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The $C\ell_n$-Valued Robin Boundary Value Problem on Lipschitz Domains in $\mathbb{R}^n$

Loredana Lanzani * (lanzani@comp.uark.edu)

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1. Introduction

In this note we present the solution in $W^{1,2}$ of the Robin boundary value problem for the Laplacian on a Lipschitz domain $\Omega \subset \mathbb{R}^n$ with $C\ell_n$- valued datum $f \in L^2(b\Omega)$ (see (R) below). This work originates from [6], where we considered the case of scalar-valued datum $f \in L^p(b\Omega)$, $1 < p \leq 2$. In the present context of the Clifford algebra $C\ell_n$, the direct relationship between the Clifford derivatives of the single layer potential and left Clifford-Cauchy integral operators allows for a more unified and direct approach to the solution of the problem. Because we are choosing the Robin coefficient b in the space $L^s(b\Omega)$ with s greater than the critical exponent $n-1$, the solution operator for the Robin problem turns out to be a compact perturbation of the solution operator of the Neumann problem. In this respect, the situation we present here bears a close affinity with the classical study of the Neu- mann problem for C^1-domains (see [3]). The treatment of the critical exponent case (namely, $b \in L^{n-1}(b\Omega)$) requires a different approach, which has been developed in [6]. The structure of this paper is as follows. In sections 2 and 3 we describe and summarize the features of the Clifford algebras, the function spaces and the singular integral operators that are involved in this work. In Section 4 we present a simple proof of the L^2-solution of the Robin problem with non-critical Robin coefficient, and we state without proof the corresponding result in L^p, with critical Robin coefficient.

2. Clifford Algebras: Notation and Basic Properties

The *real Clifford Algebra associated with the Euclidean space* $\mathbb{R}^n$, denoted $C\ell_n$, is defined as the minimal enlargement of $\mathbb{R}^n$ to a unitary algebra not generated by any proper subspace of $\mathbb{R}^n$, with the property

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that

$$x^2 = -|x|^2 = -\sum_{j=0}^{n-1} x_j^2 \tag{1}$$

for any $x \in \mathbb{R}^n$. This implies that

$$e_j e_k + e_k e_j = -2\delta_{jk}, \quad j, k \geq 1 \tag{2}$$

where $\{e_j\}_{j=0}^{n-1}$ denote the generating elements of $C\ell_n$, which are usually identified with the standard orthonormal basis of $\mathbb{R}^n$. In particular, e_0 is identified with the unit of the algebra. $C\ell_n$ is thus a 2^n-dimensional vector space over $\mathbb{R}$ and any element $a \in C\ell_n$ can be uniquely represented as

$$a = \sum_{l=0}^{n-1} \sum_{|I|=l} a_I\, e_I, \quad a_I \in \mathbb{R}, \qquad \text{where} \tag{3}$$

$$e_I = e_{i_1} e_{i_2} \ldots e_{i_l},\ 0 \leq i_1 < i_2 < \ldots i_l \leq n-1,\ I = (i_1, i_2, \ldots i_l) \tag{4}$$

In particular, we single out the *Scalar part of* a, denoted $\mathrm{Sc}(a)$, defined as

$$\mathrm{Sc}(a) = a_0\, e_0 \equiv a_0 \tag{5}$$

Clifford conjugation in $C\ell_n$ is defined as the unique (real-)linear involution on $C\ell_n$ with

$$\overline{e}_I\, e_I = e_I\, \overline{e}_I = 1 \quad \text{for all} \ \ I \tag{6}$$

Thus

$$\overline{a} = \sum_{l=0}^{n-1} \sum_{|I|=l} a_I\, \overline{e}_I, \quad \text{and} \ \ \overline{e}_I = (-1)^{\frac{l(l+1)}{2}} e_I, \quad |I| = l. \tag{7}$$

In particular, we have

$$\mathrm{Sc}(a\, \overline{a}) = \mathrm{Sc}(\overline{a}\, a) = |a|^2 = \sum_{l=0}^{n-1} \sum_{|I|=l} (a_I)^2 \tag{8}$$

