Research Article

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Area formula for regular submanifolds of low codimension in Heisenberg groups

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Abstract: We establish an area formula for the spherical measure of intrinsically regular submanifolds of low codimension in Heisenberg groups. The spherical measure is constructed by an arbitrary homogeneous distance. Among the arguments of the proof, we point out the differentiability properties of intrinsic graphs and a chain rule for intrinsically differentiable functions.

Keywords: Heisenberg group, area formula, spherical measure, centered Hausdorff measure, intrinsic differentiability, chain rule

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1 Introduction

Research in the analysis and geometry of simply connected nilpotent Lie groups has spread into several directions, especially in the last decade. Carnot groups, or stratified groups equipped with a homogeneous left invariant distance, are an important class of these nilpotent groups, which are metrically different from Euclidean spaces or Riemannian manifolds, still maintaining a rich algebraic and metric structure.

Our aim is to compute the area of submanifolds in the Heisenberg group $\mathbb{H}^n$, which represents the simplest model of a noncommutative stratified group. For different classes of C^1, smooth submanifolds area formulas are available; see [31] and the references therein. The question has new difficulties when we consider "intrinsic regular submanifolds" of $\mathbb{H}^n$ that need neither be C^1 smooth nor Lipschitz with respect to the Euclidean distance [22].

On the other hand, they are suitable level sets of continuously differentiable functions from $\mathbb{H}^n$ to $\mathbb{R}^k$. The differentiability here is understood with respect to the group operation and dilations, i.e. the so-called Pansu differentiability. Precisely, these level sets are defined when $1 \leq k \leq n$ and the differential of the defining function is surjective. They are called $\mathbb{H}$-regular surfaces of low codimension in $\mathbb{H}^n$ (Definition 2.19). These special submanifolds in $\mathbb{H}^n$ and their characterizations have been studied under different perspectives. We mention, for instance, the papers [2, 6–8, 14], along with the lecture notes [35] and the references therein. An implicit function theorem, proved in [14], states that every $\mathbb{H}$-regular surface can be locally seen as an intrinsic graph with respect to a special semidirect factorization (Definition 2.7). Although the parametrizing mapping of the $\mathbb{H}$-regular surface is not Lipschitz continuous in the Euclidean sense, in [6] Arena and Serapioni proved that it is uniformly intrinsically differentiable (Definition 2.15). Indeed, uniform intrinsic differentiability for

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maps acting between suitable factorizing homogeneous subgroups has been largely studied, also in a broader framework and from the viewpoint of nonlinear first order systems of PDEs [3, 4, 9, 12, 23].

We consider a *vertical subgroup* $\mathbb{W}$ and a *horizontal subgroup* $\mathbb{V}$ (Definition 2.2). We assume that $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ (Definition 2.3) and we fix a *parametrized* $\mathbb{H}$*-regular surface* Σ *with respect to* $(\mathbb{W}, \mathbb{V})$, parametrized by ϕ and with defining function f (Definition 2.23). The following measure μ can be associated to Σ: For every Borel set $B \subset \Sigma$, we have

$$\mu(B) = \|V \wedge N\|_g \int_{\Phi^{-1}(B)} \frac{J_H f(\Phi(n))}{J_{\mathbb{V}} f(\Phi(n))}\, d\mathcal{H}_E^{2n+1-k}(n), \tag{1.1}$$

where the Jacobians $J_H f$ and $J_{\mathbb{V}} f$ are defined in (2.9) and (2.10), respectively, $\Phi(n) = n\phi(n)$ is the graph map associated to ϕ and $\mathcal{H}_E^{2n+1-k}$ is the Euclidean Hausdorff measure. The factor $\|V \wedge N\|_g$ takes into account the "angle" between the multivectors N and V, which are associated with the domain and the codomain of the implicit mapping, respectively.

The measure in (1.1) was introduced in [14], where Franchi, Serapioni and Serra Cassano proved that it is equal to the centered Hausdorff measure restricted to Σ. Precisely, [14, Theorem 4.1] has been revised in [35, Theorem 4.50], using a metric area formula for the centered Hausdorff measure [16]. The question of finding an area formula for the spherical measure of a low codimensional $\mathbb{H}$-regular surface remained unanswered. The present paper settles this question, proving an area formula for the spherical measure of Σ in terms of the measure μ.

Let d be a fixed homogeneous distance in $\mathbb{H}^n$ and let us consider the spherical measure $\mathcal{S}^{2n+2-k}$ with respect to d, according to (2.12). We can associate a geometric constant $\beta_d(\Pi)$ to a p-dimensional subspace Π and a distance d that is called *spherical factor*. Essentially, it represents the maximal p-dimensional area of the intersections of Π with metric unit balls whose centers are suitably close to the origin (Definition 2.36). In our area formula, the spherical factor is computed for the homogeneous tangent cones $\mathrm{Tan}(\Sigma, x)$ of Σ at the points $x \in \Sigma$ (Definition 2.20). Theorem 3.2 establishes the "upper blow-up" of the measure μ, proving that the spherical factor of $\mathrm{Tan}(\Sigma, x)$ equals the $(2n+2-k)$-spherical Federer density of μ at $x \in \Sigma$ (Definition 2.34), namely

$$\theta^{2n+2-k}(\mu, x) = \beta_d(\mathrm{Tan}(\Sigma, x)).$$

The previous equality represents the central technical tool of the paper. Indeed, if we combine Theorem 3.2 and the metric area formula of Theorem 2.35, we immediately obtain our main result, that is, the following area formula for the spherical measure.

Theorem 1.1 (Area formula). *If* Σ *is a parametrized* $\mathbb{H}$*-regular surface with respect to* $(\mathbb{W}, \mathbb{V})$*, then for every Borel set* $B \subset \Sigma$ *we have*

$$\mu(B) = \int_B \beta_d(\mathrm{Tan}(\Sigma, x))\, d\mathcal{S}^{2k+2-k}(x), \tag{1.2}$$

where the measure μ *is defined in* (1.1).

The previous integral formula also shows that the measure μ does not depend on the factorization $\mathbb{W} \rtimes \mathbb{V}$ and on the defining function f appearing in (1.1). In fact, when the factors $\mathbb{W}$ and $\mathbb{V}$ of the semidirect product of $\mathbb{H}^n$ are orthogonal, then [9, Theorem 6.1] proves that the integrand in (1.1) can be written in terms of intrinsic partial derivatives of the parametrization ϕ of Σ, namely the defining function f disappears. Thus, the area formula takes the following form.

Theorem 1.2. *Let* Σ *be a parametrized* $\mathbb{H}$*-regular surface with respect to* $(\mathbb{W}, \mathbb{V})$*. Let* ϕ *be the parametrization of* Σ *with respect to* $(\mathbb{W}, \mathbb{V})$*, according to Definition 2.23. If* $\Phi(n) = n\phi(n)$ *is the graph mapping and* $\mathbb{W}$ *is orthogonal to* $\mathbb{V}$*, then for every Borel set* $B \subset \Sigma$ *we have*

$$\int_{\Phi^{-1}(B)} J^{\phi}\phi(w)\, d\mathcal{H}_E^{2n+1-k}(w) = \int_B \beta_d(\mathrm{Tan}(\Sigma, x))\, d\mathcal{S}^{2k+2-k}(x), \tag{1.3}$$

where $J^{\phi}\phi$ *is the natural intrinsic Jacobian of* ϕ *(Definition 2.30).*

At this point, it is worth giving some ideas about the proof of our main technical tool, that is, the "upper blow-up" of Theorem 3.2. This type of blow-up appeared in codimension one, to compute the spherical Federer density of the perimeter measure [30]. In our higher codimensional framework, the proof of the upper blow-up involves some new features. Three key aspects must be emphasized. First, rather unexpectedly, we realize that the intrinsic differentiability of the parametrizing map ϕ (Theorem 2.27) is crucial to establish the limit of the set (3.7). Second, we prove an "intrinsic chain rule" (Theorem 2.18) that permits us to connect the kernel of Df with the intrinsic differential of ϕ, according to (3.9). However, to make our chain rule work we have slightly modified the well-known notion of intrinsic differentiability associated to a factorization, introducing the *extrinsic differentiability* (Definition 2.16). We will pay more attention to this notion of differentiability and its associated chain rule in future investigations, since they may have an independent interest. Third, we establish a delicate algebraic lemma for computing the Jacobian of projections between two vertical subgroups that are complementary to the same horizontal subgroup (Lemma 3.1).

The area formulas (1.2) and (1.3) take a simple form when the homogeneous distance d is invariant under suitable classes of symmetries. We refer to p-vertically symmetric distances (Definition 2.37) and multiradial distances (Definition 2.40). For instance, the Cygan–Korányi distance [10], the distances constructed in [18, Theorem 2] and the distance d_∞ of [14, Section 2.1] are examples of multiradial distances. Furthermore, the sub-Riemannian distance in the first Heisenberg group is 2-vertically symmetric. By combining Theorem 1.1, Theorem 2.38 and Proposition 2.41, a simpler version of (1.2) can be immediately established.

Theorem 1.3. *Let d be either a $(2n+1-k)$-vertically symmetric distance or a multiradial distance of $\mathbb{H}^n$. Let Σ be a parametrized $\mathbb{H}$-regular surface with respect to $(\mathbb{W}, \mathbb{V})$ and let μ be defined as in (1.1). We have that*

$$\mu = \omega_d(2n+1-k)\mathcal{S}^{2k+2-k} \llcorner \Sigma,$$

where $\omega_d(2n+1-k)$ is the constant spherical factor introduced in Definition 2.39. Therefore, setting

$$\mathcal{S}_d^{2n+2-k} = \omega_d(2n+1-k)\mathcal{S}^{2n+1-k},$$

we have

$$\mathcal{S}_d^{2n+2-k} \llcorner \Sigma = \mu = \|V \wedge N\|_g \Phi_\sharp\Big(\frac{J_{\mathbb{H}}f}{J_{\mathbb{V}}f} \circ \Phi\Big)\mathcal{H}_E^{2n+1-k} \llcorner \mathbb{W}. \tag{1.4}$$

In the assumptions of the previous theorem, assuming in addition that $\mathbb{W}$ and $\mathbb{V}$ are orthogonal, formula (1.4) can be rewritten as follows:

$$\mathcal{S}_d^{2n+2-k} \llcorner \Sigma(B) = \int_{\Phi^{-1}(B)} J^\phi\phi(w)\, d\mathcal{H}_E^{2n+1-k}(w) \tag{1.5}$$

for any Borel set $B \subset \mathbb{H}^n$, where $J^\phi\phi$ is the intrinsic Jacobian of ϕ (Definition 2.30). The form of formula (1.5) clearly reminds of the Euclidean area formula. Indeed, the Euclidean Hausdorff measure $\mathcal{H}_E^{2n+1-k}$ can be replaced by the Lebesgue measure $\mathcal{L}^{2n+1-k}$.

Some additional applications of our results concern the relationship between the spherical measure and the centered Hausdorff measure. This study is treated in Section 4, where the main result is the equality between spherical measure and centered Hausdorff measure, assuming that the metric unit ball of the homogeneous distance is convex (Theorem 4.2).

Concerning the more recent literature, a general form of the area formula can be written for suitably "C^1 smooth" intrinsic graphs in stratified groups [20, Theorem 1.1], using the Hausdorff measure or the spherical measure. The proof mainly relies on a suitable application of measure theoretic area formulas [29] (see also the developments of [24]). The upper blow-up of the Hausdorff (or spherical) measure of an intrinsic graph leads to a natural notion of "area factor" [20, Lemma 3.2], which formally represents the Jacobian of the graph mapping and extends the notions of Jacobian introduced in [1, 21, 25]. The approach of [20] can be suitably adapted to obtain area formulas involving the centered Hausdorff measures, and hence using suitable "centered area factors". Then an area formula for the centered Hausdorff measure of graphs of intrinsic Lipschitz mappings can be obtained [5, Theorem 1.3], under the assumption on their a.e. intrinsic differentiability.

On the other hand, whenever a metric-algebraic notion of differentiability is available for the parametrization, it is reasonable to connect the measure of its image with its “suitable differential” by an explicit formula for the Jacobian, getting a full area formula. Connecting the Jacobian with the differential, and hence allowing for an effective computation of the Hausdorff (or spherical) measure of a set, has been completely achieved for intrinsic regular hypersurfaces in stratified groups. This result stems from the contribution of many authors; see [4] for the last version of this one-codimensional area formula, along with the full list of references. For one-codimensional intrinsic Lipschitz graphs, area formulas for the spherical measure are obtained in [11] for stratified groups of step two; see also the references therein.

