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ABSTRACT

It is shown that all minimally superintegrable Hamiltonian systems in a three-dimensional flat space derived in the work of Evans [Phys. Rev. A **41**, 5666–5676 (1990)] possess hidden symmetries leading to their linearization.

I. INTRODUCTION

The superintegrability of Hamiltonian systems is classically analyzed in terms of the separability properties of the corresponding Hamilton–Jacobi equation and the algebraic relations among the corresponding constants of the motion. However, as shown in recent works (see, e.g., Refs. 1–4 and references therein), superintegrable systems are deeply related to the linearization problem of differential equations. In this context, several types of superintegrable Hamiltonian systems can be linearized using their hidden symmetries, obtained by application of various reduction techniques and transformations that are more general than point symmetries.[5,6] The analysis of hidden symmetries and the subsequent linearization does not depend upon the degree of the constants of the motion, or the particular class of separating coordinates. Along these lines, it was shown in Ref. 3 that all maximally superintegrable systems in a flat three-dimensional space as classified in Ref. 7 by Evans are linearizable by hidden symmetries.

In this work, we continue with the analysis begun in Ref. 3 concerning the linearizability of superintegrable systems in a three-dimensional flat space, according to the classification given in Ref. 7. While the case of maximally superintegrable Hamiltonians was covered in Ref. 3, it remains to inspect the eight classes of minimally superintegrable potentials described in Ref. 7. It turns out that, of these eight systems, only one is genuinely three-dimensional, while the other seven cases correspond to a (separable) extension of the superintegrable potentials on the real plane described in Ref. 8, the linearizability of which has been shown in earlier works (see, e.g., Refs. 2 and 9 and references therein).

We briefly recall the classification in Ref. 8, where all plane systems admitting two quadratic first integrals in addition to the Hamiltonian were determined. Four classes were found, and it was proven that the corresponding Hamilton–Jacobi equation was separable in at least two different coordinate systems.

1. Type I:

$$\mathcal{H}_1 = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{\omega^2}{2}\left(w_1^2 + w_2^2\right) + \frac{\beta_1}{w_1^2} + \frac{\beta_2}{w_2^2}. \tag{1}$$

2. Type II:

$$\mathcal{H}_2 = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{\omega^2}{2}\left(4w_1^2 + w_2^2\right) + \beta_1 w_1 + \frac{\beta_2}{w_2^2}. \tag{2}$$

3. Type III:

$$\mathcal{H}_3 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{\alpha}{r} + \frac{1}{r^2}\left(\frac{\beta_1}{\cos^2\left(\frac{\theta}{2}\right)} + \frac{\beta_2}{\sin^2\left(\frac{\theta}{2}\right)}\right). \tag{3}$$

4. Type IV:

$$\mathcal{H}_4 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{\alpha}{r} + \frac{1}{\sqrt{r}}\left(\beta_1 \cos\left(\frac{\theta}{2}\right) + \beta_2 \sin\left(\frac{\theta}{2}\right)\right). \tag{4}$$

It was shown in Refs. 2 and 9 that these four types (and generalized Hamiltonians) are all linearizable by means of their hidden symmetries. These hidden symmetries are more general than the symmetries of the Hamiltonian and do not correspond to canonical transformations that preserve the Hamiltonian.

II. MINIMALLY SUPERINTEGRABLE SYSTEMS IN $\mathbb{R}^3$

Besides the maximally superintegrable systems in $\mathbb{R}^3$, Evans established in Ref. 7 the existence of eight equivalence classes of minimally superintegrable systems in $\mathbb{R}^3$ depending on an arbitrary function and possessing first integrals that are at most quadratic in the canonical momenta. If $H = \frac{1}{2}|\mathbf{P}|^2 + V(w_1, w_2, w_3)$ denotes the Hamiltonian of the system in Cartesian coordinates, with $\mathbf{P}$ the linear momentum, these potentials are given by

