} \frac{1}{m} \sum_{j=0}^{[\frac{m-1}{2}]} e^{-c_k \sin\theta_j |\rho|} \sin\left(c_k|\rho|\cos\theta_j + (\ell+1)\theta_j\right),$$

if ℓ *is even.*

Proof. Using e.g. the residue theorem we have that

$$G_k(\rho) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\rho\sigma} \frac{(-1)^m}{\sigma^{2m} + c_k^{2m}} d\sigma.$$

If $\rho \ge 0$ we have

$$G_k(\rho) = (ic_k)^{-(2m-1)}\, i \sum_{j=0}^{m-1} \frac{1}{2m\ e^{i(2m-1)\theta_j}} e^{ic_k e^{i\theta_j}\rho}.$$

If $\rho \le 0$ we have

$$G_k(\rho) = -(ic_k)^{-(2m-1)}\, i \sum_{j=m}^{2m-1} \frac{1}{2m\ e^{i(2m-1)\theta_j}} e^{ic_k e^{i\theta_j}\rho}.$$

Since $\theta_{\ell+m} = \theta_\ell + \pi$ for $0 \le \ell \le m-1$, we deduce that

$$G_k(\rho) = (ic_k)^{-(2m-1)}\, i \sum_{j=0}^{m-1} \frac{1}{2m\ e^{i(2m-1)\theta_j}} e^{ic_k e^{i\theta_j}|\rho|}. \tag{5.2.12}$$

Further we have that $e^{i(2m-1)\theta_j} = -e^{-i\theta_j}$ for every j, $0 \le j \le 2m-1$, and $\theta_{m-1-\ell} = \pi - \theta_\ell$. Plugging this into the expression of G_k we, if m is even, can sum the j-th summand and the $(m-1-j)$-th summand obtaining that

$$\begin{aligned}
&- e^{i\theta_j} e^{ic_k e^{i\theta_j}|\rho|} - e^{i\theta_{m-1-j}} e^{ic_k e^{i\theta_{m-1-j}}|\rho|} \\
&\qquad = -e^{i\theta_j} e^{ic_k e^{i\theta_j}|\rho|} + e^{-i\theta_j} e^{-ic_k e^{-i\theta_j}|\rho|} \\
&\qquad = -e^{i\theta_j + ic_k\cos\theta_j|\rho|} e^{-c_k\sin\theta_j|\rho|} + e^{-i\theta_j - ic_k\cos\theta_j} e^{-c_k\sin\theta_j|\rho|} \\
&\qquad = -2ie^{-c_k\sin\theta_j|\rho|}\left(\sin\theta_j \cos\left(c_k\cos\theta_j|\rho|\right) + \cos\theta_j \sin\left(c_k\cos\theta_j|\rho|\right)\right) \\
&\qquad = -2ie^{-c_k\sin\theta_j|\rho|} \sin\left(c_k\cos\theta_j|\rho| + \theta_j\right).
\end{aligned}$$

Plugging this into (5.2.12) and taking into account that the number of summands is even, we obtain

$$G_k(\rho) = (ic_k)^{-(2m-1)}\,\frac{1}{m}\sum_{j=0}^{\frac{m}{2}-1} e^{-c_k\sin\theta_j|\rho|}\sin\left(c_k\cos\theta_j|\rho|+\theta_j\right),$$

which is (5.2.8) for m even.

Consider now the case of m odd. The number of summands in (5.2.12) is odd, so that we may proceed as above for all but one summand, obtained when $j=\frac{m-1}{2}$. Now for this value of j, $\theta_j=\frac{\pi}{2}$, which implies that the corresponding summand is written as

$$(ic_k)^{-(2m-1)}\,\frac{1}{2m}e^{-c_k|\rho|}$$

which gives (5.2.8). Deriving (5.2.9) is straightforward.

