Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

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Tome 72, n° 6 (2022), p. 2191-2240.

https://doi.org/10.5802/aif.3503

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte

www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
72, 6 (2022) 2191-2240

MODULI OF G-COVERS OF CURVES: GEOMETRY AND SINGULARITIES

by Mattia GALEOTTI

Abstract. — We analyze the singular locus and the locus of non-canonical singularities of the moduli space $\overline{\mathcal{R}}_{g,G}$ of curves with a G-cover for any finite group G. We show that non-canonical singularities are of two types: T-curves, that is singularities lifted from the moduli space $\overline{\mathcal{M}}_g$ of stable curves, and J-curves, that is new singularities entirely characterized by the dual graph of the cover. Finally, we prove that in the case $G = S_3$, the J-locus is empty, which is the first fundamental step in evaluating the Kodaira dimension of $\overline{\mathcal{R}}_{g,S_3}$.

Résumé. — Nous analysons le lieu singulier et le lieu des singularités non-canoniques de l'espace de modules $\overline{\mathcal{R}}_{g,G}$ des courbes avec un G-recouvrement où G est un groupe fini. Nous montrons que les singularités non canoniques sont de deux types: T-courbes, c'est-à-dire des singularités relevées de l'espace de modules $\overline{\mathcal{M}}_g$ des courbes stables, et J-courbes, c'est-à-dire des singularités nouvelles caractérisées entièrement par le graphe dual du recouvrement. Enfin, nous prouvons que dans le cas $G = S_3$, le lieu J est vide, une première étage très importante dans l'évaluation de la dimension de Kodaira de $\overline{\mathcal{R}}_{g,S_3}$

1. Introduction

This is the first of two papers whose goal is to analyze the birational geometry of the moduli space of curves equipped with a G-cover, where G is any finite group. In particular we focus on the case $G = S_3$, the symmetric group of order 3.

The moduli space $\mathcal{M}_g$ of smooth curves of genus g is a widely studied object along with its Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ described for the first time in [9]. This compactification is the moduli space of genus g stable curves, that is curves admitting nodal singularities and a finite

Keywords: Moduli of curves, G-covers, curves, stable curves, curve coverings, singularities, birational geometry.
2020 *Mathematics Subject Classification:* 14H10, 14H20, 14H60, 14D20, 14D23.

automorphism group. The birational geometry of $\overline{\mathcal{M}}_g$ was first approached by Eisenbud, Harris and Mumford [10, 15, 16], proving that it is a variety of general type for genus $g > 23$. The cases $g = 22,\ 23$ were recently solved by Farkas–Jensen–Payne [11], proving that $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ are of general type too.

The present work fits in the framework of finite covers of $\overline{\mathcal{M}}_g$, whose study is motivated by the fact that in many cases the transition to the general type happens for genus lower than 22. Farkas and Verra (see [12]) focused in the case of odd spin curves; Chiodo–Eisenbud–Farkas–Schreyer work [7] analyzes the moduli of curves with a 3-torsion bundle; in both cases the moduli space is of general type for $g \geqslant 12$. For this type of results is fundamental an analysis of the singular locus. This has been done by Chiodo and Farkas in [8] for curves with an ℓ-torsion bundle, also called level ℓ curves. In his work [13], the author generalized this analysis to the case of the moduli space $\mathcal{R}^k_{g,\ell}$ of curves with a line bundle L such that $L^{\otimes \ell} \cong \omega^{\otimes k}$.

Here we propose another generalization of Chiodo and Farkas approach, by treating curves with a G-cover for any finite group G, where the case of level ℓ curves is equivalent to $G = \boldsymbol{\mu}_\ell$ a cyclic group. In order to compactify the moduli space $\mathcal{R}_{g,G}$ of genus g smooth curves with a principal G-bundle, we introduce two notions of covers: twisted G-covers and admissible G-covers. Twisted covers are treated in [8] as balanced maps $\phi\colon \mathsf{C} \to BG$ where C is a twisted curve, that is a Deligne–Mumford stack whose coarse space is a stable curve and with non-trivial cyclic stabilizer at some nodes. For a wide introduction to twisted curves and their moduli see for example [1, 2, 6]. Admissible G-covers are principal G-bundles admitting ramification points over some nodes. The two cover notions are proved equivalent in [1], as recalled here in Theorem 2.38.

The main result we propose is the description of the singular locus $\operatorname{Sing} \overline{\mathcal{R}}_{g,G}$ and the non-canonical singular locus $\operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{R}}_{g,G}$. In particular, we are interested in characterizing the singularities outside the preimage of singular points of $\overline{\mathcal{M}}_g$. In order to achieve this, for any twisted G-cover (C, ϕ) we consider the group $\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$ of ghost automorphisms, *i.e.* C automorphisms lifting to ϕ and acting trivially on the coarse curve C. As any singularity of $\overline{\mathcal{R}}_{g,G}$ is a quotient singularity, there are some tools allowing its description, such as quasireflections (see Definition 4.9) and the age invariant, in particular via the notion of *junior* group (see Definition 5.2). We denote by $\mathrm{QR} \subset \underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$ the subgroup generated by quasireflections. Moreover, if $\pi\colon \overline{\mathcal{R}}_{g,G} \to \overline{\mathcal{M}}_g$ is the natural projection, we denote by

$N_{g,G} \coloneqq \operatorname{Sing} \overline{\mathcal{R}}_{g,G} \cap \pi^{-1} \operatorname{Sing} \overline{\mathcal{M}}_g$ and $T_{g,G} \coloneqq \operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{R}}_{g,G} \cap \pi^{-1} \operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{M}}_g$ the loci of singularities lifted from $\overline{\mathcal{M}}_g$. Theorems 4.20 and 5.8, summarized below, say that the "new" singularities are characterized by their ghost structure.

THEOREM. — *If $H_{g,G} \subset \overline{\mathcal{R}}_{g,G}$ is the locus of twisted G-covers (C, ϕ) such that $\underline{\operatorname{Aut}}_C(\mathsf{C}, \phi)$ is not generated by quasireflections, then*

$$\operatorname{Sing} \overline{\mathcal{R}}_{g,G} = N_{g,G} \cup H_{g,G}.$$

If $J_{g,G} \subset \overline{\mathcal{R}}_{g,G}$ is the locus such that $\underline{\operatorname{Aut}}_C(\mathsf{C}, \phi)/\operatorname{QR}$ is a junior group, then

$$\operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{R}}_{g,G} = T_{g,G} \cup J_{g,G}.$$

In order to approach the problem of evaluating the Kodaira dimension of $\overline{\mathcal{R}}_{g,G}$, a fundamental step is proving the pluricanonical form extension result, similarly to what has been done for $\overline{\mathcal{M}}_g$ in [16]. As last result we prove in Theorem 5.13 that the J-locus is empty for $G = S_3$, and this will be the starting point to the extension of pluricanonical forms over a desingularization of $\overline{\mathcal{R}}_{g,S_3}$, because it allows the generalization of Harris–Mumford techniques.

THEOREM. — *In the case of the symmetric group $G = S_3$, the J-locus J_{g,S_3} is empty for any genus $g \geqslant 2$. Therefore* $\operatorname{Sing}^{\mathrm{nc}} \overline{\mathcal{R}}_{g,S_3} = T_{g,S_3}$.

As a direct application, in our next paper we are going to prove that the moduli space of genus g connected twisted S_3-covers is of general type for any odd genus $g \geqslant 13$.

In section Section 2 we introduce the different notions of covers and recall their equivalence. In Section 3 we review the dual graph and torsor notions, they are very important in describing the structure of twisted covers and their ghost automorphisms. In Section 4 and Section 5 we prove the main results for the loci of singularities.

Acknowledgements

This work is the completion of my PhD thesis at *Institut de Mathématiques de Jussieu*, thus I want to thank my PhD supervisor Alessandro Chiodo, for his patience and his advices. Furthermore, I want to thank Michele Bolognesi and Gavril Farkas for the careful reading and the comments on my thesis work. Finally, I am grateful to Claudio Fontanari, my post-doc tutor at *Università degli Studi di Trento*, for his important suggestions, and to Roberto Pignatelli for the fruitful conversations.

2. Moduli of curves with a G-cover

Consider G a finite group, $\mathcal{R}_{g,G}$ is the moduli space of genus g smooth curves with a principal G-bundle. The moduli $\mathcal{R}_{g,G}$ comes with a natural forgetful proper morphism $\pi\colon \mathcal{R}_{g,G} \to \mathcal{M}_g$. As shown in [9], the moduli space $\overline{\mathcal{M}}_g$ of stable curves, is a compactification of $\mathcal{M}_g$. In the case of principal G-bundles over stable curves, the nodal singularities prevent the forgetful projection π to be proper.

In order to find a compactification of $\mathcal{R}_{g,G}$ which is proper over $\overline{\mathcal{M}}_g$, we introduce two equivalent stacks: the one of twisted G-covers of genus g, denoted by $\mathcal{B}_g^{\mathrm{bal}}(G)$, and the one of admissible G-covers of genus g, denoted by Adm_g^G. These stacks are Deligne–Mumford and are proven to be isomorphic by Abramovich, Corti and Vistoli (see [1]), we introduce both of them because we will use different insights from both points of view. The coarse space $\overline{\mathcal{R}}_{g,G}$ of these spaces is a compactification of $\mathcal{R}_{g,G}$, and it comes with a proper forgetful morphism $\overline{\mathcal{R}}_{g,G} \to \overline{\mathcal{M}}_g$ which extends π.

2.1. Curves with principal G-bundles

Given any finite group G, in this section we explore the geometry of principal G-bundles over stable curves and their automorphisms.

2.1.1. Moduli of stable curves

In [9], Deligne and Mumford carry a local analysis of the stack $\overline{\mathsf{M}}_{g,n}$ of stable curves, based on deformation theory. For every n-marked stable curve $(C;p_1,\dots,p_n)$, the deformation functor is representable (see [22] and [3, §11]) and it is represented by a smooth scheme $\mathrm{Def}(C;p_1,\dots,p_n)$ of dimension $3g-3+n$ with one distinguished point q. The deformation scheme comes with a universal family $X \to \mathrm{Def}(C;p_1,\dots,p_n)$ whose central fiber X_q is identified with $(C;p_1,\dots,p_n)$. Every automorphism of the central fiber naturally extends to the whole family X by the universal property of the deformation scheme. The strict henselization of $\overline{\mathsf{M}}_{g,n}$ at the geometric point $[C;p_1,\dots,p_n]$ is the same of the Deligne–Mumford stack

$$[\mathrm{Def}(C;p_1,\dots,p_n)/\operatorname{Aut}(C;p_1,\dots,p_n)]$$

at q. As a consequence, for every geometric point $[C;p_1,\dots,p_n]$ of the coarse space $\overline{\mathcal{M}}_{g,n}$, the strict Henselization at $[C;p_1,\dots,p_n]$ is $\mathrm{Def}(C;p_1,\dots,p_n)/$

$\mathrm{Aut}(C; p_1, \ldots, p_n)$. This implies that every singularity of $\overline{\mathcal{M}}_{g,n}$ is a quotient singularity. From now on, we will refer to the strict henselization of a scheme X at a geometric point q as the *local picture* of X at q.

As showed in [3, §11.2], given a smooth curve C with n marked points $p_1, \ldots, p_n$, we have $\mathrm{Def}(C; p_1, \ldots, p_n) \cong H^1(C, T_C(-p_1 - \cdots - p_n))$, where T_C is the tangent bundle to curve C.

Remark 2.1. — Given a stable n-marked curve C, we denote by $C_1, C_2, \ldots, C_V$ its irreducible components. Let $\mathsf{nor}\colon \overline{C} \to C$ be the normalization morphism of C, and denote by $\overline{C}_i$ the normalization of component C_i for every i, then $\overline{C} = \bigsqcup_i \overline{C}_i$. We mark on $\overline{C}$ the preimage point via nor of any marked point or node. We denote by g_i the genus of $\overline{C}_i$ for any i, by D_i the divisor of marked points on $\overline{C}_i$ and by $n_i \coloneqq \deg(D_i)$ its degree. The stability condition for C is equivalent to $2g_i - 2 + n_i > 0$ for all i.

Remark 2.2. — We follow [8] to give a more explicit description of the deformation scheme. For a nodal curve C, consider $\mathrm{Def}(C; \mathrm{Sing}\, C)$ the universal deformation of curve C alongside with its nodes. We impose $n = 0$ in this for sake of simplicity, the $n > 0$ case is similar. If V is the number of irreducible components of C, there is a canonical decomposition

$$\mathrm{Def}(C; \mathrm{Sing}\, C) = \bigoplus_{i=1}^{V} \mathrm{Def}(\overline{C}_i; D_i) \cong \bigoplus_{i=1}^{V} H^1(\overline{C}_i, T_{\overline{C}_i}(-D_i)). \tag{2.1}$$

Furthermore, if δ is the number of nodes of C, the quotient $\mathrm{Def}(C)/\mathrm{Def}(C; \mathrm{Sing}\, C)$ has a canonical splitting

$$\mathrm{Def}(C)/\mathrm{Def}(C; \mathrm{Sing}\, C) = \bigoplus_{j=1}^{\delta} M_j, \tag{2.2}$$

where $M_j \cong \mathbb{A}^1$ is the deformation scheme of node q_j of C. The isomorphism $M_j \to \mathbb{A}^1$ is non-canonical and choosing one isomorphism is equivalent to choose a smoothing of the node.

2.1.2. Group actions

Given any finite group G and an element h in it, we call $c_h\colon G \to G$ the conjugation automorphism such that $c_h\colon g \mapsto h \cdot g \cdot h^{-1}$ for all g in G. The subgroup of conjugation automorphisms, inside $\mathrm{Aut}(G)$, is called group of the *inner automorphisms* and denoted by $\mathrm{Inn}(G)$. We call $\mathrm{Sub}(G)$ the set of G subgroups and, for any subgroup $H \in \mathrm{Sub}(G)$, we call $Z_G(H)$ its centralizer

$$Z_G(H) \coloneqq \{g \in G \,|\, gh = hg \;\forall h \in H\}.$$

We denote by Z_G the center of the whole group. The group $\mathrm{Inn}(G)$ acts naturally on $\mathrm{Sub}(G)$.

Definition 2.3. — *We call $\mathcal{T}(G)$ the set of the orbits of the* $\mathrm{Inn}(G)$*-action in* $\mathrm{Sub}(G)$*. Equivalently, $\mathcal{T}(G)$ is the set of conjugacy classes of G subgroups.*

Definition 2.4. — *Consider two subgroup conjugacy classes $\mathcal{H}_1, \mathcal{H}_2$ in $\mathcal{T}(G)$, we say that $\mathcal{H}_2$ is a subclass of $\mathcal{H}_1$, denoted by $\mathcal{H}_2 \leqslant \mathcal{H}_1$, if for one element $H_2 \in \mathcal{H}_2$ (and hence for all), there exists $H_1 \in \mathcal{H}_1$ such that H_2 is a subgroup of H_1. If the inclusion is strict, then $\mathcal{H}_2$ is a strict subclass of $\mathcal{H}_1$ and the notation is $\mathcal{H}_2 < \mathcal{H}_1$.*

Consider a transitive G-set $\mathcal{T}$, *i.e.* a set $\mathcal{T}$ with a transitive left G-action $\psi\colon G \times \mathcal{T} \to \mathcal{T}$. Any map $\eta\colon \mathcal{T} \to G$ induces, via ψ, a map $\mathcal{T} \to \mathcal{T}$. In particular,

$$E \mapsto \psi(\eta(E), E), \quad \forall E \in \mathcal{T}.$$

This induces a map

$$\psi_*\colon G^{\mathcal{T}} \to \mathcal{T}^{\mathcal{T}}.$$

If we denote by $S_{\mathcal{T}} \subset \mathcal{T}^{\mathcal{T}}$ the subset of $\mathcal{T}$ permutations, we obtain that $\psi_*^{-1}(S_{\mathcal{T}})$ is the subset of maps $\mathcal{T} \to G$ inducing a $\mathcal{T}$ permutation.

Consider an element E in $\mathcal{T}$. We denote by H_E its stabilizer, *i.e.* the G subgroup fixing E. Given any other element $\psi(g, E)$ for some g in G, its stabilizer is

$$H_{\psi(g,E)} = g \cdot H_E \cdot g^{-1},$$

this proves the following lemma.

Lemma 2.5. — *Given any transitive G-set $\mathcal{T}$, there exists a canonical conjugacy class $\mathcal{H}$ in $\mathcal{T}(G)$, and a canonical surjection $\mathcal{T} \twoheadrightarrow \mathcal{H}$ sending any element to its stabilizer.*

Given the transitive G-set $\mathcal{T}$ and the group G seen as a G-set with respect to the $\mathrm{Inn}(G)$-action, we consider the set of G-equivariant maps $\mathrm{Hom}^G(\mathcal{T}, G)$.

Lemma 2.6. — *For any element E in $\mathcal{T}$, and any map η in* $\mathrm{Hom}^G(\mathcal{T}, G)$*, $\eta(E) \in Z_G(H_E)$.*

Proof. — The equivariance condition means that

$$\eta(\psi(h, E)) = c_h(\eta(E)) = h \cdot \eta(E) \cdot h^{-1}$$

for all h in G. If h is in H_E, the left hand side of the equality above is simply $\eta(E)$, therefore $c_h(\eta(E)) = \eta(E)$ for all h in H_E, and this is possible if and only if $\eta(E)$ is in $Z_G(H_E)$. □

Proposition 2.7. — *Given any object E in $\mathcal{T}$, there exists a canonical isomorphism*

$$\mathrm{Hom}^G(\mathcal{T}, G) \cong Z_G(H_E).$$

Moreover, the set of equivariant maps $\mathrm{Hom}^G(\mathcal{T}, G)$ is uniquely determined by the canonical class $\mathcal{H}$ associated to $\mathcal{T}$ (see Lemma 2.5).

Proof. — The first part of the proposition follows from Lemma 2.6. We observe that if we consider another object $E' = \psi(s, E)$, then $H_{E'} = s \cdot H_E \cdot s^{-1}$ and

$$Z_G(H_{E'}) = s \cdot Z_G(H_E) \cdot s^{-1}.$$

Therefore there exists an inclusion $\mathrm{Hom}^G(\mathcal{T}, G) \hookrightarrow G$ which is determined, up to conjugation, by the class $\mathcal{H}$ of H_E. □

2.1.3. Principal G-bundles

Definition 2.8 (principal G-bundle). — *If G is a finite group, a principal G-bundle over a scheme X is a fiber bundle $F \to X$ together with a left action $\psi \colon G \times F \to F$ such that the induced morphism*

$$\widetilde{\psi} \colon G \times F \to F \times_X F,$$

is an isomorphism. Here $\widetilde{\psi} = \psi \times \pi_2$, where π_2 is the projection $G \times F \to F$.

Remark 2.9. — As a direct consequence of the definition, every geometric fiber of $F \to X$ is isomorphic to the group G itself.

Remark 2.10. — The category of principal G-bundles is denoted by BG and comes with a natural forgetful functor $BG \to Sch$.

We introduce the stack $\mathsf{R}_{g,G}$ of smooth curves of genus g with a principal G-bundle.

Definition 2.11. — *In the category $\mathsf{R}_{g,G}$, the objects are smooth S-curves $X \to S$ of genus g, equipped with a principal G-bundle $F \to X$, for any scheme S. The morphisms of $\mathsf{R}_{g,G}$ are commutative diagrams*

$$\begin{array}{ccccc} F' & \longrightarrow & X' & \longrightarrow & S' \\ {\scriptstyle b}\downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & X & \longrightarrow & S \end{array}$$

such that the two squares are cartesian and b is G-equivariant with respect to the natural G-actions. The category $\mathsf{R}_{g,G}$ comes with a forgetful functor $\pi \colon \mathsf{R}_{g,G} \to \mathsf{M}_g$, sending any object or morphism on the underlying curve or curve morphism.

Consider a connected normal scheme X and a principal G-bundle $F \to X$. We denote by $\mathrm{Aut}_{\mathrm{Cov}}(X, F)$ its automorphism group in the category of coverings, that is the automorphisms of F commuting with the projection $F \to X$. Furthermore, $\mathrm{Aut}_{BG}(X, F)$ is its automorphism group in the category of principal G-bundles, that is the covering automorphisms of F compatible with the natural G-action.

