ON THE REGULARITY OF THE SOLUTIONS AND OF ANALYTIC VECTORS FOR "SUMS OF SQUARES"
SULLA REGOLARITÁ DELLE SOLUZIONI E DEI VETTORI ANALITICI PER "SOMME DI QUADRATI"

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Abstract. We present a brief survey on some recent results concerning the local and global regularity of the solutions for some classes/models of sums of squares of vector fields with real-valued real analytic coefficients of Hörmander type. Moreover we also illustrate a result concerning the microlocal Gevrey regularity of analytic vectors for operators sums of squares of vector fields with real-valued real analytic coefficients of Hörmander type, thus providing a microlocal version, in the analytic category, of a result due to M. Derridj.

Sunto. Presentiamo una breve rassegna di alcuni recenti risultati riguardanti la regolarità locale e globale delle soluzioni per alcune classi/modelli di somme di quadrati di campi vettoriali con coefficienti reali analitici a valori reali di tipo Hörmander. Illustriamo anche un risultato riguardante la regolarità microlocale dei vettori analitici per operatori somme di quadrati di campi vettoriali con coefficienti reali analitici a valori reali di tipo Hörmander, fornendo così una versione microlocale, nel caso analitico, di un risultato dovuto a M. Derridj.

2020 MSC. 35H10, 35H20 (Primary) ; 35B65, 35A20, 35A27 Secondary .
Keywords. Sums of squares of vector fields; Global analytic hypoellipticity; Treves conjecture, Local Gevrey Hypoellipticity, Microlocal regularity, Analytic vectors.

1. Introduction

The aim of this paper is to provide an overview of some recent results on the global and local regularity of the solutions for some classes of sums of squares of vector fields with real-valued real analytic coefficients of Hörmander type. Moreover, we deal of the

Bruno Pini Mathematical Analysis Seminar, Vol. 13 (2022) pp. 90–108
Dipartimento di Matematica, Università di Bologna
ISSN 2240-2829.

microlocal regularity of the analytic vectors for a generic sum of squares of vector fields of Hörmander type with real-valued real analytic coefficients.

Let $X_1(x, D), \ldots, X_m(x, D)$ be vector fields with real-valued real analytic coefficients on Ω, open neighborhood of the origin in $\mathbb{R}^n$. Let $P(x, D)$ denote the corresponding sum of squares operator

$$P(x, D) = \sum_{j=1}^{m} X_j(x, D)^2. \tag{1}$$

We assume that the operator P satisfies the Hörmander condition: the Lie algebra generated by the vector fields and their commutators has the dimension n, equal to the dimension of the ambient space, at all points.

The celebrated theorem of Hörmander, [35], settled the problem of C^∞-hypoellipticity: if the fields X_j, $j = 1, \ldots, m$, satisfy the Hörmander condition at the step r, then P is C^∞-hypoelliptic, i.e. if for a given $u \in \mathscr{D}'(U)$, U open subset of Ω, with $Pu \in C^\infty(U)$, we have that $u \in C^\infty(U)$. Furthermore it has been proved by a number of authors (see [35],[7], [32],[46] and [38] for a short proof of a non-optimal statement) that the following *subelliptic* a priori estimate holds:

$$\|u\|_{1/r}^2 + \sum_{j=1}^{m} \|X_j u\|^2 \leq C\big(|\langle Pu, u\rangle| + \|u\|^2\big). \tag{2}$$

Here $u \in C_0^\infty(\Omega)$, $\|\cdot\|_0$ denotes the norm in $L^2(\Omega)$ and $\|\cdot\|_s$ the Sobolev norm of order s in Ω.

Since we will work both in a local setting as well in global one we recall some basic definitions in both settings. We point out that what was stated before can be reformulated in the global setting on the n-dimensional torus replacing Ω with $\mathbb{T}^n$.

Definition 1.1. *Let U be an open subset of $\mathbb{R}^n$. The space $G^s(U)$, $s \geq 1$, the class of Gevrey functions of order s in U, denotes the set of all $f \in C^\infty(U)$ such that for every compact set $K \Subset U$ there are two positive constants C_K and A such that*

$$|D^\alpha f(x)| \leq A C_K^{|\alpha|} |\alpha|^{s|\alpha|}, \qquad \forall\, \alpha \in \mathbb{Z}_+^n \ \textit{and}\ \ \forall\, x \in K.$$

$G^s(\mathbb{T}^n)$, the class of the global Gevrey functions of order s in $\mathbb{T}^n$, denotes the set of all $f \in C^\infty(\mathbb{T}^n)$ such that there exists a positive constant C, for which

$$|D^\alpha f(x)| \leq C^{|\alpha|+1}|\alpha|^{s|\alpha|}, \qquad \forall \alpha \in \mathbb{Z}^n_+, x \in \mathbb{T}^n.$$

When $s = 1$ we shall say that u is analytic in Ω, $u \in C^\omega(\Omega)$, or in $\mathbb{T}^n$, $u \in C^\omega(\mathbb{T}^n)$.