It is customary to view $\mathbb{R}^n$ as embedded into $C\ell_n$ via the obvious identification

$$x = \sum_{j=0}^{n-1} x_j\, e_j \tag{9}$$

A *$C\ell_n$-valued function* f on an open set $\Omega \subset \mathbb{R}^n$ is defined via:

$$f(x) = \sum_{l=0}^{n-1} \sum_{|I|=l} f_I(x)\, e_I, \quad \text{with} \ \ f_I : \Omega \to \mathbb{R} \tag{10}$$

Any continuity, differentiability or integrability property which is ascribed to f has to be possessed by all components f_I. In particular, the Lebesgue and Sobolev spaces of $C\ell_n$-valued functions $L^2(b\Omega, C\ell_n)$, $W^{1,2}(b\Omega, C\ell_n)$ are defined by requiring that each component f_I belong to $L^2(b\Omega)$ (resp. $W^{1,2}(b\Omega)$), with norm in L^2 defined via: $||f||_2^2 := \mathrm{Sc}(\int\limits_{b\Omega} f\,\overline{f}\,\mathrm{d}\sigma)$ (see (8)). The $W^{1,2}$-norm is defined similarly.

The *Left* and *Right Dirac derivatives* of a (differentiable) $C\ell_n-$valued function f are defined respectively as

$$D\,f := \sum_{j=0}^{n-1}\left(\sum_{l=0}^{n-1}\sum_{|I|=l}\left(\frac{\partial}{\partial x_j} f_I\right)\, e_j\, e_I\right)\;; \tag{11}$$

$$f\,D := \sum_{j=0}^{n-1}\left(\sum_{l=0}^{n-1}\sum_{|I|=l}\left(\frac{\partial}{\partial x_j} f_I\right)\, e_I\, e_j\right) \tag{12}$$

Similarly, we define

$$\overline{D}\,f := \sum_{j=0}^{n-1}\left(\sum_{l=0}^{n-1}\sum_{|I|=l}\left(\frac{\partial}{\partial x_j} f_I\right)\, \overline{e}_j\, e_I\right)\;; \tag{13}$$

$$f\,\overline{D} := \sum_{j=0}^{n-1}\left(\sum_{l=0}^{n-1}\sum_{|I|=l}\left(\frac{\partial}{\partial x_j} f_I\right)\, e_I\, \overline{e}_j\right) \tag{14}$$

It is immediate to check that

$$\overline{(D\,f)} = \overline{f}\;\overline{D} \tag{15}$$

Moreover, we have

$$D\,\overline{D}\,f = \overline{D}\,D\,f = \Delta f := \sum_{l=0}^{n-1}\sum_{|I|=l}(\Delta f_I)\, e_I. \tag{16}$$

We conclude by recalling *Stokes' Formula:*

$$\begin{aligned}&\int\limits_{b\Omega} u(x)\, n(x)\, v(x)\;\mathrm{d}\sigma(x) = \\ &\iint\limits_{\Omega}(uD)(x)\, v(x)\;\mathrm{d}x + \iint\limits_{\Omega} u(x)\,(Dv)(x)\;\mathrm{d}x,\end{aligned} \tag{17}$$

provided the the functions above are integrable.

3. Singular Integrals: Notations and Main Properties

The operators we will be mainly concerned with are the (non-tangential) boundary values of the *Left Cauchy Integral* $\mathcal{C}$, the *Single Layer Potential* $\mathcal{S}$ and its left Dirac derivative $D\mathcal{S}$; here we recall their definitions and their basic properties. The fundamental result in the context of Lipschitz domains is the following theorem, due to A. Calderon (for domains with small Lipschitz constant) and R. Coifman, A. McIntosh and Y. Meyer (for arbitrary Lipschitz domains) (see [1], [2], [4], [5], [7]):

THEOREM 3.1. *Let $\Omega \subset \mathbb{R}^n$ denote a bounded, connected domain with Lipschitz boundary.*
Then, for any $f \in L^2(b\Omega, C\ell_n)$ the Left Clifford-Cauchy Integral of f:

$$\mathcal{C}f\,(y) \;:=\; \frac{1}{\omega_n} \int\limits_{b\Omega} \frac{q-y}{|q-y|^n}\,\overline{n}(q)\,f(q)\;\mathrm{d}\sigma(q), \qquad y \in \Omega \tag{18}$$

has the following properties:

