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# Existence of Positive Eigenfunctions to an Anisotropic Elliptic Operator via Sub-Super Solutions Method

Simone Ciani, Giovany M. Figueiredo and Antonio Suárez

**Abstract.** Using the sub-supersolution method we study the existence of positive solutions for the anisotropic problem

$$-\sum_{i=1}^N \frac{\partial}{\partial x_i} \left( \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = \lambda u^{q-1} \quad (0.1)$$

where  $\Omega$  is a bounded and regular domain of  $\mathbb{R}^N$ ,  $q > 1$  and  $\lambda > 0$ .

**Mathematics Subject Classification (2010).** 35K65, 35B65, 35B45, 35K20.

**Keywords.** Anisotropic  $p$ -Laplacian, Positive Solution, Sub-Supersolution, Eigenvalues.

## 1. Introduction

In this paper the main goal is to show the existence of positive solutions of the problem

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left( \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = \lambda u^{q-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 1$ , is a bounded and regular domain,  $p_i > 1$ ,  $i = 1, \dots, N$ ,  $q > 1$  and  $\lambda$  is a real parameter. We will assume without loss of generality that the  $p_i$  are ordered increasingly, that is,  $p_1 < \dots < p_N$ .

There is a vast literature concerning to anisotropic elliptic problems. We mention here only those references most strongly related to (1.1). First, in [9] it was proved that for  $q < p_N$  for any  $\gamma > 0$  there exists  $\lambda_\gamma > 0$  and  $u_\gamma$  with  $\|u_\gamma\|_p = \gamma$  and  $u_\gamma$  solution of (1.1) with  $\lambda = \lambda_\gamma$ . As the authors themselves claim, from this result it can not be deduced the existence of solutions of (1.1) for a given  $\lambda$ . In [4], using mainly variational methods, it was proved that if  $p_1 < q < p_N$  then there exist  $0 < \lambda_* \leq \lambda^*$  such that:

- If  $\lambda \leq \lambda_*$ , (1.1) does not possess positive solution.
- If  $\lambda > \lambda^*$ , (1.1) possesses at least a positive solution.

Finally, for the general results of [14] (Corollary 1) we can deduce that for the case  $1 < q < p_1$  there exist  $0 < \lambda_* < \lambda_{**}$  such that (1.1) possesses at least a solution for  $\lambda \in (0, \lambda_*) \cup (\lambda_{**}, \infty)$ .

In this paper we complete and improve the above results. For that, we use the sub-supersolution method, see [1], [5] and [16], (see also [6], [7], [8] and references therein for the application of this method to problems with non-linear reaction function including singularities or critical exponent).

This method allows us not only to prove the existence of a solution, but also gives us lower and upper bounds of such solution. Specifically, our main result is the following.

**Theorem 1.1.**

1. Assume that  $1 < q < p_1$ . There exists a positive solution of (1.1) if and only if  $\lambda > 0$ .
2. Assume that  $p_1 \leq q < p_N$ . There exists  $\Lambda > 0$  such that (1.1) does not possess positive solutions for  $\lambda < \Lambda$  and (1.1) possesses at least one positive solution for  $\lambda > \Lambda$ .

An outline of the paper is as follows: in Section 2 we recall some definitions and some properties of the eigenvalues and eigenfunctions of the classical  $p$ -Laplacian. Next in Section 3 we enunciate the sub-supersolution method. Then in Section 4 we construct sub and super-solutions by multiplication of powers of  $p$ -Laplacian eigenfunctions to be applied in the existence theorem.

## 2. Preliminary Lemmas and Setting

Consider  $h(x, s) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  a Caratheodory function, i.e. measurable in  $x$  and continuous in the second variable  $s$ . Consider the anisotropic problem

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left( \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = h(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

The natural framework to study (2.1) is the anisotropic Sobolev Space  $W_0^{1,\mathbf{P}}(\Omega)$ , that is, the closure of  $C_0^\infty(\Omega)$  under the anisotropic norm

$$\|u\|_{W^{1,\mathbf{P}}(\Omega)} := \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}$$

where  $\frac{\partial u}{\partial x_i}$  denotes the  $i$ -th weak partial derivative of  $u$ .