1. $V_I(w_1,w_2,w_3) = \frac{k_1}{w_1^2} + \frac{k_2}{w_2^2} + \frac{k_3}{w_3^2} + F(r), \quad r^2 = w_1^2 + w_2^2 + w_3^2$,
2. $V_{II}(w_1,w_2,w_3) = k\left(w_1^2 + w_2^2\right) + \frac{k_1}{w_1^2} + \frac{k_2}{w_2^2} + F(w_3)$,
3. $V_{III}(w_1,w_2,w_3) = k\left(4w_1^2 + w_2^2\right) + \frac{k_2}{w_2^2} + F(w_3)$,
4. $V_{IV}(w_1,w_2,w_3) = \frac{k_1}{\sqrt{w_1^2+w_2^2}} + \frac{k_2}{w_2^2} + \frac{k_1 w_1}{w_2^2\sqrt{w_1^2+w_2^2}} + F(w_3)$,
5. $V_V(w_1,w_2,w_3) = k\left(w_1^2 + w_2^2 + w_3^2\right) + \frac{k_3}{w_3^2} + \frac{F(w_2/w_1)}{w_1^2+w_2^2}$,
6. $V_{VI}(w_1,w_2,w_3) = k\left(w_1^2 + w_2^2\right) + 4kw_3^2 + \frac{F(w_2/w_1)}{w_1^2+w_2^2}$,
7. $V_{VII}(w_1,w_2,w_3) = -\frac{k}{r} + \frac{k_1 w_3}{r(w_1^2+w_2^2)} + \frac{F(w_2/w_1)}{w_1^2+w_2^2}, \quad r^2 = w_1^2 + w_2^2 + w_3^2$,
8. $V_{VIII}(w_1,w_2,w_3) = \frac{k}{R} + \frac{k_1\sqrt{R+w_2}}{R} + \frac{k_2\sqrt{R-w_2}}{R} + F(w_3), \quad R^2 = w_1^2 + w_2^2$,

where k, k_1, k_2, and k_3 are arbitrary constants.

It is immediate to observe that for the potentials $V_{II} - V_{VIII}$, the Hamiltonian systems can be interpreted as an extension of a (superintegrable) system in the plane, which suggests to reduce these systems to a canonical form, which has already been shown to be linearizable.[2]

A. The potential $V_I(w_1, w_2, w_3)$

For the Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{k_1}{w_1^2} + \frac{k_2}{w_2^2} + \frac{k_3}{w_3^2} + F(r), \tag{5}$$

it is more convenient to consider spherical coordinates $w_1 = r\sin\theta\cos\psi$, $w_2 = r\sin\theta\sin\psi$, and $w_3 = r\cos\theta$ so that the Hamiltonian is rewritten as

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\psi^2}{r^2\sin^2\theta}\right) + \frac{k_1}{r^2\sin^2\theta\cos^2\psi} + \frac{k_2}{r^2\sin^2\theta\sin^2\psi} + \frac{k_3}{r^2\cos^2\theta} + F(r), \tag{6}$$

and the equations of motion are

$$\begin{aligned}
\dot{r} &= p_r, \quad \dot{p}_r = \frac{p_\theta^2\sin^2\theta + p_\psi^2}{r^3\sin^2\theta} + \frac{2(k_2\cos^2\psi + k_1\sin^2\psi)}{r^3\sin^2\theta\sin^2\psi\cos^2\psi} + \frac{2k_3}{r^3\cos^2\theta} - \frac{\mathrm{d}F(r)}{\mathrm{d}r},\\
\dot{\theta} &= \frac{p_\theta}{r^2}, \quad \dot{p}_\theta = \frac{p_\psi^2\cos\theta}{r^2\sin^3\theta} + \frac{2\cos\theta(k_2\cos^2\psi + k_1\sin^2\psi)}{r^2\sin^3\theta\sin^2\psi\cos^2\psi} - \frac{2k_3\sin\theta}{r^2\cos^3\theta},\\
\dot{\psi} &= \frac{p_\psi}{r^2\sin^2\theta}, \quad \dot{p}_\psi = \frac{2k_2\cos\psi}{r^2\sin^2\theta\sin^3\psi} - \frac{2k_1\sin\psi}{r^2\sin^2\theta\cos^3\psi}.
\end{aligned} \tag{7}$$

We begin by solving the last equation with respect to ψ, observing that

$$\dot{p}_\psi = \frac{\mathrm{d}p_\psi}{\mathrm{d}\psi}\frac{p_\psi}{r^2\sin^2\theta} = \frac{2k_2\cos\psi}{r^2\sin^2\theta\sin^3\psi} - \frac{2k_1\sin\psi}{r^2\sin^2\theta\cos^3\psi}, \tag{8}$$

which yields the separable equation

$$p_\psi \frac{\mathrm{d}p_\psi}{\mathrm{d}\psi} = \frac{2k_2 \cos\psi}{\sin^3\psi} - \frac{2k_1 \sin\psi}{\cos^3\psi}, \tag{9}$$

with the solution

$$p_\psi^2 = 2\left(C_0 - \frac{k_2}{\sin^2\psi} - \frac{k_1}{\cos^2\psi}\right) \Longrightarrow p_\psi = \sqrt{2}\sqrt{C_0 - \frac{k_2}{\sin^2\psi} - \frac{k_1}{\cos^2\psi}} \tag{10}$$

for some constant C_0. At this stage, we follow the same procedure used in Ref. 4 to analyze the remaining equations. If, now, h_0 is a constant such that the Hamiltonian (6) satisfies $H = h_0$, solving with respect to p_r taking into account identity (10) leads to

$$p_r = \frac{\sqrt{2\cos(\theta)^2 \sin(\theta)^2 r^2(2h_0 - F(r)) - \cos(\theta)^2\sin(\theta)^2 p_\theta^2 - C_0\cos(\theta)^2 - 2k_3\sin(\theta)^2}}{r\cos(\theta)\sin(\theta)}. \tag{11}$$