Parametrized intrinsic $\mathbb{H}$-regular submanifolds in Heisenberg groups essentially represent the first higher codimensional case where the differential of the parametrization is connected to the measure of the submanifold. For the centered Hausdorff measure, we refer to the works [9, 35]. The case of the spherical measure is more delicate and relies on the techniques described above, which lead to the upper blow-up of Theorem 3.2. Although the problem of computing the “area factor" does not seem an easy task, due to the low regularity of intrinsic graphs, we believe however that our scheme for the area formula in Heisenberg groups has actually a wider scope of applications. For this reason, we have left such developments for subsequent investigations.

Finally, we wish to mention that, by combining Theorem 1.1 and the deep Rademacher’s theorem for intrinsic Lipschitz mappings in Heisenberg groups [36, Theorem 1.1], our area formula extends to intrinsic Lipschitz graphs [36, Theorem 1.3]. A similar extension is not automatic in general, since an interesting example of a nowhere intrinsically differentiable Lipschitz graph can be constructed [19].

2 Definitions and preliminary results

The next sections will introduce notions, notations and basic tools that will be used throughout the paper.

2.1 Coordinates in Heisenberg groups

The purpose of this section is to introduce $(2n+1)$-dimensional Heisenberg groups, along with the special coordinates that allow us to identify $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$. The Heisenberg group $\mathbb{H}^n$ can be represented as a direct sum of two linear subspaces

$$\mathbb{H}^n = H_1 \oplus H_2$$

with $\dim(H_1)= 2n$ and $\dim(H_2)= 1$, endowed with a symplectic form ω on H_1 and a fixed nonvanishing element e_{2n+1} of H_2. We denote by π_{H_1} and π_{H_2} the canonical projections on H_1 and H_2, which are associated with the direct sum.

We can give to $\mathbb{H}^n$ a structure of Lie algebra by setting

$$[p, q] = \omega(\pi_{H_1}(p), \pi_{H_1}(q))e_{2n+1}.$$

Then the Baker–Campbell–Hausdorff formula ensures that

$$pq = p + q + \frac{[p, q]}{2}$$

defines a Lie group operation on $\mathbb{H}^n$. For $t > 0$, the linear mapping $\delta_t : \mathbb{H}^n \to \mathbb{H}^n$ such that

$$\delta_t(w) = t^k w \quad \text{if } w \in H_k, \qquad k = 1, 2,$$

defines *intrinsic dilation*.

Given $p \in \mathbb{H}^n$, we denote by l_p the translation by p. Any left invariant vector field on $\mathbb{H}^n$ is of the form $X_v(p) = dl_p(0)(v)$ for any $p \in \mathbb{H}^n$ and some $v \in \mathbb{H}^n$, where we have identified $\mathbb{H}^n$ with $T_0\mathbb{H}^n$. Through the Baker–Campbell–Hausdorff formula, one can check that the Lie algebra of left invariant vector fields $\mathrm{Lie}(\mathbb{H}^n)$ is isomorphic to the given Lie algebra $(\mathbb{H}^n, [\cdot, \cdot])$.

We fix a symplectic basis $(e_1, \dots, e_{2n})$ of (H_1, ω), namely

$$\omega(e_i, e_{n+j}) = \delta_{ij}, \quad \omega(e_i, e_j) = \omega(e_{n+i}, e_{n+j}) = 0$$

for every $i, j = 1, \dots, n$, where δ_{ij} is the Kronecker delta. Thus, we have obtained a *Heisenberg basis*

$$\mathcal{B} = (e_1, \dots, e_{2n+1})$$

that allows us to identify $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$. The associated linear isomorphism is defined by

$$\pi_{\mathcal{B}} : \mathbb{H}^n \to \mathbb{R}^{2n+1}, \quad \pi_{\mathcal{B}}(p) = (x_1, \dots, x_{2n+1}),$$

for $p = \sum_{j=1}^{2n+1} x_j e_j$. We can read the given Lie product on $\mathbb{R}^{2n+1}$ as follows:

$$\begin{aligned}[(x_1, \dots, x_{2n+1}), (y_1, \dots, y_{2n+1})] &= \pi_{\mathcal{B}}\Big(\Big[\sum_{i=1}^{2n+1} x_i e_i, \sum_{i=1}^{2n+1} y_i e_i\Big]\Big) \\ &= \Big(0, \dots, 0, \sum_{i=1}^{n}(x_i y_{i+n} - x_{i+n} y_i)\Big).\end{aligned}$$

Then the group product takes the following form on $\mathbb{R}^{2n+1}$:

$$(x_1, \dots, x_{2n+1})(y_1, \dots, y_{2n+1}) = \Big(x_1 + y_1, \dots, x_{2n+1} + y_{2n+1} + \sum_{i=1}^{n} \frac{x_i y_{i+n} - x_{i+n} y_i}{2}\Big).$$

Taking into account the previous formula, in our coordinates we obtain the following basis of left invariant vector fields:

$$\begin{aligned} X_j(p) &= \partial_{x_j} - \frac{1}{2} x_{j+n} \partial_{x_{2n+1}}, \quad j = 1, \dots, n, \\ Y_j(p) &= \partial_{x_{n+j}} + \frac{1}{2} x_j \partial_{x_{2n+1}}, \quad j = 1, \dots, n, \\ T(p) &= \partial_{x_{2n+1}}. \end{aligned}$$

They clearly constitute a basis $(X_1, \dots, X_{2n+1})$ of $\mathrm{Lie}(\mathbb{H}^n)$ such that $X_j(0) = e_j$ for every $j = 1, \dots, 2n+1$. Any linear combination of $X_1, \dots, X_{2n}$ is called a *left invariant horizontal vector field of* $\mathbb{H}^n$.

2.2 Metric structure

We fix a scalar product $\langle \cdot, \cdot \rangle$ that makes our Heisenberg basis $\mathcal{B} = (e_1, \dots, e_{2n+1})$ orthonormal. In the sequel, any Heisenberg basis will be understood to be orthonormal. We denote by $|\cdot|$ both the Euclidean metric on $\mathbb{R}^{2n+1}$ and the norm induced by $\langle \cdot, \cdot \rangle$ on $\mathbb{H}^n$. The symmetries of the Heisenberg group $\mathbb{H}^n$ are detected through the isometry

$$J : H_1 \to H_1$$

that is defined by the Heisenberg basis

$$J(e_i) = e_{n+i} \quad \text{and} \quad J(e_{n+i}) = -e_i$$

for all $i = 1, \dots, n$. It is then easy to check that

$$\langle p, q \rangle = \omega(p, Jq) \quad \text{and} \quad J^2 = -I$$

for all $p, q \in H_1$.

A *homogeneous distance* d on $\mathbb{H}^n$ is a function $d : \mathbb{H}^n \times \mathbb{H}^n \to [0, +\infty)$ such that

$$d(zx, zy) = d(x, y) \quad \text{and} \quad d(\delta_t(x), \delta_t(y)) = t\,d(x, y)$$

for every $x, y, z \in \mathbb{H}^n$ and $t > 0$. Any two homogeneous distances are bi-Lipschitz equivalent. We also introduce the *homogeneous norm* $\|x\| = d(x, 0)$, $x \in \mathbb{H}^n$, associated to a homogeneous distance d. Notice that this norm satisfies

$$\|xy\| \leq \|x\| + \|y\| \quad \text{and} \quad \|\delta_r x\| = r\|x\|$$

for $x, y \in \mathbb{H}^n$ and $r > 0$.

By identifying $T_0\mathbb{H}^n$ with $\mathbb{H}^n$ and by left translating the fixed scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{H}^n$, we obtain a left invariant Riemannian metric g on $\mathbb{H}^n$. Its associated Riemannian norm is denoted by $\|\cdot\|_g$. We may restrict the identification of $T_0\mathbb{H}^n$ with $\mathbb{H}^n$ to the so-called horizontal subspace, by identifying H_1 with

$$H_0\mathbb{H}^n \subset T_0\mathbb{H}^n.$$

Then the horizontal fiber at $p \in \mathbb{H}^n$ is $H_p\mathbb{H}^n = dl_p(0)(H_0\mathbb{H}^n)$. The collection of all horizontal fibers constitutes the so-called *horizontal subbundle* $H\mathbb{H}^n$. If we restrict the left invariant metric g to the horizontal subbundle $H\mathbb{H}^n$, we obtain a scalar product on each horizontal fiber, that is, the sub-Riemannian metric. This leads in a standard way to the so-called *Carnot–Carathéodory distance*, or *sub-Riemannian distance* [17], which is an example of homogeneous distance.

2.3 Differentiability and factorizations

We have different notions of differentiability in $\mathbb{H}^n$ and general Carnot groups, starting from the notion of Pansu differentiability [33]. Throughout the paper, we fix a homogeneous distance d. Let $\Omega \subset \mathbb{H}^n$ be an open set, let $f : \Omega \to \mathbb{R}^k$, $x \in \Omega$ and $v \in H_1$. If there exists

$$\lim_{t \to 0} \frac{f(x(tv)) - f(x)}{t} \in \mathbb{R}^k,$$

then we say that it is the *horizontal partial derivative* at x along X_v, that is, the unique left invariant vector field such that $X_v(0) = v$. The above limit is denoted by $X_v f(x)$. Notice that X_v is precisely a left invariant horizontal vector field. We say that $f \in C_h^1(\Omega, \mathbb{R}^k)$ if for every $x \in \Omega$ and every horizontal vector field $X \in \mathrm{Lie}(\mathbb{H}^n)$ the horizontal derivative $Xf(x)$ exists and is continuous with respect to $x \in \Omega$.

A linear mapping $L : \mathbb{H}^n \to \mathbb{R}^k$ that is homogeneous, i.e. $tL(v) = L(\delta_t v)$ for all $t > 0$ and $v \in \mathbb{H}^n$, is an *h-homomorphism*, which stands for "homogeneous homomorphism". If there exists an h-homomorphism $L : \mathbb{H}^n \to \mathbb{R}^k$ that satisfies

$$|f(xw) - f(x) - L(w)| = o(d(w, 0)) \quad \text{as } d(w, 0) \to 0,$$

then it is unique and is called the *h-differential*, or *Pansu differential*, of f at x. We denote it by $Df(x)$. Notice that $f \in C_h^1(\Omega, \mathbb{R}^k)$ if and only if it is everywhere Pansu differentiable and $x \to Df(x)$ is continuous as a map from Ω to the space of h-homomorphisms; see, for instance, [27, Section 3].

Definition 2.1. Let $\Omega \subset \mathbb{H}^n$ be an open set and let $f \in C_h^1(\Omega, \mathbb{R})$. We call the unique vector $\nabla_H f(x)$ of H_1 such that $Df(x)(z) = \langle \nabla_H f(x), z \rangle$ for every $z \in \mathbb{H}^n$ the *horizontal gradient* of f at $x \in \Omega$.

When differentiability meets the factorizations of Heisenberg groups, the notion of intrinsic differentiability comes up naturally; see [35] for more information. Now, we introduce some algebraic properties of factorizations in $\mathbb{H}^n$ in order to define intrinsic differentiability and its basic properties.

Definition 2.2. If a Lie subgroup of $\mathbb{H}^n$ is closed under intrinsic dilations, we call it a *homogeneous subgroup*. Homogeneous subgroups of $\mathbb{H}^n$ containing H_2 are called *vertical subgroups*. Homogeneous subgroups contained in H_1 are called *horizontal subgroups*.

It is easy to realize that any homogeneous subgroup of $\mathbb{H}^n$ is either horizontal or vertical. We also notice that normal homogeneous subgroups of $\mathbb{H}^n$ coincide with vertical subgroups.

Definition 2.3. Let $\mathbb{W}$ and $\mathbb{V}$ be a vertical subgroup and a horizontal subgroup of $\mathbb{H}^n$, respectively. We say that $\mathbb{H}^n$ is the *semidirect product* of $\mathbb{W}$ and $\mathbb{V}$ if $\mathbb{H}^n = \mathbb{W}\mathbb{V}$ and $\mathbb{W} \cap \mathbb{V} = \{0\}$. In symbols, we write $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$.