Recall that if we denote

$$\sum_{i=1}^N \frac{1}{p_i} > 1, \quad p_i > 1 \quad \forall i = 1, \dots, N, \quad p^* := \frac{N}{\sum \frac{1}{p_i} - 1}, \quad p_\infty := \max\{p^*, p_N\}, \quad (2.2)$$

then for every  $r \in [1, p_\infty]$  the embedding

$$W_0^{1,\mathbf{P}}(\Omega) \subset L^r(\Omega)$$

is continuous, and it is compact if  $r < p_\infty$ . More precisely, it holds the following directional Poincaré-type inequality for any  $u \in C_c^1(\Omega)$  (see for instance [9])

$$\|u\|_r \leq \frac{d^i r}{2} \left\| \frac{\partial u}{\partial x_i} \right\|_r, \quad \forall r \geq 1, \quad d^i = \sup_{x,y \in \Omega} \langle x - y, e_i \rangle, \quad (2.3)$$

denoting by  $\{e_1, \dots, e_N\}$  the canonical basis of  $\mathbb{R}^N$ .

The theory of embeddings of this kind of anisotropic Sobolev spaces is vast and we refer to [9] for directional Poincaré-type inequality and to [12] for Sobolev and Morrey's embeddings of the whole  $W^{1,\mathbf{P}}(\Omega)$  space, obtained with an important geometric condition on the domain  $\Omega$ , namely that it must be semi-rectangular. It is not a case that this semi-rectangular condition reflects in our construction of the solution: the existence of traces for this kind of functions is heavily depending on the geometry of the domain as shown in [12]. Regularity theory for orthotropic operators as the one defined by equation (1.1) is still a challenging open problem, see for example [3].

We also recall the following definition.

**Definition 2.1.** *A function  $u \in W^{1,\mathbf{P}}(\Omega)$  is defined to be a sub-(super-) solution to the problem (2.1) if  $u \leq (\geq) 0$  in  $\partial\Omega$  and  $\forall 0 \leq \phi \in W_0^{1,\mathbf{P}}(\Omega)$  it satisfies*

$$\int_{\Omega} \left[ \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_i} - h(x, u(x)) \phi \right] dx \leq (\geq) 0. \quad (2.4)$$

Finally, a solution  $u \in W_0^{1,\mathbf{P}}(\Omega)$  to (2.1) has to satisfy

$$\int_{\Omega} \left[ \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_i} - h(x, u(x)) \phi \right] dx = 0 \quad \forall \phi \in W_0^{1,\mathbf{P}}(\Omega).$$

Now, we recall some well-known results concerning the eigenvalue problem for the  $p$ -Laplacian. Specifically, the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.5)$$

where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right)$$

The following result is well-known:

**Lemma 2.1.** *The eigenvalue problem (2.5) has a unique eigenvalue  $\lambda = \lambda_1$  with the property of having a positive associated eigenfunction  $\varphi_1 \in W_0^{1,\mathbf{P}}(\Omega)$ ,*

called principal eigenfunction. Moreover,  $\lambda_1$  is simple, isolated and is defined by

$$\lambda_1 = \inf \left\{ \int_{\Omega} |\nabla u|^p : u \in W_0^{1,p}(\Omega), \int_{\Omega} |u|^p dx = 1 \right\}.$$

Furthermore,  $\varphi_1 \in C^{1,\beta}(\overline{\Omega})$  for some  $\beta \in (0, 1)$  and  $\partial\varphi_1/\partial n < 0$  on  $\partial\Omega$ , where  $n$  is the outward unit normal on  $\partial\Omega$ . Finally, for  $N = 1$  we have that

$$|\nabla\varphi_1|^{p-2}\nabla\varphi_1 \in W^{1,2}(\Omega), \quad (2.6)$$

and in fact

$$-\Delta_p\varphi_1(x) = \lambda_1\varphi_1(x) \quad \text{a.e. } x \in \Omega.$$

**Remark 2.1.** The existence of  $\lambda_1$  and main properties of  $\varphi_1$  are well-known, see [10], [13], [15]. Property (2.6) holds in  $N = 1$ , see for instance [11], and for  $N \geq 2$  is some specific domains, for example for  $\Omega$  convex, see [2].