We now solve the fourth equation of (7) observing that

$$\dot{p}_\theta = \frac{\mathrm{d}p_\theta}{\mathrm{d}\theta}\frac{p_\theta}{r^2} = \frac{p_\psi^2\cos\theta}{r^2\sin^3\theta} + \frac{2\cos\theta(k_2\cos^2\psi + k_1\sin^2\psi)}{r^2\sin^3\theta\sin^2\psi\cos^2\psi} - \frac{2k_3\sin\theta}{r^2\cos^3\theta}. \tag{12}$$

Replacing p_ψ by (10) and simplifying the resulting expression leads to the separable equation

$$p_\theta \frac{\mathrm{d}p_\theta}{\mathrm{d}\theta} = \frac{2\cos^4\theta C_0 - 2k_3\sin^4\theta}{\sin^2\theta\cos^2\theta} \tag{13}$$

with the solution

$$p_\theta = \frac{\sqrt{\cos^2\theta\sin^2\theta C_1 - \cos^2\theta C_0 - 2k_3\sin^2\theta}}{\sin\theta\cos\theta} \tag{14}$$

for some constant C_1.

With the appropriate replacements of p_r, p_θ, and p_ψ by (11), (14), and (10), respectively, the remaining three first-order equations adopt the form

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{\sqrt{2r^2h_0 - 2r^2F(r) - C_1}}{r}, \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{\sqrt{\sin^2\theta\cos^2\theta C_1 - \cos^2\theta C_0 - 2k_3\sin^2\theta}}{r^2\sin\theta\cos\theta}, \\ \frac{\mathrm{d}\psi}{\mathrm{d}t} &= \frac{\sqrt{\sin^2\psi\cos^2\psi C_0 - 2k_2\cos^2\psi - 2k_1\sin^2\psi}}{r^2\sin^2\theta\sin\psi\cos\psi}. \end{aligned} \tag{15}$$

The first of these equations is separable and, thus, in principle, formally solvable by quadratures. Considering the change of variables $\psi = 2\arctan r_2$, $\theta = 2\arctan r_3$, taking into account the relation

$$\frac{\mathrm{d}r_3}{\mathrm{d}t} = \frac{\mathrm{d}r_3}{\mathrm{d}r_2}\frac{\mathrm{d}r_2}{\mathrm{d}t}, \tag{16}$$

the combination of the transformed equations leads to the first-order differential equation

$$\frac{\mathrm{d}r_3}{\mathrm{d}r_2} = \frac{2\sqrt{2}r_3(r_2^3 - r_2)\sqrt{\left((4C_1 - 2C_0)r_3^2 - C_0(1 + r_3^4)\right)(r_3^2 - 1)^2 - 8k_3r_3^2(r_3^2 + 1)^2}}{(r_3^4 - 1)(r_2^2 + 1)\sqrt{\left(2C_0r_2^2 - k_2(r_2^2 + 1)^2\right)(r_2^2 - 1)^2 - 4k_1r_2^2(r_2^2 + 1)^2}} \tag{17}$$

We solve this expression with respect to the constant k_3 and derive once with respect to the variable r_2, which yields the second-order ordinary differential equation (ODE)

$$\frac{\mathrm{d}^2r_3}{\mathrm{d}r_2^2} = -\frac{2}{r_3^3 - r_3}\left(\frac{\mathrm{d}r_3}{\mathrm{d}r_2}\right)^2 + S_1(r_2)\frac{\mathrm{d}r_3}{\mathrm{d}r_2} + S_2(r_2)\frac{r_3(r_3^2 + 1)^4C_0 - 16C_1r_3^5}{(r_3^2 + 1)^3(r_3^2 - 1)}, \tag{18}$$

where

$$S_1(r_2) = -\frac{3k_2r_2^{12} + 4(2k_1 - k_2 - C_0)r_2^{10} + (12C_0 - 8k_1 - 5k_2)r_2^8 + (8k_2 - 40k_1 - 12C_0)r_2^6}{(r_2^5 - r_2)\big(k_2r_2^8 + (4k_1 - 2C_0)r_2^6 + (8k_1 - 2k_2 + 4C_0)r_2^4 + (4k_1 - 2C_0)r_2^2 + k_2\big)} - \frac{(k_2 - 24k_1 - 12C_0)r_2^4 - 4k_2r_2^2 + k_2}{(r_2^5 - r_2)\big(k_2r_2^8 + (4k_1 - 2C_0)r_2^6 + (8k_1 - 2k_2 + 4C_0)r_2^4 + (4k_1 - 2C_0)r_2^2 + k_2\big)},$$
$$S_2(r_2) = \frac{8(r_2^2 - 1)^2r_2^2}{(r_2^2 + 1)^2\big(k_2r_2^8 + (4k_1 - 2C_0)r_2^6 + (8k_1 - 2k_2 + 4C_0)r_2^4 + (4k_1 - 2C_0)r_2^2 + k_2\big)}, \tag{19}$$

and the functions are related by $2S_1(r_2)S_2(r_2) - \frac{dS_2}{dr_2} = 0$.