Let us now turn to the derivatives of G_k when m is odd. We have

$$\begin{aligned}
\frac{d}{d\rho}G_k(\rho) &= (ic_k)^{-(2m-1)}\frac{1}{m}\sum_{j=0}^{[\frac{m-1}{2}]} 2^{-[\sin\theta_j]}e^{-c_k\sin\theta_j|\rho|}\left[-c_k\sin\theta_j\sin\left(c_k|\rho|\cos\theta_j+\theta_j\right)\right.\\
&\qquad \left.+c_k\cos\theta_j\cos\left(c_k|\rho|\cos\theta_j+\theta_j\right)\right]\operatorname{sign}(\rho)\\
&= i^{-(2m-1)}c_k^{-(2m-2)}\frac{1}{m}\sum_{j=0}^{[\frac{m-1}{2}]} 2^{-[\sin\theta_j]}e^{-c_k\sin\theta_j|\rho|}\cos\left(c_k|\rho|\cos\theta_j+2\theta_j\right)\operatorname{sign}(\rho),
\end{aligned}$$

which is (5.2.10) when $\ell=1$. Again

$$\begin{aligned}
\frac{d^2}{d\rho^2}G_k(\rho) &= i^{-(2m-1)}c_k^{-(2m-2)}\frac{1}{m}\sum_{j=0}^{[\frac{m-1}{2}]} 2^{-[\sin\theta_j]}e^{-c_k\sin\theta_j|\rho|}\left[-c_k\sin\theta_j\cos\left(c_k|\rho|\cos\theta_j+2\theta_j\right)\right.\\
&\qquad \left.-c_k\cos\theta_j\sin\left(c_k|\rho|\cos\theta_j+2\theta_j\right)\right]\\
&\quad + i^{-(2m-1)}c_k^{-(2m-2)}\frac{1}{m}\sum_{j=0}^{[\frac{m-1}{2}]} 2^{-[\sin\theta_j]}e^{-c_k\sin\theta_j|\rho|}\cos\left(c_k|\rho|\cos\theta_j+2\theta_j\right)2\delta(\rho)\\
&= i^{-(2m-3)}c_k^{-(2m-3)}\frac{1}{m}\sum_{j=0}^{[\frac{m-1}{2}]} 2^{-[\sin\theta_j]}e^{-c_k\sin\theta_j|\rho|}\sin\left(c_k|\rho|\cos\theta_j+3\theta_j\right)\\
&\qquad + i^{-(2m-1)}c_k^{-(2m-2)}\frac{2}{m}\sum_{j=0}^{[\frac{m-1}{2}]} 2^{-[\sin\theta_j]}\cos\left(2\theta_j\right)\delta(\rho).
\end{aligned}$$

The last sum in the above formula gives zero:

$$\sum_{j=0}^{[\frac{m-1}{2}]} 2^{-[\sin\theta_j]}\cos\left(2\theta_j\right)=0.$$

In fact we have—we recall that m is odd and that $\theta_{\frac{m-1}{2}}=\frac{\pi}{2m}(1+2\frac{m-1}{2})=\frac{\pi}{2}$—

$$\sum_{j=0}^{\frac{m-1}{2}} 2^{-[\sin\theta_j]}\cos\left(2\theta_j\right)=\sum_{j=0}^{\frac{m-3}{2}}\cos\left(2\theta_j\right)+\frac{1}{2}\cos(2\theta_{\frac{m-1}{2}})$$

$$= -\frac{1}{2} + \Re\left(e^{i\frac{\pi}{m}}\sum_{j=0}^{\frac{m-3}{2}} e^{i\frac{2\pi}{m}j}\right) = -\frac{1}{2} + \Re\left(e^{i\frac{\pi}{m}}\frac{1-e^{i\pi\frac{m-1}{m}}}{1-e^{i\frac{2\pi}{m}}}\right)$$

$$= -\frac{1}{2} + \Re\left(\frac{1+e^{-i\frac{\pi}{m}}}{e^{-i\frac{\pi}{m}}-e^{i\frac{\pi}{m}}}\right) = -\frac{1}{2} + \Re\frac{1+\cos\frac{\pi}{m} - i\sin\frac{\pi}{m}}{-2i\sin\frac{\pi}{m}} = 0.$$

This proves (5.2.11) for $\ell = 2$. For ℓ odd we see that the ℓ-th derivative is a multiple of $\operatorname{sign}(\rho)$ and (5.2.10) can be derived with an argument completely analogous to the above. Assume that ℓ is even. Then the ℓ-th derivative is the sum of a term of the form (5.2.11) and a multiple of the distribution δ. The latter contains the sum

$$\sum_{j=0}^{\frac{m-1}{2}} 2^{-[\sin\theta_j]}\cos(\ell\theta_j).$$

This is zero since arguing as above we isolate the summand corresponding to $j = \frac{m-1}{2}$ obtaining the value $\frac{(-1)^{\frac{\ell}{2}}}{2}$. The remaining sum then gives the opposite using the same argument as above.