### 3. An existence sub-supersolution theorem

We start by stating an important theorem, see [1] and [5], that assures the existence of a solution between a sub and a supersolution.

**Theorem 3.1.** Suppose that  $h : \mathbb{R} \mapsto \mathbb{R}$  a continuous function and that there exist  $\underline{u}, \overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$  subsolution and supersolution of (2.1) such that  $\underline{u} \leq \overline{u}$ . Then there exists  $u \in W_0^{1,p}(\Omega)$  solution to (2.1) such that

$$\underline{u} \leq u \leq \overline{u}.$$

*Proof.* Since  $\underline{u}$  and  $\overline{u}$  belong to  $L^\infty(\Omega)$ , then  $h$  verifies condition  $(h_2)$  of [1]. This concludes the proof.  $\square$

### 4. Construction of sub and super-solutions: proof of the main result

In this section we prove Theorem 1.1. For that, we apply Theorem 3.1 to (1.1). Mainly, we construct the sub and the supersolution.

#### 4.1. Sub-solutions

Let us consider a rectangular bounded domain  $U \subseteq \Omega$  i.e.

$$U := \prod_{i=1}^N U_i, \quad \text{where } U_i = (a_i, b_i), \quad a_i, b_i \in \mathbb{R} \quad \forall i = 1, \dots, N.$$

Denote by  $v_i = v_i(x_i)$  a positive principal eigenfunction of  $-\Delta_{p_i}$  in  $U_i$ , that is,

$$\begin{cases} -\Delta_{p_i} v_i = \eta_i |v_i|^{p_i-2} v_i & \text{in } U_i, \\ v_i = 0 & \text{on } \partial U_i. \end{cases} \quad (4.1)$$

From Lemma 2.1, recall that, if  $n_i$  is the outward normal derivative to  $\partial U_i$ , we have

$$\frac{\partial v_i}{\partial n_i} < 0 \quad \text{on } \partial U_i. \quad (4.2)$$

Let us consider the function

$$\underline{u}(x) = \begin{cases} \epsilon \prod_{i=1}^N v_i^{\alpha_i}(x_i) & x \in U, \\ 0 & x \in \Omega \setminus \overline{U}, \end{cases} \quad (4.3)$$

where  $\alpha_i > 0$ ,  $i = 1, \dots, N$ , and  $\epsilon > 0$  will be chosen later.

**Remark 4.1.** We note that  $\underline{u}(x) > 0$  in  $\emptyset \neq U \subset \Omega$ .

As  $v_i$  are bounded, it is clear that  $\underline{u} \in W^{1,p}(\Omega)$  and that  $\underline{u}|_{\partial\Omega} = 0$ . Hence,  $\underline{u}$  is subsolution of (1.1) provided that

$$\int_{\Omega} \sum_{i=1}^N \left| \frac{\partial \underline{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_i} dx \leq \lambda \int_{\Omega} \underline{u}^{q-1} \phi dx \quad \forall \phi \in W_0^{1,p}(\Omega), \quad \phi \geq 0.$$

Observe that

$$\lambda \int_{\Omega} \underline{u}^{q-1} \phi dx = \lambda \epsilon^{q-1} \int_U \prod_{i=1}^N v_i^{\alpha_i(q-1)} \phi dx. \quad (4.4)$$

On the other hand, observe that

$$\frac{\partial \underline{u}}{\partial x_i} = \epsilon \alpha_i \left( \prod_{j \neq i} v_j^{\alpha_j} \right) v_i^{\alpha_i-1} \frac{\partial v_i}{\partial x_i} \quad \text{in } U_i.$$

Then, taking into account the positivity of  $v_i$ ,  $\forall i = 1, \dots, N$ ,

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial \underline{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_i} dx = \\ & \sum_{i=1}^N \int_{\prod_{j \neq i} U_j} \left\{ \int_{U_i} \left[ \epsilon \alpha_i \left( \prod_{j \neq i} v_j^{\alpha_j} \right) v_i^{\alpha_i-1} \right]^{p_i-1} \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \frac{\partial \phi}{\partial x_i} dx_i \right\} d\hat{x}^i \end{aligned}$$

with the obvious notation for  $d\hat{x}^i$ . Next, by using an integration by parts argument and the Fubini-Tonelli theorem, we get