If $C_1 = 0$, we consider the change of dependent variable $r_3 = \frac{1}{2}\big(w - \sqrt{w^2 - 4}\big)$, which transforms (18) into the linear equation

$$\frac{d^2w}{dr_2^2} = -S_1(r_2)\frac{dw}{dr_2} - C_0S_2(r_2)w,$$

from which we conclude that it has eight point symmetries. For $C_1 \neq 0$, the equation admits a three-dimensional symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and symmetry generators

$$\mathbf{X} = U(r_2)\partial_{r_2} + \frac{r_2^3 + r_2}{2(r_2^2 - 1)}\left(U(r_2)S_1(r_2) + \frac{dU}{dr_2}\right)\partial_{r_3}, \tag{20}$$

where $U(r_2)$ is a solution of the third-order linear equation

$$\frac{d^3U}{dr_2^3} + \left(2\frac{dS_1}{dr_2} - S_1^2(r_2) - 4C_0S_2(r_2)\right)\frac{dU}{dr_2} + \left(\frac{d^2S_1}{dr_2^2} - S_1(r_2)\frac{dS_1}{dr_2} - 2C_0\frac{dS_2}{dr_2}\right)U(r_2) = 0. \tag{21}$$

This fact suggests us to apply the procedure described in Ref. 6 and that is valid for any second-order ordinary differential equation possessing $\mathfrak{sl}(2,\mathbb{R})$ symmetry. To this extent, we solve Eq. (18) with respect to the constant C_1 and derive once with respect to r_2, which yields the third-order equation

$$\frac{d^3r_3}{dr_2^3} = -\frac{3(r_3^4 + 3)}{r_3^5 - r_3}\frac{d^2r_3}{dr_2^2}\frac{dr_3}{dr_2} + \frac{S_3(r_2)}{(r_2^5 - r_2)S_5(r_2)}\frac{d^2r_3}{dr_2^2} + \frac{12}{r_3^6 - r_3}\left(\frac{dr_3}{dr_2}\right)^3 + \frac{(r_3^4 + 3)S_3(r_2)}{r_2(r_2^4r_3^4 - r_2^4 - r_3^4 + 1)r_3S_5(r_2)}\left(\frac{dr_3}{dr_2}\right)^2 - \frac{3S_4(r_2)}{r_2^2(r_2^4 - 1)^2S_5(r_2)}\frac{dr_3}{dr_2}, \tag{22}$$

where

$$S_3(r_2) = 12(r_2^2 - 1)^3r_2^4C_0 - 24k_1r_2^4(r_2^2 - 3)(r_2^2 + 1)^2 - 3k_2(3r_2^2 - 1)(r_2^4 - 1)^2(r_2^2 - 1),$$
$$S_4(r_2) = -4C_0r_2^4(r_2^2 + 3)(r_2^2 - 1)^4 + 8k_1r_2^4(r_2^2 + 1)^2(r_2^6 - 5r_2^4 + 11r_2^2 + 1) + k_2(r_2^2 - 1)^4(r_2^2 + 1)^2(5r_2^4 - 2r_2^2 + 1),$$
$$S_5(r_2) = 4k_1r_2^2(r_2^2 + 1) + k_2(r_2^4 - 1)^2 - 2C_0r_2^2(r_2^2 - 1)^2.$$

The symmetry generators of the equation have the generic form

$$\mathbf{X} = V_1(r_2)\partial_{r_2} + \frac{1}{(r_2^4 - 1)}\big(V_2(r_2)r_3^3 + (r_3^5 + r_2)V_3(r_2)\big)\partial_{r_3}. \tag{23}$$

The coefficient function $V_1(r_2)$ provides three point symmetries, as it satisfies the third-order differential equation

$$\frac{d^3V_1}{dr_2^3} = \left(\frac{6S_4(r_2)}{(r_2^5 - r_2)^2S_5(r_2)} - \frac{S_3^2(r_2)}{3(r_2^5 - r_2)^2S_5^2(r_2)} + \frac{96(r_2^3 - r_2)^2C_0}{(r_2^2 + 1)^4S_5(r_2)}\right)\frac{dV_1}{dr_2} + \frac{S_6(r_2)}{(r_2^5 - r_2)^3S_5(r_2)^3}V_1(r_2), \tag{24}$$

where $S_6(r_2)$ is given by

$$S_6(r_2) = 6\big((3r_2^4 - 6r_2^2 - 1)(r_2^2 + 1)^4(r_2^2 - 3)k_2^2 + 8(5r_2^4 - 2r_2^2 - 1)(r_2^3 + r_2)^2k_2C_0 - 32r_2^8C_0^2\big)r_2^2(r_2^2 - 1)^9C_0 - 3\big(512r_2^{10}k_1^3 + (r_2^2 - 1)^9\,)k_2^3\big)(r_2^2 + 1)^9. \tag{25}$$