This proves (5.2.11) and (5.2.10) when m is odd.

Assume m even. Then the factor $2^{-[\sin\theta_j]} \equiv 1$. Again, arguing as above, we find that the distribution δ has a factor containing the sum

$$\sum_{j=0}^{\frac{m}{2}-1} \cos(\ell\theta_j) = 0.$$

This completes the proof of the lemma.

□

Lemma 5.2.5. *Let a and b be two positive constants such that $a > b$. Then there is a positive constant C_0 such that for every $R \geq C_0(j+1)$, $j \in \mathbb{Z}_+$, the following inequality holds*

$$\int_R^{+\infty} \frac{1}{\tau^j} e^{-a|\rho-\tau|-b\tau} d\tau \leq \left(\frac{1}{a} + \frac{2}{a-b}\right)\frac{e^{-b\rho}}{\rho^j}. \tag{5.2.13}$$

Proof. We write the right hand side of (5.2.13) as

$$\int_R^{\rho} e^{a\tau - a\rho}\frac{1}{\tau^j}e^{-b\tau}d\tau + \int_\rho^{+\infty} e^{-a\tau+a\rho}\frac{1}{\tau^j}e^{-b\tau}d\tau \doteq (I) + (II).$$

Since $\rho < \tau$ then $\rho^{-1} > \tau^{-1}$ and $e^{-b\tau} < e^{-b\rho}$ we have

$$(II) \leq \frac{e^{-b\rho}}{\rho^j}\int_\rho^{+\infty} e^{-a(\tau-\rho)}d\tau = \frac{e^{-b\rho}}{a\rho^j}\int_0^{+\infty} e^{-s}ds = \frac{e^{-b\rho}}{a\rho^j}.$$

Let us estimate of term (I). We take $\tau^{-j} = e^{-j\log\tau}$ and we consider the function $f(\tau) = 2^{-1}(a\tau - b\tau) - j\log\tau$. Since $f^{(1)}(\tau) = 2^{-1}(a-b) - j\tau^{-1}$, $f(\tau)$ is decreasing function for $\tau < 2(a-b)^{-1}j$ and increasing function for $\tau > 2(a-b)^{-1}j$. Since $e^{\inf f(\tau)} \leq e^{f(\tau)} \leq e^{\sup f(\tau)}$ taking $R > 2(a-b)^{-1}j$ so that $f(\tau)$ is increasing function in the region $[R,\rho]$ we have that $\sup f(\tau) = f(\rho)$. We have

$$e^{f(\tau)} \leq \frac{1}{\rho^j}e^{\frac{(a-b)}{2}\rho}.$$

For $R > 2(a-b)^{-1}j$ we have

$$(I) \leq \frac{1}{\rho^j} e^{-\frac{(a+b)}{2}\rho} \int_R^\rho e^{\frac{(a-b)}{2}\tau} d\tau \leq \frac{1}{\rho^j} e^{-\frac{(a+b)}{2}\rho} \int_0^\rho e^{\frac{(a-b)}{2}\tau} d\tau$$
$$= \frac{1}{\rho^j} e^{-\frac{(a+b)}{2}\rho} \frac{2}{a-b} \left(e^{\frac{(a-b)}{2}\rho} - 1 \right) \leq \frac{2}{a-b} \frac{1}{\rho^j} e^{-b\rho}.$$

Summing up we obtain (5.2.13). □

Remark 5.2.6. *Setting $a = c_k \sin(\pi/2m) = 2m(2m-1)^{-1} E_k^{1/2m} \sin(\pi/2m)$ and $b = c_0 \sin(\pi/2m) = 2m(2m-1)^{-1} E_0^{1/2m} (\sin \pi/2m)$, $E_k = 4k(n+1) + 2n + 1$, $k \in \mathbb{Z}_+$, in the above Lemma we have*

$$(5.2.14) \quad \int_R^{+\infty} \frac{1}{\tau^j} e^{-c_k \sin(\pi/2m)|\rho-\tau| - c_0 \sin(\pi/2m)\tau} d\tau \leq \frac{C}{(k+1)^{1/2m} \rho^j} e^{-c_0 \sin(\pi/2m)\rho},$$

for $R \geq C_0(j+1)$. Here C denotes a suitable constant depending only on m; C_0 can be chosen greater than two and independent of j and k.

