A genus 4 origami with minimal hitting time and an intersection property

Published Version:
Luca Marchese (2021). A genus 4 origami with minimal hitting time and an intersection property. ILLINOIS JOURNAL OF MATHEMATICS, 65(3), 579-596 [10.1215/00192082-9366075].

Availability:
This version is available at: https://hdl.handle.net/11585/918497 since: 2023-02-27

Published:

DOI: http://doi.org/10.1215/00192082-9366075

A GENUS 4 ORIGAMI WITH MINIMAL HITTING TIME AND AN INTERSECTION PROPERTY

LUCA MARCHESE

Abstract. In a minimal flow, the hitting time is the exponent of the power law, as r goes to zero, for the time needed by orbits to become r-dense. We show that on the so-called *Ornithorynque* origami the hitting time of the flow in an irrational slope equals the diophantine type of the slope. We give a general criterion for such equality. In general, for genus at least two, hitting time is strictly bigger than diophantine type.

1. Introduction

An *origami*, also known as *square-tiled surface*, is a surface obtained glueing copies of the square $[0,1]^2$ along the boundaries. On a given origami, any $\alpha \in \mathbb{R}$ defines a *linear flow* in slope α, whose dynamical properties are related to the diophantine properties of α. This reflects a more general principle in *Teichmüller dynamics*. [3] gives an introduction to the subject and a selection of the many relevant references. In this paper we consider a special genus 4 origami called *Ornithorynque* (see § 1.2). Our main Theorem 1.1 states that on such origami the *hitting time* in any slope α equals the diophantine type of α. This is the minimal possible value for the hitting time (Lemma 1.2), and in many cases the equality does not hold, according to [6]. We prove Theorem 1.1 stating a general criterion based on a specific intersection property, namely Theorem 4.1, and showing that the Ornithorynque satisfies the intersection property (see § 3). This extends to the Ornithorynque results previously proved in [6] for the genus 3 origami called *Eierlgende Wollmilchsau*.

1.1. **Origamis and linear flows.** Fix a finite set $\mathcal{Q}$ and a pair (h,v) of permutations of $\mathcal{Q}$ generating a transitive subgroup $\langle h,v\rangle$ of the symmetric group. Any $j \in \mathcal{Q}$ corresponds to a copy $Q_j := \{j\} \times [0,1]^2$ of the unit square and to copies l_j, r_j, b_j, t_j of the four sides

$$l := \{0\} \times [0,1]\ ,\ r := \{1\} \times [0,1]\ ,\ b := [0,1] \times \{0\}\ ,\ t := [0,1] \times \{1\}.$$

For any $j \in \mathcal{Q}$ paste the right side r_j of Q_j to the left side $l_{h(j)}$ of $Q_{h(j)}$ and the top side t_j of Q_j to the bottom side $b_{v(j)}$ of $Q_{v(j)}$. An origami X is a surface arising in this way. It is compact, connected, orientable and without boundary. We have a covering

$$\rho_X : X \to \mathbb{T}^2 \tag{1.1}$$

over the torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, ramified only over $[0] \in \mathbb{T}^2$, where $[x]$ denotes the coset of $x \in \mathbb{R}^2$. Define ρ_X on $\mathcal{Q} \times [0,1]^2$ by $\rho_X\big((j,x)\big) := [x]$. This gives a map on X because glued points have the same image. The points $p_1, \dots, p_m$ in X where ρ_X is ramified are in bijection with the cycles of the commutator $[v,h] := v^{-1}h^{-1}vh$. Let $k_1, \dots, k_m$ in $\mathbb{N}$ be such that for any $1 \le j \le m$ the cycle of $[v,h]$ corresponding to p_j has length $k_j + 1$. The surface

2010 *Mathematics Subject Classification.* Primary 37D40, secondary 37E35.

Key words and phrases. Origamis, square-tiled surfaces, hitting time, continued fractions.

X inherits a metric with a conical angle $2(k_j+1)\pi$ at any p_j and which is flat outside these points. If g is the genus of X, then $k_1+\cdots+k_m = 2g-2$. Details can be found in [3], while § 1.2 below describes an explicit example.

Fix $\alpha \in \mathbb{R}\cup\{\infty\}$ and set $e_\alpha := (\sin\theta, \cos\theta)$, where $\theta := \arctan\alpha \in (-\pi/2, \pi/2]$, that is the unit vector $e_\alpha \in \mathbb{R}\times\mathbb{R}_+$ with slope α. The *linear flow* $\phi_\alpha : \mathbb{R}\times X \to X$ on X is the continuous flow determined for any $p \in X$ and $t \in \mathbb{R}$ by

$$\rho_X\big(\phi_\alpha(t,p)\big) = \rho_X(p) + te_\alpha \mod \mathbb{Z}^2. \tag{1.2}$$

Equation (1.2) determines k_j+1 trajectories starting at any conical point p_j, which may or may not be defined for any $t \geq 0$, where the trajectory stops at $t = t_0$ if $\phi_\alpha(t_0, p_j)$ is also a conical point. Similarly we have k_j+1 trajectories ending in p_j. We call *singular leaves* such trajectories. The flow ϕ_α is a regular $\mathbb{R}$-action outside singular leaves. According to [10], we have the following dichotomy. If $\alpha \in \mathbb{Q}$ then any infinite orbit is periodic, moreover periods take finitely many values. Otherwise, if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, then ϕ_α is *uniquely ergodic*, that is the Lebesgue measure of X is the only invariant measure. This implies that any positive-infinite orbit is dense.

1.2. **The *Ornithorynque* origami.** Consider the set $\mathcal{Q} := \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and let $X_\mathcal{O}$ be the origami defined by the pair (h,v) of permutations of $\mathcal{Q}$ given by

$$h\begin{pmatrix}(i,0,0)\\(i,0,1)\\(i,1,0)\\(i,1,1)\end{pmatrix} := \begin{pmatrix}(i+1,1,0)\\(i-1,1,1)\\(i,0,0)\\(i,0,1)\end{pmatrix} \quad\text{and}\quad v\begin{pmatrix}(i,0,0)\\(i,0,1)\\(i,1,0)\\(i,1,1)\end{pmatrix} := \begin{pmatrix}(i-1,0,1)\\(i,0,0)\\(i+1,1,1)\\(i,1,0)\end{pmatrix}.$$

Figure 1 represents the origami $X_\mathcal{O}$. Half of the 24 pairs of identified sides are represented by dotted lines. The other 12 pairs are named by letters A_i, B_i, C_i, D_i with $i \in \mathbb{Z}/3\mathbb{Z}$. There are three conical points p_1, p_2, p_3 with orders $k_1 = k_2 = k_3 = 2$, that is a conical angle 6π at each conical point. Figure 1 shows 3 big squares with size 2×2. The 12 vertices of these big squares are identified to p_1, the 6 middle points of the horizontal sides correspond to p_2 and the 6 middle points of the vertical sides correspond to p_3. From the relation $2g-2 = k_1+k_2+k_3$ we get that $X_\mathcal{O}$ has genus $g = 4$.

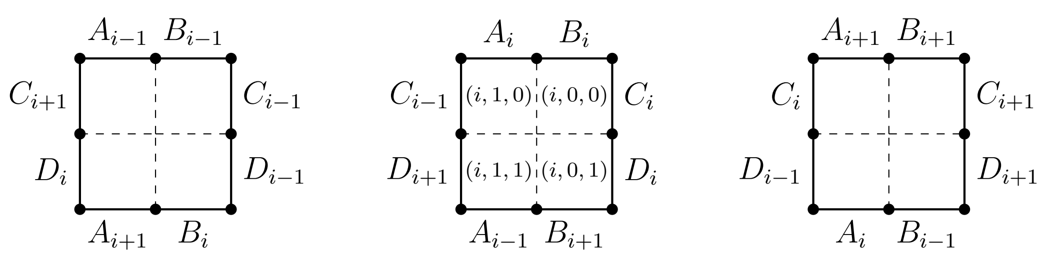

FIGURE 1. The Ornithorynque origami $X_\mathcal{O}$.