The coefficient functions $V_2(r_2)$ and $V_3(R_2)$ provide three point symmetries and one point symmetry, respectively, as they satisfy the differential equations

$$\frac{\mathrm{d}^3 V_2}{\mathrm{d}r_2^3} = \frac{S_3(r_2)}{r_2 S_5(r_2)}\frac{\mathrm{d}^2 V_2}{\mathrm{d}r_2^2} - \frac{3S_4(r_2)}{(r_2^6 - r_2^2)S_5(r_2)}\frac{\mathrm{d}V_2}{\mathrm{d}r_2}, \tag{26}$$

$$\begin{aligned}\frac{\mathrm{d}V_3}{\mathrm{d}r_2} = &\frac{\mathrm{d}^2 V_1}{\mathrm{d}r_2^2} + \frac{S_3(r_2)}{3(r_2^5 - r_2)S_5(r_2)}\frac{\mathrm{d}V_1}{\mathrm{d}r_2} + \frac{1}{(r_2^5 - r_2)^2}\left(\frac{2S_3(r_2)^2}{9S_5(r_2)^2} - \frac{3S_4(r_2)}{S_5(r_2)}\right)V_1(r_2)\\ &- \frac{32r_2^2(r_2^2-1)^2 C_0}{(r_2^2+1)^2 S_5(r_2)}V_1(r_2).\end{aligned} \tag{27}$$

Consequently, the symmetry algebra of (22) is seven-dimensional, corresponding to the maximal symmetry of third-order equations. For the particular values $V_1(r_2) = 0$, $V_2(r_2) = \lambda_2$, and $V_3(r_2) = \frac{1}{2}\lambda_3$, with λ_i constant, the vector fields

$$\mathbf{X} = \frac{1}{(r_2^4 - 1)}\left(\lambda_2 r_2^3 + \frac{1}{2}(r_2^5 + r_2)\lambda_3\right)\partial_{r_3} \tag{28}$$

generate a two-dimensional non-Abelian intransitive symmetry algebra. Following Lie's classification,[10] the symmetry generators (28) can be reduced to the normal forms

$$\mathbf{X}_1 = \partial_u, \quad \mathbf{X}_2 = u\partial_u \tag{29}$$

through the transformation $y = r_2$ and $u = \frac{r_3^4+1}{2r_3^2}$. In these canonical coordinates, the nonlinear Eq. (23) is transformed into the following linear third-order ODE:

$$\frac{\mathrm{d}^3 u}{\mathrm{d}y^3} = \frac{S_3(y)}{(y^5 - y)S_5(y)}\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} - \frac{3S_4(y)}{(y^6 - y^2)S_5(y)}\frac{\mathrm{d}u}{\mathrm{d}y}. \tag{30}$$

Summarizing, the Hamiltonian (6) hides a third-order equation, leading to the linearization of the system. The general solution can be derived by substitution and integration by quadratures, without making explicit use of the constants of the motion.

B. Potentials obtained by extension of planar systems

As commented before, for the remaining seven potentials, we show that the system is obtained by the extension of one of the four types of Hamiltonians shown in Ref. 8 to be superintegrable, the linearizability of which was proved in Refs. 9 and 2.

1. The potential $V_{II}(w_1, w_2, w_3)$

For the Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + k\left(w_1^2 + w_2^2\right) + \frac{k_1}{w_1^2} + \frac{k_2}{w_2^2} + F(w_3), \tag{31}$$

we get the equations of motion

$$\begin{cases}\dot{w}_1 = p_1, & \dot{p}_1 = -2kw_1 + \dfrac{2k_2}{w_1^3},\\ \dot{w}_2 = p_2, & \dot{p}_2 = -2kw_2 + \dfrac{2k_2}{w_2^3},\\ \dot{w}_3 = p_3, & \dot{p}_3 = -\dfrac{\mathrm{d}F(w_3)}{\mathrm{d}w_3}.\end{cases} \tag{32}$$