(i) The Non-Tangential Maximal Function of $\mathcal{C}f$, $(\mathcal{C}f)^*$ (see [4], [5], [7]) is square-integrable on $b\Omega$, and

$$||\;(\mathcal{C}f)^*\;||_2 \leq C\,||f||_2 \tag{19}$$

(ii) $\mathcal{C}f$ has Non-Tangential Limit $(\mathcal{C}f)^+(p)$ (see[4], [5], [7]) at almost every $p \in b\Omega$ and

$$(\mathcal{C}f)^+\,(p) \;=\; \frac{1}{2}\left(-f(p) + \frac{2}{\omega_n}\,\mathrm{p.v.}\int\limits_{b\Omega} \frac{q-p}{|q-p|^n}\,\overline{n}(q)\,f(q)\;\mathrm{d}\sigma(q)\right) \tag{20}$$

(Here, $\overline{n}$ denotes the Clifford-conjugate of the outer normal unit vector).

For $f \in L^2(b\Omega, C\ell_n)$ the *Single Layer Potential of f*, denoted $\mathcal{S}f$, is defined as

$$\mathcal{S}f\,(y) = \frac{-1}{\omega_n(n-1)} \int\limits_{b\Omega} \frac{1}{|y-q|^{n-2}}\,f(q)\;\mathrm{d}\sigma(q)\,, \qquad y \in \Omega \tag{21}$$

Since

$$D_y\left(\frac{-1}{(n-1)|y-q|^{n-2}}\right) = \frac{q-y}{|y-q|^n} \tag{22}$$

it follows that $D\mathcal{S}$ and $\mathcal{C}$ are related via

$$D\,(\mathcal{S}f)\,(y) = \mathcal{C}(n\,f)(y), \quad y \in \Omega, \quad f \in L^2(b\Omega, C\ell_n) \tag{23}$$

where n denotes the outer unit normal vector to $b\Omega$. The following corollary is thus a direct consequence of Theorem 3.1 (see [4], [7]):

COROLLARY 3.2. *With the same notations and hypotheses as Theorem (3.1), for any $f \in L^2(b\Omega, C\ell_n)$ we have:*

$$||(D\mathcal{S}f)^*||_2 \leq C||f||_2 \quad \textit{and} \tag{24}$$

$$\overline{n}(p)\ \left(D(\mathcal{S}f)\right)^+(p) = \frac{1}{2}\,(-I + \mathcal{K}^*)f(p), \quad \textit{a.e. } p \in b\Omega \tag{25}$$

where $\mathcal{K}^*$ denotes the $L^2(b\Omega, C\ell_n)$-adjoint of the Left Clifford-Hilbert transform $\mathcal{K}$ (see, e.g., [4] or [7]):

$$\mathcal{K}f(p) \ = \ \frac{2}{\omega_n}\text{p.v.}\int_{b\Omega} \frac{\overline{p-q}}{|p-q|^n}\,n(q)\,f(q)\ \mathrm{d}\sigma(q) \tag{26}$$

The following result, essentially due to G. Verchota, will be of great importance to us (see [4], [7], [8]):

THEOREM 3.3. *With the same notations and hypotheses as above, we have:*

(i) $\mathcal{S} : L^2(b\Omega, C\ell_n) \to W^{1,2}(b\Omega, C\ell_n)$ *is invertible;*

(ii) $-I + \mathcal{K}^*$ *is invertible on* $L^2_0(b\Omega, C\ell_n)$ *and, moreover*

$$\text{Ind}_{L^2}\,(-I + \mathcal{K}^*) = 0 \tag{27}$$

(Here, $\text{Ind}_{L^2}\,(T) := \dim \text{Ker}(T) - \dim\,\left(L^2 \setminus \text{Range}(T)\right)$, *see [9]).*

In addition, we have

$$\begin{aligned} &||(I \pm \mathcal{K}^*)g||_2 \ \leq \\ &\leq \ C(||(I \mp \mathcal{K}^*)g||_2 + ||\mathcal{S}g||_2) \end{aligned} \tag{28}$$