The surface $X_\mathcal{O}$ was discovered by Forni and Matheus in the preprint [2], and then included in a larger family of surfaces in [4]. After Delecroix and Weiss, the origami $X_\mathcal{O}$ was named *Ornithorynque* (french for Platypus), as a rare example of surface with totally degenerate *Lyapunov spectrum.* Previously, in [1], Forni discovered the only other known example with such property, which is a genus $g = 3$ surface $X_\mathcal{E}$ called in german *Eierlegende Wollmilchsau.*

The surface $X_{\mathcal{E}}$ was first introduced in [5] and its name was given by Herrlich, Möller and Schmithüsen, referring to its peculiar algebro-geometrical properties, which make $X_{\mathcal{E}}$ a source of counterexamples in Teichmüller theory.

1.3. **Main statement.** Recall that the *diophantine type* of $\alpha \in \mathbb{R}$ is

$$w(\alpha) := \sup \left\{ w > 0 : \left|\alpha - p/q\right| < \frac{1}{q^{w+1}} \text{ for infinitely many } p/q \in \mathbb{Q} \right\},$$

where fractions p/q are written with co-prime p and q. We always have $w(\alpha) \geq 1$ by Dirichlet's Theorem. Moreover $w(\alpha) = 1$ for almost any α. Fix an origami X and $\alpha \in \mathbb{R}$. For any $p \in X$ and $r > 0$, the time needed by the positive ϕ_α-orbit of p to become r-dense is

$$T(X, \alpha, p, r) := \sup \left\{ \widetilde{p} \in X : \inf \left\{ t > r : \mathrm{Dist}\big(\phi_\alpha(t,p), \widetilde{p}\big) < r \right\} \right\},$$

where $\mathrm{Dist}(\cdot,\cdot)$ is the distance on X, which equals the euclidean distance on small enough discs in $X \setminus \{p_1, \dots, p_m\}$. Minimality implies that $T(X, \alpha, p, r)$ is defined for any p outside singular leaves. In general, the scaling law of $T(X, \alpha, p, r)$ as $r \to 0$ has an irregular behaviour. Nevertheless it can be bounded from above by a power law r^{-H}, where the best exponent $H = H(X, \alpha, p)$, called *hitting time*, is defined by

$$H(X, \alpha, p) := \limsup_{r \to 0^+} \frac{\log T(X, \alpha, p, r)}{-\log r}.$$

Theorem 1.1. *Let $X_{\mathcal{O}}$ be the Ornithorynque origami. Then for any α irrational and any p outside of singular leaves we have*

$$H(X_{\mathcal{O}}, \alpha, p) = w(\alpha).$$

Theorem 4.1 below proves the identity $H(X, \alpha, p) = w(\alpha)$ in a more general setting. The non-trivial inequality is $H(X, \alpha, p) \leq w(\alpha)$, which holds for any origami X satisfying a specific intersection property. Proposition 3.1 and Corollary 3.2 below show that $X_{\mathcal{O}}$ satisfies such property. The same is true for the Eierlegende Wollmilchsau $X_{\mathcal{E}}$ (§ 8.2 in [6]). Such property fails for any genus 2 origami with one conical point (Lemma 6.5 in [6]). *Cyclic covers* in [9] are a natural candidate for testing the assumption of Theorem 4.1 and thus proving the identity between diophantine type and hitting time. The easier inequality in Theorem 4.1 and Theorem 1.1 is implicit in [6]. We state it as follows (a proof is in § A).

Lemma 1.2. *Let X be any origami and α be an irrational slope. For any p outside singular leaves we have*

$$H(X, \alpha, p) \geq w(\alpha).$$

For any origami X and any α irrational, the function $p \mapsto H(X, \alpha, p)$ is invariant under ϕ_α (Lemma 4.2 in [6]). Thus $H(X, \alpha, \cdot)$ is constant almost everywhere. Theorem 1.1 was proved on the standard torus $X = \mathbb{T}^2$ in [8]. Proposition 2.5 in [6] extends the same result to the Eierlegende Wollmilchsau $X_{\mathcal{E}}$. On the other hand, for any origami X with genus $g = 2$ and an unique conical point of order $k_1 = 2$, Theorem 2.2 in [6] proves that for any $\lambda \in [1, 2]$ there are directions α with

$$H(X, \alpha, p) = w(\alpha)^\lambda \quad \text{for almost any} \quad p \in X.$$

For X with the same topological data, we have $H(X,\alpha,p) \leq w(\alpha)^2$ for any α and any p outside singular leaves (Theorem 2.1 in [6]). Proposition 4.6 in [6] proves that for any origami X and α irrational we have

$$\liminf_{r\to 0} \frac{\log\big(\inf\{t>r : \mathrm{Dist}\big(\phi_\alpha(t,p),\widetilde{p}\big)<r\}\big)}{-\log r} = 1 \quad \text{for a. e.} \quad p,\widetilde{p}\in X.$$

The lim inf above depends a priori both on p and $\widetilde{p}$, because the orbit of p may reach the neighbourhood of different points at very different times. The fact that the result is the same for almost any $\widetilde{p}$ (and p) is a consequence of ergodicity. On the other hand $H(X,\alpha,p)$ is defined taking a supremum over $\widetilde{p}\in X$ and thus depends only on p. This is meaningful when establishing an uniform upper bound for the lim sup. Combining the last result and Theorem 1.1, and recalling that generically $w(\alpha)=1$, we get that for almost any α and almost any $p,\widetilde{p}$ in $X_{\mathcal{O}}$ there exists the limit

$$\lim_{r\to 0} \frac{\log\big(\inf\{t>r : \mathrm{Dist}\big(\phi_\alpha(t,p),\widetilde{p}\big)<r\}\big)}{-\log r} = 1.$$

The limit above was established for generic *interval exchange transformations* in [7]. Most results quoted from [6] are proved in the general setting of *translation surfaces*.

Contents of this paper. In § 2 we describe the action of $\mathrm{SL}(2,\mathbb{Z})$ over the set of origamis, which fixes $X_{\mathcal{O}}$. In § 3 we state and prove Proposition 3.1 and Corollary 3.2, which establish that $X_{\mathcal{O}}$ satisfies the intersection property in Theorem 4.1. In § 4 we revise continued fractions in terms of $\mathrm{SL}(2,\mathbb{Z})$ and use them as a renormalization tool to prove Theorem 4.1. The proof of Theorem 1.1 is resumed in § 4.1. In § A we prove Lemma 1.2.

Acknowledgements. The author is grateful to D. H. Kim, S. Marmi and C. Matheus. The author also thanks the anonymous referee for reading the paper carefully and providing thoughtful suggestions and comments.

2. Background

Let $\mathrm{SL}(2,\mathbb{Z})$ be the group of 2×2 matrices A with coefficients in $\mathbb{Z}$ and determinant $\det(A)=1$. In particular we consider the following elements

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad ; \quad V := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad ; \quad R := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

Any $A\in\mathrm{SL}(2,\mathbb{Z})$ acts projectively on points $\alpha\in\mathbb{R}\cup\{\infty\}$ by

$$A\cdot\alpha := \frac{a\alpha+b}{c\alpha+d} \quad \text{where} \quad A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

2.1. **Action of** $\mathrm{SL}(2,\mathbb{Z})$ **on origamis.** Fix an origami X, defined by permutations (h,v) of a finite set $\mathcal{Q}$. Fix $A\in\mathrm{SL}(2,\mathbb{Z})$ and consider the parallelogram $P:=A([0,1]^2)$. For $j\in\mathcal{Q}$ the j-th copy $P_j:=\{j\}\times P$ has sides

$$\widetilde{l}_j := \{j\}\times A(l) \quad ; \quad \widetilde{r}_j := \{j\}\times A(r) \quad ; \quad \widetilde{b}_j := \{j\}\times A(b) \quad ; \quad \widetilde{t}_j := \{j\}\times A(t),$$

where the sides l,r,b,t of $[0,1]^2$ are defined in § 1.1. For any $j\in\mathcal{Q}$, paste the side $\widetilde{r}_j$ of P_j to the side $\widetilde{l}_{h(j)}$ of $P_{h(j)}$ and the side $\widetilde{t}_j$ of P_j to the side $\widetilde{b}_{v(j)}$ of $P_{v(j)}$. Let $A\cdot X$ be the corresponding surface, which is compact, connected, orientable and without boundary.