Definition 1.2. *An operator P is said to be G^s-hypoelliptic, $s \geq 1$, in U, open subset of $\mathbb{R}^n$, if for any given U' open subset of U, the conditions $u \in \mathscr{D}'(U)$ and $Pu \in G^s(U)$ imply that $u \in G^s(U')$.*
P is said to be globally G^s-hypoelliptic, $s \geq 1$, in $\mathbb{T}^n$ if the conditions $u \in \mathscr{D}'(\mathbb{T}^n)$ and $Pu \in G^s(\mathbb{T}^n)$ imply that $u \in G^s(\mathbb{T}^n)$.
When $s = 1$ we shall say that P is (globally) analytic hypoelliptic.

When in 1967 Hörmander formulated the theorem on the hypoellipticity, the question on the analytic hypoellipticity for such operators was also asked. In 1971, Derridj, [27], showed that, in the real analytic case, the Hörmander condition is needed for such operators to be analytic hypoelliptic. Not all sums of squares of real analytic vector fields are analytic hypoelliptic as the following two seminal examples show. The first is due to Baouendi and Goulaouic in 1972, [4]. In the local setting they studied the operator

$$P_{BG} = D_1^2 + D_2^2 + x_1^2 D_3^2 \quad \text{in } \mathbb{R}^3. \tag{3}$$

They showed that P_{BG} is G^2-hypoelliptic and no better in any neighborhood of the origin. The second is due to Métivier in 1981, [43]. He studied the operator

$$P_M = D_x^2 + x^2 D_y^2 + (yD_y)^2 \quad \text{in } \mathbb{R}^2. \tag{4}$$

Métivier showed that P_M is G^2-hypoelliptic and no better in any neighborhood of the origin. Let Σ_{PG} and Σ_M the characteristic varieties of P_{BG} and P_M respectively:

$$\Sigma_{PG} = \left\{(x,\xi) \in T^*\mathbb{R}^3 \setminus \{0\} : \xi_1 = 0 = x_1,\ \xi_2 = 0 \text{ and } \xi_3 \neq 0\right\},$$
$$\Sigma_M = \left\{(x,y,\xi,\eta) \in T^*\mathbb{R}^2 \setminus \{0\} : \xi = 0 = x,\ y = 0 \text{ and } \eta \neq 0\right\}.$$

From the geometrical point of view the main difference between the above operators is due to the fact that in the case of the Baouendi-Goulaouic operator, (3), the Hamilton (bicharacteristic) leaf, $T\Sigma_{BG} \cap T\Sigma_{BG}^{\perp_\sigma}$ [1], on the characteristic variety lies on the base of the cotangent bundle; in the case of the Métivier operator, (4), the Hamilton (bicharacteristic) leaf, $T\Sigma_M \cap T\Sigma_M^{\perp_\sigma}$, on the characteristic variety lies along the fiber of the cotangent bundle. Now we adopt the global viewpoint and we consider the global version of the above operators

$$\tilde{P}_{BG} = D_1^2 + D_2^2 + (\sin x_1)^2 D_3^2 \quad \text{in } \mathbb{T}^3, \tag{5}$$

and

$$\tilde{P}_M = D_x^2 + (\sin x)^2 D_y^2 + (\sin(y) D_y)^2 \quad \text{in } \mathbb{T}^2. \tag{6}$$

In 1994, Cordaro and Himonas, [25], showed that the global version of the Baouendi-Goulaouic operator, $\tilde{P}_{BG}$, is globally analytic hypoellipticity, see also [49] and [11]. On the other hand the global version of the Métivier operator, $\tilde{P}_M$, is not globally analytic hypoelliptic. $\tilde{P}_M$ is G^2-globally hypoelliptic and no better as showed in [52] and in [11]. This shows that the scenario can be very different moving from the local to the global case.