Considering the last two equations and introducing the new variable $y = w_3$, with $\frac{\mathrm{d}y}{\mathrm{d}t} = p_3$, we are led to the separable equation

$$p_3\,\mathrm{d}p_3 = -F'(y)\mathrm{d}y, \tag{33}$$

from which $p_3^2 = 2(C_0 - F(y))$ follows for some constant C_0. Introducing this result into Eq. (31) leads us to the reduced two-dimensional Hamiltonian

$$\widehat{H} = \frac{1}{2}\left(p_1^2 + p_2^2\right) + k\left(w_1^2 + w_2^2\right) + \frac{k_1}{w_1^2} + \frac{k_2}{w_2^2} + C_0, \tag{34}$$

corresponding to the Hamiltonian of type (1) with $k = \omega^2$, $k_1 = \beta_1$, and $k_2 = \beta_2$. It was shown in Ref. 2 that the Hamiltonian system (34) hides a third-order linear equation, from which we conclude the linearizability of the Hamiltonian (31).

2. The potential $V_{III}(w_1, w_2, w_3)$

The Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + 4kw_1^2 + kw_2^2 + \frac{k_2}{w_2^2} + F(w_3) \tag{35}$$

yields the equations of motion

$$\begin{cases} \dot{w}_1 = p_1, & \dot{p}_1 = -8kw_1, \\ \dot{w}_2 = p_2, & \dot{p}_2 = -2kw_2 + \dfrac{2k_2}{w_2^3}, \\ \dot{w}_3 = p_3, & \dot{p}_3 = -\dfrac{\mathrm{d}F(w_3)}{\mathrm{d}w_3}. \end{cases} \tag{36}$$

As before, we take w_3 as the new variable y so that p_3 is given by $p_3^2 = 2(C_0 - F(y))$. Inserting it into (31) leads to the two-dimensional Hamiltonian

$$\widehat{H} = \frac{1}{2}\left(p_1^2 + p_2^2\right) + 4kw_1^2 + kw_2^2 + \frac{k_2}{w_2^2} + C_0, \tag{37}$$

which corresponds to the Hamiltonian of type (2) with $k = \omega^2$, $\beta_1 = 0$, and $k_2 = \beta_2$. Following Ref. 2, the system determined by (37) hides a second-order linear equation and a third-order linear equation, from which the linearizability of the Hamiltonian (35) follows. We observe that this case was already considered in Ref. 3 by direct computation.

3. The potential $V_{IV}(w_1, w_2, w_3)$

For the Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{k}{\sqrt{w_1^2 + w_2^2}} + \frac{k_2}{w_2^2} + \frac{k_1 w_1}{w_2^2\sqrt{w_1^2 + w_2^2}} + F(w_3), \tag{38}$$

the equations of motion are given by

$$\begin{cases} \dot{w}_1 = p_1, & \dot{p}_1 = \dfrac{kw_1w_2^2 + k_1w_1^2}{w_2^2\sqrt{w_1^2 + w_2^2}} - \dfrac{k_1}{w_2^2\sqrt{w_1^2 + w_2^2}}, \\ \dot{w}_2 = p_2, & \dot{p}_2 = \dfrac{kw_2^2 + k_1w_1}{w_2\sqrt{w_1^2 + w_2^2}} + \dfrac{2k_1w_1}{w_2^3\sqrt{w_1^2 + w_2^2}} + \dfrac{2k_2}{w_2^3}, \\ \dot{w}_3 = p_3, & \dot{p}_3 = -\dfrac{\mathrm{d}F(w_3)}{\mathrm{d}w_3}. \end{cases} \tag{39}$$

Solving again the last equation with respect to p_3 and the new variable $y = w_3$ leads to $p_3^2 = 2(C_0 - F(y))$ as before. The reduced two-dimensional system has the Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{k}{\sqrt{w_1^2 + w_2^2}} + \frac{k_2}{w_2^2} + \frac{k_1 w_1}{w_2^2\sqrt{w_1^2 + w_2^2}} + C_0. \tag{40}$$

which expressed in polar coordinates $\{r, \theta\}$ adopts the form

$$\widehat{H} = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{k}{r} + \frac{k_2}{r^2\sin^2\theta} + \frac{k_1\cos\theta}{r^2\sin^2\theta} + C_0. \tag{41}$$

Now, observing that $\cos\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$ and $\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$, the Hamiltonian (41) can be rewritten as

$$\widehat{H} = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{k}{r} + \frac{k_1 + k_2}{4r^2\sin^2\left(\frac{\theta}{2}\right)} + \frac{k_2 - k_1}{4r^2\cos^2\left(\frac{\theta}{2}\right)} + C_0. \tag{42}$$

This corresponds to the Hamiltonian of type (3) with $k = \alpha$, $\beta_1 = \frac{k_1 + k_2}{4}$, and $\beta_2 = \frac{k_2 - k_1}{4}$. The systems of this class were shown in Ref. 2 to hide a second-order linear equation, from which the linearizability of the Hamiltonian (38) follows.