Moreover $A \cdot X$ is an origami, corresponding to a pair $(h^{(A)}, v^{(A)})$ of permutations of $\mathcal{Q}$. It is possible to see from the commutator $[h^{(A)}, v^{(A)}]$ that $A \cdot X$ has the same number of conical points as X, with same orders $k_1, \ldots, k_m$, and thus also the same genus (see [3] for details). For the matrix T in Equation (2.1) we have

$$h^{(T)} = h \quad \text{and} \quad v^{(T)} = v \circ h^{-1},$$

while for the matrix V in Equation (2.1) we have

$$h^{(V)} = h \circ v^{-1} \quad \text{and} \quad v^{(V)} = v.$$

Since T, V generate $\mathrm{SL}(2, \mathbb{Z})$, we can compute $(h^{(A)}, v^{(A)})$ from (h, v) for any $A \in \mathrm{SL}(2, \mathbb{Z})$.

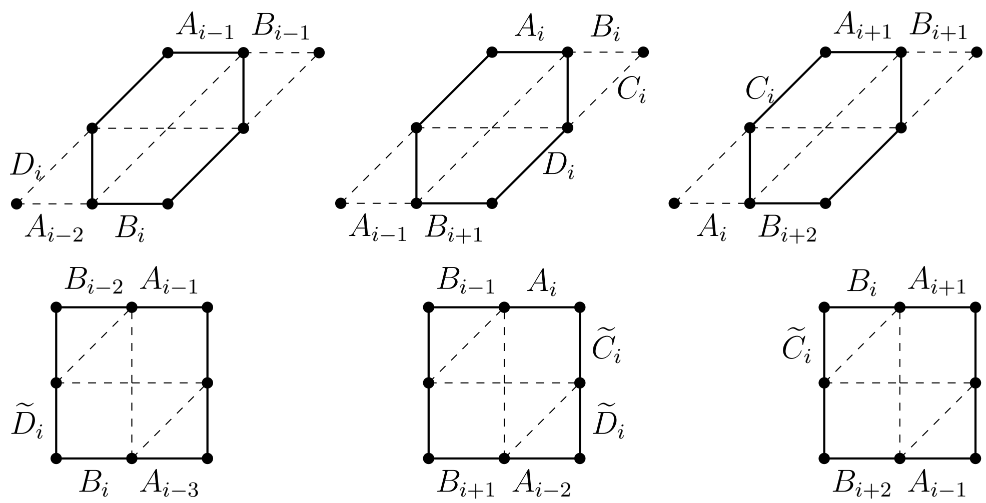

FIGURE 2. Cut the dotted triangles in the above line and paste them along the sides C_i, D_i for $i = 0, 1, 2$, as in the line below. It follows $T \cdot X_{\mathcal{O}} = X_{\mathcal{O}}$.

Proposition 2.1. *We have $A \cdot X_{\mathcal{O}} = X_{\mathcal{O}}$ for any $A \in \mathrm{SL}(2, \mathbb{Z})$.*

Proof. Recall that T, R in Equation (2.1) generate $\mathrm{SL}(2, \mathbb{Z})$. Figure 2 shows that $T \cdot X_{\mathcal{O}} = X_{\mathcal{O}}$, while it is clear from Figure 1 that $R \cdot X_{\mathcal{O}} = X_{\mathcal{O}}$. See [3] for more details. □

2.2. **Affine homeomorphisms.** Fix an origami X and $A \in \mathrm{SL}(2, \mathbb{Z})$. For $j \in \mathcal{Q}$, the affine maps $(j, x) \mapsto \big(j, A(x)\big)$ of Q_j onto P_j agree on glued sides, where we use the same notation of § 2.1. Therefore we have a globally defined homeomorphism

$$\psi_A : X \to A \cdot X \tag{2.2}$$

sending $\{p_1, \ldots, p_m\}$ bijectively onto the set of conical points of $A \cdot X$. Local inverses $\varphi : U \to X \setminus \{p_1, \ldots, p_m\}$ of the covering ρ_X in Equation (1.1), defined over simply connected open sets $U \subset \mathbb{T}^2$, give smooth charts for $X \setminus \{p_1, \ldots, p_m\}$. Change of charts are indeed translations [1]. Similar translation charts exist on $A \cdot X$ (minus its conical points). In these translation charts ψ_A is a diffeomorphism, which is locally affine. The *linear part* $D\psi_A$ is the linear part of ψ_A computed in translation charts. We have of course $D\psi_A = A$. The *automorphisms group* $\mathrm{Aut}(X)$ is the set of orientation preserving homeomorphisms

[1]Thus they are holomorphic, and one can extend them to an holomorphic atlas over the entire X, see [3].

$\psi : X \to X$ which preserve $\{p_1, \dots, p_m\}$ and are affine in translation charts, with $D\psi = \mathrm{Id}$. In general $\mathrm{Aut}(X)$ is non trivial, thus for a given $A \in \mathrm{SL}(2, \mathbb{Z})$ there exist more than one ψ_A as in Equation (2.2). We have $\mathrm{Aut}(X_{\mathcal{O}}) \simeq \mathbb{Z}/3\mathbb{Z}$, which acts by translation on the big 2×2 squares in Figure 1 (see § 3.1 in [9]).

3. The intersection property of $X_{\mathcal{O}}$

Let X be any origami and $p_1, \dots, p_m$ be its conical points. Let $\rho_X : X \to \mathbb{T}^2$ be the covering in Equation (1.1). A *straight segment* in X, or simply *segment*, is a smooth path $S : (a, b) \to X \setminus \{p_1, \dots, p_m\}$ such that there exists a vector $v \in \mathbb{R}^2$ with

$$\frac{d}{dt}\rho_X\big(S(t)\big) = v \quad \text{for any} \quad t \in (a, b).$$

If $v = (x, y) \in \mathbb{R}^2$, then the slope $\mathrm{Slope}(S) \in \mathbb{R} \cup \{\pm\infty\}$ of such S is

$$\mathrm{Slope}(S) := \frac{x}{y}.$$

The length $|S|$ of such segment is $|S| := |b - a| \cdot \|v\|$, where $\| \cdot \|$ is the euclidean norm in $\mathbb{R}^2$. Observe that segments do not contain conical points in their interior. Endpoints of straight segments can be conical points. A *saddle connection* of the surface X is a straight segment connecting conical points. Proposition 3.1 is the main result in this section. Its proof is resumed in § 3.3 below, applying the constructions developed in § 3.1 and § 3.2.

Proposition 3.1. *Let $X_{\mathcal{O}}$ be the Ornithorynque origami. Fix segments H, V in $X_{\mathcal{O}}$ with $0 < \mathrm{Slope}(V) < 1$ and $\mathrm{Slope}(H) < -1$. If both segments have length $|H|, |V| \geq \sqrt{288}$ then*

$$H \cap V \neq \emptyset.$$

Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ acting by $S(x, y) := (-x, y)$. The same construction as in § 2.1 gives a surface $S \cdot X_{\mathcal{O}}$, obtained glueing copies $\{j\} \times S\big([0, 1]^2\big)$ of the reflected square $S\big([0, 1]^2\big)$, where $j \in \mathcal{Q}$ and where identifications in $S \cdot X_{\mathcal{O}}$ are induced by identifications in $X_{\mathcal{O}}$. It is easy to see that indeed we have $S \cdot X_{\mathcal{O}} = X_{\mathcal{O}}$. As in § 2.2, there exists an orientation reversing homeomorphism $f_S : X_{\mathcal{O}} \to X_{\mathcal{O}}$ with linear part $Df_S = S$. If H, V are segments in $X_{\mathcal{O}}$ with $-1 < \mathrm{Slope}(H) < 0$ and $\mathrm{Slope}(V) > 1$, then $0 < \mathrm{Slope}\big(f_S(H)\big) < 1$ and $\mathrm{Slope}\big(f_S(V)\big) < -1$. Proposition 3.1 implies directly the next Corollary.

Corollary 3.2. *Let $X_{\mathcal{O}}$ be the Ornithorynque origami. Fix segments H, V in $X_{\mathcal{O}}$ with $-1 < \mathrm{Slope}(H) < 0$ and $\mathrm{Slope}(V) > 1$. If both segments have length $|H|, |V| \geq \sqrt{288}$ then*

$$H \cap V \neq \emptyset.$$

3.1. **Preliminary Lemmas.**

If ℓ_H and ℓ_V are lines in $\mathbb{R}^2$ with different slopes, then they intersect in a point. Lemma 3.3 below, whose proof is left to the reader, delimits the position of the intersection.