We call $\mathcal{T}(F)$ the set of connected components of any principal G-bundle $F \to X$. The group G acts transitively on $\mathcal{T}(F)$, and by abuse of notation we call $\psi\colon G \times \mathcal{T}(F) \to \mathcal{T}(F)$ this action. As explained in Section 2.1.2, this action induces a map $\psi_*\colon G^{\mathcal{T}(F)} \to \mathcal{T}(F)^{\mathcal{T}(F)}$.

Proposition 2.12. — *If X is a connected normal scheme, and $F \to X$ a principal G-bundle, then we have the following canonical identifications:*

- $\mathrm{Aut}_{\mathrm{Cov}}(X, F) = \psi_*^{-1}(S_{\mathcal{T}(F)})$;
- $\mathrm{Aut}_{BG}(X, F) = \mathrm{Hom}^G(\mathcal{T}(F), G)$.

Here we denoted by $S_{\mathcal{T}(F)}$ the set of $\mathcal{T}(F)$ permutations.

Proof. — We start by showing the identification $\mathrm{Hom}_{\mathrm{Cov}}(F/X, F/X) = G^{\mathcal{T}(F)}$, where the first one is the set of covering automorphism of a principal G-bundle $F \to X$. Consider any covering morphism $b\colon F \to F$ over X, given the isomorphism $\widetilde{\psi}\colon G \times F \to F \times_X F$ introduced in Definition 2.8, we consider the chain of maps

$$F \xrightarrow{b\times \mathrm{id}} F \times_X F \xrightarrow{\tilde{\psi}^{-1}} G \times F \xrightarrow{\pi_1} G. \tag{2.3}$$

As G is discrete, the map above is constant on the connected components and therefore it induces a map $\mathrm{Hom}_{\mathrm{Cov}}(F/X, F/X) \xrightarrow{\eta} G^{\mathcal{T}(F)}$ which moreover is bijective.

The morphism b is an automorphism if and only if $\eta(b)$ acts bijectively on $\mathcal{T}(F)$, *i.e.* if and only if $\psi_*(\eta(b)) \in S_{\mathcal{T}(G)}$.

The automorphisms of F as a principal G-bundle must moreover preserve the G-action, *i.e.* we must have

$$b_h \coloneqq b \circ \psi(-, h) = h \cdot b(-) \quad \forall h \in G,$$

where $\cdot$ is the multiplication in G. Observe that $\eta(b_h) = (\eta(b) \circ \psi(-, h)) \cdot h$, where we denoted by ψ the G-action on F and $\mathcal{T}(F)$ indistinctly. Therefore $\eta(b) \circ \psi(-, h) = \eta(b_h) \cdot h^{-1}$, and so

$$\eta(b) \circ \psi(-, h) = c_h \circ \eta(b),$$

which is the exact definition of η being in $\mathrm{Hom}^G(\mathcal{T}(F), G) \subset G^{\mathcal{T}(F)}$. □

Remark 2.13. — In the case of a connected principal G-bundle $F \to X$, the proposition above summarizes in $\mathrm{Aut}_{\mathrm{Cov}}(X, F) = G$ and $\mathrm{Aut}_{BG}(X, F) = Z_G$.

For a general G-bundle $F \to X$, the set of connected components $\mathcal{T}(F)$ has a transitive G-action. By Lemma 2.5, this induces a canonical conjugacy class $\mathcal{H}$ in $\mathcal{T}(G)$.

DEFINITION 2.14. — *We call principal $\mathcal{H}$-bundle, a principal G-bundle whose canonical associated class in $\mathcal{T}(G)$ is $\mathcal{H}$. Equivalently, the stabilizer of every connected component in $\mathcal{T}(F)$ is a G subgroup in $\mathcal{H}$.*

Remark 2.15. — By Proposition 2.7, the automorphism group of any principal $\mathcal{H}$-bundle, is isomorphic to $Z_G(H)$, where H is any G subgroup in the $\mathcal{H}$ class.

2.2. Twisted G-covers

In order to enlarge the notion of principal G-bundles we admit non-trivial stabilizers at the nodes of a stable curve, by defining twisted curves. The twisting techniques are widely discussed in [1] and [2], furthermore twisted curves are introduced in [8] in the case of a level structure on stable curves.

2.2.1. Definitions

DEFINITION 2.16 (Twisted curve). — *A twisted n-marked S-curve is a diagram*

$$\begin{array}{ccc} \Sigma_1, \Sigma_2, \ldots, \Sigma_n & \subset & \mathsf{C} \\ & & \downarrow \\ & & C \\ & & \downarrow \\ & & S. \end{array}$$

Where:

1. C *is a Deligne–Mumford stack, proper over S, and étale locally it is a nodal curve over S;*
2. *the $\Sigma_i \subset \mathsf{C}$ are disjoint closed substacks in the smooth locus of $\mathsf{C} \to S$ for all i;*
3. *$\Sigma_i \to S$ is an étale gerbe for all i;*
4. *$\mathsf{C} \to C$ exhibits C as the coarse space of C, and it is an isomorphism over C_{gen}.*

We recall that, given a scheme U and a finite abelian group $\boldsymbol{\mu}$ acting on U, the stack $[U/\boldsymbol{\mu}]$ is the category of principal $\boldsymbol{\mu}$-bundles $E \to T$, for any scheme T, equipped with a $\boldsymbol{\mu}$-equivariant morphism $f\colon E \to U$. The stack $[U/\boldsymbol{\mu}]$ is a proper Deligne–Mumford stack and has a natural morphism to its coarse scheme $U/\boldsymbol{\mu}$.

By the definition of twisted curve we get the local pictures:

- *at a marking*, morphism $\mathsf{C} \to C \to S$ is locally isomorphic to
$$[\operatorname{Spec} A[x']/\boldsymbol{\mu}_r] \to \operatorname{Spec} A[x] \to \operatorname{Spec} A$$
for some normal ring A and some integer $r > 0$. Here $x = (x')^r$, and $\boldsymbol{\mu}_r$ is the cyclic group of order r acting on $\operatorname{Spec} A[x']$ by the action $\xi\colon x' \mapsto \xi x'$ for any $\xi \in \boldsymbol{\mu}_r$;
- *at a node*, morphism $\mathsf{C} \to C \to S$ is locally isomorphic to
$$\left[\operatorname{Spec}\left(\frac{A[x',y']}{(x'y'-a)}\right)/\boldsymbol{\mu}_r\right] \to \operatorname{Spec}\left(\frac{A[x,y]}{(xy-a^\ell)}\right) \to \operatorname{Spec} A$$
for some integer $r > 0$ and $a \in A$. Here $x = (x')^\ell,\ y = (y')^\ell$. The group $\boldsymbol{\mu}_r$ acts by the action
$$\xi\colon (x',y') \mapsto (\xi x', \xi^m y')$$
where m is an element of $\mathbb{Z}/r$ and ξ is a primitive rth root of the unit. The action is called *balanced* if $m \equiv -1 \mod r$. A curve with balanced action at every node is called a balanced curve.

Definition 2.17 (Twisted G-cover). — *Given an n-marked twisted curve $(\Sigma_1, \dots, \Sigma_n;\ \mathsf{C} \to C \to S)$, a twisted G-cover is a representable stack morphism $\phi\colon \mathsf{C} \to BG$,* i.e. *an object of the category* $\operatorname{Fun}(\mathsf{C}, BG)$ *which moreover is representable.*

Definition 2.18. — *We introduce category $\mathcal{B}_{g,n}(G)$. Objects of $\mathcal{B}_{g,n}(G)$ are twisted n-marked S-curves of genus g with a twisted G-cover, for any scheme S.*

Consider two twisted G-covers $\phi'\colon \mathsf{C}' \to BG$ and $\phi\colon \mathsf{C} \to BG$ over the twisted n-marked curves C' and C respectively. A morphism $(\mathsf{C}', \phi') \to (\mathsf{C}, \phi)$ is a pair (f, α) such that $\mathsf{f}\colon \mathsf{C}' \to \mathsf{C}$ is a morphism of n-marked twisted curves, and $\alpha\colon \phi' \to \phi \circ \mathsf{f}$ is an isomorphism in $\operatorname{Fun}(\mathsf{C}', BG)$.

Following [2], the category $\mathcal{B}_{g,n}(G)$ can be defined as the 2-category of twisted stable n-pointed maps of genus g and degree 0 to the category BG. In the same paper it is observed that the automorphism group of every 1-morphism is trivial, therefore this 2-category is equivalent to the category obtained by replacing 1-morphisms with their 2-isomorphism classes. In [2] this category is denoted by $\mathcal{K}_{g,n}(BG,0)$, the notation $\mathcal{B}_{g,n}(G)$ for the case of twisted G-covers appears for example in [1].

Definition 2.19. — *A balanced twisted G-cover is a twisted G-cover over a twisted balanced curve. We call $\mathcal{B}^{\mathrm{bal}}_{g,n}(G)$ the full sub-functor of twisted balanced G-covers.*

Twisted G-covers generalize the notion of root of the trivial bundle. Indeed, for any twisted curve C and any integer $\ell > 0$, there exists a canonical bijection between the set of twisted $\boldsymbol{\mu}_\ell$-covers over C, and the set of ℓth roots of $\mathcal{O}_{\mathsf{C}}$. Here a faithful line bundle is a line bundle $\mathsf{L} \to \mathsf{C}$ such that the associated morphism $\mathsf{C} \to B\mathbb{C}^*$ is representable, and an ℓth root of $\mathcal{O}_{\mathsf{C}}$ is a faithful line bundle such that $\mathsf{L}^{\otimes \ell} \cong \mathcal{O}_{\mathsf{C}}$.

2.2.2. Local structure of twisted covers

We consider a twisted curve C over a geometric point $\mathrm{Spec}(\mathbb{C})$. For any marked or nodal point p, the local picture of C at p is the same as $[U/\boldsymbol{\mu}_r]$ at the origin, for some scheme U and positive integer r. Any principal G-bundle over C, or equivalently any object of $BG(\mathsf{C})$, is locally isomorphic at p to a principal G-bundle on $[U/\boldsymbol{\mu}_r]$.

In [1, §2.1.8] is explained how to realize twisted stable maps as twisted objects over scheme theoretic curves. In particular, a principal G-bundle on $[U/\boldsymbol{\mu}_r]$ is the same as a principal G-bundle $f\colon \widetilde{F} \to U$ with the natural G-action $\psi\colon G \times \widetilde{F} \to \widetilde{F}$, and also with a $\boldsymbol{\mu}_r$-action $\nu\colon \boldsymbol{\mu}_r \times \widetilde{F} \to \widetilde{F}$ which is compatible with the $\boldsymbol{\mu}_r$-action on U and with ψ.

In formulas we have:

(1) $f \circ \nu(\xi, -) = \xi \cdot f\colon \widetilde{F} \to U$, for all $\xi \in \boldsymbol{\mu}_r$;

(2) $\psi(h, \nu(\xi, -)) = \nu(\xi, \psi(h, -))\colon \widetilde{F} \to \widetilde{F}$, for all $h \in G$ and $\xi \in \boldsymbol{\mu}_r$.

Remark 2.20. — We consider at first the case of a marked point p of C whose local picture is $[\mathbb{A}^1/\boldsymbol{\mu}_r]$ with $\boldsymbol{\mu}_r$ acting by multiplication. By what we just said we have a principal G-bundle $\widetilde{F} \to \mathbb{A}^1$, and for any $\xi \in \boldsymbol{\mu}_r$ a morphism $\widetilde{\alpha}(\xi)\colon \widetilde{F} \to \widetilde{F}$ such that

$$\widetilde{\alpha}(\xi) := \nu(\xi, -).$$

If we fix a privileged rth root $\xi_r = \exp(2\pi/r)$, then $\widetilde{\alpha}(\xi_r)(\widetilde{p}) = \psi(h_{\widetilde{p}}, \widetilde{p})$, for all preimages $\widetilde{p}$ of p, where $h_{\widetilde{p}}$ is an element of group G depending on $\widetilde{p}$.

Remark 2.21. — In the case of a node q of C, its local picture is $[V/\boldsymbol{\mu}_r]$ for some positive integer r where $V \cong \{x'y' = 0\} \subset \mathbb{A}^2_{x',y'}$ and the $\boldsymbol{\mu}_r$-action is given by $\xi \cdot (x', y') = (\xi x', \xi^{-1} y')$ for all $\xi \in \boldsymbol{\mu}_r$.

The normalization of the node neighborhood V is naturally isomorphic to $\mathbb{A}^1_{x'} \sqcup \mathbb{A}^1_{y'} \to V$. We consider the normalization $\mathsf{nor}\colon \overline{\mathsf{C}} \to \mathsf{C}$ of the twisted curve C, the local picture of nor morphism at q is

$$[\mathbb{A}^1_{x'}/\boldsymbol{\mu}_{r_q}] \sqcup [\mathbb{A}^1_{y'}/\boldsymbol{\mu}_{r_q}] \to [V/\boldsymbol{\mu}_{r_q}].$$

We denote by $q_1 \in \mathbb{A}^1_{x'}$ and $q_2 \in \mathbb{A}^1_{y'}$ the two preimages of q. As before, a twisted G-cover on $[V/\boldsymbol{\mu}_r]$ is the same as a principal G-bundle $\widetilde{F} \to V$ plus a $\boldsymbol{\mu}_r$-action on V with the right compatibilities. This induces

- two principal G-bundles $\widetilde{F}' \to \mathbb{A}^1_{x'}$ and $\widetilde{F}'' \to \mathbb{A}^1_{y'}$ with the naturally associated $\boldsymbol{\mu}_{r_q}$-actions. We denote by $\nu'\colon \boldsymbol{\mu}_{r_q} \times \widetilde{F}' \to \widetilde{F}'$ and $\nu''\colon \boldsymbol{\mu}_{r_q} \times \widetilde{F}'' \to \widetilde{F}''$ these actions;
- a gluing isomorphism between the central fibers $\kappa_q\colon \widetilde{F}'_q \xrightarrow{\cong} \widetilde{F}''_q$. This means that
  (i) $\kappa_q(\psi(h,-)) = \psi(h,\kappa_q(-))\colon \widetilde{F}'_q \to \widetilde{F}''_q$ for any $h \in G$;
  (ii) $\kappa_q(\nu'(\xi,-)) = \nu''(\xi,\kappa_q(-))\colon \widetilde{F}'_q \to \widetilde{F}''_q$ for any $\xi \in \boldsymbol{\mu}_{r_q}$.
  And furthermore, $\widetilde{F} = (\widetilde{F}' \sqcup \widetilde{F}'')/\kappa_q$.

Following Remark 2.20, we define $\alpha'(\xi) \coloneqq \nu'(\xi,-)\colon F' \to F'$, $\alpha''(\xi) \coloneqq \nu''(\xi,-)\colon F'' \to F''$ for any $\xi \in \boldsymbol{\mu}_{r_q}$. By the balancing condition, if we have two points $\widetilde{q}_1$ and $\widetilde{q}_2$ in F'_{q_1} and F''_{q_2} respectively, such that $\kappa_q(\widetilde{q}_1) = \widetilde{q}_2$, then $h_{\tilde{q}_1} = h^{-1}_{\tilde{q}_2}$.

This local structure can be encoded in conjugation classes associated to every marked or nodal point. Consider a marked point p of C and the local picture $[\mathbb{A}^1/\boldsymbol{\mu}_r]$ at p, then the twisted G-cover $\phi\colon \mathsf{C} \to BG$ induces a morphism $\phi\colon [\mathbb{A}^1/\boldsymbol{\mu}_r] \to BG$. This induces a morphism $\widetilde{\phi}_p\colon \boldsymbol{\mu}_r \to G$ defined up to conjugation, which is an injection by the ϕ representability.

Definition 2.22. — *The conjugacy class $[\![\widetilde{\phi}_p]\!]$ of $\widetilde{\phi}_p$ is called G-type of ϕ at p.*

In the case of a node q, the composition of ϕ with the normalization induces

$$\widetilde{\phi}_{q_1}\colon \boldsymbol{\mu}_r \to G \quad\text{and}\quad \widetilde{\phi}_{q_2}\colon \boldsymbol{\mu}_r \to G,$$

and by the balancing condition the two G-types are the inverse of each other, $[\![\widetilde{\phi}_{q_1}]\!] = [\![\widetilde{\phi}_{q_2}]\!]^{-1}$. Once we choose a privileged branch of a node, we call G-type of that node the G-type with respect to the restriction of the cover to that branch. Switching the branches changes the G-type into its inverse class.

2.2.3. Local structure of $\mathcal{B}^{\mathrm{bal}}_{g,n}(G)$

The local structure of $\mathcal{B}^{\mathrm{bal}}_{g,n}(G)$ can be described with a very similar approach to what we did for $\overline{\mathsf{M}}_{g,n}$. We work the case $n = 0$ of unmarked twisted G-covers. Given a twisted G-cover (C, ϕ), its deformation functor is representable and the associated scheme $\mathrm{Def}(\mathsf{C}, \phi)$ is isomorphic to $\mathrm{Def}(\mathsf{C})$ via the forgetful functor $(\mathsf{C}, \phi) \mapsto \mathsf{C}$. The automorphism group $\mathrm{Aut}(\mathsf{C}, \phi)$ naturally acts on $\mathrm{Def}(\mathsf{C}, \phi) = \mathrm{Def}(\mathsf{C})$ and the local picture of $\mathcal{B}^{\mathrm{bal}}_{g}(G)$ at $[\mathsf{C}, \phi]$ is the same of $[\mathrm{Def}(\mathsf{C}) / \mathrm{Aut}(\mathsf{C}, \phi)]$ at the central point.

Remark 2.23. — Consider a twisted curve C whose coarse space is the curve C, we give a description of the scheme $\mathrm{Def}(\mathsf{C})$ as we did in Remark 2.2 for $\mathrm{Def}(C)$ with the notation of Remark 2.1. As C is a twisted curve, every node q_j has a possibly non-trivial stabilizer, which is a cyclic group of order r_j.

The deformation $\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\, \mathsf{C})$ of C alongside with its nodes, is canonically identified with the deformation of C alongside with its nodes $\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\, \mathsf{C}) = \mathrm{Def}(C; \mathrm{Sing}\, C)$. As in the previous case, the following quotient has a canonical splitting.

$$\mathrm{Def}(\mathsf{C}) / \mathrm{Def}(\mathsf{C}; \mathrm{Sing}\, \mathsf{C}) = \bigoplus_{j=1}^{\delta} R_j. \tag{2.4}$$

In this case $R_j \cong \mathbb{A}^1$ is the deformation scheme of the node q_j together with its stack structure. If we consider the schemes M_j of Equation (2.2) in Remark 2.2, there exists for every j a canonical morphism $R_j \to M_j$ of order r_j ramified in exactly one point.

2.3. Admissible G-covers

In order to define admissible G-covers, in the next sections we introduce admissible covers and we put a balancing condition on them.

2.3.1. Admissible covers

Definition 2.24 (Admissible cover). — *Given a nodal S-curve $X \to S$ with marked points, an admissible cover $u \colon F \to X$ is a morphism such that:*

(1) *the composition $F \to S$ is a nodal S-curve;*

(2) *given a geometric point* $\bar{s} \in S$*, every node of* $F_{\bar{s}}$ *maps via* u *to a node of* $X_{\bar{s}}$*;*
(3) *the restriction* $F|_{X_{\text{gen}}} \to X_{\text{gen}}$ *is an étale cover of degree* d*;*
(4) *given a geometric point* $\bar{s} \in S$*, the local picture of* $F_{\bar{s}} \xrightarrow{u} X_{\bar{s}}$ *at a point of* $F_{\bar{s}}$ *mapping to a marked point of* X *is isomorphic to*

$$\operatorname{Spec} A[x'] \to \operatorname{Spec} A[x] \to \operatorname{Spec} A,$$

for some normal ring A*, an integer* $r > 0$ *and* $u^* x = (x')^r$*;*
(5) *the local picture of* $F_{\bar{s}} \xrightarrow{u} X_{\bar{s}}$ *at a node of* $F_{\bar{s}}$ *is isomorphic to*

$$\operatorname{Spec}\left(\frac{A[x', y']}{(x'y' - a)}\right) \to \operatorname{Spec}\left(\frac{A[x, y]}{(xy - a^r)}\right) \to \operatorname{Spec} A,$$

for some integer $r > 0$ *and an element* $a \in A$*,* $u^* x = (x')^r$ *and* $u^* y = (y')^r$*.*

The category $\mathrm{Adm}_{g,n,d}$ of n-pointed stable curves of genus g with an admissible cover of degree d, is a proper Deligne–Mumford stack.