Lemma 5.2.7. *Let $G_0(\rho)$ be the fundamental solution of Θ_0 used in Lemma 5.2.4. Then there is a positive constant C_0 greater than two such that for every $R \geq C_0(j+1)$, $j \in \mathbb{Z}_+$, we have*

$$(5.2.15) \quad \left| \int_R^{+\infty} G_0(\rho-\tau) \frac{1}{\tau^{j+1}} e^{-c_0 \lambda_0'' \tau} d\tau - h_0(\rho) - h_{m-1}(\rho) \right| \leq C_1 \frac{e^{-c_0 \lambda_0'' \rho}}{\rho^j (j+1)},$$

where C_1 is a positive constant independent by j, $c_0 = 2m(2m-1)^{-1} E_0^{1/2m}$ and

$$h_0(\rho) = (ic_0)^{-(2m-1)} \frac{1}{2im} e^{i(c_0 \lambda_0 \rho + \theta_0)} \int_R^{+\infty} e^{ic_0 \lambda_0' \tau} \frac{1}{\tau^{j+1}} d\tau,$$
$$h_{m-1}(\rho) = (ic_0)^{-(2m-1)} \frac{1}{2im} e^{i(c_0 \lambda_{m-1} \rho + \theta_{m-1})} \int_R^{+\infty} e^{ic_0 \lambda_{m-1}' \tau} \frac{1}{\tau^{j+1}} d\tau,$$

with $\lambda_0 = \lambda_0' + i\lambda_0'' = \cos\theta_0 + i \sin\theta_0$, $\theta_0 = \frac{\pi}{2m}$, $\lambda_{m-1} = \lambda_{m-1}' + i\lambda_{m-1}'' = \cos\theta_{m-1} + i \sin\theta_{m-1}$, $\theta_{m-1} = \frac{\pi(2m-1)}{2m}$.

Proof. We recall that

$$G_0(\rho) = (ic_0)^{-(2m-1)} \frac{1}{m} \sum_{k=0}^{[\frac{m-1}{2}]} 2^{-[\sin\theta_k]} e^{-c_0 \sin\theta_k |\rho|} \sin\left(c_0 |\rho| \cos\theta_k + \theta_k\right).$$

We have

$$(5.2.16) \quad \int_R^{+\infty} G_0(\rho-\tau) \frac{1}{\tau^{j+1}} e^{-c_0 \lambda_0'' \tau} d\tau = (ic_0)^{-(2m-1)} \frac{1}{m}$$
$$\times \sum_{k=0}^{[\frac{m-1}{2}]} 2^{-[\sin\theta_k]} \int_R^{+\infty} e^{-c_0 \sin\theta_k |\rho-\tau|} \sin\left(c_0 |\rho-\tau| \cos\theta_k + \theta_k\right) \frac{1}{\tau^{j+1}} e^{-c_0 \lambda_0'' \tau} d\tau.$$

We begin to handle the term $k = 0$. Since

$$e^{-c_0 \sin\theta_0 |\rho|} \sin\left(c_0 \cos\theta_0 |\rho| + \theta_0\right)$$
$$= e^{-c_0 \sin\theta_0 |\rho|} \left(\sin\theta_0 \cos\left(c_0 \cos\theta_0 |\rho|\right) + \cos\theta_0 \sin\left(c_0 \cos\theta_0 |\rho|\right) \right)$$
$$= \frac{1}{2i} \left(e^{i\theta_0 + ic_0 \cos\theta_0 |\rho|} e^{-c_0 \sin\theta_0 |\rho|} - e^{-i\theta_0 - ic_0 \cos\theta_0} e^{-c_0 \sin\theta_0 |\rho|} \right)$$

$$\frac{1}{2i}\left(e^{i\theta_0}e^{ic_0e^{i\theta_0}|\rho|}-e^{-i\theta_0}e^{-ic_0e^{-i\theta_0}|\rho|}\right)$$
$$=\frac{1}{2i}\left(e^{i\theta_0}e^{ic_0e^{i\theta_0}|\rho|}+e^{i\theta_{m-1}}e^{ic_0e^{i\theta_{m-1}}|\rho|}\right)$$

$$=\frac{1}{2i}\left(e^{i\theta_0}e^{ic_0\lambda_0|\rho|}+e^{i\theta_{m-1}}e^{ic_0\lambda_{m-1}|\rho|},\right).$$