Lemma 3.3. *Let $Q_1 := [0, 1]^2$ and $Q_2 := [1, 2] \times [0, 1]$. Let ℓ_H and ℓ_V be two lines in $\mathbb{R}^2$ with $0 < \mathrm{Slope}(\ell_V) < 1$ and $\mathrm{Slope}(\ell_H) < -1$ and set $P := \ell_H \cap \ell_V$. If both ℓ_V and ℓ_H intersect $\{1\} \times [0, 1]$, then either $P \in Q_1$ or $P \in Q_2$.*

Lemma 3.3 is used in the proof of Lemma 3.4 below, where Q_1 and Q_2 play the role of neighbouring squares in an origami.

Lemma 3.4. *Let X be any origami labelled by a finite set $\mathcal{Q}$. Let H and V be segments in X with $0 < \mathrm{Slope}(V) < 1$ and $\mathrm{Slope}(H) < -1$. Fix a square Q_j with $j \in \mathcal{Q}$ and assume that $H \cap Q_j \neq \emptyset$ and $V \cap Q_j \neq \emptyset$, and that moreover both H, V have endpoints in $\bigcup_{l \neq j} \partial Q_l$. Then $H \cap V \neq \emptyset$.*

Proof. Let ℓ_H, ℓ_V be lines as in Lemma 3.3 and R be the matrix in Equation (2.1). The lines $R(\ell_H), R(\ell_V)$ satisfy the same assumption of Lemma 3.3, with inverted roles. Thus we get an extended version of Lemma 3.3, where Q_2 is replaced by any of the four unitary squares in $\mathbb{R}^2$ sharing a side with Q_1. Now consider an origami X and segments $H, V \subset X$ as in the statement. The assumption implies that there exists a side γ of Q_j intersecting both H and V. Let ℓ_H, ℓ_V be the maximal orbit segments containing H, V respectively. The extended version of Lemma 3.3 implies that $P := \ell_H \cap \ell_V$ belongs to $Q_j \cup Q_k$, where Q_k is the square sharing the side γ with Q_j. We have $P \in H$, indeed the assumption also implies that the endpoints of H are not in the interior of $Q_j \cup Q_k$. In other words, H is long enough to contain P. Similarly $P \in V$. Thus $H \cap V \neq \emptyset$. □

3.2. **Cutting sequences.** Recall Figure 1 and consider the twelve letters alphabet

$$\mathcal{A} := \{A_i, B_i, C_i, D_i : i = 0, 1, 2\}.$$

Geometrically, any $\gamma \in \mathcal{A}$ is a saddle connection of $X_{\mathcal{O}}$. Symbolically, elements $\gamma \in \mathcal{A}$ are letters composing words $(\gamma_1, \dots, \gamma_n)$, which arise as *cutting sequences* of straight segments in $X_{\mathcal{O}}$. Fix a segment $S \subset X$ with $\mathrm{Slope}(S) \neq 0, \infty$ and let $S : (0, 1) \to X$ be one of its two parametrizations with constant speed. Define recursively integers $k = 1, \dots, n$ and instants $0 \leq t_1 < \dots < t_n \leq 1$ by

$$t_1 := \min\{t \geq 0 : \exists \gamma \in \mathcal{A} : S(t) \in \gamma\}$$
$$t_k := \min\{t > t_{k-1} : \exists \gamma \in \mathcal{A} : S(t) \in \gamma\} \text{ for } k \geq 2,$$

where $t_n = \max\{0 \leq t \leq 1 : \exists \gamma \in \mathcal{A} : S(t) \in \gamma\}$. Then define the cutting sequence

$$[S] := (\gamma_1, \dots, \gamma_n)$$

of S as the word in the letters of $\mathcal{A}$ such that $S(t_k) \in \gamma_k$ for $k = 1, \dots, n$. The other parametrization of S gives the inverted cutting sequence $(\gamma_n, \dots, \gamma_1)$. The results in this § 3.2 only concern intersections, and none of them depends on the choice of the parametrization. Below, segments V with $0 < \mathrm{Slope}(V) < 1$ should be interpreted as segments of trajectories of ϕ_α for some $0 < \alpha < 1$. In such case Equation (1.2) provides a natural choice of parametrization. Figure 3 shows examples of cutting sequences. In the notation of § 1.2, for $i = 0, 1, 2$ define the *tile* $\mathcal{T}_i \subset X$ by

$$\mathcal{T}_i := Q_{(i,1,0)} \cup Q_{(i,0,0)} \cup Q_{(i,1,1)} \cup Q_{(i,0,1)}.$$

Lemma 3.5. *Let V be a segment with $0 < \mathrm{Slope}(V) < 1$ and assume that its cutting sequence $[V] = (\gamma_1, \dots, \gamma_n)$ contains $n \geq 6$ letters. Then $V \cap \mathcal{T}_i \neq \emptyset$ for $i = 0, 1, 2$.*

Proof. Assume without loss of generality that V does not cross the tile $\mathcal{T}_0$. The path of V can be followed in Figure 1 setting $i = 0$, $i + 1 = 1$, $i - 1 = 2$. We have

$$\gamma_k \neq A_0, B_0, C_0, D_0, C_2, D_1, A_2, B_1 \quad \text{for} \quad k = 1, \dots, n-1.$$

Observing that $\gamma_k = C_1 \Rightarrow \gamma_{k+1} = A_2$ we get

$$\gamma_k \neq C_1 \quad \text{for} \quad k = 1, \dots, n-2.$$

Since $\gamma_k = B_2 \Rightarrow \gamma_{k+1} \in \{D_1, C_1\}$ it follows

$$\gamma_k \neq B_2 \quad \text{for} \quad k = 1, \dots, n-3.$$

Moreover we have $\gamma_k = A_1 \Rightarrow \gamma_{k+1} \in \{A_2, B_2, C_2\}$, therefore

$$\gamma_k \neq A_1 \quad \text{for} \quad k = 1, \dots, n-4.$$

Finally $\gamma_k = D_2 \Rightarrow \gamma_{k+1} \in \{A_1, B_1\}$, which implies

$$\gamma_k \neq D_2 \quad \text{for} \quad k = 1, \dots, n-5.$$

Since $n \geq 6$, the conditions above imply that there is no value left for γ_1, which is absurd. □

Lemma 3.6. *Let H be a segment with* $\mathrm{Slope}(H) < -1$ *and assume that its cutting sequence* $[H] = (\gamma_1, \dots, \gamma_n)$ *contains $n \geq 6$ letters. Then $H \cap \mathcal{T}_i \neq \emptyset$ for $i = 0, 1, 2$*

Proof. The Lemma follows by an argument similar to Lemma 3.5. Alternatively consider R in Equation (2.1), observe that $V := R(H)$ satisfies the assumption of Lemma 3.5, and recall $R \cdot X_{\mathcal{O}} = X_{\mathcal{O}}$. □

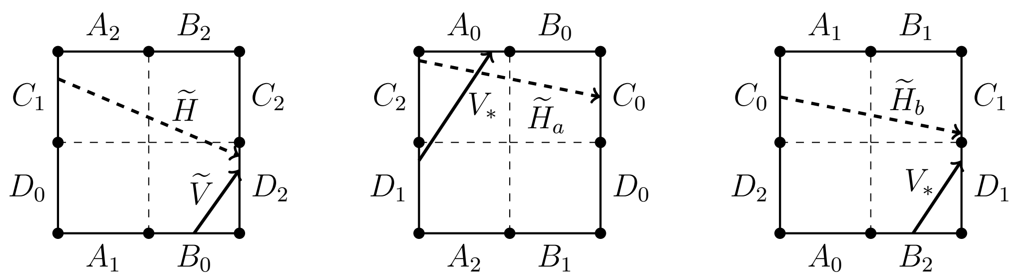

FIGURE 3. In $\mathcal{T}_2$ segments as in the proof of Lemma 3.7. In particular $n = 4$, $[\widetilde{H}] = (C_1, D_2)$ and $[\widetilde{V} \cap \mathcal{T}_2] = (B_0, D_2)$. Even if $\widetilde{V} \cap \widetilde{H} = \emptyset$, Lemma 3.4 gives $H \cap V \neq \emptyset$. In $\mathcal{T}_0 \cup \mathcal{T}_1$ segments as in the proof of Lemma 3.8. In particular $[\widetilde{H}_a] = (C_2, C_0)$, $[\widetilde{H}_b] = (C_0, C_1)$ and V_* represents the subsegment of V with cutting sequence $[V_*] = (B_2, D_1, \nu)$, where $\nu \in \{A_0, B_0\}$.