Consider $F \to C$ an admissible cover of a nodal curve C, a G-action on F such that the restriction $F|_{C_{\text{gen}}} \to C_{\text{gen}}$ is a principal G-bundle, a smooth point p of C, and a preimage $\widetilde{p} \in F$ of p. We denote by $H_{\tilde{p}} \subset G$ the stabilizer of $\widetilde{p}$. By definition of admissible cover, if $p \in C_{\text{gen}}$ (*i.e.* p is non-marked), then $H_{\tilde{p}} = (1)$. Moreover, the G-action induces a primitive character

$$\chi_{\tilde{p}} \colon H_{\tilde{p}} \to \mathrm{GL}(T_{\tilde{p}} F) = \mathbb{C}^*,$$

where $T_{\tilde{p}} F$ is the tangent space to F in $\widetilde{p}$.

Given any subgroup $H \subset G$, for any primitive character $\chi \colon H \to \mathbb{C}^*$ and for any $s \in G$, we denote by χ^s the conjugated character $\chi^s \colon sHs^{-1} \to \mathbb{C}^*$ such that $\chi^s(h) = \chi(s^{-1} h s)$ for all $h \in G$.

In the set of pairs (H, χ), with H a G subgroup and $\chi \colon H \to \mathbb{C}^*$ a character, we introduce the equivalence relation $(H, \chi) \sim (H', \chi')$ iff there exists $s \in G$ such that $H' = sHs^{-1}$ and $\chi' = \chi^s$. Consider a point $\widetilde{p}$ on F with stabilizer $H_{\tilde{p}}$ and associated character $\chi_{\tilde{p}}$. We observe that for any point $s \cdot \widetilde{p}$ of the same fiber,

$$H_{s \cdot \tilde{p}} = s H_{\tilde{p}} s^{-1} \ \text{ and } \ \chi_{s \cdot \tilde{p}} = \chi_{\tilde{p}}^s.$$

Therefore the equivalence class of the pair $(H_{\tilde{p}}, \chi_{\tilde{p}})$ only depends on the point p.

Definition 2.25. — *For any smooth point* $\widetilde{p}$ *on* F*, we call* local index *the associated pair* $(H_{\tilde{p}}, \chi_{\tilde{p}})$*. For any smooth point* $p \in C$*, the conjugacy class of the local index of any* $\widetilde{p}$ *in* F_p *is called the* G-type *at* p*, following* *the notation in [4]. We denote the G-type by $[\![H_p, \chi_p]\!]$, where H_p is the stabilizer of one of the points in F_p, and χ_p the associated character.*

The notion of G-type is equivalent to the one introduced in Definition 2.22. We will discuss this equivalence in Section 2.3.4.

2.3.2. Balancing the G-action

Lemma 2.26. — *Consider $u\colon F \to C$ an admissible cover of a nodal curve C such that the restriction $F|_{C_{\text{gen}}} \to C_{\text{gen}}$ is a principal G-bundle. If $\widetilde{p} \in F$ is one of the preimages of a node or a marked point, then the stabilizer $H_{\tilde{p}}$ is a cyclic group.*

Proof. — If $\widetilde{p}$ is the preimage of a marked point, the local picture of morphism u at $\widetilde{p}$ is

$$\operatorname{Spec} A[x'] \to \operatorname{Spec} A[x],$$

where $x' = x^r$ for some integer $r > 0$. This local description induces an action of $H_{\tilde{p}}$ on $U := \operatorname{Spec} A[x']$ which is free and transitive on $U\backslash\{\widetilde{p}\}$. The group of automorphisms of $U\backslash\{\widetilde{p}\}$ preserving r is exactly $\boldsymbol{\mu}_r$, therefore $H_{\tilde{p}}$ must be cyclic too.

In the case of a node $\widetilde{p}$ we observe that u is locally isomorphic to

$$\operatorname{Spec}\left(\frac{A[x', y']}{(x'y' - a^r)}\right) \to \operatorname{Spec}\left(\frac{A[x, y]}{(xy - a)}\right),$$

where $x' = x^r$ and $y' = y^r$, for an integer $r > 0$ and an element $a \in A$. The scheme $U' := \operatorname{Spec}\left(A[x', y']/(x'y' - a^r)\right)$ is the union of two irreducible components U_1, U_2, and we can apply the deduction above to $U_i\backslash\{\widetilde{p}\}$ for $i = 1, 2$. □

Observe that the set of characters $\chi\colon \boldsymbol{\mu}_r \to \mathbb{C}^*$ of a cyclic group, is the group $\mathbb{Z}/r\mathbb{Z}$. In particular, the character associated to $k \in \mathbb{Z}/r\mathbb{Z}$ maps $\xi \mapsto \xi^k$ for any ξ rth root of the unit.

Focusing on the case of a node $\widetilde{p} \in F$, we observe that $H_{\tilde{p}}$ acts independently on the two branches U_1 and U_2. We denote by $\chi^{(1)}_{\tilde{p}}$ and $\chi^{(2)}_{\tilde{p}}$ the characters of these actions.

Definition 2.27. — *The G-action at node $\widetilde{p}$ is balanced when $\chi^{(1)}_{\tilde{p}} = -\chi^{(2)}_{\tilde{p}}$, that is they are opposite as elements of $\mathbb{Z}/r\mathbb{Z}$ (where r depends on the $\widetilde{p}$ fiber).*

Definition 2.28 (Admissible G-cover). — *Given any finite group G, consider an admissible cover $F \to C$ of a nodal curve C, it is an admissible G-cover if*

(1) *the restriction* $F|_{C_{\text{gen}}} \to C_{\text{gen}}$ *is a principal* G*-bundle. This implies, by Lemma 2.26, that for every node or marked point* $\widetilde{p} \in F$*, the stabilizer* $H_{\tilde{p}}$ *is a cyclic group;*
(2) *the action of* $H_{\tilde{p}}$ *is balanced for every node* $\widetilde{p} \in F$*.*

This notion was firstly developed by Abramovich, Corti and Vistoli in [1], and also by Jarvis, Kaufmann and Kimura in [17].

Definition 2.29. — *We call* $\mathrm{Adm}^G_{g,n}$ *the stack of stable curves of genus* g *with* n *marked points and equipped with an admissible* G*-cover.*

Remark 2.30. — For any cyclic subgroup $H \subset G$, the image of a character $\chi\colon H \to \mathbb{C}^*$ is the group of $|H|$th roots of the unit, $\boldsymbol{\mu}_{|H|}$. We choose a privileged root in this set, which is $\exp(2\pi i/|H|)$. After this choice, The datum of (H, χ), is equivalent to the datum of the H generator $h = \chi^{-1}(e^{2\pi i/|H|})$. As a consequence, the conjugacy class $[\![H, \chi]\!]$ is identified with the conjugacy class $[\![h]\!]$ of h in G.

Definition 2.31. — *Given an admissible* G*-cover* $F \to C$ *over an* n*-marked stable curve, the series* $[\![h_1]\!]$, $[\![h_2]\!], \dots,$ $[\![h_n]\!]$*, of the* G*-types of the singular fibers over the marked points, is called* Hurwitz datum *of the cover. The stack of admissible* G*-covers of genus* g *with a given Hurwitz datum is denoted by* $\mathrm{Adm}^G_{g,[\![h_1]\!],\dots,[\![h_n]\!]}$.

Remark 2.32. — Given an admissible G-cover $F \to C$, if p is a node of C and $\widetilde{p}$ one of its preimages on F, then the local index of $\widetilde{p}$ and the G-type of p are well defined once we fix a privileged branch of p. Switching the branches sends the local index and the G-type in their inverses.

Consider a smooth curve C of genus g and n marked points $p_1, \dots, p_n$, the fundamental group of $C_{\text{gen}} = C \backslash \{p_1, \dots, p_n\}$ has $2g + n$ generators $\alpha_1, \alpha_2, \dots \alpha_g, \beta_1 \dots, \beta_g, \gamma_1, \dots, \gamma_n$. These generators respect the following relation,

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} \cdot \gamma_1 \cdots \gamma_n = 1, \tag{2.5}$$

and this is sufficient to represent the fundamental group. This is called the canonical representation of the fundamental group of a genus g smooth curve.

It is possible to describe admissible G-covers over smooth curves by the monodromy action, as done for example in [4, §2.3] and [21, §3.5]. Consider a smooth curve C, a generic point p_* on it and an admissible G-cover $F \to C$. We denote the points of the fiber F_{p_*} by $\widetilde{p}_*^{(g)}$ for any $g \in G$, in such a way that $g \cdot \widetilde{p}_*^{(1)} = \widetilde{p}_*^{(g)}$. This induces a group morphism $\pi_1(C_{\text{gen}}, p_*) \to G$.

This monodromy morphism is well defined up to relabelling the points $\widetilde{p}_*^{(i)}$, *i.e.* up to G conjugation. The following proposition is a rephrasing of [4, Lemma 2.6].

Proposition 2.33. — *Given a smooth n-marked curve $(C; p_1, \ldots, p_n)$ and a point p_* on its generic locus C_{gen}, the set of isomorphism classes of admissible G-covers on C is naturally in bijection with the set of conjugacy classes of maps*

$$\varpi\colon \pi_1(C_{\text{gen}}, p_*) \to G.$$

Remark 2.34. — We also point out that the monodromy of γ_i at any point $p_*^{(g)}$, with $g \in G$, is given by a small circular lacet around the deleted point p_i. Therefore by definition of G-type, if $[\![h_i]\!]$ is the G-type of p_i, then $[\![\varpi(\gamma_i)]\!] = [\![h_i]\!]$.

2.3.3. Admissible G-cover automorphisms

Consider an admissible G-cover $F \to C$ over a smooth n-marked curve $(C; p_1, \ldots, p_n)$. We denote by $\mathcal{T}(F)$ the set of connected components of F, which inherits the G-action ψ. For any connected component $E \subset F$, we denote by $H_E \subset G$ its stabilizer. The component $\psi(s, E)$, for some element s of G, has stabilizer $s \cdot H_E \cdot s^{-1}$. Therefore the conjugacy class of the stabilizer is independent on the choice of E. As in the case of principal G-bundles, for every admissible G-cover there exists a canonical class $\mathcal{H}$ in $\mathcal{T}(G)$ such that the stabilizer of every E in $\mathcal{T}(F)$ is a subgroup H_E in $\mathcal{H}$. Moreover, we have a canonical surjective map

$$\mathcal{T}(F) \twoheadrightarrow \mathcal{H}.$$

Definition 2.35. — *Given the set $\mathcal{T}(G)$ of subgroup conjugacy classes in G, and a class $\mathcal{H}$ in it, an admissible $\mathcal{H}$-cover is an admissible G-cover such that every connected component has stabilizer in $\mathcal{H}$.*

Definition 2.36. — *We denote by $\mathrm{Adm}_g^{G,\mathcal{H}}$ the stack of admissible $\mathcal{H}$-covers over stable curves of genus g, and we denote by $\mathrm{Adm}_{g,[\![h_1]\!],\ldots,[\![h_n]\!]}^{G,\mathcal{H}}$ the stack of admissible $\mathcal{H}$-cover with Hurwitz datum $[\![h_1]\!], \ldots, [\![h_n]\!]$ over the n marked points.*

It is possible to generalize the second point of Proposition 2.12. We denote by $\mathrm{Aut}_{\mathrm{Adm}}(C, F)$ the set of automorphisms of an admissible G-cover $F \to C$.

Proposition 2.37. — *Consider* $(C;p_1,\dots,p_n)$ *a nodal* n*-marked curve, and* $F\to C$ *an admissible* G*-cover, then*

$$\mathrm{Aut}_{\mathrm{Adm}}(C,F)=\mathrm{Hom}^G(\mathcal{T}(F),G).$$

Proof. — In the case of a smooth curve C, we consider the general locus $C_{\mathrm{gen}}=C\backslash\{p_1,\dots,p_n\}$. The restriction $F|_{C_{\mathrm{gen}}}$ is a principal G-bundle, therefore by Proposition 2.12,

$$\mathrm{Aut}_{\mathrm{Adm}}(C,F)\subset\mathrm{Aut}_{BG}(C_{\mathrm{gen}},F_{\mathrm{gen}})=\mathrm{Hom}^G(\mathcal{T}(F|_{C_{\mathrm{gen}}}),C_{\mathrm{gen}}).$$

Since $\mathcal{T}(F|_{C_{\mathrm{gen}}})=\mathcal{T}(F)$ and every automorphism of $F|_{C_{\mathrm{gen}}}\to C_{\mathrm{gen}}$ extends to the whole F, the thesis follows in this case.

In the case of a general stable curve C, with $C_1,\dots,C_V$ its connected components, and F_i the restriction $F|_{C_i}$ for any i, as a consequence of the first part, we have

$$\mathrm{Aut}_{\mathrm{Adm}}(C_i,F_i)=\mathrm{Hom}^G(\mathcal{T}(F_i),C_i).$$

The balancing condition at the nodes imposes that any automorphism in $\mathrm{Aut}_{\mathrm{Adm}}(C,F)$ acts as the same multiplicative factor on two touching components. This means that a sequence of functions in $\prod_i\mathrm{Hom}^G(\mathcal{T}(F_i),G)$, induces a global automorphism if and only if it is the sequence of restrictions of a global function $\mathrm{Hom}^G(\mathcal{T}(F),G)$. □

2.3.4. Equivalence between twisted and admissible G-covers

We introduced the two categories $\mathcal{B}_g^{\mathrm{bal}}(G)$ and Adm_g^G with the purpose of "well" defining the notion of principal G-bundle over stable non-smooth curves. These two categories are proven isomorphic in [1].

Theorem 2.38 (see [1, Theorem 4.3.2]). — *There exists a base preserving equivalence between* $\mathcal{B}_g^{\mathrm{bal}}(G)$ *and* Adm_g^G*, therefore in particular they are isomorphic Deligne–Mumford stacks.*

The proof proposed in [1] can be sketched quickly. Given a twisted G-cover $\phi\colon\mathsf{C}\to BG$, the restriction to $\mathsf{C}_{\mathrm{gen}}=C_{\mathrm{gen}}$ is a principal G-bundle $F_{\mathrm{gen}}\to C_{\mathrm{gen}}$ on the generic locus of the coarse space C, and this can be uniquely completed to an admissible G-cover $F\to C$. Conversely, given an admissible G-cover $F\to C$, it induces a quotient stack $\mathsf{C}:=[F/G]$ and therefore a representable morphism $\mathsf{C}\to BG$ with balanced action on nodes.

In what follows we will adopt the notation $\overline{\mathbf{R}}_{g,G}$ for the equivalent stacks $\mathcal{B}_g^{\mathrm{bal}}(G)$ and Adm_g^G. For every class $\mathcal{H}$ in $\mathcal{T}(G)$ we denote by $\overline{\mathbf{R}}_{g,G}^{\mathcal{H}}$ the full substack of $\overline{\mathbf{R}}_{g,G}$ whose objects are admissible $\mathcal{H}$-covers.

The correspondence of Theorem 2.38 allows the translation of every machinery we developed on twisted G-covers to admissible G-covers, and conversely. For example, the two definitions of G-type we introduced are equivalent. Precisely, consider a twisted G-cover (C, ϕ), a point p whose G-type is $[\![\widetilde{\phi}_p]\!]$, and an element $\widetilde{\phi}_p \colon \boldsymbol{\mu}_r \to G$ in the class of the G-type. Therefore $\operatorname{Im} \widetilde{\phi}_p = H \subset G$ is a cyclic subgroup and $\widetilde{\phi}_p^{-1} \colon H \to \boldsymbol{\mu}_r$ is a character. The class $[\![H, \widetilde{\phi}_p^{-1}]\!]$ is precisely the G-type at p from the admissible G-cover point of view.

Furthermore, we can use over twisted G-covers the notion of Hurwitz datum. We will denote by $\overline{\mathsf{R}}^{\mathcal{H}}_{g,[\![h_1]\!],\dots,[\![h_n]\!]}$ the stack of admissible $\mathcal{H}$-covers of genus g with Hurwitz datum $[\![h_1]\!], \dots, [\![h_n]\!]$.

If there is no risk of confusion, we will say that a twisted G-cover (C, ϕ) "is" an admissible G-cover $F \to C$ (or the other way around), meaning that $F \to C$ is the naturally associated admissible G-cover to (C, ϕ).

3. Dual graphs and torsors

In this section we introduce the important tool of dual graphs to describe subloci of the moduli of curves with a twisted G-cover. This subject was already treated by the author in [13] in the case of spin curves. Here we update this tool in order to generalize this notion to the case of G-covers. Furthermore, we introduce torsors and some of their fundamental properties.

3.1. Decorated dual graphs and G-covers

3.1.1. Basic graph theory

Consider a graph Γ with vertex set V and edge set E, we call *loop* an edge that starts and ends on the same vertex, we call *separating* an edge e such that the graph with vertex set V and edge set $E \backslash \{e\}$ is disconnected. We denote by E_{sep} the set of separating edges, and by $\mathbb{E}$ the set of oriented edges: the elements of this set are edges in E equipped with an orientation, in particular for every edge $e \in \mathbb{E}$ we denote by e_+ the head vertex and by e_- the tail. There is a canonical 2-to-1 projection $\mathbb{E} \to E$. We also introduce a conjugation in $\mathbb{E}$, such that for each $e \in \mathbb{E}$, the conjugated edge $\bar{e}$ is obtained by reversing the orientation, in particular $(\bar{e})_+ = e_-$.

For every graph Γ, when there is no risk of confusion we denote by V the cardinality of the vertex set $V(\Gamma)$ and by E the cardinality of the edge set $E(\Gamma)$.

We consider a finite group G acting on graph Γ: That is, we consider two G-actions on the vertex set and on the edge set,

$$G \times V(\Gamma) \to V(\Gamma) \quad \text{and} \quad G \times \mathbb{E}(\Gamma) \to \mathbb{E}(\Gamma).$$

We denote these actions by $h \cdot v$ and $h \cdot e$ for every h in G and every vertex v and oriented edge e. These actions must respect the following natural intersection conditions

(1) $(h \cdot e)_+ = h \cdot e_+ \ \ \forall h \in G, e \in \mathbb{E}(\Gamma)$;
(2) $\overline{h \cdot e} = h \cdot \bar{e} \ \ \forall h \in G, e \in \mathbb{E}(\Gamma)$.

Observe that there are no faithfulness conditions, therefore any vertex or edge may have a non-trivial stabilizer. We denote by H_v and H_e the stabilizers of vertex v and edge e respectively. We have $H_{s\cdot v} = s \cdot H_v \cdot s^{-1}$ for any $v \in V(\Gamma)$ and $s \in G$, and the same is true for H_e. In general, every orbit of vertices (or oriented edges) is characterized by a conjugacy class $\mathcal{H}$ in $\mathcal{T}(G)$, and every element of $\mathcal{H}$ is the stabilizer of some object in the orbit.

Definition 3.1 (Cochains). — *The group of* 0*-cochains is the group of* G*-valued functions on* $V(\Gamma)$ *compatible with the* G*-action*

$$C^0(\Gamma; G) \coloneqq \left\{a\colon V(\Gamma) \to G \,\middle|\, a(g \cdot v) = g \cdot a(v) \cdot g^{-1}\right\}.$$

The group of 1*-cochains is the group of antisymmetric functions on* $\mathbb{E}$ *with the same compatibility condition*

$$C^1(\Gamma; G) \coloneqq \left\{b\colon \mathbb{E} \to G \,\middle|\, b(\bar{e}) = b(e)^{-1},\ b(g \cdot e) = g \cdot b(e) \cdot g^{-1}\right\}.$$

These groups generalize the cochain groups defined by Chiodo and Farkas in [8]. In particular the Chiodo–Farkas groups refer to the case of a trivial G-action on Γ.