we have

$$\begin{aligned}
&\int_R^{+\infty}e^{-c_0\sin\theta_0|\rho-\tau|}\sin\left(c_0|\rho-\tau|\cos\theta_0+\theta_0\right)\frac{1}{\tau^{j+1}}e^{-c_0\lambda_0''\tau}d\tau\\
&=\frac{1}{2i}\left(e^{i\theta_0}\int_R^{+\infty}e^{ic_0\lambda_0|\rho-\tau|}\frac{1}{\tau^{j+1}}e^{-c_0\lambda_0''\tau}d\tau+e^{i\theta_{m-1}}\int_R^{+\infty}e^{ic_0\lambda_{m-1}|\rho-\tau|}\frac{1}{\tau^{j+1}}e^{-c_0\lambda_0''\tau}d\tau\right)\\
&=\frac{1}{2i}\left[e^{i\theta_0}\left(\int_R^{\rho}e^{ic_0\lambda_0(\rho-\tau)}\frac{1}{\tau^{j+1}}e^{-c_0\lambda_0''\tau}d\tau+\int_\rho^{+\infty}e^{ic_0\lambda_0(\tau-\rho)}\frac{1}{\tau^{j+1}}e^{-c_0\lambda_0''\tau}d\tau\right)\right.\\
&\left.+e^{i\theta_{m-1}}\left(\int_R^{\rho}e^{ic_0\lambda_{m-1}(\rho-\tau)}\frac{1}{\tau^{j+1}}e^{-c_0\lambda_0''\tau}d\tau+\int_\rho^{+\infty}e^{ic_0\lambda_{m-1}(\tau-\rho)}\frac{1}{\tau^{j+1}}e^{-c_0\lambda_0''\tau}d\tau\right)\right]\\
&=\frac{1}{2i}\left[e^{i(c_0\lambda_0\rho+\theta_0)}\left(\int_R^{+\infty}\frac{e^{-ic_0\lambda_0'\tau}}{\tau^{j+1}}d\tau+\int_\rho^{+\infty}\frac{e^{-ic_0\lambda_0'\tau}}{\tau^{j+1}}d\tau+\int_\rho^{+\infty}\frac{e^{ic_0\lambda_0'\tau}}{\tau^{j+1}}e^{-2c_0\lambda_0''\tau}d\tau\right)\right.\\
&\qquad+e^{i(c_0\lambda_{m-1}\rho+\theta_{m-1})}\left(\int_R^{+\infty}\frac{e^{-ic_0\lambda_{m-1}'\tau}}{\tau^{j+1}}d\tau+\int_\rho^{+\infty}\frac{e^{-ic_0\lambda_{m-1}'\tau}}{\tau^{j+1}}d\tau\right.\\
&\qquad\qquad\left.\left.+\int_\rho^{+\infty}\frac{e^{ic_0\lambda_{m-1}'\tau}}{\tau^{j+1}}e^{-2c_0\lambda_0''\tau}d\tau\right)\right],
\end{aligned}$$

where we used that $\lambda_{m-1}'=-\lambda_0'$ and $\lambda_{m-1}''=\lambda_0''$. We obtain that

$$\begin{aligned}
&(ic_0)^{-(2m-1)}\frac{1}{m}\int_R^{+\infty}e^{-c_0\sin\theta_0|\rho-\tau|}\sin\left(c_0|\rho-\tau|\cos\theta_0+\theta_0\right)\frac{1}{\tau^{j+1}}e^{-c_0\lambda_0''\tau}d\tau\\
&=h_0(\rho)+h_{m-1}(\rho)\\
&+(ic_0)^{-(2m-1)}\frac{1}{2im}\left[e^{i(c_0\lambda_0\rho+\theta_0)}\left(\int_\rho^{+\infty}\frac{e^{-ic_0\lambda_0'\tau}}{\tau^{j+1}}d\tau+\int_\rho^{+\infty}\frac{e^{ic_0\lambda_0'\tau}}{\tau^{j+1}}e^{-2c_0\lambda_0''\tau}d\tau\right)\right.\\
&\qquad\left.+e^{i(c_0\lambda_{m-1}\rho+\theta_{m-1})}\left(\int_\rho^{+\infty}\frac{e^{-ic_0\lambda_{m-1}'\tau}}{\tau^{j+1}}d\tau+\int_\rho^{+\infty}\frac{e^{ic_0\lambda_{m-1}'\tau}}{\tau^{j+1}}e^{-2c_0\lambda_0''\tau}d\tau\right)\right]\\
&\qquad\qquad=h_0(\rho)+h_{m-1}(\rho)+(I)
\end{aligned}$$