Lemma 3.7. *Fix segments H, V with* $\mathrm{Slope}(H) < -1$ *and* $0 < \mathrm{Slope}(V) < 1$ *and cutting sequences $[H] = (\gamma_1, \dots, \gamma_n)$ and $[V] = (\nu_1, \dots, \nu_m)$ with $n \geq 4$ and $m \geq 8$. Fix $i = 0, 1, 2$ and assume that there exists $2 \leq k \leq n-2$ with*

$$\gamma_k \in \{C_{i+2}, A_i\} \quad \textit{and} \quad \gamma_{k+1} \in \{B_{i+1}, D_i\}.$$

Then $H \cap V \neq \emptyset$.

Note that in Lemma 3.7 the case $(\gamma_k, \gamma_{k+1}) = (A_i, B_{i+1})$ is forbidden by $\mathrm{Slope}(H) < -1$.

Proof. Let $i \in \{0, 1, 2\}$ be as in the statement. Figure 3 shows an example with $i = 2$. Let $\widetilde{V}$ be the minimal subsegment of V with cutting sequence $[\widetilde{V}] = (\nu_2, \dots, \nu_{m-1})$. Lemma 3.5 implies $\widetilde{V} \cap \mathcal{T}_i \neq \emptyset$. Let $\widetilde{H}$ be the minimal subsegment of H with cutting sequence $[\widetilde{H}] = (\gamma_2, \dots, \gamma_{n-1})$. The assumption on $[H]$ implies that $\widetilde{H}$ intersects at least 3 of the 4 squares $Q_{(i,1,0)}$, $Q_{(i,0,0)}$, $Q_{(i,1,1)}$, $Q_{(i,0,1)}$ composing the tile $\mathcal{T}_i$, where we recall that the squares in an origami are closed and overlap along the boundaries. The square missed by $\widetilde{H}$ can only be either $Q_{(i,1,1)}$ or $Q_{(i,0,0)}$. None of these two squares can contain $\widetilde{V}\cap\mathcal{T}_i$. It follows that, among the 4 squares composing the tile $\mathcal{T}_i$, there is a square Q with $\widetilde{V}\cap Q\neq\emptyset$ and $\widetilde{H}\cap Q\neq\emptyset$. Lemma 3.4 gives $H\cap V\neq\emptyset$. □

Lemma 3.8. *Fix segments H,V with* $\mathrm{Slope}(H)<-1$ *and* $0<\mathrm{Slope}(V)<1$ *and cutting sequences* $[H]=(\gamma_1,\dots,\gamma_n)$ *and* $[V]=(\nu_1,\dots,\nu_m)$ *with* $n\geq 5$ *and* $m\geq 7$. *Fix* $i=0,1,2$ *and assume that there exists* $2\leq k\leq n-3$ *with*

$$(\gamma_k,\gamma_{k+1},\gamma_{k+2})=(C_{i+2},C_i,C_{i+1})\quad \text{or}\quad (\gamma_k,\gamma_{k+1},\gamma_{k+2})=(D_i,D_{i+2},D_{i+1}). \tag{3.1}$$

Then $H\cap V\neq\emptyset$.

Proof. Let i be as in the statement. Assume first $(\gamma_k,\gamma_{k+1},\gamma_{k+2})=(C_{i+2},C_i,C_{i+1})$. Figure 3 shows an example with $i=0$. Let $\widetilde{V}$ be the minimal subsegment of V with cutting sequence $[\widetilde{V}]=(\nu_1,\dots,\nu_{m-1})$. Let $\widetilde{H}_a,\widetilde{H}_b$ be respectively the minimal subsegments of H with $[\widetilde{H}_a]=(C_{i+2},C_i)$ and $[\widetilde{H}_b]=(C_i,C_{i+1})$, so that in particular $\widetilde{H}_a\subset\mathcal{T}_i$ and $\widetilde{H}_b\subset\mathcal{T}_{i+1}$. We have $\widetilde{V}\cap\mathcal{T}_{i+1}\neq\emptyset$ by Lemma 3.5. If $\widehat{H}_b\cap\widetilde{V}\neq\emptyset$ then obviously $H\cap V\neq\emptyset$. If $\widehat{H}_b\cap\widetilde{V}=\emptyset$ but both $\widetilde{V}$ and $\widetilde{H}_b$ intersect the square $Q_{(i+1,0,0)}$ of $\mathcal{T}_{i+1}$, then $H\cap V\neq\emptyset$ by Lemma 3.4. The last possibility is that $(\nu_j,\nu_{j+1})=(B_{i+2},D_{i+1})$ for some $1\leq j\leq m-2$. Therefore the subsegment of V encoded by (ν_{j+1},ν_{j+2}) satisfies $\nu_{j+1}=D_{i+1}$ and $\nu_{j+2}\in\{A_i,B_i\}$, and this last property implies that such subsegment intersects $\widetilde{H}_a$. Thus again $H\cap V\neq\emptyset$.

Now assume $(\gamma_k,\gamma_{k+1},\gamma_{k+2})=(D_i,D_{i+2},D_{i+1})$. Let $\widetilde{V}$ be the minimal subsegment of V with cutting sequence $[\widetilde{V}]=(\nu_2,\dots,\nu_m)$. Let $\widetilde{H}_c,\widetilde{H}_d$ be respectively the minimal subsegments of H with $[\widetilde{H}_c]=(D_i,D_{i+2})$ and $[\widetilde{H}_d]=(D_{i+2},D_{i+1})$, so that in particular $\widetilde{H}_c\subset\mathcal{T}_{i+2}$ and $\widetilde{H}_d\subset\mathcal{T}_{i+1}$. We have $\widetilde{V}\cap\mathcal{T}_{i+2}\neq\emptyset$ by Lemma 3.5. If $\widehat{H}_c\cap\widetilde{V}\neq\emptyset$ then obviously $H\cap V\neq\emptyset$. If $\widehat{H}_c\cap\widetilde{V}=\emptyset$ but both $\widetilde{V}$ and $\widetilde{H}_c$ intersect the square $Q_{(i+2,1,1)}$ of $\mathcal{T}_{i+2}$, then $H\cap V\neq\emptyset$ by Lemma 3.4. The last possibility is that $(\nu_j,\nu_{j+1})=(C_{i+1},A_{i+2})$ for some $2\leq j\leq m-1$. Therefore the subsegment of V encoded by (ν_{j-1},ν_j) satisfies $\nu_j=C_{i+1}$ and $\nu_{j-1}\in\{A_i,B_{i+1}\}$, and this last property implies that such subsegment intersects $\widetilde{H}_c$. Thus again $H\cap V\neq\emptyset$. The Lemma is proved. □

3.3. **Proof of Proposition 3.1.**

Let $[H]=(\gamma_1,\dots,\gamma_n)$ and $[V]=(\nu_1,\dots,\nu_m)$ be the cutting sequences of H,V respectively. Since $|H|,|V|\geq\sqrt{288}$, then we have both $n\geq 12$ and $m\geq 12$. Assume that the cutting sequence $[H]$ of H does not satisfy Equation (3.1) for any $i=0,1,2$. Then we must have $-6<\mathrm{Slope}(H)<-1$. Since $n\geq 12$, then H satisfies the assumption of Lemma 3.7. Proposition 3.1 follows.

4. The general criterion

For $\alpha\in\mathbb{R}$, let $w(\alpha)$ be the diophantine type. In this section we state and prove the following general criterion.

Theorem 4.1. *Let X be an origami and assume that there exists a constant $\mathcal{K}>0$ such that for any origami $Y\in\mathrm{SL}(2,\mathbb{Z})\cdot X$ and any pair of segments $H,V\subset Y$ we have $H\cap V\neq\emptyset$ whenever they have length $|H|,|V|\geq\mathcal{K}$ and satisfy*

- *either* $\mathrm{Slope}(H)<-1$ *and* $0<\mathrm{Slope}(V)<1$
- *or* $-1<\mathrm{Slope}(H)<0$ *and* $\mathrm{Slope}(V)>1$.

Then $H(X,\alpha,p)=w(\alpha)$ for any α irrational and any p outside (X,α)-singular leaves.