There exists a natural differential $\delta\colon C^0(\Gamma; G) \to C^1(\Gamma; G)$ such that

$$\delta a(e) \coloneqq a(e_+) \cdot a(e_-)^{-1}, \ \ \forall a \in C^0(\Gamma; G)\ \forall e \in \mathbb{E}.$$

Consider the set $\mathcal{T}(\Gamma)$ of the connected components of the graph, with the naturally induced G-action. The exterior differential fits into an useful exact sequence of groups

$$0 \to \operatorname{Hom}^G(\mathcal{T}(\Gamma), G) \xrightarrow{i} C^0(\Gamma; G) \xrightarrow{\delta} C^1(\Gamma; G). \tag{3.1}$$

Here the injection i sends $f \in \operatorname{Hom}^G(\mathcal{T}(\Gamma), G)$ on the cochain a such that for every component $\gamma \in \mathcal{T}(\Gamma)$, a is constantly equal to $f(\gamma)$ on γ. If Γ is a connected graph, then the first term of the exact sequence is the group G and i sends $g \in G$ to the associated constant cochain. We recall that for any group we can define a (non-associative) $\mathbb{Z}$-action via $h \cdot n := h^n$ for all h in G and $n \in \mathbb{Z}$.

Proposition 3.2. — *A* 1*-cochain b is in* $\operatorname{Im}\delta$ *if and only if, for every circuit $\mathcal{K} = (e_1, \ldots, e_k)$ in $\mathbb{E}$, we have*

$$b(\mathcal{K}) := b(e_1) \cdot b(e_2) \cdots b(e_k) = 1.$$

Proof. — If $b \in \operatorname{Im}\delta$, the condition above is easily verified. To complete the proof we show that if the condition si verified, then there exists a cochain $a \in C^0(\Gamma; G)$ such that $\delta a = b$. We choose a vertex $v \in V(\Gamma)$ and impose $a(v) = 1$, for any other vertex $w \in V(\Gamma)$ we consider a path $\mathcal{P} = (e_1, \ldots, e_m)$ starting in v and ending in w. We set

$$a(w) := b(\mathcal{P}) = b(e_1) \cdots b(e_m).$$

By the condition on circuits, the cochain a is well defined, independently of path $\mathcal{P}$, and by construction we have $b = \delta a$. □

3.1.2. Trees and tree-like graphs

Definition 3.3. — *A tree is a graph that does not contain any circuit. A* tree-like *graph is a connected graph whose only circuits are loops.*

Remark 3.4. — For every connected graph Γ, the first Betti number $b_1(\Gamma) = E - V + 1$ is the dimension rank of the homology group $H_1(\Gamma; \mathbb{Z})$. Note that, b_1 being positive, $E \geqslant V - 1$. This inequality is an equality if and only if Γ is a tree.

For every connected graph Γ with vertex set V and edge set E, we can choose a connected subgraph T with the same vertex set and edge set $E_T \subset E$ such that T is a tree.

Definition 3.5. — *The graph T is called a* spanning tree *for* Γ.

Lemma 3.6. — *If $E_{\text{sep}} \subset E$ is the set of edges in Γ that are separating, then $E_{\text{sep}} \leqslant V - 1$ with equality if and only if Γ is tree-like.*

Proof. — If T is a spanning tree for Γ and E_T its edge set, then $E_{\text{sep}} \subset E_T$. Indeed, an edge $e \in E_{\text{sep}}$ is the only path between its two extremities, therefore, since T is connected, e must be in E_T. Thus $E_{\text{sep}} \leqslant E_T = V - 1$, with equality if and only if all the edges of Γ are loops or separating edges, *i.e.* if Γ is a tree-like graph. □

3.1.3. Graph contraction and graph G-covers

Consider a graph Γ with vertex set V and edge set E, we choose a subset $D \subset E$ which is stable by the G-action.

Definition 3.7. — *Consider the graph Γ_0 such that:*

(1) *the edge set of Γ_0 is $E_0 := E \backslash D$;*
(2) *given the relation in V, $v \sim w$ if v and w are linked by an edge $e \in D$, the vertex set of Γ_0 is $V_0 := V/\sim$.*

The graph Γ_0 inherits naturally a G-action. The natural morphism $\Gamma \to \Gamma_0$ is called contraction *of D or D-contraction.*

Edge contraction will be useful, in particular we will consider the image of the exterior differential δ and its restriction over contractions of a given graph. If Γ_0 is a contraction of Γ, then $E(\Gamma_0)$ is canonically a subset of $E(\Gamma)$. As a consequence, cochains over Γ_0 are cochains over Γ with the additional condition that the values on $E(\Gamma)\backslash E(\Gamma_0)$ are all the identity. Then we have a natural immersion $C^i(\Gamma_0; G) \hookrightarrow C^i(\Gamma; G)$. Consider the two exterior differentials

$$\delta\colon C^0(\Gamma; G) \to C^1(\Gamma; G) \quad \text{and} \quad \delta_0\colon C^0(\Gamma_0; G) \to C^1(\Gamma_0; G).$$

The following proposition follows.

Proposition 3.8. — *The differential δ_0 is the restriction of δ on $C^0(\Gamma_0; G)$.*

$$\operatorname{Im}\delta_0 = C^1(\Gamma_0; \mathbb{Z}/\ell) \cap \operatorname{Im}\delta.$$

Given any graph Γ with a G-action, we define its G-quotient Γ/G by $V(\Gamma/G) := V(\Gamma)/G$ and $E(\Gamma/G) := E(\Gamma)/G$. The conditions on the G-action assure that Γ/G is well defined. Moreover, the edge contraction of a subset $D \subset E(\Gamma)$ stable under G-action, is compatible with the quotient, so that if $\Gamma \to \Gamma_0$ is the D-contraction, then $\Gamma/G \to \Gamma_0/G$ is the contraction of D/G (the G-action on the new quotiented graphs is trivial).

We call a G-graph morphism $\widetilde{\Gamma} \to \Gamma$ a graph G-cover if $\Gamma \cong \widetilde{\Gamma}/G$ and $\widetilde{\Gamma} \to \Gamma$ is the natural quotient morphism. For any vertex $\widetilde{v}$ of $\widetilde{\Gamma}$, we denote by $H_{\widetilde{v}}$ its stabilizer in G. For any vertex v of Γ, its preimages in $V(\widetilde{\Gamma})$ all have a stabilizer in the same conjugacy class $\mathcal{H}$ in $\mathcal{T}(G)$, *i.e.* for all $\widetilde{v}$ in $f^{-1}(v)$ we have $H_{\widetilde{v}} \in \mathcal{H}$. Moreover, for every subgroup H in the class $\mathcal{H}$, there exists a vertex preimage $\widetilde{v}$ of v with stabilizer exactly H. In particular the cardinality of the v fiber is $|G|/|H|$ where $|H|$ is the cardinality of any subgroup in $\mathcal{H}$. The same is true for any edge e in $E(\Gamma)$.

We observe that it is possible to give another description of the cochain groups of $\widetilde{\Gamma}$ by considering the graph G-cover $\widetilde{\Gamma} \to \Gamma$. Given a set T with a G-action and a conjugation $e \mapsto \bar{e}$, define $\overline{\mathrm{Hom}}^G(T, G)$ as the set of morphisms $f\colon T \to G$ compatible with the G-action and such that $f(\bar{e}) = f(e)^{-1}$. We extend the G-action on $\mathbb{E}(\widetilde{\Gamma})$ by defining it as a fibered product in the category of G-sets, $\mathbb{E}(\widetilde{\Gamma}) := E(\widetilde{\Gamma}) \times_{E(\Gamma)} \mathbb{E}(\Gamma)$, this prevents that $h \cdot e = \bar{e}$ for some $e \in \mathbb{E}(\widetilde{\Gamma})$ and $h \in G$.

Proposition 3.9. — *Consider a graph G-cover $f\colon \widetilde{\Gamma} \to \Gamma$. We have the identification*

$$C^0(\widetilde{\Gamma}; G) = \prod_{v \in V(\Gamma)} \mathrm{Hom}^G(f^{-1}(v), G).$$

Moreover,

$$C^1(\widetilde{\Gamma}; G) = \prod_{e \in \mathbb{E}(\Gamma)} \overline{\mathrm{Hom}}^G(f^{-1}(e), G).$$

3.1.4. Graph G-cover of an admissible G-cover

Definition 3.10 (dual graph). — *Consider a nodal curve C, its dual graph $\Gamma(C)$ is defined by*

$$V(\Gamma(C)) := \{\textit{irreducible components of } C\}$$
$$E(\Gamma(C)) := \{\textit{nodes of } C\}$$

with the natural link relations.

Remark 3.11. — We observe that the set of oriented edges $\mathbb{E}(\Gamma(C))$ is naturally identified with the set of nodes equipped with a privileged branch, or equivalently with the set of node preimages on the normalization $\overline{C}$.

For any admissible G-cover $F \to C$, consider the dual graphs $\widetilde{\Gamma} := \Gamma(F)$ and $\Gamma := \Gamma(C)$. Therefore $\Gamma = \widetilde{\Gamma}/G$ and $\widetilde{\Gamma} \to \Gamma$ is a graph G-cover. We recall the correspondence between admissible G-covers over a stable curve C, and twisted G-covers over C, treated in section 2.3.4. As a consequence, the dual graphs $\widetilde{\Gamma}$ and Γ introduced for any admissible G-cover, are well defined for the associated twisted G-cover, too.

Consider the function b_F defined on $\mathbb{E}(\widetilde{\Gamma})$ that sends any oriented edge $\widetilde{e}$ to the local index (H, χ) of the associated node, where the privileged node branch (necessary to define the local index) is given by the $\widetilde{e}$ orientation (see the Definition 2.32 of the local index). We observe that for any $h \in G$, $b_F(h \cdot \widetilde{e}) = (hHh^{-1}, \chi^h)$. Furthermore, we associate to b_F another function M_{b_F} sending any e in $\mathbb{E}(\Gamma)$ to the G-type $[\![H, \chi]\!]$ of the associated node.

Definition 3.12. — *We call* index cochain *of the admissible G-cover $F \to C$, the function b_F. Moreover, we call type function of $F \to C$, the function M_{b_F}. When there is no risk of confusion, we denote the type function of $F \to C$ simply by M.*

Remark 3.13. — Once we choose a privileged rth root of the unit $\xi_r = \exp(2\pi i/r)$ for every positive integer r, the index cochain b_F is identified with a 1-cochain in $C^1(\widetilde{\Gamma}; G)$, and the associated type function is a function $M_{b_F} : \mathbb{E}(\Gamma) \to [\![G]\!]$.

Remark 3.14. — In the case of an admissible G-cover with G abelian group, the type function uniquely determines the index cochain. In the case of $G = \boldsymbol{\mu}_\ell$, our notation reduces to the multiplicity index notation of Chiodo and Farkas [8].

We observe that the order of $M_{b_F}(e)$ is well defined for any $e \in \mathbb{E}(\Gamma)$ as the order of any element in the conjugacy class, therefore we define the function $r \colon \mathbb{E}(\Gamma(C)) \to \mathbb{Z}_{>0}$. Clearly $r(e) = r(\bar{e})$ for any e.

Definition 3.15. — *A pair $(\Gamma, r(-))$, where $r \colon \mathbb{E}(\Gamma) \to \mathbb{Z}_{>0}$ is an even function, is called* decorated graph. *The pair $(\Gamma(C), r(-))$ given by the admissible G-cover $F \to C$ (or equivalently, the associated twisted G-cover (C, ϕ)) is called decorated graph of the cover. If there is no risk of confusion, we will refer also to $\Gamma(C)$ or Γ alone as the decorated graph.*

Let $D \subset \mathbb{E}(\widetilde{\Gamma})$ be the subset of edges where the cochain b_F of local indices is trivial, that is

$$D := \{\widetilde{e} \in \mathbb{E}(\widetilde{\Gamma}) |\ b_F(\widetilde{e}) = 1\}.$$

Definition 3.16. — *The graph $\widetilde{\Gamma}_0$ is the result of the D-contraction on $\widetilde{\Gamma}$. The graph Γ_0 is the quotient $\widetilde{\Gamma}_0/G$. Equivalently, it is the graph Γ after the conctraction of the edges where the type function M has value $[\![1]\!]$.*

3.2. Basic theory of sheaves in groups and torsors

In this section we refer in particular to Calmès and Fasel paper [5] for notations and definitions. Consider a scheme S and a site $\mathbb{T}$ over the category Sch/S of S-schemes. An S-sheaf for us will be a sheaf over $(Sch/S, \mathbb{T})$. Consider $\mathcal{G}$ an S-sheaf in groups, and P an S-sheaf in sets with a left $\mathcal{G}$-action.

Definition 3.17 (torsor). — *The sheaf P is a torsor under $\mathcal{G}$, or a $\mathcal{G}$-torsor, if*

(1) *the application* $\mathcal{G} \times P \to P \times P$, *where the components are the action and the identity, is an isomorphism;*
(2) *for every covering* $\{S_i\}$ *of* S, $P(S_i)$ *is non-empty for every* i.

For example, if G is a finite group, a principal G-bundle over a scheme S, is a $\mathcal{G}$-torsor, where $\mathcal{G}$ is the S-sheaf in groups defined by $\mathcal{G}(S') \coloneqq S' \times G$ for any S-scheme S'. When we consider any S-sheaf in groups $\mathcal{G}$ as acting on itself, we get a $\mathcal{G}$-torsor called *trivial* $\mathcal{G}$*-torsor*.

PROPOSITION 3.18 (see [5, Proposition 2.2.2.4]). — *An* S*-sheaf* P *with a left* $\mathcal{G}$*-action is a torsor if and only if it is* $\mathbb{T}$*-locally isomorphic to the trivial torsor* $\mathcal{G}$.

Consider two S-sheaves P and P' with $\mathcal{G}$-action respectively on the left and on the right.

DEFINITION 3.19. — *We denote by* $P' \wedge^{\mathcal{G}} P$ *the cokernel sheaf of the two morphisms*

$$\mathcal{G} \times P' \times P \rightrightarrows P' \times P$$

given by the $\mathcal{G}$*-action on* P *and* P' *respectively. This is called* contracted product. *Equivalently,* $P' \wedge^{\mathcal{G}} P$ *is the sheafification of the presheaf of the orbits of* $\mathcal{G}$ *acting on* $P' \times P$ *by*

$$(h, (z', z)) \mapsto (z'h^{-1}, hz).$$

Remark 3.20. — If $\mathcal{G}$ is the sheaf in groups constantly equal to $\mathbb{C}^*$ and P, P' are two line bundles, then the contracted product is simply the usual tensor product $P \otimes P'$.

If another S-sheaf in groups $\mathcal{G}'$ acts on the left on P', then the contracted product $P' \wedge^{\mathcal{G}} P$ has a $\mathcal{G}'$-action on the left, too. The same is true for a $\mathcal{G}'$-action on the right on P.

LEMMA 3.21 (see [5, Lemma 2.2.2.10]). — *The* $\wedge$ *construction is associative. Consider* $\mathcal{G}$ *and* $\mathcal{G}'$ *two* S*-sheaves in groups,* P *and* P' *two* S*-sheaves with respectively left* $\mathcal{G}$*-action and right* $\mathcal{G}'$*-action, finally* P'' *an* S*-sheaf with* $\mathcal{G}'$*-action on the left and* $\mathcal{G}$*-action on the right, and the actions commute. Then there exists a canonical isomorphism*

$$(P' \wedge^{\mathcal{G}'} P'') \wedge^{\mathcal{G}} P \cong P' \wedge^{\mathcal{G}'} (P'' \wedge^{\mathcal{G}} P).$$

Moreover, we have $\mathcal{G} \wedge^{\mathcal{G}} P \cong P$ *for every* $\mathcal{G}$*-torsor* P.

Proposition 3.22 (see [5, Proposition 2.2.2.12]). — *Consider a morphism $\mathcal{G} \to \mathcal{G}'$ of S-sheaves in groups and the associated $\mathcal{G}$-action (on the right) on $\mathcal{G}'$. The map*

$$P \mapsto \mathcal{G}' \wedge^{\mathcal{G}} P,$$

from the category of $\mathcal{G}$-torsors to $\mathcal{G}'$-torsors, is a functor.

Definition 3.23. — *Given an S-scheme S' and a site $\mathbb{T}$ on Sch/S, we denote by $H^1_{\mathbb{T}}(S', \mathcal{G})$ the pointed set of $\mathcal{G}$-torsors (on the left) over S' with respect to the $\mathbb{T}$ topology. The base point of the set being the torsor $\mathcal{G}$ itself.*

We observe that if P' is a $\mathcal{G}$-bitorsor, on the left and on the right, over S', then the contracted product $P' \wedge^{\mathcal{G}} P$ is a $\mathcal{G}$-torsor (on the left) for every $\mathcal{G}$-torsor P. Therefore P' induces a map

$$P' \wedge^{\mathcal{G}} - \colon H^1_{\mathbb{T}}(S', \mathcal{G}) \to H^1_{\mathbb{T}}(S', \mathcal{G}).$$

This cohomology type notation fits with the cohomology type behavior we are going to describe. We refer for the following results to [5, §2.2.5] or [14, Chap. 3]. Consider three S-sheaves in groups fitting in a short exact sequence

$$1 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 1. \tag{3.2}$$

Theorem 3.24. — *This gives a long exact sequence in cohomology*

$$\begin{aligned}1 \to \mathcal{G}_1(S) \to \mathcal{G}_2(S) \xrightarrow{\delta} \mathcal{G}_3(S) \xrightarrow{\tau} H^1_{\mathbb{T}}(S, \mathcal{G}_1)& \\ \xrightarrow{w} H^1_{\mathbb{T}}(S, \mathcal{G}_2) \to H^1_{\mathbb{T}}(S, \mathcal{G}_3)&.\end{aligned} \tag{3.3}$$

This is an exact sequence of pointed sets, and it is exact in $\mathcal{G}_1(S)$ and $\mathcal{G}_2(S)$ as a sequence of groups.

To describe the map τ, observe that $\mathcal{G}_3 = \mathcal{G}_2/\mathcal{G}_1$. By [14, Proposition 3.1.2], the set $\mathcal{G}_3(S)$ is in bijection with the set of sub-$\mathcal{G}_1$-torsors of $\mathcal{G}_2$. Any object Q in $\mathcal{G}_3(S)$ in sent by τ on the $\mathcal{G}_1$-torsor induced by the pullback along $\mathcal{G}_2 \to \mathcal{G}_2/\mathcal{G}_1$. As a consequence $\tau(Q)$ is a $\mathcal{G}_1$-bitorsor.

Via the τ map we also have a $\mathcal{G}_3(S)$-action on $H^1_{\mathbb{T}}(S, \mathcal{G}_1)$. Indeed, for every Q in $\mathcal{G}_3(S)$ and for every $\mathcal{G}_1$-torsor P, we obtain by contracted product the $\mathcal{G}_1$-torsor $\tau(Q) \wedge^{\mathcal{G}_1} P$.

To state the next proposition, we observe that $\mathcal{G}_1$ acts trivially on the right on $\mathcal{G}_3$, therefore given any $\mathcal{G}_1$-torsor P (on the left), we have the identification of sheaves $\mathcal{G}_3 \wedge^{\mathcal{G}_1} P = \mathcal{G}_3$. Consider the map $\mathcal{G}_2 \to \mathcal{G}_3$ in the short exact sequence (3.2), and its image via the contracted product functor of Proposition 3.22,

$$\mathcal{G}_2 \wedge^{\mathcal{G}_1} P \xrightarrow{-\wedge P} \mathcal{G}_3 \wedge^{\mathcal{G}_1} P = \mathcal{G}_3.$$

We define $\mathcal{G}_2^P := \mathcal{G}_2 \wedge^{\mathcal{G}_1} P$, and $\delta^P \colon \mathcal{G}_2^P(S) \to \mathcal{G}_3(S)$.