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial \underline{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_i} dx = \\ & - \sum_{i=1}^N \int_{\prod_{j \neq i} U_j} \left( \epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-1} \int_{U_i} \frac{\partial}{\partial x_i} \left( v_i^{(\alpha_i-1)(p_i-1)} \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \right) \phi dx_i d\hat{x}^i \\ & + \sum_{i=1}^N \int_{\prod_{j \neq i} U_j} \left( \epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-1} \left\{ \int_{\partial U_i} v_i^{(\alpha_i-1)(p_i-1)} \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial n_i} \phi dx_i \right\} d\hat{x}^i. \end{aligned}$$

The second term on the right can be discarded as  $\frac{\partial v_i}{\partial n_i} < 0$  in  $\partial U_i$ , see (4.2). Considering that

$$\begin{aligned} & \frac{\partial}{\partial x_i} \left( v_i^{(\alpha_i-1)(p_i-1)} \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \right) = \\ & (\alpha_i - 1)(p_i - 1) \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + v_i^{(\alpha_i-1)(p_i-1)} \frac{\partial}{\partial x_i} \left( \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \right) \end{aligned}$$

and, from Lemma 2.1, that

$$\frac{\partial}{\partial x_i} \left( \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \right) = -\eta_i v_i^{p_i-1} \quad \text{in } U_i,$$

our sub-solution condition becomes

$$\begin{aligned} & \sum_{i=1}^N \int_{\prod_{j \neq i} U_j} \left( \epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-1} \int_{U_i} \left\{ \left[ (1 - \alpha_i)(p_i - 1) v_i^{(\alpha_i-1)(p_i-1)-1} \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + \right. \right. \\ & \left. \left. + v_i^{(\alpha_i-1)(p_i-1)} \eta_i v_i^{p_i-1} \right] - \lambda \epsilon^{q-1} \left( \prod_{k=1}^N v_k^{\alpha_k(q-1)} \right) \right\} \phi \, dx_i \, d\hat{x}^i \leq 0. \end{aligned} \quad (4.5)$$

Let us require a condition on the pointwise integrand

$$\lambda \geq$$

$$\begin{aligned} & \sum_{i=1}^N \left( \epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-q} v_i^{(\alpha_i-1)(p_i-1)-1-\alpha_i(q-1)} \left[ (1 - \alpha_i)(p_i - 1) \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + \eta_i v_i^{p_i} \right] \\ & = \sum_{i=1}^N \left( \epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-q} v_i^{\alpha_i(p_i-q)-p_i} \left[ (1 - \alpha_i)(p_i - 1) \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + \eta_i v_i^{p_i} \right]. \end{aligned}$$

Now we consider various cases.

- If  $1 < q < p_1$ , then by choosing  $\alpha_i > \frac{p_i}{(p_i-q)} > 1$ , letting  $\epsilon \rightarrow 0^+$  we obtain that  $\underline{u}$  is a subsolution provided  $\lambda > 0$ .
- Assume that some  $i_0 \in \{1, \dots, N\}$  we have  $p_{i_0+1} > q \geq p_{i_0}$  taking  $p_{N+1} = \infty$ . Then  $\underline{u}$  is a subsolution if

$$\lambda \geq \lambda_* := \max_U \mathcal{S}$$

where

$$\mathcal{S} = \sum_{i=1}^N \left( \epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-q} v_i^{\alpha_i(p_i-q)-p_i} \left[ (1 - \alpha_i)(p_i - 1) \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + \eta_i v_i^{p_i} \right].$$

We show that  $\lambda_*$  is finite. Observe that  $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1$  where

$$\mathcal{S}_0 = \sum_{i=1}^{i_0} \left( \epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-q} v_i^{\alpha_i(p_i-q)-p_i} \left[ (1 - \alpha_i)(p_i - 1) \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + \eta_i v_i^{p_i} \right].$$



and

$$\mathcal{S}_1 = \sum_{i=i_0+1}^N \left( \epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i - q} v_i^{\alpha_i(p_i - q) - p_i} \left[ (1 - \alpha_i)(p_i - 1) \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + \eta_i v_i^{p_i} \right].$$

It is clear that  $\mathcal{S}_1$  is finite, taking  $\alpha_i > p_i/(p_i - q)$ .