4.1. **Proof of Main Theorem 1.1.** Recall that $\mathrm{SL}(2,\mathbb{Z}) \cdot X_{\mathcal{O}} = X_{\mathcal{O}}$ by Proposition 2.1. Therefore Proposition 3.1 and Corollary 3.2 imply that $X_{\mathcal{O}}$ satisfies the assumption of Theorem 4.1. Then Theorem 1.1 follows as a particular case of Theorem 4.1.

4.2. **Continued fractions.** Let T, V be as in Equation (2.1). Consider positive integers $a_1, \dots, a_n$ and define $g(a_1, \dots, a_n) \in \mathrm{SL}(2,\mathbb{Z})$ by

$$g(a_1, \dots, a_n) := \begin{cases} V^{a_1} \circ \cdots \circ V^{a_{n-1}} \circ T^{a_n} & \text{for even } n; \\ V^{a_1} \circ \cdots \circ T^{a_{n-1}} \circ V^{a_n} & \text{for odd } n. \end{cases} \tag{4.1}$$

Let $[\alpha] := \max\{k \in \mathbb{Z}, k \leq \alpha\}$ be the *integer part* and $\{\alpha\} := \alpha - [\alpha]$ be the *fractional part* of $\alpha \in \mathbb{R}$, where $0 \leq \{\alpha\} < 1$. The *Gauss map* $G : (0,1) \to [0,1)$ is defined by

$$G(\alpha) := \{\alpha^{-1}\} \quad \text{for} \quad \alpha \in (0,1).$$

Any irrational $\alpha \in (0,1)$ admits an unique *continued fraction expansion*

$$\alpha = [a_1, a_2, \dots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}, \tag{4.2}$$

where $\alpha_0 := \alpha$ and $\alpha_n := G(\alpha_{n-1})$ for $n \geq 1$, and the n-th *partial quotient* of α is given by

$$a_n := \left[\frac{1}{\alpha_{n-1}}\right] \quad \text{that is} \quad \frac{1}{\alpha_{n-1}} = a_n + \alpha_n.$$

The n-th convergent $p_n/q_n := [a_1, \dots, a_n]$ of α is obtained truncating Equation (4.2) to its n-th partial quotient a_n. We get

$$g(a_1, \dots, a_{2n-1}) = \begin{pmatrix} p_{2n-1} & p_{2n-2} \\ q_{2n-1} & q_{2n-2} \end{pmatrix} \quad \text{and} \quad g(a_1, \dots, a_{2n}) = \begin{pmatrix} p_{2n-1} & p_{2n} \\ q_{2n-1} & q_{2n} \end{pmatrix} \tag{4.3}$$

from the recursive relations $q_n = a_n q_{n-1} + q_{n-2}$ and $p_n = a_n p_{n-1} + p_{n-2}$. Therefore

$$p_n/q_n = \begin{cases} g(a_1, \dots, a_n) \cdot 0 & \text{for even } n \\ g(a_1, \dots, a_n) \cdot \infty & \text{for odd } n. \end{cases} \tag{4.4}$$

We have $\alpha_n^{-1} = a_{n+1} + \alpha_{n+1} \Leftrightarrow \alpha_n = V^{a_{n+1}} \cdot \alpha_{n+1}^{-1} \Leftrightarrow \alpha_n^{-1} = T^{a_{n+1}} \cdot \alpha_{n+1}$. Hence

$$\alpha = g(a_1, \dots, a_{2k}) \cdot \alpha_{2k} = g(a_1, \dots, a_{2k}, a_{2k+1}) \cdot \frac{1}{\alpha_{2k+1}} \quad \text{for any} \quad k \in \mathbb{N}. \tag{4.5}$$

4.3. **Proof of Theorem 4.1.** We need Proposition 4.2 below, whose proof is postponed to § 4.5.

Proposition 4.2. *Let X and $\mathcal{K} > 0$ be an origami and a constant as in Theorem 4.1. Fix a slope $\alpha = [a_1, a_2, \dots] \in (0,1)$. For any $p \in X$ outside singular leaves and $n \in \mathbb{N}$ we have*

$$T(X, \alpha, p, r_n) \leq 4\mathcal{K} \cdot q_n \quad \textit{where} \quad r_n := \frac{2(\mathcal{K}+1)}{q_n}$$

Now we prove Theorem 4.1. Let X be an origami as in Theorem 4.1. Let α be an irrational slope. It is enough to prove $H(X, \alpha, p) \leq w(\alpha)$, indeed Lemma 1.2 gives the other inequality. Assume first $\alpha \in (0,1)$. Set $w := w(\alpha)$, so that $q_n \leq K \cdot q_{n-1}^w$ for some K and all n. Fix

$p \in X$ outside singular leaves. For any $r > 0$ small enough consider n with $r_n < r \leq r_{n-1}$. Proposition 4.2 gives

$$\frac{\log T(X_{\mathcal{O}}, \alpha, p, r)}{|\log r|} \leq \frac{\log T(X_{\mathcal{O}}, \alpha, p, r_n)}{|\log r_{n-1}|} \leq \frac{\log 4\mathcal{K} + \log q_n}{\log q_{n-1} - \log 2(\mathcal{K}+1)}$$
$$\leq \frac{\log 4\mathcal{K} + \log K + w \cdot \log q_{n-1}}{\log q_{n-1} - \log 2(\mathcal{K}+1)} \to w \quad \text{for} \quad n \to +\infty.$$

Hence $H(X, \alpha, p) \leq w$. Now consider any $\alpha \in \mathbb{R}$ irrational and write $\alpha = a + \widetilde{\alpha}$, where $a := [\alpha]$ and $\widetilde{\alpha} := \{\alpha\}$ are the integer and fractional part respectively. Set $Y := T^{-a} \cdot X$ and let $\psi : X \to Y$ be an affine homeomorphism with $D\psi = T^{-a}$ as in § 2.2. We have $\kappa > 0$ with

$$\phi_{\widetilde{\alpha}}\big(t, \psi(p)\big) = \psi\big(\phi_\alpha(\kappa t, p)\big) \quad \text{for any} \quad t \in \mathbb{R} \text{ and } p \in X.$$

Thus $H\big(Y, \widetilde{\alpha}, \psi(p)\big) = H(X, \alpha, p)$. Obviously any $Y \in \mathrm{SL}(2, \mathbb{Z}) \cdot X$ satisfies the same assumption as X. Therefore $H\big(Y, \widetilde{\alpha}, \psi(p)\big) \leq w(\widetilde{\alpha})$, because $\widetilde{\alpha} \in (0, 1)$. On the other hand we have obviously $w(\widetilde{\alpha}) = w(\alpha)$. Theorem 4.1 is proved. $\square$

4.4. **Cylinder decompositions.** Let X be any origami. A *closed geodesic* is a straight segment $\sigma : [a, b] \to X$ with $\sigma(a) = \sigma(b)$, where such point is not conical. If ρ_X is the covering in Equation (1.1), then $\rho_X \circ \sigma$ is a closed geodesic in $\mathbb{T}^2$ and must have rational slope. Thus $\mathrm{Slope}(\sigma) \in \mathbb{Q} \cup \{\infty\}$. Given any p/q rational, a *cylinder* in slope p/q is a maximal open and connected subset $C \subset X$ foliated by closed geodesics σ with same length and $\mathrm{Slope}(\sigma) = p/q$. Set $\mathrm{Slope}(C) := p/q$ and $|C| := |\sigma|$, where σ is any closed geodesic as above. The boundary ∂C is union of saddle connections with slope p/q.

Referring to Figure 1, the vertical path $\sigma : [0, 6] \to X_{\mathcal{O}}$ such that $\sigma(2i)$ is the middle point of A_i for $i = 0, 1, 2$ is an example of closed geodesic in $X_{\mathcal{O}}$. We have $|\sigma| = 6$ and $\mathrm{Slope}(\sigma) = 0$. The two vertical cylinders of $X_{\mathcal{O}}$ are

$$C_0^{(+)} := \bigcup_{i=0,1,2} Q_{(i,1,1)} \cup Q_{(i,1,0)} \quad \text{and} \quad C_0^{(-)} := \bigcup_{i=0,1,2} Q_{(i,0,1)} \cup Q_{(i,0,0)}$$

We have a decomposition $X_{\mathcal{O}} = C_0^{(+)} \cup C_0^{(-)}$, where the boundaries of the two cylinders are made by vertical saddle connections.