PROPOSITION 3.25 (see [14, Proposition 3.3.3]). — *For every P in $H^1_{\mathbb{T}}(S, \mathcal{G}_1)$, the stabilizer of P with respect to the $\mathcal{G}_3(S)$-action induced by τ, is the image of $\delta^P \colon \mathcal{G}_2^P(S) \to \mathcal{G}_3(S)$.*

4. Singularities of $\overline{\mathcal{R}}_{g,G}$

4.1. Ghost automorphisms of a twisted curve

Consider a twisted G-cover (C, ϕ), its automorphism group is

$$\operatorname{Aut}(\mathsf{C}, \phi) := \left\{ (\mathsf{f}, \rho) \,\middle|\, \mathsf{f} \in \operatorname{Aut}(\mathsf{C}),\ \rho \colon \phi \xrightarrow{\cong} \mathsf{f}^*\phi \right\}.$$

We observe that this group does not act faithfully on the universal deformation $\operatorname{Def}(\mathsf{C}, \phi)$. Indeed, Proposition 2.37 describes the group $\operatorname{Aut}_{\mathsf{C}}(\mathsf{C}, \phi)$ of automorphisms of (C, ϕ) acting trivially on C, and these automorphisms are the ones acting trivially on $\operatorname{Def}(\mathsf{C}, \phi)$, too. It becomes natural to consider the group

$$\begin{aligned} \underline{\operatorname{Aut}}(\mathsf{C}, \phi) &:= \operatorname{Aut}(\mathsf{C}, \phi) / \operatorname{Aut}_{\mathsf{C}}(\mathsf{C}, \phi) \\ &= \{ \mathsf{f} \in \operatorname{Aut}(\mathsf{C}) \mid \mathsf{f}^*\phi \cong \phi \text{ as twisted } G\text{-covers} \}. \end{aligned}$$

Remark 4.1. — The local description of $\overline{\mathcal{R}}_{g,G}$ at $[\mathsf{C}, \phi]$ could be rewritten

$$\operatorname{Def}(\mathsf{C}) / \underline{\operatorname{Aut}}(\mathsf{C}, \phi).$$

The coarsening $\mathsf{C} \to C$ induces moreover a group morphism $\underline{\operatorname{Aut}}(\mathsf{C}, \phi) \to \operatorname{Aut}(C)$. We denote the kernel and the image of this morphism by $\underline{\operatorname{Aut}}_C(\mathsf{C}, \phi)$ and $\operatorname{Aut}'(C)$ (see also [8, Chap. 2]). They fit into the following short exact sequence,

$$1 \to \underline{\operatorname{Aut}}_C(\mathsf{C}, \phi) \to \underline{\operatorname{Aut}}(\mathsf{C}, \phi) \to \operatorname{Aut}'(C) \to 1. \tag{4.1}$$

DEFINITION 4.2. — *The group $\underline{\operatorname{Aut}}_C(\mathsf{C}, \phi)$ is called the group of ghost automorphisms of (C, ϕ).*

To describe the ghost automorphisms of a twisted G-cover, we start by describing $\mathrm{Aut}_C(\mathsf{C})$, the group of ghost automorphisms of the curve (not necessarily lifting to the cover). Consider a node q of C whose local picture is $[\{x'y'=0\}/\boldsymbol{\mu}_r]$. Given an automorphism $\eta \in \mathrm{Aut}_C(\mathsf{C})$, the local action of η at q can be represented by an automorphism of $V=\{x'y'=0\}\subset \mathbb{A}^2$ such that

$$(x',y')\mapsto(\xi x',y'),$$

with ξ a primitive root in $\boldsymbol{\mu}_r$. We observe moreover that $(\xi x',y')\equiv(\xi^{u+1}x',\xi^{-u}y')$ for any integer u, by the $\boldsymbol{\mu}_r$-action on V. Anyway, when it is not specified otherwise, we will use the lifting that acts trivially on the y' coordinate. Consider the dual decorated graph $(\Gamma(C),r(-))$ associated to the twisted G-cover (C,ϕ), by definition $r(e)$ is the order of the q-stabilizer where q is the node associated to edge $e\in E(\Gamma(C))$. We naturally extend the function r over $\mathbb{E}(\Gamma)$. As a consequence of the definition of $\mathrm{Aut}_C(\mathsf{C})$, the action of η outside the C nodes is trivial. Then the whole group $\mathrm{Aut}_C(\mathsf{C})$ is generated by automorphisms of the form $(x',y')\mapsto(\xi x',y')$ on a node, and trivial elsewhere. We are interested in representing $\mathrm{Aut}_C(\mathsf{C})$ as acting on the edges of the dual graph, thus we introduce the following group.

Definition 4.3. — *Consider a decorated graph* $(\Gamma,r(-))$, *we denote by* r_{lcm} *the least common multiple of all the orders* $r(e)$ *of the edges of* Γ. *We define the group*

$$S(\Gamma;r(-)) := \{f\colon \mathbb{E}(\Gamma)\to\mathbb{Z}/r_{\mathrm{lcm}} \mid f(e)=f(\bar e)\in\mathbb{Z}/r(e)\subset\mathbb{Z}/r_{\mathrm{lcm}}\}\,.$$

We recall that $E(\Gamma)$ is the set of Γ edges while $\mathbb{E}(\Gamma)$ is the set of Γ edges with an orientation. If $e\in\mathbb{E}(\Gamma)$ is an oriented Γ edge, then we denote by $\bar e$ the same edge with reversed orientation.

If $(\Gamma(\mathsf{C}),r(-))$ is the decorated dual graph of the twisted G-cover (C,ϕ), we define a morphism $S(\Gamma(\mathsf{C});r(-))\to\mathrm{Aut}_C(\mathsf{C})$, sending any function a on the automorphism whose action at the node associated to $e\in E(\Gamma)$ is

$$(x',y')\mapsto(\mathsf{a}(e)\cdot x',y').$$

The morphism above is a canonical isomorphism, and we have the following identification

$$\underline{\mathrm{Aut}}_C(\mathsf{C}) = S(\Gamma(\mathsf{C});r(-)) = \bigoplus_{e\in E(\Gamma)}\boldsymbol{\mu}_{r(e)}.$$

Clearly the action is trivial on nodes with order $r=1$, so $S(\Gamma(\mathsf{C});r(-)) = S(\Gamma_0(\mathsf{C});r(-))$. We observe again that by choosing a privileged rth root $\exp(2\pi i/r)$ for any positive integer r, $\boldsymbol{\mu}_r$ is identified to $\mathbb{Z}/r$, and then $S(\Gamma_0(\mathsf{C});r(-))\cong\bigoplus_{e\in E(\Gamma_0)}\mathbb{Z}/r(e)$.

The group of ghost automorphisms

$$\underline{\mathrm{Aut}}_C(\mathsf{C},\phi)=\{\mathsf{a}\in \mathrm{Aut}_C(\mathsf{C})|\ \mathsf{a}^*\phi\cong\phi\}$$

is a subset of $\mathrm{Aut}_C(\mathsf{C})$. To describe it we will characterize the automorphisms in $\mathrm{Aut}_C(\mathsf{C})$ lifting to the twisted G-cover ϕ.

If C is the coarse space of C, we consider the admissible G-cover $F\to C$ associated to (C,ϕ), and the normalization morphism $\mathsf{nor}\colon \overline{C}\to C$. We denote by C_i the irreducible components of C, by $\overline{C}_i$ their normalizations and by $F_i:=F|_{C_i}$ the F restrictions. For any open subscheme $U\hookrightarrow C$, $F|_U\to U$ is an admissible G-cover. Finally we define the pullbacks $\overline{F}:=\mathsf{nor}^*F$ and $\overline{U}:=\mathsf{nor}^*U$. Consider the category Sch/C of C-schemes and the Zarisky site $\mathbb{T}_{\mathrm{Zar}}$ on it. Given the definition of automorphisms for admissibile G-covers as stated in Section 2.3.3, we introduce the following definition.

Definition 4.4. — *The C-sheaf in groups $\mathfrak{H}^F$ is defined for any open C-scheme $U\hookrightarrow C$ by,*

$$\mathfrak{H}^F(U):=\mathrm{Aut}_{\mathrm{Adm}}(U,F|_U).$$

We observe that F is a C-sheaf with a left $\mathfrak{H}^F$-action, and we have a short exact sequence of C-sheaves in groups,

$$1\to\mathfrak{H}^F\to \mathsf{nor}_*\mathsf{nor}^*\mathfrak{H}^F\xrightarrow{t}\mathfrak{H}^F|_{\mathrm{Sing}\,C}\to 1. \tag{4.2}$$

The central sheaf is defined over any open subscheme $U\hookrightarrow C$ as

$$\mathsf{nor}_*\mathsf{nor}^*\mathfrak{H}^F(U)=\mathrm{Aut}_{\mathrm{Adm}}(\overline{U},\overline{F}|_{\overline{U}}).$$

There exists a $2:1$ cover $\overline{F}|_{\mathrm{Sing}\,C}\to F|_{\mathrm{Sing}\,C}$. If ε is a section of $\mathsf{nor}_*\mathsf{nor}^*\mathfrak{H}^F(U)$, its image via t is obtained on every point p of $F|_{\mathrm{Sing}}(U)$ by taking the difference between the actions of ε on the two preimages, and therefore $t(\varepsilon)$ is well defined up to ordering the branches of every node.

We pass to the associated long exact sequence. We observe that $\mathfrak{H}^F(C)=\mathrm{Hom}^G(\mathcal{T}(F),G)=\mathrm{Hom}^G(\mathcal{T}(\widetilde{\Gamma}),G)$ by Proposition 2.12. Moreover,

$$\mathsf{nor}_*\mathsf{nor}^*\mathfrak{H}^F(C)=\mathrm{Aut}_{\mathrm{Adm}}(\overline{C},\overline{F})=\prod_i \mathrm{Hom}^G(\mathcal{T}(F_i),G),$$

because the $\overline{C}_i$ are the connected components of $\overline{C}$. If we denote by $f\colon\widetilde{\Gamma}\to\Gamma$ the graph G-cover associated to $F\to C$, by Proposition 3.9 we have

$$\mathsf{nor}_*\mathsf{nor}^*\mathfrak{H}^F(C)=C^0(\widetilde{\Gamma};G).$$

Finally, if $q_1,\dots,q_\delta$ are the nodes of C, then

$$\mathfrak{H}^F|_{\mathrm{Sing}\,C}(C)=\prod_j \mathrm{Aut}_{\mathrm{Adm}}(q_j,F_{q_j})=\prod_j \mathrm{Hom}^G(F_{q_j},G).$$

By the definition of the dual graph $\widetilde{\Gamma}$, the right hand side of the equality above is identified with $\prod_{e\in E(\Gamma)} \overline{\mathrm{Hom}}^G(f^{-1}(e), G)$, and by Proposition 3.9 we have $\mathfrak{H}^F|_{\mathrm{Sing}\, C}(C) \cong C^1(\widetilde{\Gamma}; G)$.

We consider the long exact sequence (3.3) associated to any short exact sequence of C-sheaves. In this case, the site over Sch/C is the Zariski site $\mathbb{T}_{\mathrm{Zar}}$ and taking the long exact sequence associated to (4.2), we get

$$\begin{aligned} 1 \longrightarrow \mathrm{Hom}^G(\mathcal{T}(\widetilde{\Gamma}), G) \xrightarrow{\quad i \quad} C^0(\widetilde{\Gamma}; G) \xrightarrow{\quad \delta \quad} \\ C^1(\widetilde{\Gamma}; G) \xrightarrow{\quad \tau \quad} H^1_{\mathbb{T}_{\mathrm{Zar}}}(C; \mathfrak{H}^F) \xrightarrow{\quad w \quad} \\ H^1_{\mathbb{T}_{\mathrm{Zar}}}(C; \mathsf{nor}_*\mathsf{nor}^*\mathfrak{H}^F) \longrightarrow 1, \end{aligned} \tag{4.3}$$

where $H^1_{\mathbb{T}_{\mathrm{Zar}}}(C; \mathfrak{H}^F)$ is the set of $\mathfrak{H}^F$-torsors, $H^1_{\mathbb{T}_{\mathrm{Zar}}}(C; \mathsf{nor}_*\mathsf{nor}^*\mathfrak{H}^F)$ is the set of $\mathsf{nor}_*\mathsf{nor}^*\mathfrak{H}^F$-torsors on C, and it is identified with $H^1_{\mathbb{T}_{\mathrm{Zar}}}(\overline{C}; \mathsf{nor}^*\mathfrak{H}^F)$. Moreover, the only object of $H^1_{\mathbb{T}_{\mathrm{Zar}}}(C; \mathfrak{H}^F|_{\mathrm{Sing}})$ is the trivial torsor. The first part of this sequence is exactly the sequence (3.1). To describe explicitly the map w, consider the normalization $\mathsf{nor}\colon \overline{C} \to C$. Then,

$$w\colon (F \to C) \mapsto (\overline{F} = \mathsf{nor}^* F \to \overline{C}).$$

Given any cochain $b \in C^1(\widetilde{\Gamma}; G)$, by what we saw in Section 3.2 we know that $\mathfrak{H}^F$ acts on the right on $\tau(b)$. Therefore we can define an admissible G-cover by the contracted product $\tau(b) \wedge^{\mathfrak{H}^F} F$ (see Definition 3.19).

Recall that to every admissible G-cover $F \to C$ is assigned an index cochain b_F (see Definition 3.12). Now consider an automorphism

$$\mathsf{a} \in \mathrm{Aut}_C(\mathsf{C}) = S(\Gamma(\mathsf{C}); r(-)).$$

We define a 1-cochain $b_F \cdot \mathsf{a} \in C^1(\widetilde{\Gamma}(\mathsf{C}); G)$: for every oriented edge $\widetilde{e}$ of $\widetilde{\Gamma}(\mathsf{C})$, $b_F(\widetilde{e}) = (H, \chi)$ that is a character $\chi\colon H \to \mathbb{C}$ where $H \subset G$ is a cyclic subgroup. If $e \in \mathbb{E}(\Gamma(\mathsf{C}))$ is the projection of $\widetilde{e}$, we define

$$(b_F \cdot \mathsf{a})(\widetilde{e}) := \chi^{-1}(\mathsf{a}(e)) \in H \subset G.$$

Proposition 4.5. — *Given a finite group G and a twisted G-cover (C, ϕ) consider the associated admissible G-cover $F \to C$, whose index cochain is b_F. If $\mathsf{a} \in \underline{\mathrm{Aut}}_C(\mathsf{C}) = S(\Gamma(\mathsf{C}), r(-))$ is a ghost automorphism of C, the pullback twisted G-cover $(\mathsf{C}, \mathsf{a}^*\phi)$, where $\mathsf{a}^*\phi = \phi \circ \mathsf{a}$, is associated to the admissible G-cover*

$$\tau(b_F \cdot \mathsf{a}) \wedge^{\mathfrak{H}^F} F.$$

Proof. — Consider a node q of the twisted curve C. In Remark 2.21 we observed that the local picture of (C, ϕ) at q can be seen as a twisted object on $V \cong \{x'y'=0\}$. This is equivalent to a principal G-bundle $\widetilde{F} \to V$ with a compatible $\boldsymbol{\mu}_r$-action and the other conditions of the same Remark. We remark that $\widetilde{F}/\boldsymbol{\mu}_r \to V/\boldsymbol{\mu}_r \cong V$ is isomorphic to the local picture of $F \to C$ around q.

We start by characterizing $\mathsf{a}^*\phi$ with respect to ϕ. Again from Remark 2.21 we know that $\widetilde{F} = (\widetilde{F}' \sqcup \widetilde{F}'')/\kappa_q$ locally at node q, where $\widetilde{F}'$ and $\widetilde{F}''$ are the two pullbacks of $\widetilde{F}$ on the node branches $\mathbb{A}^1_{x'}$, $\mathbb{A}^1_{y'}$, and $\kappa_q \colon \widetilde{F}'_q \to \widetilde{F}''_q$ is the gluing morphism of the central fibers. We consider the oriented edge $e \in \mathbb{E}(\Gamma(\mathsf{C}))$ associated to q with privileged branch $\mathbb{A}^1_{x'}$. We can lift the a-action to the twisted object by acting trivially on $\mathbb{A}^1_{y'}$ and by $\mathsf{a}(e)$ multiplication on $\mathbb{A}^1_{x'}$. By the same Remark 2.21 we can lift the action to $\widetilde{F}'$,

$$\begin{array}{ccc} \widetilde{F}' & \xrightarrow{\alpha'(\mathsf{a}(e))} & \widetilde{F}' \\ \downarrow & & \downarrow \\ \mathbb{A}^1_{x'} & \xrightarrow{\mathsf{a}(e)\cdot -} & \mathbb{A}^1_{x'}. \end{array}$$

We observe that $\mathsf{a}^*\widetilde{F}' \cong \widetilde{F}'$ and $\mathsf{a}^*\widetilde{F}'' \cong \widetilde{F}''$, what really changes is the gluing morphism. Indeed,

$$\mathsf{a}^*\widetilde{F} = (\widetilde{F}' \sqcup \widetilde{F}'')/(\kappa_q \circ \alpha'(\mathsf{a}(e))).$$

By definition of α', for any point $\widetilde{q}'$ on the fiber $\widetilde{F}'_q$, we have $\alpha'(\mathsf{a}(e))(\widetilde{q}') = \nu'(\mathsf{a}(e), \widetilde{q}')$. Again by the α' definition, $\alpha'(\mathsf{a}(e))(\widetilde{q}') = \psi(h_{\widetilde{q}'}, \widetilde{q}')$, where (H, χ) is the local index at $\widetilde{q}'$ and $h_{\widetilde{q}'} = \chi^{-1}(\mathsf{a}(e))$, that is $h_{\widetilde{q}'} = (b_F \cdot \mathsf{a})(\widetilde{e})$, where $\widetilde{e} \in \mathbb{E}(\widetilde{\Gamma})$ is the edge associated to $\widetilde{q}'$ and the privileged branch associated to $\widetilde{e}$ orientation is $\mathbb{A}^1_{x'}$.

By the definition of contracted product, if we denote by a^*F the admissible G-cover associated to $\mathsf{a}^*\widetilde{F}$, then $\mathsf{a}^*F = \tau(b_F \cdot \mathsf{a}) \wedge^{\mathfrak{H}^F} F$ as we wanted to prove. □

Theorem 4.6. — *Given a twisted G-cover (C, ϕ) with associated admissible G-cover $F \to C$, any ghost automorphism $\mathsf{a} \in \underline{\mathrm{Aut}}_C(\mathsf{C})$ lifts to a ghost automorphism of (C, ϕ) if and only if the 1-cochain $b_F \cdot \mathsf{a}$ is in $\operatorname{Ker}\tau = \operatorname{Im}\delta$ of sequence (4.3).*

Proof. — After the proposition above, we have that $\phi \cong \mathsf{a}^*\phi$ if and only if $\tau(b_F \cdot \mathsf{a})$ acts trivially via the contracted product on F. We consider the restriction $F_{\mathrm{gen}} \to C_{\mathrm{gen}}$ over the generic locus. We observe that F_{gen} is an $\mathfrak{H}^F$-torsor on C_{gen}, then we apply Proposition 3.25 to obtain that $\tau(b_F \cdot \mathsf{a}) \wedge^{\mathfrak{H}^F} F_{\mathrm{gen}} = F_{\mathrm{gen}}$ if and only if $b_F \cdot \mathsf{a} \in \operatorname{Im}\delta^F$. This is a necessary condition to have $\tau(b_F \cdot \mathsf{a}) \wedge^{\mathfrak{H}^F} F = F$, but it is also sufficient because F_{gen} completes uniquely to F.

It remains to prove that $\operatorname{Im}\delta^F = \operatorname{Im}\delta$. In particular we observe that the contracted product does not act on the δ morphism, so $\delta^F = \delta$ and the proof is concluded. □

Remark 4.7. — Given the dual graph G-cover $\widetilde{\Gamma} \to \Gamma$ associated to $F \to C$, and the contracted decorated graph $(\widetilde{\Gamma}_0, r(-))$, we recall the subcomplexes inclusion $C^i(\widetilde{\Gamma}_0; G) \subset C^i(\widetilde{\Gamma}; G)$ for $i = 0, 1$. We also consider the exterior differential δ_0 on $C^0(\widetilde{\Gamma}; G)$, *i.e.* the restriction of the δ operator to this group. Because of Proposition 3.8, we have $\operatorname{Im}(\delta_0) = C^1(\widetilde{\Gamma}_0; G) \cap \operatorname{Im}\delta$.

Remark 4.8. — Previously we obtained a characterization of the cochains in $\operatorname{Im}(\delta)$ that we could restate in our new setting. Indeed, because of Proposition 3.2, an automorphism $\mathsf{a} \in S(\widetilde{\Gamma}_0; r(-))$ is an element of $\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$ if and only if for every circuit $(\widetilde{e}_1, \ldots, \widetilde{e}_k)$ in $\widetilde{\Gamma}_0$ we have $\prod_{i=1}^k (b_F \cdot \mathsf{a})(\widetilde{e}_i) = 1$.