On the other hand, observe that the behaviour next to  $\partial U$  is controlled: when  $v_i \rightarrow 0^+$  then as  $\frac{\partial}{\partial n_i} v_i < 0$  on  $\partial U_i$  we have that there exists  $\delta > 0$  small enough such that the quantity

$$\left[ (1 - \alpha_i)(p_i - 1) \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + \eta_i v_i^{p_i} \right] < 0 \quad \text{in } U_i^\delta, \text{ for all } i = 1, \dots, p_{i_0},$$

where

$$U_i^\delta := \{x_i \in U_i : \text{dist}(x_i, \partial U_i) \geq \delta\}.$$

Moreover,  $\mathcal{S}_0$  is bounded in  $U \cap U_i^\delta$ . Then,  $\mathcal{S}_0$  is bounded in  $U$  and we can conclude that  $\lambda_*$  is finite.

## 4.2. Supersolutions

Since  $\Omega$  is bounded, we can choose a domain  $U$  such that

$$\Omega \subset U = \prod_{i=1}^N U^i, \quad U^i = (a_i, b_i), \quad a_i, b_i \text{ in } \mathbb{R}.$$

Now for  $M > 0$  we consider the function

$$\bar{u}(x) := M \prod_{i=1}^N v_i(x_i), \quad x \in \Omega,$$

where  $v_i$  are the first eigenfunctions to the  $p_i$ -Laplacian in  $U^i$ , whose first eigenvalue we denote by  $\eta^i$ . Observe that

$$\bar{u}_{\partial\Omega} > 0.$$

Then,  $\bar{u}$  is a supersolution to (1.1) for all  $0 \leq \phi \in W_0^{1,\mathbf{P}}(\Omega)$  holds

$$\lambda \int_{\Omega} M^{q-1} \prod_{i=1}^N v_i^{q-1} \phi \, dx \leq \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial \bar{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \bar{u}}{\partial x_i} \frac{\partial \phi}{\partial x_i} \, dx.$$

It is clear that

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial \bar{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \bar{u}}{\partial x_i} \frac{\partial \phi}{\partial x_i} \, dx = \\ & \int_{\Omega} \sum_{i=1}^N \left[ M \left( \prod_{j \neq i} v_j \right) \right]^{p_i-1} \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \frac{\partial \phi}{\partial x_i} \, dx = \\ & - \sum_{i=1}^N \int_{\Omega} \left( M \prod_{j \neq i} v_j \right)^{p_i-1} \frac{\partial}{\partial x_i} \left( \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \right) \phi \, dx = \\ & \sum_{i=1}^N \eta^i \int_{\Omega} \left( M \prod_{j=1}^N v_j \right)^{p_i-1} \phi \, dx \end{aligned}$$

Thus we may ask for the strong condition

$$\lambda^* := \sum_{i=1}^N \eta^i \left( M \prod_{j=1}^N v_j \right)^{p_i - q} \geq \lambda. \quad (4.6)$$

Hence, if  $1 < q < p_N$  by letting  $M \rightarrow \infty$  we have that  $\bar{u}$  is a super solution  $\forall \lambda > 0$ .

*Proof of Theorem 1.1.*

1. Assume  $1 < q < p_1$ . Fix  $\lambda > 0$ . Then, we can choose  $\epsilon > 0$  small and  $M$  large enough such that  $\underline{u}, \bar{u}$  are sub-supersolution of (1.1) and  $\underline{u} \leq \bar{u}$  in  $\Omega$ . Theorem 3.1 assures the existence of a solution  $u$  of (1.1) such that  $\underline{u} \leq u \leq \bar{u}$ . This completes this case.
2. Assume  $p_1 \leq q < p_N$ . In this case, taking for example  $\epsilon = 1$ , we have that  $\underline{u}$  is subsolution provided that  $\lambda \geq \lambda_*$  for some  $\lambda_*$ . On the other hand, we can take  $M$  large such that  $\bar{u}$  is supersolution and  $\underline{u} \leq \bar{u}$ . Thus, there exists a positive solution for  $\lambda \geq \lambda_*$ .