4. The potential $V_V(w_1, w_2, w_3)$

The Hamiltonian

$$H_5 = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + k\left(w_1^2 + w_2^2 + w_3^2\right) + \frac{k_3}{w_3^2} + \frac{F(w_2/w_1)}{w_1^2 + w_2^2} \tag{43}$$

is more conveniently expressed in cylindrical coordinates $w_1 = r\cos\theta$, $w_2 = r\sin\theta$, and $w_3 = z$ as

$$H_5 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta}{r^2} + p_z^2\right) + \frac{k_3}{z^2} + k(r^2 + z^2) + \frac{F(\tan(\theta))}{r^2}, \tag{44}$$

leading to the Hamiltonian equations

$$\begin{aligned} \dot{r} &= p_r, \quad \dot{p}_r = -2kr + \frac{p_\theta^2 + 2F(\tan(\theta))}{r^3}, \\ \dot{\theta} &= \frac{p_\theta}{r^2}, \quad \dot{p}_\theta = \frac{-\sec^2(\theta)}{r^2}\frac{\mathrm{d}F(\tan(\theta))}{\mathrm{d}\theta}, \\ \dot{z} &= p_z, \quad \dot{p}_z = \frac{2k_3}{z^3} - 2kz. \end{aligned} \tag{45}$$

Introducing the new variable $y = \theta$ such that $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{p_\theta}{r^2}$, the fourth equation in (45) adopts the separable form

$$p_\theta \,\mathrm{d}p_\theta = -\sec^2(y)F'(\tan(y))\mathrm{d}y, \tag{46}$$

from which we obtain $p_\theta^2 = 2(C_0 - F(\tan(y)))$. Substitution into (44) leads to the two-dimensional Hamiltonian

$$\widehat{H}_5 = \frac{1}{2}\left(p_r^2 + p_z^2\right) + \frac{k_3}{z^2} + k(r^2 + z^2) + \frac{C_0}{r^2}, \tag{47}$$

corresponding to the Hamiltonian of type (1) with $k = \omega^2$, $C_0 = \beta_1$, and $k_3 = \beta_2$, the linearizability of which follows from the analysis in Ref. 2.

5. The potential $V_{VI}(w_1, w_2, w_3)$

In Cartesian coordinates, the Hamiltonian is given by

$$H_6 = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + k\left(w_1^2 + w_2^2\right) + 4kw_3^2 + \frac{F(w_2/w_1)}{w_1^2 + w_2^2}. \tag{48}$$

Again, it is convenient to consider cylindrical coordinates $w_1 = r\cos\theta$, $w_2 = r\sin\theta$, and $w_3 = z$ so that H_6 can be rewritten as

$$H_6 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right) + kr^2 + 4kz^2 + \frac{F(\tan\,\theta)}{r^2}, \tag{49}$$

with canonical equations given by

$$\begin{aligned} \dot{r} &= p_r, \quad \dot{p}_r = \frac{2F(\tan\,\theta)}{r^3} - 2kr + \frac{p_\theta^2}{r^3}, \\ \dot{\theta} &= \frac{p_\theta}{r^2}, \quad \dot{p}_\theta = -\frac{\sec^2\theta\, F'(\tan\,\theta)}{r^2}, \\ \dot{z} &= p_z, \quad \dot{p}_z = -8kz. \end{aligned} \tag{50}$$

If we take $y = \theta$ as the new variable, the same calculation as before gives $p_\theta^2 = 2(C_0 - F(\tan(y)))$, which substituted into (49) leads to the two-dimensional Hamiltonian

$$\widehat{H}_6 = \frac{1}{2}\left(p_r^2 + p_z^2\right) + kr^2 + 4kz^2 + \frac{C_0}{r^2}, \tag{51}$$

corresponding to the Hamiltonian of type (2) with $k = \omega^2$, $\beta_1 = 0$, and $C_0 = \beta_2$.

6. The potential $V_{VII}(w_1, w_2, w_3)$

In cylindrical coordinates $\{r, \theta, z\}$, the Hamiltonian is given by

$$H_7 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right) - \frac{k}{\sqrt{r^2 + z^2}} + \frac{k_1 z}{r^2\sqrt{r^2 + z^2}} + \frac{F(\tan\,\theta)}{r^2}, \tag{52}$$

with canonical equations

$$\begin{aligned} \dot r &= p_r, \quad \dot p_r = \frac{p_\theta^2}{r^3} + \frac{2k_1 z}{r^2\sqrt{r^2+z^2}} + \frac{k_1 z - kr^2}{(r^2+z^2)^{\frac{3}{2}}} + \frac{2F(\tan(\theta))}{r^2},\\ \dot\theta &= \frac{p_\theta}{r^2}, \quad \dot p_\theta = -\frac{\sec^2\theta\, F'(\tan\theta)}{r^2},\\ \dot z &= p_z, \quad \dot p_z = -\frac{k_1 + kz}{(r^2+z^2)^{\frac{3}{2}}}. \end{aligned} \tag{53}$$