4.2. Smooth points

In Remark 4.1 we discussed the fact that every point $[\mathsf{C}, \phi] \in \overline{\mathcal{R}}_{g,G}$ has a local picture isomorphic to $\mathrm{Def}(\mathsf{C})/\underline{\mathrm{Aut}}(\mathsf{C}, \phi)$. This is a quotient of the form $\mathbb{C}^n/\mathfrak{G}$ where $\mathfrak{G}$ is a finite subgroup of $\mathrm{GL}(\mathbb{C}^n)$. In this setting we introduce some automorphisms called quasireflections.

Definition 4.9 (Quasireflection). — *Any finite order complex automorphism $\mathsf{h} \in \mathrm{GL}(\mathbb{C}^n)$ is called a quasireflection if its fixed locus has dimension exactly $n - 1$. Equivalently, h is a quasireflection if, for an opportune choice of the basis, we can diagonalize it as*

$$\mathsf{h} = \mathrm{Diag}(\xi, 1, 1, \ldots, 1),$$

where ξ is a primitive root of the unit of order equal to the order of h. Given a finite group $\mathfrak{G} \subset \mathrm{GL}(\mathbb{C}^n)$, we denote by $\mathrm{QR}(\mathfrak{G})$ the subgroup generated by quasireflections.

Quasireflections have the interesting property that any complex vector space, quotiented by them, keeps being a smooth variety. In particular if $\mathsf{h} \in \mathrm{GL}(\mathbb{C}^n)$ is a quasireflection, the variety $\mathbb{C}^n/\mathsf{h}$ is isomorphic to $\mathbb{C}^n$.

Proposition 4.10 (see [19]). — *Consider any vector space quotient $V' := V/\mathfrak{G}$, where $V \cong \mathbb{C}^n$ is a complex vector space and $\mathfrak{G} \subset \mathrm{GL}(V)$ is a finite group. The variety V' is smooth if and only if $\mathfrak{G}$ is generated by quasireflections.*

Therefore, to find the smooth points of $\overline{\mathcal{R}}_{g,G}$, by Proposition 4.10 we need to know when $\underline{\mathrm{Aut}}(\mathsf{C},\phi)$ is generated by quasireflections. We start by recalling the quasireflection analysis in the case of stable scheme theoretic curves.

Definition 4.11. — *Within a stable curve C, an elliptic tail is an irreducible component of geometric genus 1 that meets the rest of the curve in only one point called an elliptic tail node. Equivalently, E is an elliptic tail if and only if its algebraic genus is 1 and $E \cap \overline{C\backslash E} = \{q\}$.*

An element $i \in \mathrm{Aut}(C)$ is an elliptic tail automorphism if there exists an elliptic tail E of C such that i fixes E and his restriction to $\overline{C\backslash E}$ is the identity. An elliptic tail automorphism of order 2 is called an elliptic tail quasireflection (ETQR). In the literature ETQRs are called elliptic tail involutions (or ETIs), we changed this convention in order to generalize the notion.

Remark 4.12. — Every scheme theoretic curve of algebraic genus 1 with one marked point has exactly one involution i. Then there is a unique ETQR associated to every elliptic tail.

More precisely an elliptic tail E could be of two types. The first type is a smooth curve of geometric genus 1 with one marked point, *i.e.* an elliptic curve: in this case we have $E = \mathbb{C}/\Lambda$, for Λ integral lattice of rank 2, the marked point is the origin, and the only involution is the map induced by $x \mapsto -x$ on $\mathbb{C}$. The second type is the rational line with one marked point and an autointersection point: in this case we can write $E = \mathbb{P}^1/\{0 \equiv \infty\}$, the marked point is the origin, and the only involution is the map induced by $z \mapsto 1/z$ on $\mathbb{P}^1$.

From Remark 2.2 we have a coordinate system on $\mathrm{Def}(C)$ and on the canonical subscheme $\mathrm{Def}(C; \mathrm{Sing}\, C)$. Furthermore, the quotient of these two schemes has a splitting

$$\mathrm{Def}(C)/\,\mathrm{Def}(C; \mathrm{Sing}\, C) \cong \bigoplus_{j=1}^{\delta} \mathbb{A}^1_{t_j}.$$

These coordinates systems on the space $\mathrm{Def}(C; \mathrm{Sing}\, C)$ and $\mathrm{Def}(C)/\,\mathrm{Def}(C; \mathrm{Sing}\, C)$ allow the detection of quasireflections. Indeed, the diagonalizations of the a-action on the two spaces determines a diagonalization of the a-action on the whole $\mathrm{Def}(C)$. Therefore, a is a quasireflection if it acts non-trivially on exactly one coordinate of scheme $\mathrm{Def}(C; \mathrm{Sing}\, C)$ or $\mathrm{Def}(C)/\,\mathrm{Def}(C; \mathrm{Sing}\, C)$. The following theorem by Harris and Mumford describes the action of the automorphism group $\mathrm{Aut}(C)$ on $\mathrm{Def}(C)$.

THEOREM 4.13 (See [16, Theorem 2]). — *Consider a stable curve C of genus $g \geqslant 4$. An element of* $\mathrm{Aut}(C)$ *acts as a quasireflection on* $\mathrm{Def}(C)$ *if and only if it is an ETQR. In particular, if $\eta \in \mathrm{Aut}(C)$ is an ETQR acting non trivially on the tail E with elliptic tail node q_j, then η acts trivially on* $\mathrm{Def}(C; \mathrm{Sing}\, C)$*, and its action on* $\mathrm{Def}(C)/\mathrm{Def}(C; \mathrm{Sing}\, C)$ *is $t_j \mapsto -t_j$ on the coordinate associated to q_j, and the identity $t \mapsto t$ on the remaining coordinates.*

In Remark 2.23 we have seen that the deformations $\mathrm{Def}(C; \mathrm{Sing}\, C)$ and $\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\, \mathsf{C})$ are canonically identified. For the deformation of the nodes, the description is slightly different. We denote by δ the number of nodes, by $r_1, \dots, r_\delta$ the order of the cyclic stabilizers in C of the nodes $q_1, \dots, q_\delta$ respectively. Then,

$$\mathrm{Def}(\mathsf{C})/\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\, \mathsf{C}) \cong \bigoplus_{j=1}^{\delta} \mathbb{A}^1_{\tilde{t}_j},$$

and every node comes with a flat representable morphism of Deligne–Mumford stacks, isomorphic to

$$[\{x'y' = \widetilde{t}_j\}/\boldsymbol{\mu}_{r_j}] \to \mathbb{A}^1_{\tilde{t}_j},$$

where the local stabilizer $\boldsymbol{\mu}_{r_j}$ acts by $\xi \cdot (x', y', \widetilde{t}_j) = (\xi x', \xi^{-1} y', \widetilde{t}_j)$. Also there exists a canonical morphism $\mathbb{A}^1_{\tilde{t}_j} \to \mathbb{A}^1_{t_j}$ such that $(\widetilde{t}_j)^{r_j} = t_j$.

Remark 4.14. — Consider a stack-theoretic curve E whose coarse space E is a genus 1 curve with a marked point. In the case of an elliptic tail of a curve C, the marked point is the point of intersection between E and $\overline{\mathsf{C} \setminus \mathsf{E}}$.

If E is an elliptic curve, then $\mathsf{E} = E$ and the curve has exactly one involution i_0. In case E is rational, its normalization is the stack $\bar{\mathsf{E}} = [\mathbb{P}^1/\boldsymbol{\mu}_r]$, with $\boldsymbol{\mu}_r$ acting by multiplication, and $\mathsf{E} = \bar{\mathsf{E}}/\{0 \equiv \infty\}$. There exists a canonical involution i_0 in this case too: the pushforward of the inverse involution on $\mathbb{P}^1$, *i.e.* $z \mapsto 1/z$. We consider the autointersection node of E and its local picture $[\{x'y' = 0\}/\boldsymbol{\mu}_r]$, then the local picture of the same node in E is $\{xy = 0\}$ with $x = (x')^r$ and $y = (y')^r$. Therefore the i_0-action is represented locally by $(x', y') \mapsto (y', x')$ and the product $x'y'$ is unchanged, so i_0 acts trivially on the smoothing coordinate $\widetilde{t}$ associated to this node. We observe that of all the possible liftings of the canonical involution i_0 of E, i_0 is the only E involution acting trivially on $\widetilde{t}$.

Given any twisted curve C with an elliptic tail E whose elliptic tail node is called q, the construction above defines a canonical involution i_0 on E up to non-trivial action on q.

Definition 4.15. — *An element* $\mathsf{i} \in \underline{\mathrm{Aut}}(\mathsf{C}, \phi)$ *is an ETQR if there exists an elliptic tail* E *of* C *with elliptic tail node* q*, such that the action of* i *on* $\mathsf{C} \backslash \mathsf{E}$ *is trivial, and the action on* E*, up to non-trivial action on* q*, is the canonical involution* i_0*.*

Lemma 4.16. — *Consider an element* h *of* $\underline{\mathrm{Aut}}(\mathsf{C}, \phi)$*. It acts as a quasireflection on* $\mathrm{Def}(\mathsf{C})$ *if and only if one of the following is true:*

(1) *the automorphism* h *is a ghost quasireflection,* i.e. *an element of* $\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$ *which moreover operates as a quasireflection;*
(2) *the automorphism* h *is an ETQR, using the generalized Definition 4.15.*

Proof. — We first prove the "only if" part. If h acts trivially on certain coordinates of $\mathrm{Def}(\mathsf{C})$, *a fortiori* we have that its coarsening h acts trivially on the corresponding coordinates of $\mathrm{Def}(C)$. Therefore h acts as the identity or as a quasireflection on $\mathrm{Def}(C)$. In the first case, h is a ghost automorphism and we are in case (1). If h acts as a quasireflection, then it is a classical ETQR as we pointed out on Theorem 4.13, and it acts non-trivially on the coordinate associated to an elliptic tail node q.

As we know that the action of h is trivial on $\mathrm{Def}(C; \mathrm{Sing}\, C)$, so is the action of h. It remains to know the action of h on the nodes with non-trivial stabilizer and other than q. If the elliptic tail where h operates non trivially is a rational component with an autointersection node q_1, by hypothesis h acts trivially on the universal deformation $\mathbb{A}^1_{\tilde{t}_1}$ of this node. Therefore, the h restriction to the elliptic tail has to be the canonical involution i_0 (see Remark 4.14). For every node other than q and q_1, if the local picture is $[\{x'y' = 0\}/\boldsymbol{\mu}_{r_j}]$, the action of h must be of the form

$$(x', y') \mapsto (\xi x', y') \equiv (x', \xi y') \quad \text{for some} \;\; \xi \in \boldsymbol{\mu}_{r_j}.$$

If $\xi \neq 1$ this gives a non-trivial action on the associated universal deformation $\mathbb{A}^1_{\tilde{t}_j}$, against our hypothesis. By Definition 4.15 this implies that h is an ETQR of (C, ϕ).

For the "if" part, we observe that a ghost quasireflection is automatically a quasireflection. It remains to prove the case of point (2). By definition of ETQR, its action on $\mathrm{Def}(\mathsf{C})$ can be non-trivial only on the components associated to the separating node q of the tail. As a consequence h acts as the identity or as a quasireflection. The local coarse picture at q is $\{xy = 0\}$, where $y = 0$ is the branch lying on the elliptic tail. Then the action of h on the coarse space is $(x, y) \mapsto (-x, y)$. Therefore the action is *a fortiori* non trivial on the coordinate associated to the stack node q in $\mathrm{Def}(\mathsf{C})$. □

Definition 4.17. — *For any stable curve C we denote by* $\mathrm{QR}(C)$ *the subgroup of* $\mathrm{Aut}(C)$ *generated by classical ETQRs. For any twisted G-cover* (C,ϕ) *we denote by* $\mathrm{QR}(\mathsf{C},\phi)$ *the subgroup of* $\underline{\mathrm{Aut}}(\mathsf{C},\phi)$ *generated by ETQRs, and by* $\mathrm{QR}_C(\mathsf{C},\phi)$ *the subgroup of* $\underline{\mathrm{Aut}}(\mathsf{C},\phi)$ *generated by ETQRs which moreover are ghosts.*

Lemma 4.18. — *Any element* $h\in\mathrm{QR}(C)$ *which could be lifted to* $\underline{\mathrm{Aut}}(\mathsf{C},\phi)$*, has a lifting in* $\mathrm{QR}(\mathsf{C},\phi)$*, too.*

Proof. — By definition, $\underline{\mathrm{Aut}}(\mathsf{C},\phi)$ is the set of automorphisms $\mathsf{s}\in\mathrm{Aut}(\mathsf{C})$ such that $\mathsf{s}^*\phi\cong\phi$. Consider $h\in\mathrm{QR}(C)$ such that its decomposition in ETQRs is $h=i_0i_1\cdots i_m$, and every i_k acts non-trivially on an elliptic tail E_k. Any lifting of h is in the form $\mathsf{h}=\mathsf{i}_0\mathsf{i}_1\cdots\mathsf{i}_m\cdot\mathsf{a}$, where i_t is an ETQR acting non-trivially on a twisted elliptic tail E_k, and a is a ghost acting non-trivially only on nodes other than the elliptic tail nodes of the E_k. We observe that every i_k is a lifting in $\mathrm{Aut}(\mathsf{C})$ of i_k. Moreover, by construction, $\mathsf{h}^*\phi\cong\phi$ if and only if $\mathsf{i}_k^*\phi\cong\phi$ for every k and $\mathsf{a}^*\phi\cong\phi$. This implies that every i_k lies in $\underline{\mathrm{Aut}}(\mathsf{C},\phi)$, and therefore $\mathsf{h}\cdot\mathsf{a}^{-1}$ is a lifting of h lying in $\mathrm{QR}(\mathsf{C},\phi)$. □

We recall the short exact sequence (4.1),

$$1\to\underline{\mathrm{Aut}}_C(\mathsf{C},\phi)\to\underline{\mathrm{Aut}}(\mathsf{C},\phi)\xrightarrow{\beta}\mathrm{Aut}'(C)\to 1$$

and introduce the group $\mathrm{QR}'(C)\subset\mathrm{Aut}'(C)$, generated by liftable quasi-reflections, *i.e.* by those quasireflections $h\in\mathrm{Aut}(C)$ lying in $\mathrm{Im}\,\beta$. By Lemma 4.18, $\mathrm{QR}'(C)=\beta(\mathrm{QR}(\mathsf{C},\phi))$. Using also Lemma 4.16, we obtain that the following is a short exact sequence

$$1\to\mathrm{QR}_C(\mathsf{C},\phi)\to\mathrm{QR}(\mathsf{C},\phi)\to\mathrm{QR}'(C)\to 1.$$

Theorem 4.19. — *The group* $\underline{\mathrm{Aut}}(\mathsf{C},\phi)$ *is generated by quasireflections if and only if both* $\underline{\mathrm{Aut}}_C(\mathsf{C},\phi)$ *and* $\mathrm{Aut}'(C)$ *are generated by quasireflections.*

Proof. — By combining the previous sequences,

$$1\to\frac{\underline{\mathrm{Aut}}_C(\mathsf{C},\phi)}{\mathrm{QR}_C(\mathsf{C},\phi)}\to\frac{\underline{\mathrm{Aut}}(\mathsf{C},\phi)}{\mathrm{QR}(\mathsf{C},\phi)}\to\frac{\mathrm{Aut}'(C)}{\mathrm{QR}'(C)}\to 1.$$

The theorem follows. □

This gives a first important result for the moduli space of twisted G-covers $\overline{\mathcal{R}}_{g,G}$. As we know that any point $[\mathsf{C},\phi]\in\overline{\mathcal{R}}_{g,G}$ is smooth if and only if the group $\underline{\mathrm{Aut}}(\mathsf{C},\phi)$ is generated by quasireflections, then the following theorem is straightforward.

THEOREM 4.20. — *Given a twisted G-cover $\phi\colon \mathsf{C} \to BG$ over a twisted curve C of genus $g \geqslant 4$ whose coarse space is C, the point $[\mathsf{C}, \phi]$ of the moduli space $\overline{\mathcal{R}}_{g,G}$ is smooth if and only if the group $\mathrm{Aut}'(C)$ is generated by ETQRs and the group of ghost automorphisms $\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$ is generated by quasireflections.*

We introduce two closed loci of $\overline{\mathcal{R}}_{g,G}$,

$$N_{g,G} := \left\{[\mathsf{C}, \phi] \,\middle|\, \mathrm{Aut}'(C) \text{ is not generated by ETQRs}\right\},$$
$$H_{g,G} := \{[\mathsf{C}, \phi] \mid \underline{\mathrm{Aut}}_C(\mathsf{C}, \phi) \text{ is not generated by quasireflections}\}.$$

We have by Theorem 4.20 that the singular locus $\mathrm{Sing}\,\overline{\mathcal{R}}_{g,G}$ is their union

$$\mathrm{Sing}\,\overline{\mathcal{R}}_{g,G} = N_{g,G} \cup H_{g,G}. \tag{4.4}$$

Remark 4.21. — Consider the natural moduli space projection $\pi\colon \overline{\mathcal{R}}_{g,G} \to \overline{\mathcal{M}}_g$, then we have the inclusion $N_{g,G} \subset \pi^{-1}\,\mathrm{Sing}\,\overline{\mathcal{M}}_g$. Indeed, we saw that $\mathrm{QR}'(C) = \mathrm{Aut}'(C) \cap \mathrm{QR}(C)$ and therefore $\mathrm{Aut}(C) = \mathrm{QR}(C)$ implies $\mathrm{Aut}'(C) = \mathrm{QR}'(C)$. This means that $\left(\pi^{-1}\,\mathrm{Sing}\,\overline{\mathcal{M}}_g\right)^c \subset (N_{g,G})^c$, and taking the complementary we obtain the result.

We can interpret equality (4.4) as the fact that the singular locus is the union of two subloci: one coming from "old" singularities, the other coming from data encoded only in the ghost structure of the twisted G-covers.

The following lemma allows to characterize quasireflections in the ghost group. Consider the decorated graph $(\Gamma(C), r(-))$ associated to a twisted G-cover (C, ϕ), and its contraction $(\Gamma_0, r(-))$ (see Definition 3.16).

LEMMA 4.22. — *Consider a ghost automorphism a in the group $\underline{\mathrm{Aut}}_C(\mathsf{C}) = S(\Gamma_0; r(-))$. If a is a quasireflection in $\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$ then $\mathsf{a}(e) = 1$ for all edges but one that is a separating edge of $\Gamma_0(\mathsf{C})$.*

Proof. — If a is a quasireflection in $\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$, the value on all but one of the coordinates must be 0. Therefore $\mathsf{a}(e) = 1 \in \boldsymbol{\mu}_{r(e)}$ on all the edges but one, say e_1. If there exists a preimage $\widetilde{e}_1$ in $\mathbb{E}(\widetilde{\Gamma}_0)$ that is in any circuit $(\widetilde{e}_1, \ldots, \widetilde{e}_k)$ of $\widetilde{\Gamma}_0$ with $k \geqslant 1$, then we have, by Remark 4.8, that $\prod(b_F \cdot \mathsf{a})(\widetilde{e}_i) = 1$. As $\mathsf{a}(e_1) \neq 1$, then $(b_F \cdot \mathsf{a})(\widetilde{e}_1) \neq 1$ and therefore there exists $i > 1$ such that $(b_F \cdot \mathsf{a})(\widetilde{e}_i) \neq 1$ too. This would imply that, if e_i is the image in Γ_0 of $\widetilde{e}_i$, then $\mathsf{a}(e_i) \neq 1$, contradiction. Thus $\widetilde{e}_1$ is not in any circuit, then it is a separating edge and so is e_1.

Reciprocally, consider an automorphism $\mathsf{a} \in S(\Gamma_0; r(-))$ such that there exists an oriented separating edge e_1 with the property that $\mathsf{a}(e) = 1$ for every e in $\mathbb{E}\backslash\{e_1, \bar{e}_1\}$ and $\mathsf{a}(e_1)$ is a non-zero element of $\boldsymbol{\mu}_{r(e_1)}$. Then for every circuit $(\widetilde{e}'_1, \dots, \widetilde{e}'_k)$ of $\mathbb{E}(\widetilde{\Gamma}_0)$, we have $\prod(b_F \cdot \mathsf{a})(\widetilde{e}'_i) = 1$ and so a is in $\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$ by Theorem 4.6. □

5. Non-canonical singularities

5.1. Characterization of the non-canonical locus

In order to detect the singularity canonicity, we need a tool called age invariant. After its introduction we will be able to prove the bipartition of $\mathrm{Sing}^{\mathrm{nc}}\,\overline{\mathcal{R}}_{g,G}$.