Definition 2.4. Let $\mathbb{M}$, $\mathbb{W}$ and $\mathbb{V}$ be homogeneous subgroups of $\mathbb{H}^n$ such that

$$\mathbb{H}^n = \mathbb{M} \rtimes \mathbb{V} = \mathbb{W} \rtimes \mathbb{V}. \tag{2.1}$$

The semidirect product $\mathbb{W} \rtimes \mathbb{V}$ automatically yields the unique projections

$$\pi_{\mathbb{W}} : \mathbb{H}^n \to \mathbb{W} \quad \text{and} \quad \pi_{\mathbb{V}} : \mathbb{H}^n \to \mathbb{V}$$

such that $x = \pi_{\mathbb{W}}(x)\pi_{\mathbb{V}}(x)$ for every $x \in \mathbb{H}^n$. If necessary, to emphasize the dependence on the semidirect factorization we will also introduce the notations $\pi_{\mathbb{W}}^{\mathbb{W},\mathbb{V}} = \pi_{\mathbb{W}}$ and $\pi_{\mathbb{V}}^{\mathbb{W},\mathbb{V}} = \pi_{\mathbb{V}}$. The same holds for $\mathbb{M} \rtimes \mathbb{V}$. We define the following restrictions:

$$\pi_{\mathbb{W},\mathbb{M}}^{\mathbb{W},\mathbb{V}} = \pi_{\mathbb{W}}^{\mathbb{W},\mathbb{V}}|_{\mathbb{M}} \colon \mathbb{M} \to \mathbb{W} \quad \text{and} \quad \pi_{\mathbb{M},\mathbb{W}}^{\mathbb{M},\mathbb{V}} = \pi_{\mathbb{M}}^{\mathbb{M},\mathbb{V}}|_{\mathbb{W}} \colon \mathbb{W} \to \mathbb{M}.$$

Remark 2.5. The uniqueness of the factorizations (2.1) implies that both restrictions $\pi_{\mathbb{W},\mathbb{M}}^{\mathbb{W},\mathbb{V}}$ and $\pi_{\mathbb{M},\mathbb{W}}^{\mathbb{M},\mathbb{V}}$ are invertible and

$$\pi_{\mathbb{W},\mathbb{M}}^{\mathbb{W},\mathbb{V}} = (\pi_{\mathbb{M},\mathbb{W}}^{\mathbb{M},\mathbb{V}})^{-1}. \tag{2.2}$$

If $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$, then, by the local compactness of $\mathbb{H}^n$, it is immediate to observe that there exists a constant $c_0 \in (0,1)$, possibly depending on $\mathbb{W}$ and $\mathbb{V}$, such that for all $w \in \mathbb{W}$ and $v \in \mathbb{V}$ the following holds:

$$c_0(\|w\| + \|v\|) \le \|wv\| \le \|w\| + \|v\|. \tag{2.3}$$

Remark 2.6. Whenever two homogeneous subgroups $\mathbb{W}$ and $\mathbb{V}$ of $\mathbb{H}^n$ satisfy

$$\mathbb{H}^n = \mathbb{W}\mathbb{V} \quad \text{and} \quad \mathbb{W} \cap \mathbb{V} = \{0\},$$

then one of them must be necessarily vertical and the other one must be horizontal.

Now, we recall some results and definitions about intrinsic graphs of functions between two homogeneous subgroups. In the sequel, $\mathbb{W}$ and $\mathbb{V}$ denote a vertical subgroup and a horizontal subgroup, respectively, such that $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$. For more information, see [35].

Definition 2.7. For a nonempty set $U \subset \mathbb{W}$ and $\phi : U \to \mathbb{V}$, we define the *intrinsic graph of* ϕ as the set

$$\operatorname{graph}(\phi) = \{w\phi(w) : w \in U\}.$$

We also introduce the *graph map* $\Phi : U \to \Sigma$ *of* ϕ by $\Phi(w) = w\phi(w)$ for all $w \in U$.

Remark 2.8. It is important to observe that the notion of intrinsic graph is invariant with respect to both translations and dilations.

To study the action of translations on intrinsic graphs, we need the following definition.

Definition 2.9. Let us consider $x \in \mathbb{H}^n$. We define $\sigma_x : \mathbb{W} \to \mathbb{W}$ as follows:

$$\sigma_x(w) = \pi_{\mathbb{W}}(l_x(w)) = xw(\pi_{\mathbb{V}}(x))^{-1}$$

for every $w \in \mathbb{W}$. Given a set $U \subset \mathbb{W}$ and a function $\phi : U \to \mathbb{V}$, the *translation of* ϕ *at* x, $\phi_x : \sigma_x(U) \to \mathbb{V}$, is defined by

$$\phi_x(w) = \pi_{\mathbb{V}}(x)\phi(x^{-1}w\pi_{\mathbb{V}}(x)) = \pi_{\mathbb{V}}(x)\phi(\sigma_{x^{-1}}(w)). \tag{2.4}$$

Remark 2.10. The map σ_x is invertible on $\mathbb{W}$:

$$\sigma_{x^{-1}}(w) = x^{-1}w\pi_{\mathbb{V}}(x^{-1})^{-1} = x^{-1}w\pi_{\mathbb{V}}(x) = \sigma_x^{-1}(w).$$

Then, for $w \in \sigma_x(U)$, we may also write

$$\phi_x(w) = \pi_{\mathbb{V}}(x)\phi(\sigma_x^{-1}(w)).$$

Next, we recall the content of [6, Propositions 3.6].

Proposition 2.11. *Let $U \subset \mathbb{W}$ be an open set and let $\phi : U \to \mathbb{V}$ be a function. Then we have*

$$l_x(\text{graph}(\phi)) = \{w\phi_x(w) : w \in \sigma_x(U)\}.$$

Definition 2.12. Let $U \subset \mathbb{W}$ be an open set and let $\phi : U \to \mathbb{V}$ be a function. Let us take $\bar{w} \in U$ and define $x = \bar{w}\phi(\bar{w})$. The function ϕ is intrinsically differentiable at $\bar{w}$ if there exists an h-homomorphism $L : \mathbb{W} \to \mathbb{V}$ such that

$$d(L(w), \phi_{x^{-1}}(w)) = o(\|w\|) \tag{2.5}$$

as $w \to 0$. The function L is called the *intrinsic differential* of ϕ at $\bar{w}$, it is uniquely defined and we denote it by $d\phi_{\bar{w}}$.

Remark 2.13. By virtue of [6, Proposition 3.23], in our setting any intrinsic linear function is actually an h-homomorphism. We also observe that the assumption $\bar{w} \in U$ implies that $0 \in \sigma_{x^{-1}}(U)$. In addition, $\sigma_{x^{-1}}(U)$ is an open set, and hence the limit (2.5) is entirely justified.

Remark 2.14. By [6, Proposition 3.25], condition (2.5) is equivalent to ask that for all $w \in U$,

$$\|d\phi_{\bar{w}}(\bar{w}^{-1}w)^{-1}\phi(\bar{w})^{-1}\phi(w)\| = o(\|\phi(\bar{w})^{-1}\bar{w}^{-1}w\phi(\bar{w})\|),$$

as $\|\phi(\bar{w})^{-1}\bar{w}^{-1}w\phi(\bar{w})\| \to 0$.

Definition 2.15. Let $U \subset \mathbb{W}$ be an open set and let $\phi : U \to \mathbb{V}$ be a function. The map ϕ is uniformly intrinsically differentiable on U if for any point $\bar{w} \in U$ there exists an h-homomorphism $d\phi_{\bar{w}} : \mathbb{W} \to \mathbb{V}$ such that

$$\lim_{\delta\to 0} \sup_{\|\bar{w}^{-1}w'\|<\delta} \sup_{0<\|w\|<\delta} \frac{d(d\phi_{\bar{w}}(w), \phi_{\Phi(w')^{-1}}(w))}{\|w\|} = 0,$$

where Φ is the graph map of ϕ.

The following definition is a slight modification of the notion of intrinsic differentiability.

Definition 2.16. Let $U \subset \mathbb{W}$ be an open set and let $F : U \to \mathbb{R}^k$ with $u \in U$. We choose $v \in \mathbb{V}$ and define $x = uv \in \mathbb{H}^n$ and the corresponding translated function

$$F^{\mathbb{V}}_{x^{-1}}(w) = F(\sigma_x(w)) - F(u)$$

for $w \in \sigma_{x^{-1}}(U)$. We say that F is *extrinsically differentiable at u with respect to* $(\mathbb{V}, x)$ if there exists an h-homomorphism $L : \mathbb{W} \to \mathbb{R}^k$ such that

$$\frac{|F^{\mathbb{V}}_{x^{-1}}(w) - L(w)|}{\|w\|} \to 0 \quad \text{as } w \to 0. \tag{2.6}$$

The uniqueness of L allows us to denote it by $d^{\mathbb{V}}_x F$.

The terminology *extrinsic differentiability* arises from the fact that the subgroup $\mathbb{V}$ and the point x cannot be detected from the information we have on F. In a sense, they are “artificially added from outside”.

Remark 2.17. If in the previous definition we embed $\mathbb{R}^k$ in $\mathbb{H}^n$, hence replacing it by $\mathbb{V}$, and choose $v = F(u)$, then $x = uF(u) \in \mathbb{H}^n$ and we have the equalities

$$F^{\mathbb{V}}_{x^{-1}}(w) = F(\sigma_x(w)) - v = F_{x^{-1}}(w).$$

Thus, the numerator of (2.6) becomes equivalent to $d(F_{x^{-1}}(w), L(w))$, and the extrinsic differentiability of F at u with respect to $(\mathbb{V}, x)$ coincides with the intrinsic differentiability of F at u.

Extrinsic and intrinsic differentiability compensate each other in the following theorem.

Theorem 2.18 (Chain rule). *Let us consider two open sets $U \subset \mathbb{W}$, $\Omega \subset \mathbb{H}^n$ and two functions $f : \Omega \to \mathbb{R}^k$, $\phi : U \to \mathbb{V}$. Assume $\Phi(U) \subset \Omega$, where Φ is the graph function of ϕ. Let us consider $x_{\mathbb{W}} \in U$ and set $x = \Phi(x_{\mathbb{W}})$.*

If f and ϕ are h-differentiable at x and intrinsically differentiable at $x_{\mathbb{W}}$, respectively, then the composition $F = f \circ \Phi : U \to \mathbb{R}^k$, given by

$$F(u) = f(u\phi(u)) \quad \text{for all } u \in U,$$

is extrinsically differentiable at $x_{\mathbb{W}}$ with respect to $(\mathbb{V}, x)$. For every $w \in \mathbb{W}$, the formula

$$d_x^{\mathbb{V}} F(w) = Df(x)(w\, d\phi_{x_{\mathbb{W}}}(w)) \tag{2.7}$$

holds. If in addition $f(w\phi(w)) = c$ for every $w \in U$ and some $c \in \mathbb{R}$, then we obtain

$$\ker(Df(x)) = \operatorname{graph}(d\phi_{x_{\mathbb{W}}}). \tag{2.8}$$

Proof. Let us first show that F is extrinsically differentiable at $x_{\mathbb{W}}$ with respect to $(\mathbb{V}, x)$. We define

$$L(w) = Df(x)(w\, d\phi_{x_{\mathbb{W}}}(w)) = Df(x)(w) + Df(x)(d\phi_{x_{\mathbb{W}}}(w))$$

for $w \in \mathbb{W}$, which is an h-homomorphism. For w small enough, we have

$$\begin{aligned}
\frac{|F_{x^{-1}}^{\mathbb{V}}(w) - L(w)|}{\|w\|} &= \frac{|f(xwx_{\mathbb{V}}^{-1}\phi(xwx_{\mathbb{V}}^{-1})) - f(x) - L(w)|}{\|w\|} \\
&= \frac{|f(xw\phi_{x^{-1}}(w)) - f(x) - Df(x)(w\, d\phi_{x_{\mathbb{W}}}(w))|}{\|w\|} \\
&\le \frac{|f(xw\phi_{x^{-1}}(w)) - f(x) - Df(x)(w\phi_{x^{-1}}(w))|}{\|w\|} + \frac{|Df(x)(w\phi_{x^{-1}}(w)) - Df(x)(w\, d\phi_{x_{\mathbb{W}}}(w))|}{\|w\|}.
\end{aligned}$$

Let us consider the last two addends separately:

$$\frac{|f(xw\phi_{x^{-1}}(w)) - f(x) - Df(x)(w\phi_{x^{-1}}(w))|}{\|w\|} = \frac{|f(xw\phi_{x^{-1}}(w)) - f(x) - Df(x)(w\phi_{x^{-1}}(w))|}{\|w\phi_{x^{-1}}(w)\|} \frac{\|w\phi_{x^{-1}}(w)\|}{\|w\|} \to 0$$

as $\|w\| \to 0$, by the Pansu differentiability of f at x and by the validity of

$$\frac{\|w\phi_{x^{-1}}(w)\|}{\|w\|} \le 1 + \frac{\|\phi_{x^{-1}}(w)\|}{\|w\|} = 1 + \Big\| d\phi_{x_{\mathbb{W}}}\Big(\frac{w}{\|w\|}\Big)\Big\| + \frac{\|d\phi_{x_{\mathbb{W}}}(w)^{-1}\phi_{x^{-1}}(w)\|}{\|w\|} \le C_x$$

for all $w \neq 0$ sufficiently small. It is indeed a consequence of the intrinsic differentiability of ϕ at $x_{\mathbb{W}}$. For the second addend, the previous intrinsic differentiability yields

$$\frac{|Df(x)(d\phi_{x_{\mathbb{W}}}(w)^{-1}\phi_{x^{-1}}(w))|}{\|w\|} = \Big|Df(x)\Big(\frac{d\phi_{x_{\mathbb{W}}}(w)^{-1}\phi_{x^{-1}}(w)}{\|w\|}\Big)\Big| \to 0$$

as $w \to 0$. This complete the proof of the first claim and also establishes formula (2.7).