The reduction with respect to the new variable $y = \theta$ with $p_\theta^2 = 2(C_0 - F(\tan(y))$ leads to the two-dimensional Hamiltonian

$$\widehat{H}_7 = \frac{1}{2}\left(p_r^2 + p_z^2\right) - \frac{k}{\sqrt{r^2+z^2}} + \frac{k_1 z}{r^2\sqrt{r^2+z^2}} + \frac{C_0}{r^2}, \tag{54}$$

which is of the same type as the Hamiltonian (40) and, thus, corresponds to type (3) with $k = -\alpha$, $\beta_1 = \frac{k_1 + C_0}{4}$, and $\beta_2 = \frac{C_0 - k_1}{4}$.

7. The potential $V_{VIII}(w_1, w_2, w_3)$

Considering cylindrical coordinates $\{r, \theta, z\}$ defined as $w_1 = r\sin\theta$, $w_2 = r\cos\theta$, and $w_3 = z$, the Hamiltonian is expressed as

$$H_8 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right) + \frac{k}{r} + \frac{k_1\sqrt{1+\cos\theta}}{\sqrt{r}} + \frac{k_2\sqrt{1-\cos\theta}}{\sqrt{r}} + F(z), \tag{55}$$

with canonical equations

$$\begin{aligned} \dot r &= p_r, \quad \dot p_r = \frac{p_\theta^2 + kr}{r^3} + \frac{k_1\sqrt{1+\cos\theta} + k_2\sqrt{1-\cos\theta}}{2r^{\frac{3}{2}}},\\ \dot\theta &= \frac{p_\theta}{r^2}, \quad \dot p_\theta = \frac{k_1 \sin\theta}{2\sqrt{r}\sqrt{1+\cos\theta}} - \frac{k_2\sin\theta}{2\sqrt{r}\sqrt{1+\cos\theta}},\\ \dot z &= p_z, \quad \dot p_z = -F'(z). \end{aligned} \tag{56}$$

Considering $z = y$ as the new variable, we deduce from the last two equations that $p_z^2 = 2(C_0 - F(y))$ for some constant. Inserting it into (55) leads to the two-dimensional Hamiltonian

$$\widehat{H}_8 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{k}{r} + \frac{k_1\sqrt{1+\cos\theta}}{\sqrt{r}} + \frac{k_2\sqrt{1-\cos\theta}}{\sqrt{r}} + C_0. \tag{57}$$

Observing now that $\sqrt{1+\cos\theta} = \sqrt{2}\cos\left(\frac{\theta}{2}\right)$ and $\sqrt{1-\cos\theta} = \sqrt{2}\sin\left(\frac{\theta}{2}\right)$, $\widehat{H}_8$ can be rewritten as

$$\widehat{H}_8 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{k}{r} + \frac{k_1\sqrt{2}}{\sqrt{r}}\cos\left(\frac{\theta}{2}\right) + \frac{k_2\sqrt{2}}{\sqrt{r}}\sin\left(\frac{\theta}{2}\right) + C_0, \tag{58}$$

showing that it corresponds to the Hamiltonian of type (4) with $k = \alpha$, $\beta_1 = \sqrt{2}k_1$, and $\beta_2 = \sqrt{2}k_2$. The linearization of such systems, which follows as a subcase of a more general class of Hamiltonians, was proved in Ref. 9.

III. CONCLUDING REMARKS

Using the Lie symmetry method in combination with the procedure of Ref. 6 to linearize second-order ordinary differential equations with symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$, as well as the reduction method used in Ref. 4, we have completed the linearizability analysis of superintegrable systems in a three-dimensional flat space begun in Ref. 3. Specifically, it has been shown that all minimally superintegrable systems classified in Ref. 7 possess a hidden symmetry that allows us to linearize the system. For seven of the eight potentials presented in Ref. 7, the system can be seen as an extension of one of the types classified in Ref. 8, the linearizability of which has been shown in a previous work.[2] For the first potential, which constitutes the only system that is not obtained by the extension of a planar one, the linearization is proved applying the method used in Ref. 4 to determine the linearizability of large classes of two-dimensional systems in non-Euclidean spaces.

The linearization process does neither depend on the separating coordinates of the system nor the degree of the first integrals. In this context, it should be mentioned that the Hamiltonians with potentials V_I–V_{VIII} can be linearized in various ways. In this work, we have chosen the simplest solution, showing that the potentials V_{II}–V_{VIII} are actually obtained from superintegrable systems in the plane that have already been proved elsewhere to be linearizable.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

M. C. Nucci: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **R. Campoamor-Stursberg**: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Writing – original draft (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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