The first general result on analytic hypoellipticity in the local setting is due to Tartakoff ([48]) and Treves ([50]): let P be as in (1), i.e. a sum of squares of vector fields with analytic coefficients satisfying the Hörmander condition, assume further that the symbol of P, $P(x,\xi) = \sum_{j=1}^m X_j(x,\xi)^2$, vanishes "exactly" [2] of order 2 on a symplectic submanifold of $T^*\Omega$, then P is analytic hypoelliptic. We recall that the more general result concerning the local regularity of solutions to equations $Pu = f$, f real analytic, was obtained in 1973 by Derridj and Zuily, [28]. They showed: let P be as in (1) and suppose that the Hörmander condition is satisfied using commutators of length up to r and that $Pu \in C^\omega(\Omega)$, then $u \in G^r(\Omega)$. The Gevrey regularity theorem of Derridj and Zuily is not sharp in general, since it is easy to exhibit examples having better regularity, like the Heisenberg Laplacian or more generally any operator satisfying the hypothesis in the Tartakoff-Treves' theorem. However if no additional assumption is made on the operator P, the Gevrey regularity of

[1]Here $T\Sigma_{BG}^{\perp_\sigma}$ denotes the orthogonal complement of $T\Sigma_{BG}$ with respect to the symplectic form σ.

[2]Here with "exactly" we mean that if $\Sigma = Char(P) = P^{-1}(0) = \{(x,\xi) \in T^*\Omega \setminus 0 : P(x,\xi) = 0\}$, for $(x,\xi) \in \Sigma$, we have that $\ker dH_P(x,\xi) = T_{(x,\xi)}\Sigma$, where $H_P(x,\xi) = (\nabla_\xi P(x,\xi), -\nabla_x P(x,\xi))$.

theorem is (locally) optimal as shown by the Baouendi-Goulaouic operator P_{BG}, (3), and by the Métivier operator, (4).

In order to better understand the problem of the analytic hypoellipticity, in 1999 Treves first introduced in [51] and subsequently redefined in [52] the *Poisson-Treves stratification.* In [52] Treves formulated the following conjectures that link the microlocal and the global analytic hypoellipticity of the operator (1) to precise geometrical properties of its characteristic variety:

***(micro-)Local Treves Conjecture*:** the operator $P(x, D)$ in (1) is analytic hypoelliptic if and only if every stratum of the Poisson-Treves stratification is symplectic;

***Global Treves Conjecture*:** let $P(x, D)$ be as in (1) on $\mathbb{T}^n$; then $P(x, D)$ is globally analytic hypoelliptic if and only if the closure in $T^*\mathbb{T}^n$ of every Hamilton (bicharacteristic) leaf of every stratum of the Poisson-Treves stratification is compact.

In 2018 and in 2017, Albano, Bove and Mughetti, [3], and Bove and Mughetti, [12], showed that the sufficient part of the (micro-)local Treves' conjecture does not hold. They produced and studied the first models which are not consistent with the (micro-)local Treves' conjecture. More precisely in [3] the authors produced the following model

$$P_{ABM} = D_1^2 + D_2^2 + x_1^{2(r-1)}\left(D_3^2 + D_4^2\right) + x_2^{2(p-1)} D_3^2 + x_2^{2(q-1)} D_4^2, \qquad \text{in } \mathbb{R}^4, \tag{7}$$

with $1 < r < p < q$, having a single symplectic stratum, according to the Treves conjecture, and which is Gevrey hypoelliptic of order $s_0 = (r(q-1))/(q-1+(r-1)(p-1))$ and no better in any neighborhood of the origin. As remarked by the authors in this case the co-dimension of the single symplectic stratum is 4. Indeed in the presence of single symplectic stratum of codimention 2 the Treves conjecture was proved by Okaji, [44], Cordaro and Hanges [24] and Albano and Bove [1].

In [12] the authors studied the following model

$$P_{BM} = D_1^2 + x_1^{2(\ell+r-1)}\left(D_3^2 + D_4^2\right) + x_1^{2\ell}\left(D_2^2 + x_2^{2(p-1)} D_3^2 + x_2^{2(q-1)} D_4^2\right), \qquad \text{in } \mathbb{R}^4, \tag{8}$$

with $\ell, r, p, q \in \mathbb{N}$ and $1 < r < p < q$, which has two symplectic strata. They showed that even if the codimention of the characteristic manifold of P_{BM} is 2, the operator is not analytic hypoelliptic. In fact, it is Gevrey hypoelliptic of order $s_1 = [(\ell+r)(q-1))]/[(\ell+1)(q-1)+(r-1)(p-1)]$ and no better.

Currently the necessary part of the Treves' conjecture is still an open problem: if there is a non-symplectic stratum, so that Hamiltonian leaves appear, the operator P is not analytic hypoelliptic.

The discovery of these two interesting models, (7) and (8), the first ones to be not consistent with the local Treves conjecture, has de facto completely reopened, in dimension greater than or equal to 4, the question concerning the identification and nature of the sufficient conditions in order that an operator be analytic hypoelliptic. In dimension three no counterexamples are known, it is believed that the Treves conjecture does not hold, in [13] a candidate has been produced that should violate the conjecture. On the contrary, there are good reasons to surmise that the conjecture of Treves holds in two variables. We refer to [51], [16] for more details on the statement of the conjecture, as well as to [13] for a discussion of both the 3- and 2-dimensional cases. Concerning the global case, in [21] (see also [11] for a more general result), it was showed that the global version on the torus of the operators (7) and (8) are globally analytic hypoellitic. At the present, unlike the local case, the global Treves' conjecture has not yet been proved or disproved.