5.1.1. The age invariant

Consider the case of a vector space quotient $V/\mathfrak{G}$. In the case of the group $\mathfrak{G}$ not being generated by quasireflections, we need another tool to distinguish between canonical singularities and non-canonical singularities. The age is a positive function $\mathfrak{G} \to \mathbb{Q}$.

Definition 5.1 (Age). — *Consider a $\mathfrak{G}$-representation $\rho\colon \mathfrak{G} \to \mathrm{GL}(V)$. For any element $\mathsf{h} \in \mathfrak{G}$ of order r, there exists a diagonalization $\mathsf{h} = \mathrm{Diag}(\xi_r^{a_1}, \xi_r^{a_2}, \dots, \xi_r^{a_n})$, where $\xi_r = \exp(2\pi i/r)$ is a privileged rth root of the unit and $0 \leqslant a_i < r$ for any $i = 1, \dots n$. In this setting*

$$\mathrm{age}(\mathsf{h}) = \frac{1}{r}\sum_{i=1}^{n} a_i.$$

Definition 5.2 (Junior group). — *A finite group $\mathfrak{G} \subset \mathrm{GL}(\mathbb{C}^m)$ that contains no quasireflections is called junior if the image of the age function intersects the open interval $]0,1[$,*

$$\mathrm{age}\,\mathfrak{G} \cap \,]0,1[\; \neq \varnothing.$$

The group $\mathfrak{G}$ is called senior if the intersection is empty.

Remark 5.3. — The definition of age depends on the non-canonical choice of a privileged root ξ_r, but the image $\mathrm{age}(\mathfrak{G}) \subset \mathbb{Q}$ does not depend on this choice. Therefore junior and senior group are well defined.

Proposition 5.4 (Age criterion, see [20]). — *Consider any vector space quotient $V' := V/\mathfrak{G}$, where $V \cong \mathbb{C}^n$ is a complex vector space and $\mathfrak{G} \subset \mathrm{GL}(V)$ is a finite group containing no quasireflections. Then V' has a non-canonical singularity if and only if $\mathfrak{G}$ is junior.*

We will use the Age Criterion to find non-canonical singularities by the study of group $\underline{\mathrm{Aut}}(\mathsf{C},\phi)$ action on $\mathrm{Def}(\mathsf{C},\phi)$. We point out that to satisfy the hypothesis of Age Criterion, it is necessary for $\underline{\mathrm{Aut}}(\mathsf{C},\phi)$ to be quasireflection free. As this is often not the case, the following lemma is necessary to represent the same singularity by a group with no quasireflections.

Proposition 5.5 (see [19]). — *Consider a finite subgroup $\mathfrak{G}\subset \mathrm{GL}(\mathbb{C}^n)$. There exists an isomorphism $u\colon \mathbb{C}^n/\,\mathrm{QR}(\mathfrak{G})\to\mathbb{C}^n$ and a finite subgroup $\mathfrak{K}\subset\mathrm{GL}(\mathbb{C}^n)$ isomorphic to the quotient $\mathfrak{G}/\,\mathrm{QR}(\mathfrak{G})$, such that the following diagram is commutative.*

$$\begin{array}{ccccc}
\mathbb{C}^n & \longrightarrow & \mathbb{C}^n/\,\mathrm{QR}(\mathfrak{G}) & \xrightarrow{\;u\;} & \mathbb{C}^n \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{C}^n/\mathfrak{G} & \xrightarrow{\;\cong\;} & (\mathbb{C}^n/\,\mathrm{QR}(\mathfrak{G}))/(\mathfrak{G}/\,\mathrm{QR}(\mathfrak{G})) & \xrightarrow{\;\cong\;} & \mathbb{C}^n/\mathfrak{K}
\end{array}$$

5.1.2. T-curves and J-curves

We introduce two closed loci which are central in our description.

Definition 5.6 (T-curve). — *A twisted G-cover (C,ϕ) is a T-curve if there exists an automorphism $\mathsf{a}\in\underline{\mathrm{Aut}}(\mathsf{C},\phi)$ such that its coarsening a is an elliptic tail automorphism of order 6. The locus of T-curves in $\overline{\mathcal{R}}_{g,G}$ is denoted by $T_{g,G}$.*

Definition 5.7 (J-curve). — *A twisted G-cover (C,ϕ) is a J-curve if the group*

$$\underline{\mathrm{Aut}}_C(\mathsf{C},\phi)/\,\mathrm{QR}_C(\mathsf{C},\phi),$$

which is the group of ghosts quotiented by its subgroup of quasireflections, is junior. The locus of J-curve in $\overline{\mathcal{R}}_{g,G}$ is denoted by $J_{g,G}$.

Theorem 5.8. — *For $g\geqslant 4$, the non-canonical locus of $\overline{\mathcal{R}}_{g,G}$ is the union*

$$\mathrm{Sing}^{\mathrm{nc}}\,\overline{\mathcal{R}}_{g,G}=T_{g,G}\cup J_{g,G}.$$

Remark 5.9. — We observe that [8, Theorem 2.44], affirms exactly that in the case $G=\boldsymbol{\mu}_\ell$ with $\ell\leqslant 6$ and $\ell\neq 5$, the J-locus $J_{g,\boldsymbol{\mu}_\ell}$ is empty for every genus g, and therefore $\mathrm{Sing}^{\mathrm{nc}}\,\overline{\mathcal{R}}_{g,\boldsymbol{\mu}_\ell}$ coincides with the T-locus for these values of ℓ.

We introduce the notion of $\star$-smoothing, following [16] and [18].

Definition 5.10. — *Consider a twisted G-cover (C,ϕ) and a junior automorphism $\mathsf{a}\in\underline{\mathrm{Aut}}(\mathsf{C},\phi)/\,\mathrm{QR}(\mathsf{C},\phi)$, we say that the triple $(\mathsf{C},\phi,\mathsf{a})$ is $\star$-smoothable if*

- *on the coarse curve C there exists a cycle of m non-separating nodes $q_0,\dots,q_{m-1}$,* i.e. *we have $\mathsf{a}(q_i)=q_{i+1}$ for all $i=0,1\dots,m-2$ and $\mathsf{a}(q_{m-1})=q_0$;*
- *the action of a^m over the coordinate associated to every node is trivial. Equivalently, $\mathsf{a}^m(\widetilde{t}_{q_i})=\widetilde{t}_{q_i}$ for all $i=0,1,\dots,m-2$, where $\widetilde{t}_{q_i}$ is the coordinate on $\mathrm{Def}(\mathsf{C},\phi)$ associated to the q_i-smoothing (see Remark 2.23).*

If $(\mathsf{C},\phi,\mathsf{a})$ is $\star$-smoothable, there exists a deformation $(\mathsf{C}',\phi',\mathsf{a}')$ that smooths the m nodes and with $\mathsf{a}'\in\underline{\mathrm{Aut}}(\mathsf{C}',\phi')$. Moreover, this deformation preserves the age of the a-action on $\mathrm{Def}(\mathsf{C},\phi)/\,\mathrm{QR}$. Indeed, the eigenvalues of a are a discrete and locally constant set, thus constant by deformation. The T-locus and the J-locus are closed by $\star$-smoothing, *i.e.* if the deformation (C',ϕ') above is a T-curve or a J-curve, then (C,ϕ) is a T-curve or a J-curve.

Therefore in proving Theorem 5.8, we can suppose that every triple $(\mathsf{C},\phi,\mathsf{a})$ that we consider is $\star$*-rigid*, *i.e.* non-$\star$-smoothable. Indeed, if there exists a junior $\star$-smoothable automorphism $\mathsf{a}\in\underline{\mathrm{Aut}}(\mathsf{C},\phi)$, we smooth it until we obtain a rigid triple $(\mathsf{C}',\phi',\mathsf{a}')$ of the same age. Then, if (C',ϕ') is a T-curve or a J-curve, the same is true for (C,ϕ).

Proof of Theorem 5.8. — We will show in eight steps that if the group $\underline{\mathrm{Aut}}(\mathsf{C},\phi)/\,\mathrm{QR}\,(\mathsf{C},\phi)$ is junior, and (C,ϕ) is not a J-curve, then it is a T-curve. After the Age Criterion 5.4 and Proposition 5.5, this will prove Theorem 5.8. From now on we work under the hypothesis that $\mathsf{a}\in\underline{\mathrm{Aut}}_C(\mathsf{C},\phi)/$ QR is a non-trivial automorphism aged less than 1, that (C,ϕ) is not a J-curve and $(\mathsf{C},\phi,\mathsf{a})$ is $\star$-rigid.

In Steps 1 and 2 we fix the setting and prove two useful lemmata. In Step 3 we prove that all the nodes of C are fixed by a except at most 2 of them which are exchanged. In Step 4 we show that every irreducible component $\mathsf{Z}\subset\mathsf{C}$ is fixed by a. In Step 5 we can therefore conclude that there are no couple of exchanging nodes. In Step 6 and 7 we study the action of a on the irreducible components of C and the contributions to age a. Finally we prove the result in Step 8.

Step 1. — Consider the contracted decorated graph $(\Gamma_0,r(-))$ of (C,ϕ). As before, we call E_{sep} the set of separating edges of Γ_0. As stated in

Remark 2.23, we have the following splitting,

$$\mathrm{Def}(\mathsf{C},\phi) \cong \mathrm{Def}(\mathsf{C};\mathrm{Sing}\,\mathsf{C}) \oplus \bigoplus_{e\in E_{\mathrm{sep}}} \mathbb{A}_{\tilde{t}_e} \oplus \bigoplus_{e'\in E\setminus E_{\mathrm{sep}}} \mathbb{A}_{\tilde{t}_{e'}}, \tag{5.1}$$

where $\widetilde{t}_e$ is a coordinate parametrizing the smoothing of the node associated to the edge e. In particular for every vector subspace $V \subset \mathrm{Def}(\mathsf{C},\phi)$ and every automorphism a of (C,ϕ), we denote by $\mathrm{age}(\mathsf{a}|V)$ the age of the restriction $\mathsf{a}|_V$. If Z is a subcurve of C, then there exists a canonical inclusion $\mathrm{Def}(\mathsf{Z}) \subset \mathrm{Def}(\mathsf{C})$, and we define $\mathrm{age}(\mathsf{a}|\mathsf{Z}) := \mathrm{age}(\mathsf{a}|\mathrm{Def}(\mathsf{Z}))$.

Every ghost automorphism in $\underline{\mathrm{Aut}}(\mathsf{C},\phi)$ fixes the three summands of (5.1). Moreover, every quasireflection acts only on the summand $\bigoplus_{e\in E_{\mathrm{sep}}} \mathbb{A}_{\tilde{t}_e}$ by Lemmata 4.16 and 4.22. As a consequence, by Propostion 5.5, the group $\underline{\mathrm{Aut}}(\mathsf{C},\phi)/\,\mathrm{QR}$ acts on

$$\frac{\mathrm{Def}(\mathsf{C},\phi)/\,\mathrm{QR}}{\mathrm{Def}(\mathsf{C};\mathrm{Sing}\,\mathsf{C})} \cong \left(\frac{\bigoplus_{e\in E_{\mathrm{sep}}} \mathbb{A}_{\tilde{t}_e}}{\mathrm{QR}(\mathsf{C},\phi)}\right) \oplus \bigoplus_{e'\in E\setminus E_{\mathrm{sep}}} \mathbb{A}_{\tilde{t}_{e'}}. \tag{5.2}$$

Every quasireflection acts on exactly one coordinate $\widetilde{t}_e$ with $e \in E_{\mathrm{sep}}$. We rescale all the coordinates $\widetilde{t}_e$ by the action of $\mathrm{QR}(\mathsf{C},\phi)$. We call τ_e, for $e \in E(\Gamma_0)$, the new set of coordinates. Obviously $\tau_{e'} = \widetilde{t}_{e'}$ if $e' \in E(\Gamma_0)\setminus E_{\mathrm{sep}}$.

Step 2. — We show two lemmata about the age contribution of the a-action on nodes, that we call *aging* on nodes.

Definition 5.11 (coarsening order). — *If* $\mathsf{a} \in \underline{\mathrm{Aut}}(\mathsf{C},\phi)$ *and* a *is its coarsening, then we define*

$$\text{c-ord}\,\mathsf{a} := \mathrm{ord}\,a.$$

The coarsening order is the least integer n for which a^n is a ghost automorphism.

Lemma 5.12. — *Suppose that* $\mathsf{Z} \subset \mathsf{C}$ *is a subcurve of* C *such that* $\mathsf{a}(\mathsf{Z}) = \mathsf{Z}$ *and* $q_0, \dots q_{m-1}$ *is a cycle, by* a*, of nodes in* Z*. Then we have the following inequalities:*

1. $\mathrm{age}(\mathsf{a}|\mathsf{Z}) \geqslant \frac{m-1}{2}$;
2. *if the nodes* $q_0, \dots, q_{m-1}$ *are non-separating,* $\mathrm{age}(\mathsf{a}) \geqslant \frac{m}{\mathrm{ord}(\mathsf{a}|\mathsf{Z})} + \frac{m-1}{2}$;
3. *if* $\mathsf{a}^{\text{c-ord}\,\mathsf{a}}$ *is a senior ghost, we have* $\mathrm{age}(\mathsf{a}) \geqslant \frac{1}{\text{c-ord}(\mathsf{a})} + \frac{m-1}{2}$.

Proof. — We call $\tau_0, \tau_1, \dots, \tau_{m-1}$ the coordinates associated to nodes $q_0, \dots, q_{m-1}$ respectively. By hypothesis, $\mathsf{a}(\tau_0) = c_1 \cdot \tau_1$ and $\mathsf{a}^i(\tau_0) = c_i \cdot \tau_i$

for all $i = 2, \dots, m-1$, where the c_i are complex numbers. If $n' = \mathrm{ord}(\mathsf{a}|_{\mathsf{Z}})$, we have

$$\mathsf{a}^m(\tau_0) = \xi_{n'}^{um} \cdot \tau_0$$

where $\xi_{n'}$ is a primitive n'th root of the unit and u is an integer such that $0 \leqslant u < n'/m$. The integer u is called *exponent* of the cycle $(q_0, \dots, q_{m-1})$ with respect to the curve Z. Observe that $\mathsf{a}(\tau_{i-1}) = (c_i/c_{i-1}) \cdot \tau_i$ and $\mathsf{a}^m(\tau_i) = \xi_{n'}^{um} \cdot \tau_i$ for every i.

We can explicitly write the eigenvectors for the action of a on the coordinates $\tau_0, \dots, \tau_{m-1}$. Set $d := n'/m$ and $b := sd + u$ with $0 \leqslant s < m$, and consider the vector

$$\mathfrak{v}_b := (\tau_0 = 1,\ \tau_1 = c_1 \cdot \xi_{n'}^{-b}, \dots,\ \tau_i = c_i \cdot \xi_{n'}^{-ib}, \dots).$$

Then $\mathsf{a}(\mathfrak{v}_b) = \xi_{n'}^{b} \cdot \mathfrak{v}_b$. The contribution to the age of the eigenvalue $\xi_{n'}^b$ is b/n', thus we have

$$\mathrm{age}\,\mathsf{a} \geqslant \sum_{s=0}^{m-1} \frac{sd+u}{n'} = \frac{mu}{n'} + \frac{m-1}{2},$$

proving point (1).

If the nodes are non-separating, as we are supposing that $(\mathsf{C}, \phi, \mathsf{a})$ is $\star$-rigid, we have $u \geqslant 1$ and the point (2) is proved.

Suppose that a has order $n = \mathrm{ord}\,\mathsf{a}$ and its action on C has j nodes cycles of order $m_1, m_2, \dots, m_j$ and exponents respectively $u_1, \dots, u_j$ with respect to C. If $k = \text{c-ord}\,\mathsf{a}$, then a^k fixes every node, then we consider the coordinate τ_i of a node of the first cycle and we have

$$\mathsf{a}^k(\tau_i) = \xi_n^{w \cdot k} \cdot \tau_i,$$

where w is an integer such that $0 \leqslant w < n/k$. Repeating the same operation for every cycle we obtain another series of integers $w_1, w_2, \dots, w_j$. Therefore the age of a^k is

$$\mathrm{age}\,\mathsf{a}^k = \sum_{i=1}^{j} \frac{m_i w_i k}{n},$$

and it is greater or equal to 1 by hypothesis.

We observe that m_i divides k for all $i = 1, \dots, j$, and

$$u_i \cdot m_i \cdot \frac{k}{m_i} \equiv w_i \cdot k \mod n.$$

This implies that $u_i \geqslant w_i$ for every i.

By the point (2), the age of a on the ith cycle is bounded from below by $m_iu_i/n + (m_i-1)/2$. As a consequence

$$\begin{aligned} \operatorname{age} \mathsf{a} &\geqslant \sum_{i=1}^{j} \left(\frac{m_iu_i}{n} + \frac{m_i-1}{2} \right) \\ &\geqslant \sum_{i=1}^{j} \left(\frac{m_iw_i}{n} + \frac{m_i-1}{2} \right) \\ &\geqslant \frac{1}{k} + \frac{m_1-1}{2}. \end{aligned}$$ □

Step 3. — Because of Lemma 5.12, if the automorphism a induces a cycle of m nodes, then this cycles contributes by at least $\frac{m-1}{2}$ to the aging of a. Therefore, as a is junior, all the nodes of C are fixed except at most two of them, that are exchanged. Moreover, if a pair of non-fixed nodes exists, they contribute by at least $1/2$.

Step 4. — Consider an irreducible component $\mathsf{Z} \subset \mathsf{C}$, we want to prove $\mathsf{a}(\mathsf{Z}) = \mathsf{Z}$. Suppose there is a cycle of irreducible components $\mathsf{C}_1, \dots, \mathsf{C}_m$ with $m \geqslant 2$ such that $\mathsf{a}(\mathsf{C}_i) = \mathsf{C}_{i+1}$ for $i = 1, \dots, m-1$, and $\mathsf{a}(\mathsf{C}_m) = \mathsf{C}_1$. We call $\overline{\mathsf{C}}_i$ the normalizations of these components, and D_i the preimages of C nodes on $\overline{\mathsf{C}}_i$. We point out that this construction implies that $(\overline{\mathsf{C}}_i, D_i) \cong (\overline{\mathsf{C}}_j, D_j)$ for all i, j. Then, an argument of [16, p. 34] shows that the action of a on $\operatorname{Def}(\mathsf{C}; \operatorname{Sing} \mathsf{C})$ gives a contribution of at least $k \cdot (m-1)/2$ to $\operatorname{age} \mathsf{a}$, where

$$k = \dim H^1(\overline{\mathsf{C}}_i, T_{\overline{\mathsf{C}}_i}(-D_i)) = 3g_i - 3 + |D_i|.$$

This gives us two cases for which m could be greater than 1 with still a junior age: $k = 1$ and $m = 2$ or $k = 0$.

If $k = 1$ and $m = 2$, we have $g_i = 0$ or 1 for $i = 1, 2$. Moreover, the aging of at least $1/2$ sums to another aging of $1/2$ if there is a pair of non-fixed nodes. As a is junior, we conclude that $\mathsf{C} = \mathsf{C}_1 \cup \mathsf{a}(\mathsf{C}_1)$ but this implies $g(\mathsf{C}) \leqslant 3$, contradiction.

If $k = 0$, we have $g_i = 1$ or $g_i = 0$, the first is excluded because it implies $|D_i| = 0$ but the component must intersect the curve somewhere. Thus, for every component in the cycle, the normalization $\overline{\mathsf{C}}_i$ is the projective line $\mathbb{P}^1$ with 3 marked points. We have two cases: the component C_i intersects $\overline{\mathsf{C} \setminus \mathsf{C}_i}$ in 3 points or in 1 point, in the second case C_i has an autointersection node and $\mathsf{C} = \mathsf{C}_1 \cup \mathsf{a}(\mathsf{C}_1)$, which is a contradiction because $g(\mathsf{C}) < 4$. It remains the case in the image below.