Referring to [3], recall that any origami X admits a *cylinder decomposition* in the vertical slope $p/q = 0$, with a number $l \geq 1$ of cylinders $C_0^{(1)}, \dots, C_0^{(l)}$. For $i = 1, \dots, l$ any cylinder has $\mathrm{Slope}(C_0^{(i)}) = 0$, integer length $L_i := |C_0^{(i)}|$ and integer *width* W_i, where W_i is defined as the length of an horizontal segment in $C_0^{(i)}$ with endpoints in $\partial C_0^{(i)}$. Fix $p/q \in \mathbb{Q} \cup \{\infty\}$, take $A \in \mathrm{SL}(2, \mathbb{Z})$ with $A \cdot 0 = p/q$ and an origami Y with $A \cdot Y = X$. Let $\psi : Y \to X$ be an affine homeomorphism with $D\psi = A$, as in § 2.2. The vertical cylinder decomposition $Y = C_0^{(1)} \cup \dots \cup C_0^{(l)}$ induces the cylinder decomposition of X in slope p/q, that is

$$X = C_{p/q}^{(1)} \cup \dots \cup C_{p/q}^{(l)} \quad \text{where} \quad C_{p/q}^{(i)} := \psi(C_0^{(i)}) \quad \text{for} \quad i = 1, \dots, l. \tag{4.6}$$

Lemma 4.3. *Consider an origami X, a slope $p/q \in \mathbb{Q} \cup \{\infty\}$ and the decomposition in Equation* (4.6)*. Let H be a segment in X crossing the cylinders $C_{p/q}^{j_1}, \dots, C_{p/q}^{j_n}$. We have*

$$|H| \leq \frac{W_{j_1} + \dots + W_{j_n}}{\sqrt{q^2 + p^2} \cos \big| \arctan\big(\mathrm{Slope}(H)\big) - \arctan(-q/p) \big|}.$$

Proof. Any cylinder in Equation (4.6) has length $|C^{(j)}_{p/q}| = L_j\sqrt{q^2+p^2}$ and euclidean area L_jW_j. Let $\widetilde{H}_j \subset C^{(j)}_{p/q}$ be a segment with endpoints in $\partial C^{(j)}_{p/q}$. If $\mathrm{Slope}(\widetilde{H}_j) = -q/p$, which is orthogonal to p/q, then $|\widetilde{H}_j| = W_j(q^2+p^2)^{-1/2}$. If $\widetilde{H}_j$ has a different slope, then its length increases by the inverse of the cosinus of the angle between $\mathrm{Slope}(\widetilde{H}_j)$ and $-q/p$. The segment H is union of n segments $\widetilde{H}_{j_1}, \dots, \widetilde{H}_{j_n}$ as above. The Lemma follows. □

4.5. **Proof of Proposition 4.2.** Let X be an origami as in Theorem 4.1 and $\alpha = [a_1, a_2, \dots]$ irrational. Fix any two points $p, \widetilde{p}$ in X, with p outside (X,α)-singular leaves.

Consider first the case $n = 2k$. Set $A := g(a_1, \dots, a_{2k})$ and let $X_k \in \mathrm{SL}(2,\mathbb{Z}) \cdot X$ be the surface with $A \cdot X_k = X$. Let $\psi : X_k \to X$ be an affine homeomorphism with $D\psi = A$, as in § 2.2. Set $\alpha_{2k} := A^{-1} \cdot \alpha$ and $p_{2k}/q_{2k} := A \cdot 0$ as in Equations (4.5) and (4.4). Recalling Equation (4.6), for $i = 1, \dots, l$ let $C^{(i)}_0$ be the cylinder in the decomposition of X_k in vertical slope $p/q = 0$. Let W_i be the width of $C^{(i)}_0$. The cylinder decomposition of X in slope p_{2k}/q_{2k} is $X = C^{(1)}_{p_{2k}/q_{2k}} \cup \dots \cup C^{(1)}_{p_{2k}/q_{2k}}$, where $C^{(i)}_{p_{2k}/q_{2k}} := \psi(C^{(i)}_0)$. Consider β irrational such that

$$\begin{cases} A^{-1} \cdot \beta < -1 \\ \cos\left|\arctan(\beta) - \arctan(-q_{2k}/p_{2k})\right| > 1/2. \end{cases}$$

The slope $\widetilde{\beta} = -q_{2k}/p_{2k}$ is orthogonal to p_{2k}/q_{2k} and satisfies the first condition above, indeed recalling Equation (4.3) we have

$$A^{-1} \cdot \frac{-q_{2k}}{p_{2k}} = \begin{pmatrix} q_{2k} & -p_{2k} \\ -q_{2k-1} & p_{2k-1} \end{pmatrix} \cdot \frac{-q_{2k}}{p_{2k}} = \frac{-(q_{2k}^2 + p_{2k}^2)}{q_{2k}q_{2k-1} + p_{2k}p_{2k-1}} < -a_{2k} < -1.$$

The same condition is satisfied by some irrational slope β close to $\widetilde{\beta}$, by continuity of the projective action of A. The second condition on β is easily satisfied.

Let $\widetilde{H} \subset X$ be a straight segment passing through $\widetilde{p}$ with $\mathrm{Slope}(\widetilde{H}) = \beta$. Consider the segment $H := \psi^{-1}(\widetilde{H}) \subset X_k$. We have $\mathrm{Slope}(H) = A^{-1} \cdot \beta$, which is irrational since β is irrational. Since H has irrational slope, it is not a subsegment of a saddle connection of X_k. Therefore H can be extended along the slope $A^{-1} \cdot \beta$ and we can assume that it has length $|H| = \mathcal{K}$. If H crosses the vertical cylinders $C^{(j_1)}_0, \dots, C^{(j_n)}_0$ of X_k, then we have $W_{j_1} + \dots + W_{j_n} \leq \mathcal{K} + 1$. The second condition on β and Lemma 4.3 imply

$$|\widetilde{H}| \leq \frac{2(\mathcal{K}+1)}{\sqrt{q_{2k}^2 + p_{2k}^2}} \leq \frac{2(\mathcal{K}+1)}{q_{2k}} = r_{2k}.$$

Since p does not belong to any (X,α)-singular leaf, then $\psi^{-1}(p)$ does not belong to any (X_k, α_{2k})-singular leaf and it has infinite positive orbit. Let V be a segment in X_k which has an endpoint in $\psi^{-1}(p)$, with $\mathrm{Slope}(V) = \alpha_{2k}$, length $|V| = \mathcal{K}$. By assumption V intersects H. In other words, we have $t > 0$ with

$$\phi_{\alpha_{2k}}\big(t, \psi^{-1}(p)\big) \in H \quad \text{and} \quad 0 \leq t \leq |V|,$$

Consider $T > 0$ such that $\psi \circ \phi_{\alpha_{2k}}(t, \cdot) = \phi_\alpha(T, \cdot) \circ \psi$, so that we have

$$\phi_\alpha(T,p) = \phi_\alpha\big(T, \psi\big(\psi^{-1}(p)\big)\big) = \psi\big(\phi_{\alpha_{2k}}\big(t, \psi^{-1}(p)\big)\big) \in \psi(H) = \widetilde{H}.$$

Both $\widetilde{p}$ and $\phi_\alpha(T,p)$ belong to $\widetilde{H}$. Hence $\left|\phi_\alpha(T,p) - \widetilde{p}\right| \leq |\widetilde{H}| \leq r_{2k}$. We have

$$T \leq |\psi(V)| \leq \|A\| \cdot |V| \leq (p_{2k} + q_{2k} + p_{2k-1} + q_{2k-1}) \cdot |V| \leq 4\mathcal{K} \cdot q_{2k}.$$

Since $\widetilde{p}$ is arbitrary, we get

$$T(X,\alpha,p,r_{2k}) \leq 4\mathcal{K}\cdot q_{2k}.$$

Now consider the case $n=2k-1$. Set $A:=g(a_1,\dots,a_{2k-2},a_{2k-1})$ and let $X_k\in \mathrm{SL}(2,\mathbb{Z})\cdot X$ be the surface with $A\cdot X_k=X$. Let $\psi:X_k\to X$ be an affine homeomorphism with $D\psi=A$. Let α_{2k-1} be the slope related to α by Equation (4.5), that is $\alpha=A\cdot\alpha_{2k-1}^{-1}$. We have $A\cdot\infty=p_{2k-1}/q_{2k-1}$ by Equation (4.4). Moreover $-1<A^{-1}\cdot(-q_{2k-1}/p_{2k-1})<0$, indeed Equation (4.3) gives

$$A^{-1}\cdot\frac{-q_{2k-1}}{p_{2k-1}}=\begin{pmatrix} q_{2k-2} & -p_{2k-2}\\ -q_{2k-1} & p_{2k-1}\end{pmatrix}\cdot\frac{-q_{2k-1}}{p_{2k-1}}=\frac{-q_{2k-2}q_{2k-1}-p_{2k-2}p_{2k-1}}{q_{2k-1}^2+p_{2k-1}^2}.$$