2. Local and Global regularity for some classes of of sums of squares of vector fields of Hörmander type

In this section we present a couple of results: the first concerning the sharp Gevrey hypoellipticity for a generalization of the Métivier operator, the second, obtained in a joint work with Bove, about the global analytic hypoellipticity for a class of sums of squares.

2.1. **On the sharp Gevrey regularity for a generalization of the Métivier operator.** In order to better understand the problem of the regularity of sums of squares in dimension two and three, it is of crucial importance to know whether a certain Gevrey regularity, which may be relatively easy to obtain by using L^2 a priori estimates, is optimal or not. In two variables this becomes difficult because if the Poisson-Treves strata is not symplectic, the Hamilton leaves corresponding to the kernel of the symplectic form have injective projection onto the fibers of the cotangent bundle, thus causing technical complication in the construction of a singular solution, which is the method of proving optimality.

We consider the operator

$$M_{n,m} = D_x^2 + \left(x^{2n+1} D_y\right)^2 + \left(x^n y^m D_y\right)^2, \tag{9}$$

in Ω, open neighborhood of the origin in $\mathbb{R}^2$, where n and m are positive integers. The operator is a non trivial generalization of the Métivier operator (4). The operator (9) has a symplectic characteristic manifold and a non-symplectic stratum according to the Poisson-Treves stratification. According to the Treves conjecture in dimension two, it turns out not to be analytic hypoelliptic. We have the following theorem.

Theorem 2.1. [22] *The operator $M_{n,m}$, (9), is $G^{\frac{2m}{2m-1}}$-hypoelliptic and no better in any neighborhood of the origin.*

A few remarks are in order.

(a) The Gevrey regularity obtained for the operator $M_{n,m}$ is consistent with that predicted in [15] using L^2 a priori estimates, i.e. using the sub-elliptic estimate (2). The Gevrey regularity obtained for the operator $M_{n,m}$ remains the same if we perturb the operator adding a pseudodifferential operator of order less then $(2n+2)^{-1}$ as showed in [10].

(c) As in the case of the Métivier operator (4), the operator $M_{n,m}$ in (9) is not globally analytic hypoelliptic on the two dimensional torus. This means that if, for instance, we consider in $\mathbb{T}^2$ the operator

$$D_x^2 + \left(\sin^{2n+1}(x) D_y\right)^2 + \left(\sin^n(x)\sin^m(y) D_y\right)^2, \tag{10}$$

the latter is not globally analytic hypoelliptic in $\mathbb{T}^2$, it is G^s-globally hypoelliptic for every $s \geq \frac{2m}{2m-1}$. This can be obtained via Theorem 2.1 and either following the proof of the Theorem 2.1 in [52] (p. 325) or following the proof of the Proposition 1.1 in [11].

Idea of the proof of the Theorem 2.1. Following the ideas in [43] we first construct a formal solution to the problem $M_{n,m}u = 0$. The construction of the formal solution in [43], for the operator in (4), uses the spectral theory of the harmonic oscillator, i.e. of the operator $D_x^2 + x^2$, the variable y being reduced to a parameter. For the eigenfunctions of the harmonic oscillator there are three terms recurrence formulas relating the derivative and the multiplication by x of an eigenfunction to those up and down one level. In the case of an anharmonic oscillator, which occurs if one considers cases vanishing of order higher than 2, Gundersen has shown in [33] that such recurrence formulas do not exist, so that the optimality for the operators

$$D_x^2 + x^{2(q-1)} D_y^2 + (y^k D_y)^2,$$

is not known.

In 2001 Bender and Wang, [6], studied the following class of eigenvalue problems

$$-u''(t) + t^{2N+2} u(t) = t^N E\, u(t), \qquad N = -1, 0, 1, 2, \dots, \tag{11}$$

on the interval $-\infty < t < +\infty$. The eigenfunctions u are confluent hypergeometric functions and moreover can be written as a product of a polynomial and a function exponentially vanishing at infinity. As in the case of the harmonic oscillator, also the eigenfunctions of the operator (11) satisfies a "good" recurrence relation. This allows us to obtain a formal solution of $M_{n,m}$:

$$\mathscr{K}[u](x,y) = \int_0^{+\infty} e^{iy\rho^{\frac{2m}{2m-1}}} \rho^{r+\frac{2m}{(n+1)(2m-1)}} \left[u(t,\rho)\right]\Big|_{t=\rho^{\frac{m}{(n+1)(2m-1)}} x} d\rho.$$