Let us now assume the constancy of $w \to f(w\phi(w))$ on U. Since we have proved that F is extrinsically differentiable at $x_{\mathbb{W}}$ with respect to $(\mathbb{V}, x)$, being in this case $F_{x^{-1}}^{\mathbb{V}}$ identically vanishing, we obtain

$$d_x^{\mathbb{V}} F(w) = o(\|w\|)$$

as $w \to 0$. Therefore, for any $u \in \mathbb{W}$, we have

$$\|Df(x)(\delta_t u\, d\phi_{x_{\mathbb{W}}}(\delta_t u))\| = o(t)$$

as $t \to 0$. Due to the h-linearity, it follows that

$$Df(x)(u\, d\phi_{x_{\mathbb{W}}}(u)) = 0.$$

We have proved the inclusion $\operatorname{graph}(d\phi_{x_{\mathbb{W}}}) \subset \ker(Df(x))$ of homogeneous subgroups with the same dimension, and hence formula (2.8) is established. □

The notion of $\mathbb{H}$-regular surface in $\mathbb{H}^n$ was first given in [14].

Definition 2.19. Let $\Sigma \subset \mathbb{H}^n$ be a set and let $1 \le k \le n$. We say that Σ is an $\mathbb{H}$-*regular surface of low codimension*, or a k-*codimensional* $\mathbb{H}$-*regular surface*, if for every $x \in \Sigma$ there exist an open set Ω containing x and a function $f = (f_1, \dots, f_k) \in C_h^1(\Omega, \mathbb{R}^k)$ such that the following conditions hold:
(i) $\Sigma \cap \Omega = \{y \in \Omega : f(y) = 0\}$.
(ii) $\nabla_H f_1(y) \wedge \cdots \wedge \nabla_H f_k(y) \neq 0$ for all $y \in \Omega$.

We can characterize the metric tangent cone of an $\mathbb{H}$-regular surface of codimension k.

Definition 2.20. For $A \subset \mathbb{H}^n$ and $x \in A$, the *homogeneous tangent cone* is the set

$$\operatorname{Tan}(A, x) = \{v \in \mathbb{H}^n : v = \lim_{h\to\infty} \delta_{r_h}(x^{-1}x_h),\ r_h > 0,\ x_h \in A,\ x_h \to x\}.$$

From [14, Proposition 3.29], we have the following characterization.

Proposition 2.21. *If Σ is an $\mathbb{H}$-regular surface of low codimension and $f \in C_h^1(\Omega, \mathbb{R}^k)$ is as in Definition 2.19, then*

$$\ker Df(x) = \operatorname{Tan}(\Sigma, x)$$

for all $x \in \Sigma \cap \Omega$.

Given an open subset $\Omega \subset \mathbb{H}^n$, a function $f \in C_h^1(\Omega, \mathbb{R}^k)$ and $x \in \Omega$, we define the *horizontal Jacobian*

$$J_H f(x) = \|\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\|_g, \tag{2.9}$$

where the norm is given through our fixed left invariant metric g.

If $f \in C_h^1(\Omega, \mathbb{R})$ and $\mathbb{V} \subset H_1$ is a k-dimensional subspace, we set $\nabla_{\mathbb{V}} f(x)$ as the unique vector of $\mathbb{V}$ such that $Df(x)(z) = \langle \nabla_{\mathbb{V}} f(x), z\rangle$ for every $z \in \mathbb{V}$. As a consequence, we can also define the *Jacobian with respect to* $\mathbb{V}$, namely

$$J_{\mathbb{V}} f(x) = \|\nabla_{\mathbb{V}} f_1(x) \wedge \cdots \wedge \nabla_{\mathbb{V}} f_k(x)\|_g. \tag{2.10}$$

The next implicit function theorem is proved in [14, Theorem 3.27]. Its general version in the framework of homogeneous groups is given in [28, Theorem 1.3].

Theorem 2.22 (Implicit function theorem). *Let $\Omega \subset \mathbb{H}^n$ be an open set, let $f \in C_h^1(\Omega, \mathbb{R}^k)$ be a function and consider a point $x_0 \in \Omega$ such that $J_{\mathbb{V}} f(x_0) > 0$. We define the level set*

$$\Sigma = \{x \in \Omega : f(x) = f(x_0)\}.$$

Setting $\pi_{\mathbb{W}}(x_0) = \eta_0$ and $\pi_{\mathbb{V}}(x_0) = \upsilon_0$, there exist an open set $\Omega' \subset \Omega \subset \mathbb{H}^n$ with $x_0 \in \Omega'$, an open set $U \subset \mathbb{W}$ with $\eta_0 \in U$, and a unique continuous function $\phi : U \to \mathbb{V}$ such that $\phi(\eta_0) = \upsilon_0$ and

$$\Sigma \cap \Omega' = \{w\phi(w) : w \in U\}.$$

Definition 2.23 (Parametrized $\mathbb{H}$-regular surface). Let Σ be an $\mathbb{H}$-regular surface. We assume that there exist a semidirect factorization $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$, an open set $U \subset \mathbb{W}$ and a uniformly intrinsically differentiable $\phi : U \to \mathbb{V}$ such that

$$\Sigma = \{u\phi(u) \in \mathbb{H}^n : u \in U\}.$$

We say that Σ is a *parametrized* $\mathbb{H}$-*regular surface with respect to* $(\mathbb{W}, \mathbb{V})$, where ϕ is the *parametrization of* Σ. If $\Omega \subset \mathbb{H}^n$ is open, $\Sigma \subset \Omega$ and we have $f \in C_h^1(\Omega, \mathbb{R}^k)$ and $x_0 \in \Sigma$ such that $f^{-1}(f(x_0)) \cap U\mathbb{V} = \Sigma$ and $Df(x) : \mathbb{H}^n \to \mathbb{R}^k$ is surjective for every $x \in \Sigma$, then we say that f is a *defining function of* Σ.

Proposition 2.24. *Let $\Omega \subset \mathbb{H}^n$ be open and let $f \in C_h^1(\Omega, \mathbb{R}^k)$ be such that $f^{-1}(f(x_0)) = \Sigma$ for some $x_0 \in \Omega$. If $J_{\mathbb{V}} f(x) > 0$ for all $x \in \Sigma$, then Σ is a parametrized $\mathbb{H}$-regular surface with respect to $(\mathbb{W}, \mathbb{V})$ and f is a defining function.*

Proof. We may apply the implicit function theorem of [14, Proposition 3.13] at any point $x \in \Sigma$. Then locally Σ is an intrinsic graph, and by the uniqueness of the implicit mapping, we can conclude that Σ actually is entirely parametrized by a unique graph mapping. The uniform intrinsic differentiability of this parametrization follows from [6, Theorem 4.2]. As a result, Σ is a parametrized $\mathbb{H}$-regular surface with respect to $(\mathbb{W}, \mathbb{V})$ and f is its defining function. □

A simple application of Theorem 2.18 is the following proposition.

Proposition 2.25. *Let $U \subset \mathbb{W}$ be open and assume that $\phi : U \to \mathbb{V}$ is everywhere intrinsically differentiable. Let $\Sigma = \{n\phi(n) : n \in U\}$ and let $\Omega \subset \mathbb{H}^n$ be open such that $\Sigma \subset \Omega$. If $f : \Omega \to \mathbb{R}^k$ is everywhere h-differentiable with*

$$\Sigma = f^{-1}(f(x_0)) \cap (U\mathbb{V})$$

for some $x_0 \in U\mathbb{V}$ and $J_H f(x) > 0$ for all $x \in \Sigma$, then $J_{\mathbb{V}} f(x) > 0$ for all $x \in \Sigma$.

Proof. We consider $x = w\phi(w)$, so by Theorem 2.18 the function $F = f \circ \Phi$ is extrinsically differentiable at w with respect to $(\mathbb{V}, x)$ and

$$0 = d_x^{\mathbb{V}} F(v) = Df(x)(v d\phi_{x_{\mathbb{W}}}(v)) = D_{\mathbb{W}} f(x)(v) + D_{\mathbb{V}} f(x)(d\phi_{x_{\mathbb{W}}}(v)),$$

where $v \in \mathbb{W}$ and $D_S f(x) = Df(x)|_S$ for any homogeneous subgroup S of $\mathbb{H}^n$. If by contradiction

$$D_{\mathbb{V}} f(x) : \mathbb{V} \to \mathbb{V}$$

would not be a isomorphism, then its image T would have linear dimension less than k. Then the previous equalities would imply that the image of $D_{\mathbb{W}} f(x)$ would be contained in T, and hence the same would hold for the image of $Df(x)$. This conflicts with the fact that $Df(x)$ is surjective. □

The following corollary is a straightforward consequence of the previous proposition.

Corollary 2.26. *If $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ is a semidirect product and f is a defining function of a parametrized $\mathbb{H}$-regular surface Σ with respect to $(\mathbb{W}, \mathbb{V})$, then $J_{\mathbb{V}} f(x) > 0$ for every $x \in \Sigma$.*

We conclude this section by pointing out that the intrinsic graph in the implicit function theorem is suitably differentiable.

Theorem 2.27 ([6, Theorem 4.2]). *In the assumption of Theorem 2.22, ϕ is uniformly intrinsically differentiable on U.*

2.4 Intrinsic derivatives

In this section, we recall some results about uniform intrinsic differentiability in Heisenberg groups. Throughout this section, we assume that $\mathbb{H}^n$ is a semidirect product $\mathbb{W} \rtimes \mathbb{V}$ with $\mathbb{W}$ orthogonal to $\mathbb{V}$. The following proposition ensures that we can always find a Heisenberg basis which is adapted to this factorization.

Proposition 2.28. *We assume that $\mathbb{V}$ is spanned by an orthonormal basis $(v_1, \dots, v_k)$. Then $k \le n$ and there exists an orthonormal basis*

$$(v_{k+1}, \dots, v_n, w_1, \dots, w_n, e_{2n+1})$$

of $\mathbb{W}$ such that

$$(v_1, \dots, v_n, w_1, \dots, w_n, e_{2n+1})$$

is a Heisenberg basis of $\mathbb{H}^n$.