Now, we define

$$\Lambda := \inf\{\lambda : (1.1) \text{ possesses at least a positive solution}\}.$$

We have proved that  $\Lambda < \infty$ . For  $p_1 < q < p_N$ , in [4] it was proved that  $0 < \Lambda$ . We show now that this is also true for  $q = p_1$ . Indeed, let now consider  $q = p_1$  and let us multiply the equation (1.1) by  $u$  and integrate it on  $\Omega$  to obtain

$$\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx = \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}^{p_i} = \lambda \|u\|_{p_1}^{p_1} = \lambda \int_{\Omega} |u|^{p_1} dx$$

Now we use the embedding (2.3) on  $r = p_1$  to get

$$\left( \frac{d^1 p_1}{2} \right)^{-p_1} \|u\|_{p_1}^{p_1} \leq \left\| \frac{\partial u}{\partial x_1} \right\|_{p_1}^{p_1} + \sum_{i=2}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}^{p_i} = \lambda \|u\|_{p_1}^{p_1}$$

and thus

$$\|u\|_{p_1}^{p_1} \left[ \lambda - \left( \frac{2}{d^1 p_1} \right)^{p_1} \right] \geq 0.$$

But if  $\lambda < \left( \frac{2}{d^1 p_1} \right)^{p_1}$  this quantity is negative and we arrive to the absurd of declaring  $\|u\|_{p_1} \neq 0$ .

We prove now that for all  $\lambda > \Lambda$  we have the existence of positive solution. Indeed, fix  $\lambda_0 > \Lambda$ . Then, by definition of  $\Lambda$ , there exists  $\mu \in (\Lambda, \lambda_0)$  and a positive solution, denoted by  $u_{\mu}$ , of (1.1) for  $\lambda = \mu$ . Since  $\mu < \lambda_0$ , it is clear that  $u_{\mu}$  is subsolution of (1.1) for  $\lambda = \lambda_0$ . On the other hand, for  $M$  large, there exists  $\bar{u}$  supersolution of (1.1) for  $\lambda = \lambda_0$ . Finally, thanks to regularity results, see for instance Proposition 4.1 in [1] or Lemma 2.4 in [5], we have that  $u_{\mu} \in L^{\infty}(\Omega)$ . Hence, for  $M$  large  $u_{\mu} \leq \bar{u}$ , and we can conclude the existence of positive solution for  $\lambda = \lambda_0$ . This completes the proof.

□

**Remark 4.2.** Since our subsolution  $\underline{u}$  is strictly positive in  $U$ , we have by Theorem 3.1 that  $u \geq \underline{u} > 0$  in a non empty open set contained in  $\Omega$ . In the case  $p_1 \geq 2$  by the result of Corollary 4.4 [4] we have  $u > 0$  in  $\Omega$ .

**Remark 4.3.** We comment a possible further generalization.

Let  $x = (x_1, \dots, x_N)$ , where  $x_i \in \Omega_i \subset \mathbb{R}^{N_i}$ , being  $\Omega_i$  an open, bounded and convex domain. Denote with  $\nabla_{x_i}$  the gradient along the vector  $x_i$  and  $\text{div}_{x_i}$  its divergence, and let

$$\Delta_{p_i} u = \text{div}_{x_i} (|\nabla_{x_i} u|^{p_i-2} \nabla_{x_i} u) = \sum_{j=1}^{N_i} \frac{\partial}{\partial x_{ij}} \left( \left| \frac{\partial u}{\partial x_{ij}} \right|^{p_i-2} \frac{\partial u}{\partial x_{ij}} \right)$$

be the  $p_i$ -Laplacian acting on the  $x_i$  vector. Problems of the kind of

$$\begin{cases} -\sum_{i=1}^N \Delta_{p_i} u = \lambda u^{q-1} & \text{in } \Omega = \prod \Omega_i, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.7)$$

can be faced with the same technique, using properties of the  $p_i$ -Laplacian principal eigenfunctions, and owing integrability condition as (2.6) to recent regularity results obtained in [2].

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