where

$$u(t,\rho) = \sum_{\ell \geq 0} \sum_{p=0}^{\ell} g_{\ell,p}(\rho) v_p(t).$$

Here $v_p(t)$ are the even-parity eigenfunctions of the problem (11) with $N = 2n$ and $g_{\ell,p}(\rho)$ are suitable functions such that

$$|g_{\ell,p}^{(k)}(\rho)| \leq C^\ell \frac{(\ell+1)^{\ell\left(1-\frac{1}{2m}\right)}}{\rho^\ell} e^{-c_0\rho}, \quad 0 \leq p \leq \ell \text{ and } k \leq 2m-1,$$

for $\rho \geq C_0(\ell+1)$.

With the aid of a suitable family of smooth cutoff functions, we can turn the formal solution, $\mathscr{K}[u](x,y)$, into a true solution, $\mathscr{K}[\tilde{u}](x,y)$. Then $M_{n,m}\mathscr{K}[\tilde{u}]$ belongs to a suitable function space. The result is obtained by contradiction, i.e. by assuming that $M_{n,m}$ is G^s-hypoelliptic in a neighborhood of the origin for $s < 2m/(2m-1)$. This second step of the proof is based on the Theorem 3.1 in [41] and a suitable estimate from below of $|D_y^k \mathscr{K}[\tilde{u}](0,0)|$.

2.2. **On a Class of Globally Analytic Hypoelliptic Sums of Squares.**

In the joint work with Bove [11], we consider sums of squares operators globally defined on the torus. We show that if some assumptions are satisfied the operators are globally analytic hypoelliptic. The purpose of the assumptions is to rule out the existence of a Hamilton leaf on the characteristic variety lying along the fiber of the cotangent bundle, i.e. the case of the (global) Métivier operator (6), or the case of the global version of its generalization (10).

More precisely let n, m be two positive integers and P be a sum of squares operator of the form

$$P(t,x,D_t,D_x) = \sum_{j=1}^{N} X_j(t,x,D_t,D_x)^2 \text{ on } \mathbb{T}_t^n \times \mathbb{T}_x^m. \tag{12}$$

We assume:

(A-1) The X_j are real analytic vector fields defined on the torus $\mathbb{T}^{n+m}$. We denote the variable as (t,x) where $t \in \mathbb{T}^n$, $x \in \mathbb{T}^m$.

(A-2) The vector fields X_j, $1 \le j \le N$, satisfy the Hörmander condition.

(A-3) Let $n' < n$ and consider $X_j(t,x,D_t,D_x)$ for $1 \le j \le n'$. We assume that

$$X_j = \sum_{i=1}^{n'} a_{ji}(t') D_{t_i}, \tag{13}$$

where $t = (t',t'')$ with $t' \in \mathbb{T}^{n'}$, $t'' \in \mathbb{T}^{n-n'}$. Furthermore, we assume that the vector fields X_j, $1 \le j \le n'$, are linearly independent for every $t' \in \mathbb{T}^{n'}$.

(A-4) Consider now X_j for $n'+1 \le j \le N$. We assume that $N \ge n$ and that X_j has the form

$$X_j = a_j(t') D_{t_{q(j)}} + \sum_{k=1}^{m} b_{jk}(t) D_{x_k}, \tag{14}$$

where a_j, b_{jk} are real analytic funtions defined in $\mathbb{T}^{n'}$, $\mathbb{T}^n$ respectively and q is a surjective map from $\{n'+1,\dots,N\}$ onto $\{n'+1,\dots,n\}$. Hence $q^{-1}(\{j\})$ is a partition of $\{n'+1,\dots,N\}$ with non empty subsets.

Furthermore we assume that for each $j=n'+1,\dots,n$, there exists $\lambda_j \in q^{-1}(\{j\})$, such that

$$\sum_{r=n'+1}^{N}\sum_{k=1}^{m}|b_{rk}(t)| \le C|a_\ell(t')|, \tag{15}$$

for every $\ell \in \{\lambda_j \mid j \in \{n'+1,\dots,n\}\}$.

We also assume that

$$\sum_{r=n'+1}^{N} |a_r(t')| \le C|a_\ell(t')|, \tag{16}$$

for every $\ell \in \{\lambda_j \mid j \in \{n'+1,\dots,n\}\}$.

Remark 2.1. *We could also consider the vector fields above in the case when $n=n'$ with no condition* (15)*. Then the corresponding operator is in a subclass of that considered by Cordaro and Himonas in* [25].