Proof. Since $\mathbb{V}$ is commutative, an element $v = J(w)$ with $v, w \in \mathbb{V}$ satisfies

$$|v|^2 = \langle v, J(w)\rangle = -\omega(v, w) = 0,$$

and therefore $\mathbb{V} \cap J(\mathbb{V}) = \{0\}$. We set $w_i = J(v_i) \in \mathbb{W}$ for $i = 1, \dots, k$ and define the $2k$-dimensional subspace

$$\mathbb{S}_1 = \mathbb{V} \oplus J(\mathbb{V}) \subset H_1.$$

We notice that $\dim(\mathbb{S}_1^\perp \cap H_1) = 2(n - k)$. If $k < n$, we pick a vector $v_{k+1} \in \mathbb{S}_1^\perp \cap H_1$ of unit norm and define $w_{k+1} = Jv_{k+1}$. It is easily observed that both w_{k+1} and v_{k+1} are orthogonal to $\mathbb{S}_1$, so that

$$(v_1, \dots, v_{k+1}, w_1, \dots, w_{k+1}, e_{2n+1})$$

is a Heisenberg basis of

$$\mathbb{S}_2 \oplus \operatorname{span}\{e_{2n+1}\},$$

where we have defined

$$\mathbb{S}_2 = \mathbb{V} \oplus \operatorname{span}\{v_{k+1}\} \oplus J(\mathbb{V} \oplus \operatorname{span}\{v_{k+1}\}).$$

Indeed, the previous subspace has the structure of a $(2k+3)$-dimensional Heisenberg group. One can iterate this process until a Heisenberg basis of $\mathbb{H}^n$ is found. □

From now on, we assume that $(v_1, \dots, v_n, w_1, \dots, w_n, e_{2n+1})$ is the Heisenberg basis provided by Proposition 2.28. We can identify $\mathbb{V}$ with $\mathbb{R}^k$ and $\mathbb{W}$ with $\mathbb{R}^{2n+1-k}$ through the following diffeomorphisms:

$$\begin{aligned}
&i_{\mathbb{V}} : \mathbb{V} \to \mathbb{R}^k, \quad i_{\mathbb{V}}\Big(\sum_{i=1}^k x_i v_i\Big) = (x_1, \dots, x_k),\\
&i_{\mathbb{W}} : \mathbb{W} \to \mathbb{R}^{2n+1-k},\\
&i_{\mathbb{W}}\Big(z e_{2n+1} + \sum_{i=k+1}^n (x_i v_i + y_i w_i) + \sum_{i=1}^k \eta_i v_i\Big) = (x_{k+1}, \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, z).
\end{aligned}$$

We identify any function from an open subset $U \subset \mathbb{W}$, $\phi : U \to \mathbb{V}$, with the corresponding function from an open subset $\widetilde{U} \subset \mathbb{R}^{2n+1-k}$, $\widetilde{\phi} : \widetilde{U} \to \mathbb{R}^k$:

$$\widetilde{\phi}(w) = i_{\mathbb{V}}(\phi(i_{\mathbb{W}}^{-1}(w))) \quad \text{for all } w \in \widetilde{U} = i_{\mathbb{W}}(U) \subset \mathbb{R}^{2n+1-k}.$$

Any h-homomorphism $L : \mathbb{W} \to \mathbb{V}$ can be identified with the linear map $\widetilde{L} : \mathbb{R}^{2n+1-k} \to \mathbb{R}^k$ with respect to the fixed basis. So it can be identified with a $k \times (2n-k)$ matrix M_L with real coefficients such that

$$L(w) = M_L \pi(w)^T$$

for every $w \in \mathbb{R}^{2n+1-k}$, where $\pi : \mathbb{R}^{2n+1-k} \to \mathbb{R}^{2n-k}$ is the canonical projection on the first $2n-k$ components.

If $U \subset \mathbb{W}$ is an open set and $\phi : U \to \mathbb{V}$ is intrinsically differentiable at a point $w \in U$, we denote by $D^\phi\phi(w)$ the matrix associated to $d\phi_w$ and we call it *intrinsic Jacobian matrix* of ϕ at w. If $\mathcal{U} \subset \mathbb{R}^{2n+1-k}$ is an open set and $\psi = (\psi_1, \dots, \psi_k) : \mathcal{U} \to \mathbb{R}^k$ is a function, we define the family of $2n-k$ vector fields:

$$W_j^\psi = \begin{cases} (i_{\mathbb{W}})_*(X_{j+k}), & j = 1, \dots, n-k\\ \nabla^{\psi_{j-n+k}} = \partial_{\eta_{j-n+k}} + \psi_{j-n+k}\partial_z, & j = n-k+1, \dots, n,\\ (i_{\mathbb{W}})_*(Y_{j+k}), & j = n+1, \dots, 2n-k. \end{cases}$$

Definition 2.29 (Intrinsic derivatives). Let $U \subset \mathbb{W}$ be an open set and let $\bar{w}$ be a point of U. Let $\phi : U \to \mathbb{V}$ be a continuous function. For each $j = 1, \dots, 2n-k$, we say that ϕ has ∂^{ϕ_j}*-derivative at* $\bar{w}$ if and only if there exists $(\alpha_{1,j}, \dots, \alpha_{k,j}) \in \mathbb{R}^k$ such that for all integral curves $\gamma^j : (-\delta, \delta) \to U$ of W_j^ϕ with $\gamma^j(0) = \bar{w}$ the limit

$$\lim_{t\to 0} \frac{\phi(\gamma^j(t)) - \phi(\bar{w})}{t}$$

exists and is equal to $(\alpha_{1,j}, \dots, \alpha_{k,j})$. For all $j = 1, \dots, 2n-k$, we denote it by

$$\partial^{\phi_j}\phi(\bar{w}) = \begin{pmatrix} \partial^{\phi_j}\phi_1(\bar{w}) \\ \vdots \\ \partial^{\phi_j}\phi_k(\bar{w}) \end{pmatrix} = \begin{pmatrix} \alpha_{1,j} \\ \vdots \\ \alpha_{k,j}. \end{pmatrix}.$$

The existence of continuous intrinsic derivatives actually characterizes the uniform intrinsic differentiability [9, Theorem 5.7].

Definition 2.30. Let $U \subset \mathbb{W}$ be an open set. Let $\phi : U \to \mathbb{V}$ be an intrinsically differentiable function at $\bar{w} \in U$. We define the *intrinsic Jacobian of* ϕ *at* $\bar{w}$ by

$$J^\phi\phi(\bar{w}) = \sqrt{1 + \sum_{\ell=1}^k \sum_{I \in \mathcal{I}_\ell} (M_I^\phi(\bar{w}))^2},$$

where we have defined $\mathcal{I}_\ell$ as the set of multiindexes

$$\{(i_1,\dots,i_\ell,j_1,\dots,j_\ell)\in\mathbb{N}^{2l}:1\le i_1<i_2<\dots<i_\ell\le 2n-k,\ 1\le j_1<j_2\dots<j_\ell\le k\}.$$

We have also introduced the minors

$$M_I^\phi(\bar w)=\det\begin{pmatrix}\partial^{\phi_{i_1}}\phi_{j_1}(\bar w) & \cdots & \partial^{\phi_{i_\ell}}\phi_{j_1}(\bar w)\\ \vdots & \ddots & \vdots\\ \partial^{\phi_{i_1}}\phi_{j_\ell}(\bar w) & \cdots & \partial^{\phi_{i_\ell}}\phi_{j_\ell}(\bar w)\end{pmatrix}.$$

2.5 Measures and area formulas

If $\mathbb{H}^n$ is endowed with a homogeneous distance d, we set

$$\mathbb{B}(x,r)=\{y\in\mathbb{H}^n:d(x,y)\le r\},$$

and for $S\subset\mathbb{H}^n$ we set

$$\operatorname{diam}(S)=\sup\{d(x,y):x,y\in S\}.$$

Notice that $\operatorname{diam}(\mathbb{B}(x,r))=2r$ for all $x\in\mathbb{H}^n$ and $r>0$.

Definition 2.31 (Carathéodory's construction). Let $\mathcal{F}\subset\mathcal{P}(\mathbb{H}^n)$ be a nonempty family of closed subsets of $\mathbb{H}^n$, equipped with a homogeneous distance d. Let $\alpha>0$. If $\delta>0$ and $A\subset\mathbb{H}^n$, we define

$$\phi_\delta^\alpha(A)=\inf\Big\{\sum_{j=0}^\infty c_\alpha\operatorname{diam}(B_j)^\alpha:A\subset\bigcup_{j=0}^\infty B_j,\ \operatorname{diam}(B_j)\le\delta,\ B_j\in\mathcal{F}\Big\}.\tag{2.11}$$

If $\mathcal{F}$ coincides with the family $\mathcal{F}_b$ of closed balls with respect to the distance d and we choose $c_\alpha=2^{-\alpha}$ in (2.11), then

$$\mathcal{S}^\alpha(A)=\sup_{\delta>0}\phi_\delta^\alpha(A)\tag{2.12}$$

is the *α-spherical measure* of $A\subset\mathbb{H}^n$.

In the case that $\mathcal{F}$ is the family of all closed sets and $k\in\{1,\dots,2n+1\}$, we define

$$c_k=\frac{\mathcal{L}^k(\{x\in\mathbb{R}^k:|x|\le1\})}{2^k},$$

where $\mathcal{L}^k$ denotes the Lebesgue measure. Then the corresponding k-dimensional Hausdorff measure is given by

$$\mathcal{H}_E^k(A)=\sup_{\delta>0}\phi_\delta^k(A),$$

where $\mathbb{H}^n$ is equipped with the Euclidean distance induced through the identification with $\mathbb{R}^{2n+1}$. These measures are Borel regular on subsets of $\mathbb{H}^n$. For our purposes, it is useful to recall a less known Hausdorff-type measure, first introduced in [34]. Given $\alpha\in[0,\infty)$ and $\delta\in(0,\infty)$, we define the *α-dimensional centered Hausdorff measure* $\mathcal{C}^\alpha$ of a set $A\subset\mathbb{H}^n$ by

$$\mathcal{C}^\alpha(A)=\sup_{E\subset A}\mathcal{D}^\alpha(E),$$

where $\mathcal{D}^\alpha(E)=\lim_{\delta\to0^+}\mathcal{C}_\delta^\alpha(E)$, and, in turn, $\mathcal{C}_\delta^\alpha(E)=0$ if $E=\emptyset$, and if $E\neq\emptyset$, we have

$$C_\delta^\alpha(E)=\inf\Big\{\sum_{i=0}^\infty r_i^\alpha:E\subset\bigcup_{i=0}^\infty\mathbb{B}(x_i,r_i),\ x_i\in E,\ \operatorname{diam}(\mathbb{B}(x_i,r_i))\le\delta\Big\}.$$

Definition 2.32. Let $\alpha>0$, let $x\in\mathbb{H}^n$ and let μ be a Borel regular measure on $\mathbb{H}^n$. We define the *upper α-density of μ at x* by

$$\Theta^{*\alpha}(\mu,x)=\limsup_{r\to0}\frac{\mu(\mathbb{B}(x,r))}{r^\alpha}.\tag{2.13}$$

The previous definition and terminology follow [13, 2.10.19].

Theorem 2.33 ([16, Theorem 3.1]). *Let $\alpha > 0$ and let μ be a Borel regular measure on $\mathbb{H}^n$ such that there exists a countable open covering of $\mathbb{H}^n$, whose elements have μ-finite measure. Let $B \subset A \subset \mathbb{H}^n$ be Borel sets. If $\mathcal{C}^\alpha(A) < \infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{C}^\alpha \llcorner A$, then we have that $\Theta^{*\alpha}(\mu, \cdot)$ is a Borel function on A and*

$$\mu(B) = \int_B \Theta^{*\alpha}(\mu, x)\, d\mathcal{C}^\alpha(x).$$

We introduce now a crucial definition of density.

Definition 2.34. Let $\mathcal{F}_b$ be the family of closed balls with positive radius in $\mathbb{H}^n$ endowed with a homogeneous distance d. Let $\alpha > 0$, let $x \in \mathbb{H}^n$ and let μ be a Borel regular measure on $\mathbb{H}^n$. We call the real number

$$\theta^\alpha(\mu, x) = \inf_{\epsilon>0} \sup\left\{ \frac{2^\alpha \mu(\mathbb{B})}{\operatorname{diam}(\mathbb{B})^\alpha} : x \in \mathbb{B} \in \mathcal{F}_b,\ r < \epsilon \right\}$$

the *spherical α-Federer density of μ at x*.

This density naturally appears in representing a Borel regular measure that is absolutely continuous with respect to the α-dimensional spherical measure.

Theorem 2.35 ([31, Theorem 7.2]). *Let $\alpha > 0$ and let μ be a Borel regular measure on $\mathbb{H}^n$ such that there exists a countable open covering of $\mathbb{H}^n$ whose elements have μ-finite measure. If $B \subset A \subset \mathbb{H}^n$ are Borel sets, then $\theta^\alpha(\mu, \cdot)$ is a Borel function on A. If in addition $\mathcal{S}^\alpha(A) < \infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{S}^\alpha \llcorner A$, then*

$$\mu(B) = \int_B \theta^\alpha(\mu, x)\, d\mathcal{S}^\alpha(x).$$

Definition 2.36 (Spherical factor). Let d be a homogeneous distance in $\mathbb{H}^n$. If $\Pi \subset \mathbb{H}^n$ is a linear subspace of topological dimension p, then the *spherical factor* of Π with respect to d is

$$\beta_d(\Pi) = \max_{z \in \mathbb{B}(0,1)} \mathcal{H}_E^p(\Pi \cap \mathbb{B}(z, 1)).$$

When we deal with a homogeneous distance d that preserves some symmetries, then the spherical factor can become a geometric constant. The following definition detects those homogeneous distances giving a constant spherical factor. It extends [30, Definition 6.1] to higher codimension.