The absolute value of (I) can be estimated by

$$C'\frac{e^{-c_0\lambda_0''\rho}}{\rho^j j},$$

where C' is a positive constant independent by j.
Concerning the other terms in sum (5.2.16),

$$\int_R^{+\infty}e^{-c_0\sin\theta_k|\rho-\tau|}\sin\left(c_0|\rho-\tau|\cos\theta_k+\theta_k\right)\frac{1}{\tau^{j+1}}e^{-c_0\lambda_0''\tau}d\tau,\quad k=2,\dots,[\frac{m-1}{2}],$$

we can apply the Lemma 5.2.5. Since $\sin\theta_k > \sin\theta_0 = \lambda_0''$ the absolute value of the above quantity can be bonded by

$$\int_R^{+\infty} e^{-c_0 \sin\theta_k|\rho-\tau|}\frac{1}{\tau^{j+1}}e^{-c_0\lambda_0''\tau}d\tau \le C''\frac{e^{-c_0\lambda_0''\rho}}{\rho^{j+1}},$$

where C'' is a positive constant independent by j.
Putting the above considerations together we obtain the (5.2.15). □

Remark 5.2.8. *The functions $h_0(\rho)$ and $h_{m-1}(\rho)$ are both solutions of the problem $\Theta_0 h = 0$.*

Lemma 5.2.9. *Let a, s be two positive real numbers and $j \in \mathbb{Z}_+$. Then for every $\varepsilon > 0$ there exist a positive constant C_ε, independent of j, such that*

$$\int_0^{+\infty} e^{-a\rho\log\rho}\rho^{sj}d\rho \le C_\varepsilon \varepsilon^j j^{js}. \tag{5.2.17}$$

Proof. We have

$$\int_0^{+\infty} e^{-a\rho\log\rho}\rho^{sj}d\rho = \int_0^{+\infty} e^{-a\rho\log\rho+c\rho}\rho^{sj}e^{-(c-1)\rho}e^{-\rho}d\rho,$$

where c is a positive real number such that $s(2^{-1}\varepsilon)^{-1/s}+1 > c > s\varepsilon^{-1/s}+1$. We have

$$\sup_\rho \rho^{sj}e^{-(c-1)\rho} = \left[\left(\frac{s}{c-1}\right)^s\right]^j e^{-js}j^{sj} \le \varepsilon^j j^{js}.$$

On the other hand $f(\rho) = -a\rho\log\rho + c\rho$ take its maximum in $\rho_0 = e^{(c-a)/a}$. We have that $e^{f(\rho_0)} \le \exp(ae^{(s(2\varepsilon^{1/s})^{-1}-a+1)a^{-1}}) \doteq C$. Setting $C = C_\varepsilon$ we have the assertion. □

Lemma 5.2.10. *Let s_1 and s_2 two positive real number such that $s_1 < s_2$ then $G^{s_1}(\Omega) \subset \gamma^{s_2}(\Omega)$.*

Proof. We have that $AC^{|\alpha|}|\alpha|^{s|\alpha|} = AC^{|\alpha|}|\alpha|^{(s_1-s_2)|\alpha|}|\alpha|^{s_2|\alpha|}$. We want to see that

$$C^{|\alpha|}|\alpha|^{(s_1-s_2)|\alpha|} \le C_\varepsilon \varepsilon^{|\alpha|} \qquad \forall\,\varepsilon > 0.$$

We observe that since $s_2 > s_1$ than $C^{|\alpha|}|\alpha|^{(s_1-s_2)|\alpha|} \to 0$ for $|\alpha| \to +\infty$.
We take the logarithm of both side of above inequality

$$|\alpha|\log\left(\frac{C}{\varepsilon}\right) - (s_2-s_1)\log|\alpha| \le \log C_\varepsilon.$$

Consider the function $f(t) = t\log(C/\varepsilon) - (s_2-s_1)t\log t$ in $(1,+\infty)$. We have $f'(t) = 0$ for $t_0 = e^{-1}(C/\varepsilon)^{1/(s_2-s_1)}$. Setting $C_\varepsilon = \exp(f(t_0))$ we have the assertion. □

Declarations

The author states that there is no conflict of interest.

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