Therefore we can chose an irrational slope β such that

$$\begin{cases} -1<A^{-1}\cdot\beta<0\\ \cos\left|\arctan(\beta)-\arctan(-q_{2k-1}/p_{2k-1})\right|>1/2.\end{cases}$$

Let $\widetilde{H}\subset X$ be a segment passing through $\widetilde{p}$ with $\mathrm{Slope}(\widetilde{H})=\beta$ such that $H:=\psi^{-1}(\widetilde{H})$ is a segment in X_k with length $|H|=\mathcal{K}$. Let $V\subset X_k$ be a segment having an endpoint in $\psi^{-1}(p)$, with $\mathrm{Slope}(V)=1/\alpha_{2k-1}$ and length $|V|=\mathcal{K}$. By assumption we have $H\cap V\neq\emptyset$. The remaining part of the argument is as in case $n=2k$ and is left to the reader. Proposition 4.2 is proved. □

Appendix A. Proof of Lemma 1.2

Let X,α,p be as in the statement of Lemma 1.2. Recall that we always have $w(\alpha)\geq 1$. We first prove the Lemma for those α with $w(\alpha)=1$. In this case, if $H(X,\alpha,p)<1$, then there exists $0\leq w<1$ such that for any $r>0$ small enough, the r-neighbourhood of the orbit segment $\{\phi_\alpha(t,p):0\leq t\leq r^{-w}\}$ is the entire surface X. This is absurd because such r-neighbourhood has area bounded by $2\cdot r^{1-w}=o(1)$. Now assume $w(\alpha)>1$ and take any w with $1<w<w(\alpha)$. We can assume $0<\alpha<1$, as in § 4.3. Write $\alpha=[a_1,a_2,\dots]$. There exist infinitely many n with

$$a_{n+1}\geq q_n^{w-1}. \tag{A.1}$$

It is not a loss of generality to assume that all n as above are even, that is $n=2k$ (otherwise repeat the proof replacing the vertical slope $p/q=0$ by the horizontal $p/q=\infty$). Modulo subsequences, assume that there exists X_0 in the orbit $\mathrm{SL}(2,\mathbb{Z})\cdot X$ such that

$$g(a_1,\dots,a_{2k})\cdot X_0=X \quad \text{ for any } k.$$

Recall Equation (4.6) and let $X_0=C_0^{(1)}\cup\dots\cup C_0^{(l)}$ be the cylinder decomposition of X_0 in vertical slope $p/q=0$, where any $C_0^{(i)}$ has width W_i and length L_i. Let $\widetilde{p}\in X_0$ be a point in the boundary of some vertical cylinder and not on any (X_0,α_{2k})-singular leaf. According to Equation (6.9) in [6], if $\alpha_{2k}<\min_{1\leq i\leq l}L_i^{-1}$ then there exists i with

$$\phi_{\alpha_{2k}}(t,\widetilde{p})\in C_0^{(i)} \quad \text{for} \quad 0<t<W_i\cdot\frac{\sqrt{1+\alpha_{2k}^2}}{\alpha_{2k}}. \tag{A.2}$$

Since $\alpha_{2k}=(a_{2k+1}+\alpha_{2k})^{-1}\ll 1$, then Equation (A.2) holds. Equation (A.1) gives

$$W_i\cdot\frac{\sqrt{1+\alpha_{2k}^2}}{\alpha_{2k}}\geq\frac{W_i}{\alpha_{2k}}\geq a_{2k+1}W_i\geq a_{2k+1}\geq q_{2k}^{w-1}. \tag{A.3}$$

Set $r_0 := 1/4$. Equation (A.2) and Equation (A.3) imply that for any $\widetilde{p} \in X_0$ there exists a cylinder $C_0^{(i)}$ and vertical closed geodesic $\sigma \subset C_0^{(i)}$ such that

$$\phi_{\alpha_{2k}}(t, \widetilde{p}) \notin N(\sigma, r_0) \quad \text{for} \quad 0 \le t \le q_{2k}^{w-1}/2, \tag{A.4}$$

where $N(\sigma, r)$ is the r-neighbourhood of σ. Set $A := g(a_1, \dots, a_{2k})$ and let $\psi : X_0 \to X$ be an affine homomorphism with $D\psi = A$. Recall that $p_{2k}/q_{2k} = A \cdot 0$ and $\alpha = A \cdot \alpha_{2k}$. Moreover $\psi \circ \phi_{\alpha_{2k}}(t, \cdot) = \phi_\alpha(\kappa t, \cdot) \circ \psi$, where the stretching factor of A on vectors with slope α_{2k} satisfies $\kappa > q_{2k}/\sqrt{2}$ (Equation (6.11) in [6]). Equation (A.4) implies that for any $p \in X$ there exists a cylinder $C \subset X$ with $\mathrm{Slope}(C) = p_{2k}/q_{2k}$ and a closed geodesic $\widetilde{\sigma} \subset C$ with

$$\phi_\alpha(t, p) \notin N\big(\widetilde{\sigma}, r_0 \cdot (q_{2k}^2 + p_{2k}^2)^{-1/2}\big) \quad \text{for} \quad 0 \le t \le (q_{2k}^{w-1}/2) \cdot (q_{2k}/\sqrt{2}),$$

where the size of the neighbourhood of $\widetilde{\sigma}$ is derived from Lemma 4.3. Since $\alpha < 1$ and thus $p_{2k} < q_{2k}$, setting $r_k := (q_{2k}\sqrt{32})^{-1}$ we obtain $T(X, \alpha, p, r_k) \ge q_{2k}^w/\sqrt{8}$ and thus

$$H(X, \alpha, p) \ge \limsup_{k\to\infty} \frac{\log T(X, \alpha, p, r_k)}{|\log r_k|} \ge \limsup_{k\to\infty} \frac{w \log q_{2k} - \log\sqrt{8}}{\log q_{2k} + \log\sqrt{32}} = w.$$

Therefore $H(X, \alpha, p) \ge w(\alpha)$ since $w < w(\alpha)$ is arbitrary. Lemma 1.2 is proved. □

References

[1] G. Forni: *On the Lyapunov exponents of the Kontsevich-Zorich cocycle.* Handbook of Dynamical Systems (eds. B. Hasselblatt and A. Katok), Vol. 1B, Elsevier (2006), 549-580.

[2] G. Forni, C. Matheus: *An example of a Teichmüller disk in genus 4 with degenerate Kontsevich-Zorich spectrum.* Preprint, arXiv:0810.0023.

[3] G. Forni, C. Matheus: *Introduction to Teichmüller theory and its applications to dynamics of interval exchange transformations, flows on surfaces and billiards*, Journal of Modern Dynamics, Vol. 8 (2014) 271-436.

[4] G. Forni, C. Matheus, A. Zorich: *Square-tiled cyclic covers.* Journal of Modern Dynamics, Vol. 5 (2011), 285-318.

[5] F. Herrlich and G. Schmithüsen: *An extraordinary origami curve*, Math. Nachr. 281 (2008), 219-237.

[6] D. H. Kim, L. Marchese, S. Marmi: *Long hitting time for translation flows and L-shaped billiards.* Journal of Modern Dynamics, Vol. 14, 2019, 291-353.

[7] D. H. Kim, S. Marmi: *The recurrence time for interval exchange maps.* Nonlinearity, 21 (2008), 2201-2210.

[8] D.H. Kim, B.K. Seo: *The waiting time for irrational rotations.* Nonlinearity, 16 (2003), 1861-1868.

[9] C. Matheus, J.-C. Yoccoz: *The action on the affine diffeomorphisms on the relative homology group of certain exceptionnally symmetric origamis.* Journal of Modern Dynamics, Vol. 4, 2010, 453-486.

[10] W. A. Veech: *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards.* Inventiones Mathematicae 97 (1989), no. 3, 553-583.

Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40126, Bologna, Italia

E-mail address: luca.marchese4@unibo.it