We have

Theorem 2.2. [11] *Let P be a sum of squares operator as in* (12)*. Assume that the conditions **(A-1)**, **(A-2)**, **(A-3)** and **(A-4)** above are satisfied. Then P is globally analytic hypoelliptic.*

We point out that the global analog of some well-known examples belong to the class of sums of squares characterized by the above assumptions.

Example 1. Let

$$P_0 = \sum_{j=1}^{3} X_j(t, D_t, D_x)^2 = D_{t_1}^2 + a(t_1)^2 D_{t_2}^2 + b(t_1)^2 D_x^2,$$

where a, b are non identically vanishing real analytic functions defined on $\mathbb{T}^1$. Assume that $|b(t_1)| \le C|a(t_1)|$. In this case $n'=1$, $n=2$ and $m=1$. The vector fields X_j, $j=1,2,3$, satisfy the assumptions of the Theorem 2.2. Then P_0 is globally analytic hypoelliptic.

We can distinguish two cases. If b vanishes only where a vanishes, P_0 is the global version of the Oleĭnik–Radkeviç operator. We recall that the local version of the Oleĭnik–Radkeviç operator is given by $D_{t_1}^2 + t_1^{2(p-1)} D_{t_2}^2 + t_1^{2(q-1)} D_x^2$, $p, q \in \mathbb{Z}_+$, $1 < p \le q$ (for more details see [45], [14], for a generalization see [18], [20].)
If b vanishes and a does not, P_0 is a globally defined version of the Baouendi-Goulaouic operator, (3). The global analytic hypoellipticity in this case was showed for the first time by Cordaro and Himonas in 1994, [25] (see also [49] for an alternative proof.)
Example 2. Let
(17)

$$P_1 = \sum_{j=1}^{6} X_j(t, D_t, D_x)^2 = D_{t_1}^2 + a(t_1)^2 D_{t_2}^2 + b(t_1)^2 \left(D_{x_1}^2 + D_{x_2}^2\right) + c(t_1, t_2)^2 D_{x_1}^2 + d(t_1, t_2)^2 D_{x_2}^2,$$

where a, b, c and d are real analytic functions. Assume that a vanishes at the origin, then $a(t_1) = t_1^{\ell} \tilde{a}(t_1)$, where $\ell \in \mathbb{N}$, $\tilde{a}(0) \neq 0$. Assume that $b = \mathscr{O}(t_1^{\ell+r-1})$ for a certain $r \in \mathbb{N}$, $r > 1$, that $t_1^{-\ell} c(t) = \mathscr{O}(t_2^{p-1})$ and that $t_1^{-\ell} d(t) = \mathscr{O}(t_2^{q-1})$, for certain p, $q \in \mathbb{N}$, with $1 < r < p < q$. Then the operator (17) is the global analog of the operator (8). In [12] it was showed that (8) violates the local Treves conjecture, i.e. it is not locally analytic hypoelliptic at the origin even if all the Poisson-Treves strata are symplectic. From the global point of view the vectors fields X_j, $j = 1, \dots, 6$, in (17) satisfy the assumptions of the Theorem 2.2 (in this case $n' = 1$, $n = 2$ and $m = 2$.) Then P_1 is globally analytic hypoelliptic on $\mathbb{T}^4$. We point out that the operator (17) does not belong to the class studied in [25].
Example 3. Let

$$P_3 = \sum_{j=1}^{3} X_j(t, D_t, D_x)^2 = D_{t_1}^2 + a_2(t_1)^2 D_{t_2}^2 + \left(a_3(t_1) D_{t_3} + b(t) D_x\right)^2. \tag{18}$$

where a_2, a_3, b are real analytic functions in $\mathbb{T}_t^3$. Assume that $a_2 = \mathscr{O}(t_1^{p-1})$ for $t_1 \to 0$, $a_3 = \mathscr{O}(t_1^{p-1})$ for $t_1 \to 0$, and that $b = \mathscr{O}(t_1^{p-1})$ for $t_1 \to 0$, i.e. $b(t) = \mathscr{O}(t_1^{p-1}) \tilde{b}(t)$. The vector fields X_j in (18) satisfy the assumptions of the Theorem 2.2. Then P_3 is globally analytic hypoelliptic.

3. On the microlocal regularity of the analytic vectors for "sums of squares" of vector fields of Hörmander type

In the joint work with Derridj, [23], we deal with the microlocal regularity of the analytic vectors for sums of squares of vector fields.