As $\mathsf{C}_1, \mathsf{C}_2, \dots, \mathsf{C}_m$ are moved by a, every node on C_1 is transposed with another one or is fixed with its branches interchanged. If at least two

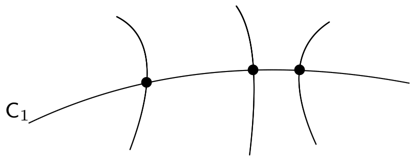


Figure 5.1. Case with $\mathsf{C}_1 \cong \mathbb{P}^1$ *and* 3 *marked points*

nodes are transposed we have an age contribution bigger or equal to 1 by Lemma 5.12. If only one node is transposed we have two cases. In the first case $\mathsf{C} = \mathsf{C}_1 \cup \mathsf{a}(\mathsf{C}_1) \cup \mathsf{C}_2 \cup \mathsf{a}(\mathsf{C}_2)$, where C_2 intersects only the component C_1 and in exactly one point. If $g(\mathsf{C}_2) \geqslant 2$, then the age is bigger than 1, if $g(\mathsf{C}_2) < 2$, then $g(\mathsf{C}) \leqslant 3$, contradiction.

In the second case, $\mathsf{C} = \mathsf{C}_1 \cup \mathsf{a}(\mathsf{C}_1) \cup \mathsf{C}_2$ where C_2 intersects C_1 and $\mathsf{a}(\mathsf{C}_1)$, both in exactly one point. If $g(\mathsf{C}_2) < 2$ we have another genus contradiction. By the results of [16, p. 28], to have $\operatorname{age}(\mathsf{a}|\mathsf{C}_2) < 1$ we must have $g(\mathsf{C}_2) = 2$ and the coarsening of a has order 2. Therefore by Lemma 5.12 point (3), a has age bigger or equal to 1.

Step 5. — We prove that every node is fixed by a. Consider the normalization $\mathsf{nor}\colon \bigsqcup_i \overline{\mathsf{C}}_i \to \mathsf{C}$ already introduced. If the age of a is lower than 1, *a fortiori* we have $\operatorname{age}(\mathsf{a}|\overline{\mathsf{C}}_i) < 1$ for all i. In [16, p.28] there is a list of those smooth stable curves for which there exists a non-trivial junior action.

(i) The projective line $\mathbb{P}^1$ with $\mathsf{a}\colon z \mapsto (-z)$ or $(\xi_4 z)$;
(ii) an elliptic curve with a of order $2,\ 3,\ 4$ or 6;
(iii) an hyperelliptic curve of genus 2 or 3 with a the hyperelliptic involution;
(iv) a bielliptic curve of genus 2 with a the canonical involution.

We observe that the order of the a-action on these components is always $2,\ 3,\ 4$ or 6. As a consequence, if a is junior, then $n = \text{c-ord}\,\mathsf{a} = 2, 3, 4, 6$ or 12, as it is the greatest common divisor between the $\text{c-ord}\big(\mathsf{a}|\overline{\mathsf{C}}_i\big)$.

First we suppose $\operatorname{ord}\mathsf{a} > \text{c-ord}\,\mathsf{a}$, thus $\mathsf{a}^{\text{c-ord}\,\mathsf{a}}$ is a ghost and it must be senior. Indeed, if $\mathsf{a}^{\text{c-ord}\,\mathsf{a}}$ is aged less than 1, then (C, ϕ) admits junior ghosts, contradicting our assumption. By point (3) of Lemma 5.12, if there exists a pair of non-fixed nodes, we obtain an aging of $1/n{+}1/2$ on node coordinates. If $\operatorname{ord}\mathsf{a} = \text{c-ord}\,\mathsf{a}$ the bound is even greater. As every component is fixed by a, the two nodes are non-separating, and by point (2) of Lemma 5.12 we obtain an aging of $2/n + 1/2$.

If $\overline{\mathsf{C}}_i$ admits an automorphism of order $3,\ 4$ or 6, by a previous analysis of Harris and Mumford (see [16] again), this yields an aging of, respectively, $1/3$, $1/2$ and $1/3$ on $H^1(\overline{\mathsf{C}}_i, T_{\overline{\mathsf{C}}_i}(-D_i))$.

These results combined, show that a non-fixed pair of nodes gives an age greater than 1. Thus, if a is junior, every node is fixed.

Step 6. — We study the action of a separately on every irreducible component. The a-action is non-trivial on at least one component C_i, and this component must lie in the list above.

In case (i), $\overline{\mathsf{C}}_i$ has at least 3 marked points because of the stability condition. Actions of type $x \mapsto \xi x$ have two fixed points on $\mathbb{P}^1$, thus at least one of the marked points is non-fixed. A non-fixed preimage of a node has order 2, thus the coarsening a of a is the involution $z \mapsto -z$. Moreover, C_i is the autointersection of the projective line and a exchanges the branches of the node. Therefore $\mathsf{a}^2|_{\overline{\mathsf{C}}_i}$ is a ghost automorphism of $\overline{\mathsf{C}}_i$. As a direct consequence of Theorem 4.6 and Remark 4.8, the action of a^2 on the coordinate associated to the autointersection node, is trivial. Therefore the action of a^2 on the same coordinate gives an aging of 0 or 1/2, by $\star$-rigidity it is 1/2.

The analysis for cases (iii) and (iv) is identical to that developed in [16]: the only possibility of a junior action is the case of an hyperelliptic curve E of genus 2 intersecting $\overline{\mathsf{C}\backslash\mathsf{E}}$ in exactly one point, whose hyperinvolution gives an aging of 1/2 on $H^1(\overline{\mathsf{C}}_i, T_{\overline{\mathsf{C}}_i}(-D_i))$.

Finally, in case (ii), we use again the analysis of [16]. The elliptic component E has 1 or 2 point of intersection with $\overline{\mathsf{C}\backslash\mathsf{E}}$. If there is 1 point of intersection, elliptic tail case, for a good choice of coordinates the coarsening a acts as $z \mapsto \xi_n z$, where n is $2, 3, 4$ or 6. The aging is, respectively, $0,\ 1/3,\ 1/2,\ 1/3$. If there are 2 points of intersection, elliptic ladder case, the order of a on E must be 2 or 4 and the aging respectively 1/2 or 3/4.

Step 7. — Resuming what we saw until now, if a is a junior automorphism of (C, ϕ), a its coarsening and C_1 an irreducible component of C, then we have one of the following:

(A) component C_1 is an hyperelliptic tail, crossing the curve in one point, with a acting as the hyperelliptic involution and aging 1/2 on $H^1(\overline{C}_1, T_{\overline{\mathsf{C}}_i}(-D_1))$;

(B) component C_1 is a projective line $\mathbb{P}^1$ autointersecting itself, crossing the curve in one point, with a the involution which fixes the nodes, and aging 1/2;

(C) component C_1 is an elliptic ladder, crossing the curve in two points, with a of order 2 or 4 and aging respectively 1/2 or 3/4;

(D) component C_1 is an elliptic tail, crossing the curve in one point, with a of order $2, 3, 4$ or 6 and aging $0, 1/3, 1/2$ or $1/3$;

(E) automorphism a acts trivially on C_1 with no aging.

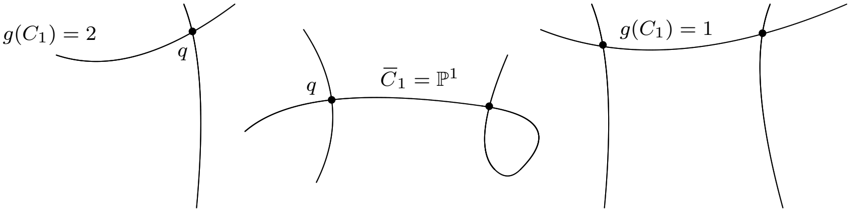


Figure 5.2. Components of type A, B and C.

We rule out cases (A), (B) and (C). At first we suppose there is a component of type (A) or (B). For genus reasons, the component intersected in both cases must be of type (E). We study the local action on the separating node q. The local picture at q is $[\{x'y' = 0\}/\boldsymbol{\mu}_r]$. The smoothing of the node is given by the stack $\left[\{xy = \widetilde{t}_q\}/\boldsymbol{\mu}_r\right]$. Consider the action of the automorphism a at the node, as the coarsening of a has order 2, then $\mathsf{a}\colon \widetilde{t}_q \mapsto \varsigma \cdot \widetilde{t}_q$ and $\varsigma^2 \in \boldsymbol{\mu}_r$. Therefore a^2 acts as the identity or as a quasireflection of factor ς^2. Thus $\tau_q = \widetilde{t}_q^{r'}$ where $r'|r$ is the order of ς^2. Therefore the action of a on $\mathbb{A}^1_{\widetilde{t}_q} / \mathrm{QR} = \mathbb{A}^1_{\tau_q}$ is $\tau_q \mapsto \varsigma^{r'} \cdot \tau_q = -\tau_q$. The additional age contribution is $1/2$, ruling out this case.

In case there is a component of type (C), if its nodes are separating, then one of them must intersect a component of type (E) and we use the previous idea. In case nodes are non-separating, we use Lemma 5.12. If $\operatorname{ord} \mathsf{a} > \operatorname{c-ord} \mathsf{a}$, then $\mathsf{a}^{\operatorname{c-ord} \mathsf{a}}$ is a senior ghost because (C, ϕ) is not a J-curve, thus by point (3) of the lemma there is an aging of $(1/\operatorname{c-ord} \mathsf{a})$ on the node coordinates. If $\operatorname{ord} \mathsf{a} = \operatorname{c-ord} \mathsf{a}$, the bound is even greater, as by point (2) we have an aging of $(2/\operatorname{c-ord} \mathsf{a})$. We observe that $\operatorname{c-ord} \mathsf{a} = 2, 4$ or 6, and in case $\operatorname{c-ord} \mathsf{a} = 6$ there must be a component of type (E). Using additional contributions listed above we rule out the case (C).

Step 8. — We proved that C contains components of type (D) or (E), *i.e.* the automorphism a acts non-trivially only on elliptic tails. If q is the elliptic tail node, there are two quasireflections acting on the coordinate $\widetilde{t}_q$: a ghost automorphism associated to this node and the elliptic tail quasireflection. If the order of the local stabilizer is r, then $\tau_q = \widetilde{t}_q^{2r}$.

If $\operatorname{ord} \mathsf{a} = 2$ we are in the ETQR case, this action is a quasireflection and it contributes to rescaling the coordinate $\widetilde{t}_q$.

If $\operatorname{ord} \mathsf{a} = 4$, the action on the (coarse) elliptic tail is $z \mapsto \xi_4 z$. The space $H^1(\overline{C}_i, T_{\overline{C}_i}(-D_i))$ is the space of 2-forms $H^0(\overline{C}_i, \omega_{\overline{C}_i}^{\otimes 2})$: this space is generated by $dz^{\otimes 2}$ and the action of a is $dz^{\otimes 2} \mapsto \xi_4^2 dz^{\otimes 2}$. Moreover, if the local picture of the elliptic tail node is $[\{x'y' = 0\}/\boldsymbol{\mu}_r]$, then $\mathsf{a}\colon (x', y') \mapsto$

$(\zeta x', \varrho y')$ such that $\zeta^r = \xi_4$ and $\varrho^r = 1$. As a consequence $\mathsf{a}\colon \widetilde{t}_q \mapsto \zeta \cdot \varrho \cdot \widetilde{t}_q$ and therefore $\tau_q \mapsto \xi_2 \tau_q$. Then, age $\mathsf{a} = 1/2 + 1/2$, proving the seniority of a.

If E admits an automorphism a of order 6, the action on the (coarse) elliptic tail is $\mathsf{a}\colon z \mapsto \xi_6^k z$. Then $dz^{\otimes 2} \mapsto \xi_3^k dz^{\otimes 2}$ and $\tau_q \mapsto \xi_3^k \tau_q$. For $k = 1, 4$ we have age lower than 1.

If (C, ϕ) is not a J-curve, we have shown that the only case where an automorphism a in $\underline{\mathrm{Aut}}(\mathsf{C}, \phi)/\,\mathrm{QR}$ is junior, is when its coarsening a is an elliptic tail automorphism of order 6. □

5.2. The J-locus in the case S_3

We consider the case of J_{g,S_3} and prove, thanks to the tools we developed, that this locus is empty.

Theorem 5.13. — *If G is the symmetric group S_3, then the non-canonical locus coincides with the T-locus,*

$$\mathrm{Sing}^{\mathrm{nc}}\, \overline{\mathcal{R}}_{g,S_3} = T_{g,S_3}.$$

In particular, a point $[\mathsf{C}, \phi]$ is a non-canonical singular point if and only if there exists an automorphism $\mathsf{a} \in \underline{\mathrm{Aut}}(\mathsf{C}, \phi)$ whose coarsening is an elliptic tail automorphism of order 6.

In order to prove this, we start with a lemma about an admissible G'-cover $F \to C$ over a 2-marked stable curve $(C; p_1, p_2)$, where G' is an abelian group. We observe that any conjugacy class in an abelian group contains exactly one element, therefore a G'-type (see Definition 2.25) is an element of G'. Moreover, if $\widetilde{p}_i$ is a preimage in F of a marked point p_i, then the local index at $\widetilde{p}_i$ equals the G'-type at p_i.

Lemma 5.14. — *If G' is an abelian group, $(C; p_1, p_2)$ a 2-marked stable curve, and $F \to C$ and admissible G'-cover over $(C; p_1, p_2)$, then the G'-types h_1 and h_2 at p_1 and p_2 respectively, are inverses, $h_1 = h_2^{-1}$.*

Proof. — We consider at first the case of a smooth 2-marked curve $(C; p_1, p_2)$. Because of the monodromy description given in Proposition 2.33 and Remark 2.34, the product $h_1 h_2$ is in the commutators subgroup of G', which is trivial because G' is abelian. Therefore $h_1 h_2 = 1$.

In the case of a general stable curve C, we denote by $\widetilde{p}_1^{(i)}, \ldots, \widetilde{p}_{m_i}^{(i)}$ the marked points on $\overline{C}_i$, *i.e.* the preimages of p_1, p_2 or the C nodes. By the previous point, if $h_j^{(i)}$ is the G'-type of F at the marked point $\widetilde{p}_j^{(i)}$, then

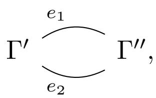
$\prod_{j=1}^{m_i} h_j^{(i)} = 1$, for every i. By the balancing condition, for every G'-type $h_j^{(i)}$ coming from a C-node, there exists another marked point on $\overline{C}$ with G'-type $h_{j'}^{(i')} = (h_j^{(i)})^{-1}$. Therefore

$$1 = \prod_{j,i} h_j^{(i)} = h_1 \cdot h_2. \qquad \square$$

Lemma 5.15. — *If* (C, ϕ) *is a twisted* S_3*-cover and* $\mathsf{a} \in \frac{\underline{\mathrm{Aut}}_C(\mathsf{C},\phi)}{\mathrm{QR}(\mathsf{C},\phi)}$ *is a ghost automorphism, then* $\mathrm{age}(\mathsf{a}) \geqslant 1$.

Proof. — Given a twisted S_3-cover (C, ϕ), we denote by $F \to C$ the associated admissible S_3-cover and by $\widetilde{\Gamma} \to \Gamma$ the associated graph S_3-cover. We recall that b_F is the index cochain of F.

We prove that if a is a ghost automorphism in $\underline{\mathrm{Aut}}_C(\mathsf{C}, \phi)$ such that $\mathsf{a}(e) = 1$ for every separating edge of Γ, then $\mathrm{age}\,\mathsf{a} \geqslant 1$. By Lemma 4.22 this implies the thesis. By Remark 4.8, we have the cycle condition that for any cycle $(\widetilde{e}_1, \dots, \widetilde{e}_k)$ of $\widetilde{\Gamma}$, $\prod (b_F \cdot \mathsf{a})(\widetilde{e}_i) = 1$. As any $\mathsf{a}(e)$ has order 2 or 3 for any e, and thus gives an aging of at least 1/2 or 1/3 respectively, the only case where $\mathrm{age}\,\mathsf{a} < 1$ is if there exist two edges $e_1, e_2 \in \mathbb{E}(\Gamma)$ such that $\mathsf{a}(e) = 1$ if $e \notin \{e_1, e_2\}$ and $\mathsf{a}(e_1) = \mathsf{a}(e_2) \in \boldsymbol{\mu}_3$. In order to respect the cycles condition, we have a dual graph Γ of the type

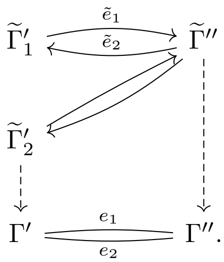


where Γ_1 and Γ_2 are two subgraphs of Γ such that $\mathsf{a}(e) = 1$ for every edge in $E(\Gamma_1)$ or $E(\Gamma_2)$. These two subgraphs are associated to two components C_1, C_2 of C such that $C = C_1 \cup C_2$ and they intersect in exactly two nodes q_1, q_2, corresponding to edges e_1, e_2.

We denote by $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ the restrictions of $\widetilde{\Gamma}$ over Γ_1 and Γ_2 respectively. If both $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ are connected, we denote by $\widetilde{e}_1$ and $\widetilde{e}_2$ two preimages of e_1 and e_2 in $\mathbb{E}(\widetilde{\Gamma})$ pointing at $\widetilde{\Gamma}_2$ and $\widetilde{\Gamma}_1$ respectively. By the cycle condition, $(b_F \cdot \mathsf{a})(\widetilde{e}_1) \cdot (b_F \cdot \mathsf{a})(\widetilde{e}_2) = 1$, but for the same reason $(b_F \cdot \mathsf{a})(\widetilde{e}_1) \cdot (b_F \cdot \mathsf{a})(g \cdot \widetilde{e}_2) = 1$ for any g in S_3, but this is impossible because $(b_F \cdot \mathsf{a})(\widetilde{e}_2)$ is non-trivial.

If one between $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$, say the first, is non-connected, we denote by $\widetilde{\Gamma}_1', \widetilde{\Gamma}_1''$ its two components (as $r(e_i) = 3$, there are no more than two components). This means that the restriction $F|_{C'} \to C'$ is an admissible N-cover, which means that $F|_{C'}$ is the union of two admissible $\boldsymbol{\mu}_3$-covers over the 2-marked curve $(C_1; p_1, p_2)$. We denote by $\widetilde{e}_1, \widetilde{e}_2$ the two oriented edges over e_1 and e_2, both touching $\widetilde{\Gamma}'_1$, and pointing to $\widetilde{\Gamma}_2$ and $\widetilde{\Gamma}'_1$ respectively. By Lemma 5.14, $(b_F \cdot \mathsf{a})(\widetilde{e}_1) = (b_F \cdot \mathsf{a})(\widetilde{e}_2)$ and as $\mathsf{a}(e_1)$ has order 3, then $(b_F \cdot \mathsf{a})(\widetilde{e}_1)$ and $(b_F \cdot \mathsf{a})(\widetilde{e}_2)$ have order 3 too.

The oriented edges $\widetilde{e}_1$ and $\widetilde{e}_2$ touch the same connected components of $\widetilde{\Gamma}''$. Indeed, if $\widetilde{\Gamma}''$ is non-connected, by local index considerations, both edges have to touch the same component. Therefore there exists a cycle passing through $\widetilde{e}_1$ and $\widetilde{e}_2$ and whose other edges have $\mathsf{a}(\widetilde{e}) = 1$.



Finally, again by the cycle condition we have

$$(b_F \cdot \mathsf{a})(\widetilde{e}_1) \cdot (b_F \cdot \mathsf{a})(\widetilde{e}_2) = (b_F \cdot \mathsf{a})(\widetilde{e}_1)^2 = 1,$$

but this is a contradiction because $(b_F \cdot \mathsf{a})(e_1)$ has order 3. □

We proved that, as in the case of G abelian group, also for $G = S_3$ the non-canonical locus $\mathrm{Sing}^{\mathrm{nc}}\,\overline{\mathcal{R}}_{g,G}$ coincides with the T-locus. This is a fundamental result to approach the extension of pluricanonical forms over a desingularization $\widehat{\mathcal{R}}_{g,G} \to \overline{\mathcal{R}}_{g,G}$.

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Manuscrit reçu le 27 mai 2019,
révisé le 20 avril 2021,
accepté le 23 juin 2021.

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