Definition 2.37. We refer to the fixed graded scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{H}^n$ and we assume that there exists a family $\mathcal{F} \subset O(H_1)$ of isometries such that for any couple of $(p-1)$-dimensional subspaces $S_1, S_2 \subset H_1$, there exists $L \in \mathcal{F}$ that satisfies the condition

$$L(S_1) = S_2.$$

Let d be a homogeneous distance on $\mathbb{H}^n$ and let $p = 1, \dots, 2n$. We say that d is *p-vertically symmetric* if $p = 1$ or $p \geq 2$ and the following conditions hold. Taking into account that H_1 and H_2 are orthogonal, we introduce the class of isometries

$$\mathcal{O} = \{T \in O(\mathbb{H}^n) : T|_{H_2} = \mathrm{Id}|_{H_2},\ T|_{H_1} \in \mathcal{F}\}.$$

We also assume the following conditions:

- $\pi_{H_1}(\mathbb{B}(0, 1)) = \mathbb{B}(0, 1) \cap H_1 = \{h \in H_1 : \theta(|\pi_{H_1}(h)|) \leq r_0\}$ for some monotone non-decreasing function $\theta : [0, +\infty) \to [0, +\infty)$ and $r_0 > 0$.
- $T(\mathbb{B}(0, 1)) = \mathbb{B}(0, 1)$ for all $T \in \mathcal{O}$.

More information on p-vertically symmetric distances can be found in [32]. For instance, the sub-Riemannian distance in the Heisenberg group is vertically symmetric. Vertically symmetric distances were already introduced in [30].

The next theorem specializes [32, Theorem 1.1] to Heisenberg groups.

Theorem 2.38. *If $p = 1, \dots, 2n+1$ and d is a p-vertically symmetric distance on $\mathbb{H}^n$, then the spherical factor $\beta_d(\mathbb{W})$ is constant on every p-dimensional vertical subgroup $\mathbb{W} \subset \mathbb{H}^n$.*

The previous theorem motivates the following definition.

Definition 2.39 (Notation for constant spherical factors). Let $\mathcal{N}_p$ be the family of all p-dimensional vertical subgroups of $\mathbb{H}^n$. We consider a homogeneous distance d. We assume that the spherical factor $\beta_d(S)$ remains constant as S varies in $\mathcal{N}_p$ (this means that d is *rotationally symmetric* with respect to $\mathcal{N}_p$). We denote the constant spherical factor by $\omega_d(p)$, without indicating the class $\mathcal{N}_p$.

Definition 2.40 ([31, Definition 8.5]). Let d be a homogeneous distance on $\mathbb{H}^n$. We say that d is *multiradial* if there exists a function $\theta : [0, +\infty)^2 \to [0, +\infty)$, which is continuous and monotone non-decreasing on each single variable, with

$$d(x, 0) = \theta(|\pi_{H_1}(x)|, |\pi_{H_2}(x)|).$$

The function θ is also assumed to be coercive in the sense that $\theta(x) \to +\infty$ as $|x| \to +\infty$.

Proposition 2.41. *If $d : \mathbb{H}^n \times \mathbb{H}^n \to [0, \infty)$ is multiradial, then it is also p-vertically symmetric for every $p = 1, \dots, 2n+1$.*

A more general statement can be found in [32]. One may also check that both d_∞ and the Cygan–Korányi distance are multiradial. One can find conditions under which the spherical factor has a simpler representation. The next theorem is established in [32, Theorem 1.4].

Theorem 2.42. *If $p = 1, \dots, 2n+1$ and d is a homogeneous distance in $\mathbb{H}^n$ whose unit ball $\mathbb{B}(0,1)$ is convex, then for every p-dimensional vertical subgroup $\mathbb{W}$ we have*

$$\beta_d(\mathbb{W}) = \mathcal{H}_E^p(\mathbb{W} \cap \mathbb{B}(0,1)).$$

3 Upper blow-up of low codimensional $\mathcal{H}$-regular surfaces

In this section, we prove the main technical tool of the paper, that is, the equality between spherical Federer density and spherical factor, established in Theorem 3.2. The next lemma will be important for the proof of our technical result. It gives a formula of how the area transforms under a suitable linear isomorphism between two vertical groups.

Lemma 3.1. *We consider two vertical subgroups $\mathbb{M}$, $\mathbb{W}$ of $\mathbb{H}^n$ and a k-dimensional horizontal subgroup $\mathbb{V} \subset \mathbb{H}^n$ such that*

$$\mathbb{H}^n = \mathbb{M} \rtimes \mathbb{V} = \mathbb{W} \rtimes \mathbb{V}.$$

We introduce the multivectors

$$V = v_1 \wedge \cdots \wedge v_k, \quad N = w_1 \wedge \cdots \wedge w_{2n-k} \wedge e_{2n+1}, \quad M = m_1 \wedge \cdots \wedge m_{2n-k} \wedge e_{2n+1},$$

where

$$(v_1, \dots, v_k), \quad (w_1, \dots, w_{2n-k}, e_{2n+1}), \quad (m_1, \dots, m_{2n-k}, e_{2n+1})$$

are orthonormal bases of $\mathbb{V}$, $\mathbb{W}$ and $\mathbb{M}$, respectively. Then for every Borel set $B \subset \mathbb{M}$, we have

$$(\pi^{\mathbb{M},\mathbb{V}}_{\mathbb{M},\mathbb{W}})_\sharp \mathcal{H}_E^{2n+1-k}(B) = \mathcal{H}_E^{2n+1-k}(\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}}(B)) = \frac{\|V \wedge M\|_g}{\|V \wedge N\|_g} \mathcal{H}_E^{2n+1-k}(B),$$

where the projections $\pi^{\mathbb{M},\mathbb{V}}_{\mathbb{M},\mathbb{W}}$ and $\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}}$ have been introduced in Definition 2.4. The norms of $V \wedge M$ and $V \wedge N$ are taken with respect to the Hilbert structure of $\Lambda_{2n+1}(\mathbb{H}^n)$ induced by our scalar product on $\mathbb{H}^n$.

Proof. It is clearly not restrictive to relabel the bases of $\mathbb{M}$ and $\mathbb{W}$ as

$$w_{k+1}, \dots, w_{2n}, e_{2n+1} \quad \text{and} \quad m_{k+1}, \dots, m_{2n}, e_{2n+1},$$

respectively. We define the isomorphisms

$$
\begin{aligned}
&i_{\mathbb{W}} : \mathbb{W} \to \mathbb{R}^{2n+1-k}, \quad && i_{\mathbb{W}}\Big(x_{2n+1}e_{2n+1} + \sum_{i=k+1}^{2n} x_i w_i\Big) = (x_{k+1}, \dots, x_{2n+1}),\\
&i_{\mathbb{M}} : \mathbb{M} \to \mathbb{R}^{2n+1-k}, \quad && i_{\mathbb{M}}\Big(x_{2n+1}e_{2n+1} + \sum_{i=k+1}^{2n} x_i m_i\Big) = (x_{k+1}, \dots, x_{2n+1}),\\
&i_{\mathbb{V}} : \mathbb{V} \to \mathbb{R}^{k}, && i_{\mathbb{V}}\Big(\sum_{i=i}^{k} x_i v_i\Big) = (x_1, \dots, x_k).
\end{aligned}
$$

We introduce

$$
\Psi_1 : \mathbb{R}^{2n+1} \to \mathbb{H}^n, \quad \Psi_1(x_1, \dots, x_{2n+1}) = \Big(x_{2n+1}e_{2n+1} + \sum_{i=k+1}^{2n} x_i w_i\Big)\Big(\sum_{j=1}^{k} x_i v_i\Big). \tag{3.1}
$$

We now notice that $J\Psi_1(x) = \|V \wedge N\|_g$ for every $x = (x_1, \dots, x_{2n+1}) \in \mathbb{R}^{2n+1}$. It suffices to observe that

$$
J\Psi_1 = \|\partial_{x_1}\Psi_1 \wedge \cdots \partial_{x_{2n+1}}\Psi_{2n+1}\|_g
$$

and use the explicit form of (3.1). We define another map

$$
\Psi_2 : \mathbb{R}^{2n+1} \to \mathbb{H}^n, \quad \Psi_2(x_1, \dots, x_{2n+1}) = \Big(x_{2n+1}e_{2n+1} + \sum_{i=k+1}^{2n} x_i m_i\Big)\Big(\sum_{j=1}^{k} x_i v_i\Big),
$$

and we observe in the same way that $J\Psi_2(x) = \|V \wedge M\|_g$. We introduce the embedding

$$
q : \mathbb{R}^{2n+1-k} \to \mathbb{R}^{2n+1}, \quad q(x_1, \dots, x_{2n+1-k}) = (0, \dots, 0, x_1, \dots, x_{2n+1-k}),
$$

and the projection

$$
p : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1-k}, \quad p(x_1, \dots, x_{2n+1}) = (x_{k+1}, \dots, x_{2n+1}).
$$

For every $z \in \mathbb{H}^n$, we observe that

$$
\Psi_1^{-1}(z) = (i_{\mathbb{V}} \circ \pi_{\mathbb{V}}(z), i_{\mathbb{W}} \circ \pi_{\mathbb{W}}(z)).
$$

It follows that

$$
i_{\mathbb{W}}^{-1} \circ p \circ \Psi_1^{-1} = \pi_{\mathbb{W}}.
$$

If we take any $m \in \mathbb{M}$, then

$$
\begin{aligned}
\pi_{\mathbb{W}}(m) &= i_{\mathbb{W}}^{-1} \circ p \circ \Psi_1^{-1} \circ \Psi_2 \circ \Psi_2^{-1}(m)\\
&= i_{\mathbb{W}}^{-1} \circ p \circ \Psi_1^{-1} \circ \Psi_2 \circ q \circ i_{\mathbb{M}}(m)\\
&= \pi_{\mathbb{W},\mathbb{M}}^{\mathbb{W},\mathbb{V}}(m).
\end{aligned} \tag{3.2}
$$

The second equality follows by the identity

$$
\Psi_2^{-1} = (i_{\mathbb{V}} \circ \pi_{\mathbb{V}}, i_{\mathbb{M}} \circ \pi_{\mathbb{M}}),
$$

and hence $\Psi_2^{-1}(m) = (0, i_{\mathbb{M}}(m))$ for all $m \in \mathbb{M}$. We notice that $\Psi_1^{-1} \circ \Psi_2$ is a polynomial diffeomorphism, whose Jacobian matrix at x has the following form:

$$
\begin{pmatrix} I & R_1 & 0\\ 0 & R_2 & 0\\ \ell_1(x) & \ell_2(x) & 1 \end{pmatrix} \in \mathbb{R}^{(2n+1)\times(2n+1)},
$$

where $I \in \mathbb{R}^{k\times k}$, $R_1 \in \mathbb{R}^{k\times(2n-k)}$, $R_2 \in \mathbb{R}^{(2n+1-k)\times(2n+1-k)}$ and the functions

$$
\ell_1 : \mathbb{R}^{2n+1} \to \mathbb{R}^k \quad \text{and} \quad \ell_2 : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n-k}
$$

are affine. From definitions of $q : \mathbb{R}^{2n+1-k} \to \mathbb{R}^{2n+1}$ and of $p : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1-k}$, by explicit computation, it follows that

$$J(\Psi_1^{-1} \circ \Psi_2)(q(y)) = |\det R_2| = J(p \circ \Psi_1^{-1} \circ \Psi_2 \circ q)(y) \tag{3.3}$$

for every $y \in \mathbb{R}^{2n+1-k}$. As a consequence, taking into account (3.2), (3.3) and

$$\frac{\|V \wedge M\|_g}{\|V \wedge N\|_g} = J(\Psi_1^{-1} \circ \Psi_2),$$

the following equalities hold:

$$\begin{aligned}
\mathcal{H}_E^{2n+1-k}(B) &= \mathcal{L}^{2n+1-k}(i_{\mathbb{M}}(B)) \\
&= \frac{\|V \wedge N\|_g}{\|V \wedge M\|_g} \mathcal{L}^{2n+1-k}((p \circ \Psi_1^{-1} \circ \Psi_2 \circ q)(i_{\mathbb{M}}(B))) \\
&= \frac{\|V \wedge N\|_g}{\|V \wedge M\|_g} \mathcal{H}_E^{2n+1-k}((i_{\mathbb{W}}^{-1} \circ p \circ \Psi_1^{-1} \circ \Psi_2 \circ q \circ i_{\mathbb{M}})(B)) \\
&= \frac{\|V \wedge N\|_g}{\|V \wedge M\|_g} \mathcal{H}_E^{2n+1-k}(\pi_{\mathbb{W},\mathbb{M}}^{\mathbb{W},\mathbb{V}}(B))
\end{aligned}$$

for every Borel set $B \subset \mathbb{M}$. □

We are now in the position to present our main technical result.