Definition 3.1. *Let $P_m(x, D)$ be a differential operator of order m in Ω. We denote by $G^s(\Omega; P_m)$ the space of the s-Gevrey vectors of P_m in Ω. A function $u \in L^2(\Omega)$ belongs to $G^s(\Omega; P_m)$ if and only if for every compact subset K of Ω there is a constant $C_K > 0$ such that*

$$P_m^k u \in L^2(\Omega) \text{ and } \|P_m^k u\|_{L^2(K)} \leq C_K^{k+1}(k!)^{ms}, \qquad \forall\, k \in \mathbb{Z}_+.$$

When $s = 1$, we shall say that u is an analytic vector for P_m, $\mathscr{A}(\Omega, P_m)$.

We remark that $G^s(\Omega) \subset G^s(\Omega; P_m)$.

The question on the regularity of the analytic/Gevrey vectors goes back to the result of Kotake and Narasimhan, [39]. In [39] the authors showed, in the local setting, that if P_m is a linear elliptic operator in Ω, if $u \in \mathscr{A}(\Omega, P_m)$ then $u \in C^\omega(\Omega)$. The result is also true for s-Gevrey vectors of elliptic operators with s-Gevrey coefficients, $s > 1$. In [40], Métivier proved that, in general, the result obtained by Kotake and Narasimhan can not be extended to non-elliptic operators, i.e. that if P_m is not an elliptic operator with analytic coefficients, then an s-Gevrey vector for P_m is not necessarily a Gevrey function of order s. More precisely in [40] Métivier showed that the following conditions are equivalent:

i) $G^s(\Omega; P_m) \subset G^s(\Omega)$, $s > 1$;
ii) P_m is an elliptic operator on Ω.

Moreover in [40] it has been showed that if P_m is a formally self-adjoint operator on Ω, then if for $s \geq 1$ we have that $G^s(\Omega; P_m) \subset G^s(\Omega)$ then P_m is elliptic on Ω. In 1982 Baouendi and Métivier, [5], studied the regularity of the s-Gevrey vectors for the class of hypoelliptic operators of principal type with analytic coefficients, showing a difference between the case $s = 1$ and $s > 1$. These results give rise to a natural question: let P_m be a differential operator with analytic coefficients not-elliptic in Ω and u a s-Gevrey vector of P_m, what is the optimal $s'-$Gevrey regularity of u?

For a large survey on the subject until 1987 we refer to [8] and for a more recent one to [29].

We focus our attention on the case of sums of squares. Let P be as in (1) and let $X_j(x,\xi)$ be the symbol of the vector field $X_j(x,D)$. Write $\{X_i, X_k\}$ the Poisson bracket of the symbols of the vector fields X_i, X_k:

$$\{X_i, X_k\}(x,\xi) = \sum_{\ell=1}^{n} \left(\frac{\partial X_i}{\partial \xi_\ell}\frac{\partial X_k}{\partial x_\ell} - \frac{\partial X_k}{\partial \xi_\ell}\frac{\partial X_i}{\partial x_\ell} \right)(x,\xi).$$

Definition 3.2. *Let (x_0,ξ_0) be a point in the characteristic set of P:*

$$Char(P) = \{(x,\xi) \in T^*U \setminus \{0\} \,:\, X_j(x,\xi) = 0,\, j = 1, \dots m\}. \tag{19}$$

Consider all the iterated Poisson brackets $\{X_i, X_k\}$, $\{X_p, \{X_i, X_k\}\}$ etcetera. We define $\nu(x_0,\xi_0)$ as the length of the shortest iterated Poisson bracket of the symbols of the vector fields which is non zero at (x_0,ξ_0).

We recall that concerning systems of vector fields with real analytic coefficients satisfying the Hörmander condition the problem of the local regularity of the analytic vectors for such systems was first studied in [26] followed by a more refined version in [34].

In a couple of recent works ([30] and [31]) Derridj studied the problem of the local regularity for the Gevrey vectors for operators of Hörmander type of first kind, i.e. sums of squares, and of the second kind or degenerate elliptic parabolic. In a joint work with Derridj we prove the minimal microlocal version of the result in [30] in the case of analytic vectors:

Theorem 3.1. [23] *Let P be as in (1). Let u be an analytic vector for P, $u \in \mathscr{A}(U;P)$. Let (x_0,ξ_0) be a point in the characteristic set of P and $\nu(x_0,\xi_0)$ its length. Then $(x_0,\xi_0) \notin WF_{\nu(x_0,\xi_0)}(u)$.*

Here $WF_s(u)$, $s \geq 1$, denotes the Gevrey wave front set of order s of the distribution u; we recall the definition via FBI-transform.