4. The Robin Problem in L^p

We are finally ready to state and prove the main result of this note:

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^n$ denote a bounded, connected domain with Lipschitz boundary. Let b denote a given, scalar-valued function, $b \in L^s(b\Omega, \mathbb{R})$, $s > n-1$, $b \geq 0$ (b positive on some subset of $b\Omega$ with positive measure).*

With the same notations as in Section 3, for $f \in L^2(b\Omega, C\ell_n)$ define

$$\mathcal{T} f(p) := \left(\frac{1}{2}(-I + \mathcal{K}^*) f\right)(p) + b(p)\, \mathcal{S} f(p), \quad a.e.\ p \in b\Omega \tag{29}$$

Then, we have that $\mathcal{T}$ is bounded and invertible in $L^2(b\Omega, C\ell_n)$.

Moreover, the Robin Problem for $C\ell_n$-valued harmonic functions:

$$(R) \quad \begin{cases} \Delta u = 0 & in\ \Omega \\ (Du)^* \in L^2(b\Omega) & \\ \overline{n} Du + b\, u = f & on\ b\Omega, \end{cases}$$

is uniquely solvable in $L^2(b\Omega, C\ell_n)$, and the solution u is represented via

$$u(x) = \mathcal{S}\left(\mathcal{T}^{-1} f\right)(x), \qquad x \in \Omega \tag{30}$$

Proof. By (i) in Theorem 3.3 and the Rellich-Kondrachev compact embedding theorem (see [9]) it follows that the point-wise multiplication operator $b\, \mathcal{S}(f)\,(p) := b(p)\, \mathcal{S} f(p)$ is compact in $L^2(b\Omega, C\ell_n)$. This, and (ii) in Theorem 3.3 imply at once that $\mathcal{T}$ is bounded in $L^2(b\Omega, C\ell_n)$. The fact that (30) gives the solution of (R) (provided $\mathcal{T}$ is invertible in $L^2(b\Omega, C\ell_n)$) is an immediate consequence of (24) and (25).

We are thus left to show that $\mathcal{T}$ is invertible in $L^2(b\Omega, C\ell_n)$. We begin by showing that $\mathcal{T}$ is one-to-one. Indeed, if we let

$$\mathcal{T} f = 0 \qquad \text{for some} \qquad f \in L^2(b\Omega, C\ell_n) \tag{31}$$

and apply Stokes' formula (17) to $v := \mathcal{S} f$ and $u := \overline{D\, v} = \overline{v}\, \overline{D}$ (see (15)) we obtain (as $uD = \overline{v}\, \overline{D}\, D = \Delta \overline{v} = \Delta v = 0$)

$$\iint_\Omega (\overline{Dv}) Dv = \int_{b\Omega} \overline{Dv}\, n\, v = \int_{b\Omega} \overline{(\overline{n}\, Dv)}\, v = -\int_{b\Omega} b\, \overline{v}\, v \tag{32}$$

In particular, by considering the scalar components of (32) we obtain (see (8))

$$\iint_\Omega |Dv|^2 = -\int_{b\Omega} b\, |v|^2 \tag{33}$$

It follows $Dv = 0$ in Ω, i.e. $v = const$ and, since $b \geq 0$ (and b is positive on a subset of $b\Omega$ with positive measure) it must be $v = 0$ in Ω. Since we have set $v = \mathcal{S} f$, by (i) in Theorem 2.3 we conclude $f = 0$.

Next, we show that $\mathcal{T}$ has dense range in $L^2(b\Omega, C\ell_n)$. Indeed, the compactness of $b\,\mathcal{S}$ and (27) imply (see [9])

$$\mathrm{Ind}_{L^2}(\mathcal{T}) \,=\, \mathrm{Ind}_{L^2}(-I+\mathcal{K}^*) \,=\, 0 \tag{34}$$

But we just proved that $\mathcal{T}$ is one-to-one, thus

$$\mathrm{Ind}_{L^2}(\mathcal{T}) \,=\, \dim\left(L^2(b\Omega, C\ell_n)\,/\mathrm{Range}\,(\mathcal{T})\right) \tag{35}$$

and the dense range property is proved.