Theorem 3.2 (Upper blow-up). *We consider a semidirect factorization $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$, an open set $\Omega \subset \mathbb{H}^n$, a function $f \in C_h^1(\Omega, \mathbb{R}^k)$ and a homogeneous distance d. We fix $x_0 \in \Omega$ and the level set $\Sigma = f^{-1}(f(x_0))$, assuming that $J_{\mathbb{V}} f(x) > 0$ for all $x \in \Sigma$. We choose the orthonormal bases $(v_1, \dots, v_k)$ of $\mathbb{V}$ and $(w_{k+1}, \dots, w_{2n}, e_{2n+1})$ of $\mathbb{W}$, setting*

$$V = v_1 \wedge \cdots \wedge v_k \quad \text{and} \quad N = w_{k+1} \wedge \cdots \wedge w_{2n} \wedge e_{2n+1}.$$

Then the following conditions hold:

(i) *Σ is a parametrized $\mathbb{H}$-regular surface with respect to $(\mathbb{W}, \mathbb{V})$.*

(ii) *If we denote by $\phi : U \to \mathbb{V}$ the parametrization of Σ and introduce the measure*

$$\mu(B) = \|V \wedge N\|_g \int_{\Phi^{-1}(B)} \frac{J_H f(\Phi(n))}{J_{\mathbb{V}} f(\Phi(n))} \, d\mathcal{H}_E^{2n+1-k}(n) \tag{3.4}$$

for every Borel set $B \subset \mathbb{H}^n$, where $\Phi(n) = n\phi(n)$, then for every $x \in \Sigma$ we have

$$\theta^{2n+2-k}(\mu, x) = \beta_d(\operatorname{Tan}(\Sigma, x)). \tag{3.5}$$

Proof. The first part of our claim is a consequence of Proposition 2.24. Then our thesis follows once we have proved (3.5). By formula (3.4), for any $y \in \Omega$, taking $t > 0$ sufficiently small, we can write

$$\mu(\mathbb{B}(y, t)) = \|V \wedge N\|_g \int_{\Phi^{-1}(\mathbb{B}(y,t))} \frac{J_H f(\Phi(n))}{J_{\mathbb{V}} f(\Phi(n))} \, d\mathcal{H}_E^{2n+1-k}(n).$$

We denote by $\zeta \in U$ the element such that

$$x = \Phi(\zeta) = \zeta\phi(\zeta).$$

We now perform the change of variables

$$n = \sigma_x(\Lambda_t(\eta)) = x(\Lambda_t \eta)(\pi_{\mathbb{V}}(x))^{-1} = x(\Lambda_t \eta)(\phi(\zeta))^{-1},$$

where $\Lambda_t = \delta_t|_{\mathbb{W}}$. The Jacobian of Λ_t is t^{2n+2-k}. It is well known that σ_x has unit Jacobian (see, for instance, [15, Lemma 2.20]). Setting $\alpha(x) = J_H f(x)/J_{\mathbb{V}} f(x)$, we obtain that

$$\frac{\mu(\mathbb{B}(y, t))}{t^{2n+2-k}} = \|V \wedge N\|_g \int_{\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t))))} (\alpha \circ \Phi)(\sigma_x(\Lambda_t(\eta))) \, d\mathcal{H}_E^{2n+1-k}(\eta).$$

By the general definition of spherical Federer density, we obtain that

$$\theta^{2n+2-k}(\mu, x) = \inf_{r>0} \sup_{\substack{y\in\mathbb{B}(x,t)\\ 0<t<r}} \frac{\mu(\mathbb{B}(y,t))}{t^{2n+2-k}}$$
$$= \inf_{r>0} \sup_{\substack{y\in\mathbb{B}(x,t)\\ 0<t<r}} \|V\wedge N\|_g \int_{\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t))))} (\alpha\circ\Phi)(\sigma_x(\Lambda_t(\eta)))\, d\mathcal{H}_E^{2n+1-k}(\eta).$$

There exists $R_0 > 0$ such that for $t > 0$ and $y \in \mathbb{B}(x, t)$ we have the following inclusion:

$$\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t)))) \subset \mathbb{B}_{\mathbb{W}}(0, R_0), \tag{3.6}$$

where the translated function $\phi_{x^{-1}}$ is defined according to formula (2.4) and we have set

$$\mathbb{B}_{\mathbb{W}}(0, R_0) = \mathbb{B}(0, R_0) \cap \mathbb{W}.$$

To see (3.6), we write more explicitly $\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t))))$, that is,

$$\{\eta \in \Lambda_{1/t}(\sigma_x^{-1}(U)) : \|y^{-1}x(\Lambda_t\eta)\phi(\zeta)^{-1}\phi(x(\Lambda_t\eta)\phi(\zeta)^{-1})\| \le t\}.$$

It can be written as follows:

$$\Big\{\eta \in \Lambda_{1/t}(\sigma_x^{-1}(U)) : \Big\|(\delta_{1/t}(y^{-1}x))\eta\Big(\frac{\phi(\zeta)^{-1}\phi(x(\Lambda_t\eta)\phi(\zeta)^{-1})}{t}\Big)\Big\| \le 1\Big\}.$$

According to (2.4), the translated function of ϕ at x^{-1} is

$$\phi_{x^{-1}}(\eta) = \pi_{\mathbb{V}}(x^{-1})\phi(x\eta\pi_{\mathbb{V}}(x^{-1})) = \phi(\zeta)^{-1}\phi(x\eta\phi(\zeta)^{-1}).$$

We finally get

$$\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t)))) = \Big\{\eta \in \Lambda_{1/t}(\sigma_x^{-1}(U)) : \Big\|(\delta_{1/t}(y^{-1}x))\eta\Big(\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\Big)\Big\| \le 1\Big\}, \tag{3.7}$$

and hence for $\eta \in \Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t))))$, taking into account the previous equality, we have established that

$$\eta\Big(\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\Big) \in \mathbb{B}(0,2).$$

From estimate (2.3), we know that

$$c_0\Big(\|\eta\| + \Big\|\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\Big\|\Big) \le \Big\|\eta\Big(\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\Big)\Big\| \le 2,$$

and hence the inclusion (3.6) holds with $R_0 = 2/c_0$. As a consequence, we have that

$$\theta^{2n+2-k}(\mu, x) < \infty.$$

There exist a positive sequence t_p converging to zero and $y_p \in \mathbb{B}(x, t_p)$ such that

$$\|V\wedge N\|_g \int_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))} \frac{J_H f(\Phi(\sigma_x(\Lambda_{t_p}(\eta)))}{J_{\mathbb{V}} f(\Phi(\sigma_x(\Lambda_{t_p}(\eta)))} \, d\mathcal{H}_E^{2n+1-k}(\eta) \to \theta^{2n+2-k}(\mu, x)$$

as $p \to \infty$. Up to extracting a subsequence, since $y_p \in \mathbb{B}(x, t_p)$ for every p, there exists $z \in \mathbb{B}(0, 1)$ such that

$$\lim_{p\to\infty} \delta_{1/t_p}(x^{-1}y_p) = z.$$

For the sake of simplicity, we use the notation

$$\mathbb{M}_x = \ker Df(x).$$

Using the projection introduced in Definition 2.4, we set

$$S_z = \pi_{\mathbb{W},\mathbb{M}_x}^{\mathbb{W},\mathbb{V}}(\mathbb{M}_x \cap \mathbb{B}(z,1)) \subset \mathbb{W}.$$

Claim 1. *For each $\omega \in \mathbb{W} \setminus S_z$, there exists*

$$\lim_{p\to\infty} \mathbf{1}_{\Lambda_{1/tp}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))}(\omega) = 0.$$

By contradiction, if we had a subsequence of the integers p such that

$$(\delta_{1/t_p}(y_p^{-1}x))\omega\Big(\frac{\phi_{x^{-1}}(\Lambda_{t_p}\omega)}{t}\Big) \in \mathbb{B}(0,1),$$

then, by a slight abuse of notation, we could still denote by t_p the sequence such that

$$(\delta_{1/t_p}(y_p^{-1}x))\omega d\phi_\zeta(\omega)\Big(\frac{(d\phi_\zeta(\Lambda_{t_p}\omega))^{-1}\phi_{x^{-1}}(\Lambda_{t_p}\omega)}{t_p}\Big) \in \mathbb{B}(0,1) \tag{3.8}$$

for all p, where we have used the homogeneity of the intrinsic differential $d\phi_\zeta$ of ϕ; see Definition 2.12 for the notion of intrinsic differential. Indeed, by Theorem 2.27, the function ϕ is in particular intrinsically differentiable at ζ. Due to the intrinsic differentiability, taking into account (3.8) as $p \to \infty$, it follows that

$$\omega d\phi_\zeta(\omega) \in \mathbb{B}(z,1).$$

It is now interesting to observe that the chain rule of Theorem 2.18 yields

$$\operatorname{graph}(d\phi_\zeta) = \ker(Df(x)) = \mathbb{M}_x. \tag{3.9}$$

As a consequence, $\omega d\phi_\zeta(\omega) \in \mathbb{B}(z,1) \cap \mathbb{M}_x$, and thus

$$\omega = \pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\omega d\phi_\zeta(\omega)) \in \pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x \cap \mathbb{B}(z,1)) = S_z, \tag{3.10}$$

which is not possible by our assumption. This concludes the proof of Claim 1.

Now, we introduce the density function

$$\alpha(t,\eta) = \frac{J_H f(\Phi(\sigma_x(\Lambda_t(\eta))))}{J_{\mathbb{V}} f(\Phi(\sigma_x(\Lambda_t(\eta))))}$$

to write

$$\|V \wedge N\|_g \int\limits_{\Lambda_{1/tp}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))} \alpha(t_p,\eta)\, d\mathcal{H}_E^{2n+1-k}(\eta) = I_p + J_p.$$

The sequence I_p, defined as follows, satisfies the estimate

$$\begin{aligned} I_p &= \|V \wedge N\|_g \int\limits_{S_z \cap \Lambda_{1/tp}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))} \alpha(t_p,\eta)\, d\mathcal{H}_E^{2n+1-k}(\eta) \\ &\le \|V \wedge N\|_g \int\limits_{S_z} \alpha(t_p,\eta)\, d\mathcal{H}_E^{2n+1-k}(\eta). \end{aligned}$$

Analogously for J_p, we find

$$\begin{aligned} J_p &= \|V \wedge N\|_g \int\limits_{\Lambda_{1/tp}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))\setminus S_z} \alpha(t_p,\eta)\, d\mathcal{H}_E^{2n+1-k}(\eta) \\ &\le \|V \wedge N\|_g \int\limits_{\mathbb{B}_{\mathbb{W}}(0,R_0)\setminus S_z} \mathbf{1}_{\Lambda_{1/tp}(\sigma_x^{-1}((\Phi^{-1}(\mathbb{B}(y_p,t_p))))}(\eta)\alpha(t_p,\eta)\, d\mathcal{H}_E^{2n+1-k}(\eta). \end{aligned}$$

Claim 1 joined with the dominated convergence theorem proves that $J_p \to 0$ as $p \to \infty$, and therefore $I_p \to \theta^{2n+2-k}(\mu, x)$. To study the asymptotic behavior of I_p, we first observe that

$$\alpha(t_p,\eta) \to \frac{J_H f(x)}{J_{\mathbb{V}} f(x)} = c(x)$$

as $p \to \infty$. It follows that

$$\theta^{2n+2-k}(\mu,x) = \lim_{p\to\infty} I_p \le \|V \wedge N\|_g c(x) \mathcal{H}_E^{2n+1-k}(S_z). \tag{3.11}$$

Claim 2. *We set* $\mathbb{M}_x = \ker(Df(x))$ *and consider*

$$N_x = m_{k+1} \wedge \cdots \wedge m_{2n} \wedge e_{2n+1}$$

such that

$$(m_{k+1}, \ldots, m_{2n}, e_{2n+1})$$

is an orthonormal basis of $\mathbb{M}_x$. *We have that*

$$c(x) = \frac{J_H f(x)}{J_{\mathbb{V}} f(x)} = \frac{1}{\|V \wedge N_x\|_g}. \tag{3.12}$$

Since $\operatorname{span}\{\nabla_H f_1(x), \ldots, \nabla_H f_k(x)\}$ is orthogonal to $\mathbb{M}_x$, it is a standard fact that

$$m_{k+1} \wedge \cdots \wedge m_{2n} \wedge e_{2n+1} = *(\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x))\lambda$$

for some $\lambda \in \mathbb{R}$; see, for instance, [26, Lemma 5.1]. Here we have defined the Hodge operator $*$ in $\mathbb{H}^n$ with respect to the fixed orientation

$$\mathbf{e} = e_1 \wedge \ldots e_{2n} \wedge e_{2n+1}$$

and the fixed scalar product $\langle \cdot, \cdot \rangle$. Precisely, we are referring to an orthonormal Heisenberg basis

$$(e_1, \ldots, e_{2n}, e_{2n+1}),$$

according to Sections 2.1 and 2.2. Therefore, $*\eta$ is the unique $(2n + 1 - k)$-vector such that

$$\xi \wedge *\eta = \langle \xi, \eta \rangle \mathbf{e} \tag{3.13}$$

for all k-vectors ξ. Since the Hodge operator is an isometry, we get

$$|\lambda| = \frac{1}{\|\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\|_g}. \tag{3.14}$$

Due to (3.14) and (3.13), we have

$$\begin{aligned}
\|V \wedge N_x\|_g &= |\lambda| \|v_1 \wedge \cdots \wedge v_k \wedge (*(\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)))\|_g \\
&= \frac{\|\langle v_1 \wedge \cdots \wedge v_k, \nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\rangle \mathbf{e}\|_g}{\|\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\|_g} \\
&= \frac{|\langle v_1 \wedge \cdots \wedge v_k, \nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\rangle|}{\|\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\|_g} \\
&= \frac{\|\nabla_{\mathbb{V}} f_1(x) \wedge \cdots \wedge \nabla_{\mathbb{V}} f_k(x)\|_g}{\|\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\|_g} \\
&= \frac{J_{\mathbb{V}} f(x)}{J_H f(x)},
\end{aligned}$$

and hence establishing Claim 2.