Definition 3.3. *Let u be a compactly supported distribution on $\mathbb{R}^n$. Let $(x_0,\xi_0) \in T^*\mathbb{R}^n\backslash 0$. We say that $(x_0,\xi_0) \notin WF_s(u)$, $s \geq 1$, if there exist a neighborhood Ω of $x_0 - i\xi_0 \in \mathbb{C}^n$*

and positive constants C, ε such that

$$|e^{-\lambda\phi_0(z)}Tu(z,\lambda)| \leq Ce^{-\varepsilon\lambda^{1/s}},$$

for every $z \in \Omega$ and $\lambda > 1$. Here T denotes the FBI transform of u with classical phase function

$$Tu(z,\lambda) = \int_{\mathbb{R}^n} e^{-\frac{\lambda}{2}(z-y)^2} u(y)dy.$$

For more details on the FBI-transform see [47]. For a different but equivalent definition of the wave front set see [36] or [37].

A few remarks are in order.

i) The result obtained in Theorem 3.1 allow to recover the result obtained in the local analytic case by Derridj in [30]. Let V denote a neighborhood of the point x_0 and

$$r = \sup_{x\in V, |\xi|=1} \nu(x,\xi). \qquad \text{Then } \mathscr{A}(V;P) \subset G^r(V).$$

ii) The method used to gain the Theorem 3.1 can be extended to a class of Hörmander type operators not strictly sums of squares; we consider operators of the form $P(x,D) + \sum_{i=1}^m b_j(x)X_j(x,D) + c(x)$ where P is as in (1), $b_j(x)$ are real-valued real analytic functions and $c(x)$ is a real analytic complex function.

iii) The strategy to obtain the Theorem 3.1 can be carried over to the case of s-Gevrey vectors with $s \in \mathbb{Z}_+$;

iv) The result is optimal, as showed by the following example given in [17]: consider the periodic version of the Baouendi-Grushin operator:

$$Q_0 = \partial_t^2 + (\sin t)^2\, \partial_x^2 \qquad \text{in } \mathbb{T}^2;$$

we have that $G^s(\mathbb{T}^2; Q_0) \not\subset G^\sigma(\mathbb{T}^N)$ for every $s \geq 1$ and every $s < \sigma < 2s$. ([17] studies the problem of the regularity of the global Gevrey vector for the class of sums of squares introduced in [25] on the n-dimensional torus, see also [19] for other results on the subject.)

Idea of the proof of the Theorem 3.1. The result is obtained by taking advantage from the result in [2], where the authors give, via FBI technique, the microlocal version of the result obtained by Derridj and Zuily in [28] concerning the local Gevrey regularity of the solutions to the problem $Pu = f$, P as in (1) and f analytic function in Ω. We use the method of adding an extra variable. The method essentially allows to relate the study of the regularity of the analytic vectors of the operator $P(x, D)$, in Ω, to the study of the regularity of the solutions of a new operator $Q(t, x, D_t, D) = \sum_{j=0}^{m} X_j^2 = D_t^2 + P(x, D)$, obtained by adding the variable $t \in \mathbb{R}$, in the open set $\tilde{\mathcal{O}} =]-\delta_0, \delta_0[\, \times \mathcal{O} \in \mathbb{R}^{1+n}$, $\delta_0 > 0$. We point out that the new operator Q satisfies the Hörmander condition. So, we study the microlocal properties of the solutions of the problem $Qv = f$, $f \in C^\omega(\tilde{\mathcal{O}})$. We denote by $\tilde{\Sigma}$ and Σ the characteristic set of Q and P respectively. We remark that the point $(t_0, x_0, 0, \xi_0) \in \tilde{\Sigma}$ have the same length as that of the point $(x_0, \xi_0) \in \Sigma$. Let ν be the length of the point $(0, x_0, 0, \xi_0) \in \tilde{\Sigma}$. Taking advantage from the result in [2], we have $(0, x_0, 0, \xi_0) \notin WF_\nu(v)$. Now we consider the problem

$$\begin{cases} (D_t^2 + P(x, D))\, U(t, x) = 0, \\ U(0, x) = u(x), \end{cases}$$

in $\tilde{\mathscr{O}} =]-\delta_0, \delta_0[\, \times \mathscr{O}$, $\delta_0 > 0$, where $u(x)$ is an analytic vector for $P(x, D)$: $\|P^k u\|_0 \leq C^{2k+1}(2k)!$. The function

$$U(t, x) = \sum_{k \geq 0} \frac{t^{2k}}{2k!} P^k u(x)$$

is a solution of the above problem. We choose $\delta_0 < \sqrt{2}C$.

The result in the Theorem 3.1 is obtained showing, via FBI transform, that if u and U are as above, then $(0, x_0, 0, \xi_0) \notin WF_s\,(U)$ if and only if $(x_0, \xi_0) \notin WF_s(u)$, $s \geq 1$. This last statement generalized a result obtained in 1990 by Bolley, Camus and Métivier, [9]; they show it in the analytic case, $s = 1$, via Fourier transform.

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