Finally, we show that $\mathcal{T}$ has closed range. To this end, have

$$(y_n)_{n\in\mathbb{N}} \subset L^2(b\Omega, C\ell_n), \quad y_n \to y \text{ in } L^2(b\Omega, C\ell_n), \quad y_n = \mathcal{T}\,x_n \tag{36}$$

We distinguish two cases: if $||x_n||_2 \le C$ for each n, then by the Banach-Alaoglou theorem (see [9]) we have (modulo a subsequence)

$$x_n \rightharpoonup x, \quad \text{for some } x \in L^2(b\Omega, C\ell_n) \quad \text{(weak convergence)} \tag{37}$$

By the uniqueness of weak limits (and the boundedness of $\mathcal{T}$) we conclude $y \,=\, \mathcal{T}\,x$.

If, instead, $(x_n)_{n\in\mathbb{N}}$ contains an unbounded subsequence, we consider

$$z_n \,:=\, \frac{x_n}{||x_n||_2} \tag{38}$$

In this case it is not difficult to show that (modulo a subsequence)

$$z_n \rightharpoonup 0 \text{ (weak)}, \; ||\mathcal{T}z_n||_2 \to 0, \; ||b\,\mathcal{S}z_n||_2 \to 0 \text{ and } ||\mathcal{S}z_n||_2 \to 0 \tag{39}$$

By combining (28) with the triangle inequality we obtain:

$$\begin{aligned} &||(I+\mathcal{K}^*+b\,\mathcal{S})z_n||_2 \le \\ &\le C\,\left(||\mathcal{T}z_n||_2 + ||\mathcal{S}z_n||_2 + 2||b\,\mathcal{S}z_n||_2\right) \to 0 \end{aligned} \tag{40}$$

This leads to the following contradiction:

$$1 = ||z_n||_2 \le ||\mathcal{T}z_n||_2 \,+\, ||(I+\mathcal{K}^*+b\,\mathcal{S})z_n||_2 \;\to 0 \tag{41}$$

The proof of Theorem 4.1 is concluded. □

Even though Corollary 3.2 and Theorem 3.3 extend to the case $f \in L^p(b\Omega)$, $1 < p \le 2$, the solution of the Robin problem in the general case: $b \in L^{n-1}(b\Omega)$, $f \in L^p(b\Omega)$ requires a more sophisticated approach than Theorem 4.1 because the operator $b\,\mathcal{S}$ now fails to be compact in $L^p(b\Omega)$ (even though it still bounded, by the Sobolev embedding theorem). Moreover, Stokes' formula can no longer be applied to show uniqueness since, in this case, the functions involved may not be integrable. Nevertheless, the result is maintained. We have:

THEOREM 4.2. *Let $\Omega \subset \mathbb{R}^n$ denote a bounded, connected domain with Lipschitz boundary. Let b denote a given, scalar-valued function, $b \in L^{n-1}(b\Omega)$, $b \geq 0$ (b positive on some subset of $b\Omega$ with positive measure). With the same notations as in Section 3, for $f \in L^p(b\Omega)$, $1 < p \leq 2$, define*

$$\mathcal{T}f(p) := \left(\frac{1}{2}(-I+\mathcal{K}^*)f\right)(p) + b(p)\,\mathcal{S}f(p), \quad a.e.\ p \in b\Omega \tag{42}$$

Then, we have that $\mathcal{T}$ is bounded and invertible in $L^p(b\Omega)$.

Moreover, the Robin Problem for $C\ell_n$-valued harmonic functions:

$$(R) \quad \begin{cases} \Delta u = 0 & in\ \Omega \\ (Du)^* \in L^p(b\Omega) & \\ \overline{n}Du + b\,u = f & on\ b\Omega, \end{cases}$$

is uniquely solvable in $L^p(b\Omega)$, and the solution u is represented via

$$u(x) = \mathcal{S}\left(\mathcal{T}^{-1}f\right)(x), \qquad x \in \Omega \tag{43}$$

The proof of Theorem 4.2 in the case of scalar-valued coefficients will appear in [6].

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Department of Mathematics

University of Arkansas
Fayetteville, AR 72701
USA