As a result, taking into account (3.11), we have proved that

$$\theta^{2n+2-k}(\mu, x) \le \frac{\|V \wedge N\|_g}{\|V \wedge N_x\|_g} \mathcal{H}_E^{2n+1-k}(S_z).$$

By Lemma 3.1, for $B = \mathbb{M}_x \cap \mathbb{B}(z, 1)$, the following formula holds:

$$\mathcal{H}_E^{2n+1-k}(\pi_{\mathbb{W},\mathbb{M}_x}^{\mathbb{W},\mathbb{V}}(\mathbb{M}_x \cap \mathbb{B}(z, 1))) = \frac{\|V \wedge N_x\|_g}{\|V \wedge N\|_g} \mathcal{H}_E^{2n+1-k}(\mathbb{M}_x \cap \mathbb{B}(z, 1)). \tag{3.15}$$

It follows that

$$\theta^{2n+2-k}(\mu, x) \le \mathcal{H}_E^{2n+1-k}(\mathbb{M}_x \cap \mathbb{B}(z, 1)) \le \beta_d(\mathbb{M}_x).$$

To prove the opposite inequality, we follow the approach of [30, Theorem 3.1]. We choose $z_0 \in \mathbb{B}(0, 1)$ such that

$$\beta_d(\mathbb{M}_x) = \mathcal{H}_E^{2n+1-k}(\mathbb{M}_x \cap \mathbb{B}(z_0, 1)) \tag{3.16}$$

and consider a specific family of points $y_t^0 = x\delta_t z_0 \in \mathbb{B}(x,t)$. For a fixed $\lambda > 1$, we have

$$\sup_{0<t<r} \frac{\mu(\mathbb{B}(y_t^0, \lambda t))}{(\lambda t)^{2n+2-k}} \le \sup_{\substack{y\in\mathbb{B}(x,t),\\ 0<t<\lambda r}} \frac{\mu(\mathbb{B}(y,t))}{t^{2n+2-k}}$$

for every $r > 0$ sufficiently small, and therefore

$$\limsup_{t\to 0^+} \frac{\mu(\mathbb{B}(y_t^0, \lambda t))}{(\lambda t)^{2n+2-k}} \le \theta^{2n+2-k}(\mu, x). \tag{3.17}$$

We introduce the set

$$\begin{aligned} A_t^0 &= \Lambda_{1/\lambda t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_t^0, \lambda t))) \\ &= \Big\{\eta \in \Lambda_{1/\lambda t}(\sigma_x^{-1}(U)) : \eta\Big(\frac{\phi_{x^{-1}}(\Lambda_{\lambda t}\eta)}{\lambda t}\Big) \in \mathbb{B}(\delta_{1/\lambda} z_0, 1)\Big\}. \end{aligned}$$

The second equality can be deduced from (3.7). Then we can rewrite

$$\begin{aligned} \frac{\mu(\mathbb{B}(y_t^0, \lambda t))}{(\lambda t)^{2n+2-k}} &= \|V \wedge N\|_g \int_{A_t^0} \alpha(\lambda t, \eta)\, d\mathcal{H}_E^{2n+1-k}(\eta) \\ &= \frac{\|V \wedge N\|_g}{\lambda^{2n+2-k}} \int_{\delta_\lambda A_t^0} \alpha(\lambda t, \delta_{1/\lambda}\eta)\, d\mathcal{H}_E^{2n+1-k}(\eta). \end{aligned} \tag{3.18}$$

The domain of integration satisfies

$$\delta_\lambda A_t^0 = \Big\{\eta \in \Lambda_{1/t}(\sigma_x^{-1}(U)) : \eta\Big(\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\Big) \in \mathbb{B}(z_0, \lambda)\Big\}.$$

Due to (3.6) and the definition of A_t^0, we get

$$\delta_\lambda A_t^0 \subset \mathbb{B}_{\mathbb{W}}(0, \lambda R_0).$$

Claim 3. *For every $\eta \in \pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x \cap B(z_0, \lambda))$, we have*

$$\lim_{t\to 0^+} \mathbf{1}_{\delta_\lambda A_t^0}(\eta) = 1. \tag{3.19}$$

The intrinsic differentiability of ϕ at ζ shows that

$$\eta\Big(\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\Big) \to \eta d\phi_\zeta(\eta) \quad \text{as } t \to 0.$$

Taking into account (2.2) and (3.10), we get

$$\pi^{\mathbb{M}_x,\mathbb{V}}_{\mathbb{M}_x,\mathbb{W}}(\eta) = \eta d\phi_\zeta(\eta).$$

Hence, our assumption on η can be written as follows:

$$d(\eta d\phi_\zeta(\eta), z_0) < \lambda.$$

We conclude that $\eta \in \delta_\lambda A_t^0$ for any $t > 0$ sufficiently small. Therefore, the limit (3.19) holds and the proof of Claim 3 is complete.

By Fatou's lemma, taking into account (3.17) and (3.18), we get

$$\frac{\|V \wedge N\|_g}{\lambda^{2n+2-k}} \int_{\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x\cap B(z_0,\lambda))} \liminf_{t\to 0}(\mathbf{1}_{\delta_\lambda A_t^0}(\eta)\alpha(\lambda t, \delta_{1/\lambda}\eta))\, d\mathcal{H}_E^{2n+1-k}(\eta) \le \theta^{2n+2-k}(\mu, x).$$

Claim 3 joined with (3.12) yields

$$\frac{1}{\lambda^{2n+2-k}} \frac{\|V \wedge N\|_g}{\|V \wedge N_x\|_g} \mathcal{H}_E^{2n+1-k}(\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x \cap \mathbb{B}(z_0, 1))) \le \theta^{2n+2-k}(\mu, x).$$

Applying again (3.15), we obtain

$$\frac{1}{\lambda^{2n+2-k}}\mathcal{H}_E^{2n+1-k}(\mathbb{M}_x \cap \mathbb{B}(z_0, 1)) \le \theta^{2n+2-k}(\mu, x).$$

Taking the limit as $\lambda \to 1^+$, considering (3.16) and taking into account Proposition 2.21, the proof of (3.5) is complete. □

The computation of the upper density (2.13) is simpler than computing the spherical Federer density. In a sense, we have less degrees of freedom, since the center of the ball for this density is fixed. As a byproduct of our approach, the following theorem can be achieved by some simplifications in the proof of Theorem 3.2, getting a "centered blow-up".

Theorem 3.3. *In the assumptions of Theorem 3.2, for every $x \in \Sigma$, we have*

$$\Theta^{*2n+2-k}(\mu, x) = \mathcal{H}_E^{2n+1-k}(\mathrm{Tan}(\Sigma, x) \cap \mathbb{B}(0, 1)),$$

where the metric ball $\mathbb{B}(0, 1)$ refers to the fixed homogeneous distance d.

4 Some special cases for the area formula

In this section, we analyze some consequences of the upper blow-up (Theorem 3.2). We consider two cases: when the factors $\mathbb{W}$ and $\mathbb{V}$ are orthogonal and when the metric unit ball of the homogeneous distance is convex. In the first case, the measure μ can be represented by the intrinsic derivatives of the parametrization, according to Theorem 1.2.

Proof of Theorem 1.2. Since $\mathbb{W}$ and $\mathbb{V}$ are orthogonal, by Proposition 2.28 we can fix a Heisenberg basis

$$(v_1, \dots, v_k, v_{k+1}, \dots, v_n, w_1, \dots w_{2n}, e_{2n+1})$$

such that

$$\mathbb{V} = \mathrm{span}\{v_1, \dots, v_k\} \quad \text{and} \quad \mathbb{W} = \mathrm{span}\{v_{k+1}, \dots, v_n, w_i, \dots, w_n, e_{2n+1}\}.$$

Our claim follows by representing the measure μ in terms of the intrinsic partial derivatives of the parametrization ϕ of Σ, arguing as in the proof [9, Theorem 6.1]. For the reader's convenience, we report the main points of the proof.

By taking into account Theorem 2.27, $\Sigma = \Phi(\Omega)$ is the graph of a uniformly intrinsically differentiable function ϕ. Arguing as in the proofs of [12, Theorem 4.1] or [6, Theorem 4.2], there exist an open set $\Omega' \subset \mathbb{H}^n$ and a function $g \in C_h^1(\Omega', \mathbb{R}^k)$ such that $\Sigma \subset g^{-1}(0)$ and for every $m \in U$ the following holds:

$$Dg(\Phi(m)) = \begin{bmatrix} \nabla_H g_1(\Phi(m)) \\ \vdots \\ \nabla_H g_k(\Phi(m)) \end{bmatrix} = \begin{bmatrix} \mathbb{I}_k & -D^\phi\phi(m) & 0 \end{bmatrix}, \tag{4.1}$$

where 0 denotes the vanishing column in the previous matrix. By Theorem 1.1, for any Borel set $B \subset \Sigma$,

$$\mu(B) = \int_B \beta_d(\mathrm{Tan}(\Sigma, x))\, d\mathcal{S}^{2k+2-k}(x) = \int_{\Phi^{-1}(B)} \frac{J_H g(\Phi(n))}{J_{\mathbb{V}} g(\Phi(n))}\, d\mathcal{H}_E^{2n+1-k}(n).$$

Notice that $J_{\mathbb{V}} g(\Phi(m)) = 1$ for every $m \in U$. By Definition 2.30, taking into account the form of $Dg(\Phi(m))$ in (4.1), the proof is achieved. □

Combining Theorem 2.33 and Theorem 3.3, we also get the area formula for the centered Hausdorff measure. It is the analogue of Theorem 1.1, where the spherical measure is replaced by the centered Hausdorff measure. For the distance d_∞, the following theorem coincides with [14, Theorem 4.1].

Theorem 4.1. *In the assumptions of Theorem 3.2, for any Borel set $B \subset \Sigma$ we have*

$$\mu(B) = \int_B \mathcal{H}_E^{2n+1-k}(\mathrm{Tan}(\Sigma, x) \cap \mathbb{B}(0,1))\, d\mathcal{C}^{2k+2-k}(x), \tag{4.2}$$

where the metric ball $\mathbb{B}(0,1)$ refers to the fixed homogeneous distance d.

As a consequence, using the previous formula, along with Theorem 1.1 and Theorem 2.42, we can show the equality between spherical measure and centered Hausdorff measure.

Theorem 4.2. *Let d be a homogeneous distance on $\mathbb{H}^n$ such that $\mathbb{B}(0,1)$ is convex. Let Σ be a parametrized $\mathbb{H}$-regular surface with respect to $(\mathbb{W}, \mathbb{V})$. Then for every $x \in \Sigma$ we obtain $\Theta^{*2n+2-k}(\mu, x) = \theta^{2n+2-k}(\mu, x)$ and, in particular,*

$$\mathcal{C}^{2n+2-k} \llcorner \Sigma = \mathcal{S}^{2n+2-k} \llcorner \Sigma.$$

Proof. By Theorem 2.42 and Theorem 3.3, for every $x \in \Sigma$ we have

$$\beta_d(\ker(Df(x))) = \mathcal{H}^{2n+1-k}(\ker(Df(x)) \cap \mathbb{B}(0,1)) = \Theta^{*2n+2-k}(\mu, x).$$

Then the area formulas (1.2) and (4.2) conclude the proof. □

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