Generalized Symmetries, First Integrals, and Exact Solutions of Chains of Differential Equations

C Muriel[1] and M C Nucci[2]

[1] *Departamento de Matemáticas, Universidad de Cádiz, 11510 Puerto Real, Spain*

[2] *Dept. Math. Inf. Phys. Earth Sci. (MIFT), University of Messina, 98166 Messina, & INFN, Section of Perugia, 06123 Perugia, Italy*

Abstract

New integrability properties of a family of sequences of ordinary differential equations, which contains the Riccati and Abel chains as the most simple sequences, are studied. The determination of n generalized symmetries of the nth-order equation in each chain provides, without any kind of integration, $n-1$ functionally independent first integrals of the equation. A remaining first integral arises by a quadrature by using a Jacobi last multiplier that is expressed in terms of the preceding equation in the corresponding sequence. The complete set of n first integrals is used to obtain the exact general solution of the nth-order equation of each sequence. The results are applied to derive directly the exact general solution of any equation in the Riccati and Abel chains.

1 Introduction

For a given smooth function $g = g(u)$ defined on some open interval $J \subset \mathbb{R}$, let us define the differential operator

$$\mathbb{D}_g = \mathbf{D}_t + g(u), \tag{1}$$

where $\mathbf{D}_t$ denotes the total derivative operator

$$\mathbf{D}_t = \partial_t + u_1 \partial_u + \cdots + u_k \partial_{u_{k-1}} + \cdots, \tag{2}$$

and $u_i = \dfrac{d^i u}{dt^i}$, for $i \in \mathbb{N}$.

The differential operator $\mathbb{D}_g$ acts on the set of smooth functions $u = u(t)$, defined in some open interval $I \subset \mathbb{R}$ such that $u(t) \in J$ for $t \in I$. We set $\mathbb{D}_g^0 u = u$ and, for $j \geq 1$, $\mathbb{D}_g^j(u) = \mathbb{D}_g(\mathbb{D}_g^{j-1}(u))$.

We consider the sequence of ordinary differential equations (ODEs)

$$\mathbb{D}_g^n u = 0, \qquad (n \in \mathbb{N}) \tag{3}$$

obtained by applying successively the differential operators in the sequence $\{\mathbb{D}_g^n\}_{n \in \mathbb{N}}$ to an unknown function $u = u(t)$. In what follows the sequence of ODEs $\mathbb{E}_g := \{\mathbb{D}_g^n(u) = 0\}_{n \in \mathbb{N}}$ will be called *the chain generated by the function g*.

Recursion operators firstly appeared in the context of evolution equations in two independent variables [18] and are traditionally applied to partial differential equations. M Euler et al. [6] adapted them to ordinary differential equations, by considering a 1+1 evolution equation and a known recursion operator free of t. One the most well known sequences of the form (3) is the chain generated by the function $g(u) = ku$, where $k \in \mathbb{R}$, which is known in the literature as the Riccati chain of parameter $k \in \mathbb{R}$ and whose elements are usually called the higher Riccati equations [2, 4, 6, 9]. Riccati sequence arises from a known recursion operator of a second-order evolution equation (Class VIII in [8]) linearizable by a x-generalized hodograph transformation [7]. The Riccati sequence exhibits interesting properties from both physical and mathematical point of view, and a large variety of methods (Darboux factors, Jacobi multipliers, extended Prelle-Singer methods, nonlocal symmetries, etc.) have been applied to its study [1, 2, 4].

For different functions $g = g(u)$ the differential operator (1) generates new sequences $\mathbb{E}_g$, some of whose properties have been obtained in [13]. In particular, it was proved the existence of a common $\mathcal{C}^\infty$−symmetry [10] for all the equations of the sequence $\mathbb{E}_g$: the pair $(\mathbf{v}, \lambda)$, where

$$\mathbf{v} = \partial_u \quad \text{and} \quad \lambda = \frac{u_1}{u} - u\, g'(u) \tag{4}$$

defines a $\mathcal{C}^\infty$−symmetry of the nth-order equation $\mathbb{D}_g^n u = 0$, for $n \in \mathbb{N}$ [13, Theorem 2.1]. This $\mathcal{C}^\infty$−symmetry was used to connect any sequence $\mathbb{E}_g$ with the Riccati chain of parameter $k \in \mathbb{R}$.

In this paper we aim to investigate some integrability properties of the class of chains generated by functions of the form $g(u) = ku^m$, where $k \in \mathbb{R}$ and $m \in \mathbb{Z}$. The corresponding elements of this type of sequences will be denoted by $P_n = 0$, where

$$P_0 := u, \quad P_n := (\mathbf{D}_t + ku^m)^n(u), \qquad n = 1, 2, \ldots. \tag{5}$$

It should be observed that the Riccati chain is contained in the study, as well as the chain generated by the function $g(u) = ku^2$, $u \in \mathbb{R}$, which is known as the Abel chain of parameter $k \in \mathbb{R}$ and that has also been extensively studied in the literature [3, 5]. Both chains contain as particular cases well-known families of equations in Mathematical Physics, such as Emden equations, generalized Van-der Pol oscillators, Chazy equations, etc.

For the purposes of this paper, a notable property of the nth-order equation $P_n = 0$ is the existence of a Jacobi last multiplier [16] that can be expressed in terms of the previous element of the sequence as follows [13, Theorem 3.3]:

$$M_n := (P_{n-1})^{-(n+m)}. \tag{6}$$

Both elements, the Jacobi last multiplier (6) and the common $\mathcal{C}^\infty$−symmetry (4), will be exploited in the study performed in this paper, which is organized as follows. In Section 2,

we firstly use the corresponding common $\mathcal{C}^\infty-$symmetry (4) to derive n generalized symmetries of the nth-order equation $P_n = 0$. In Section 3 it is proved that these generalized symmetries have the property that any ratio of their characteristics is a first integral of the equation $P_n = 0$; furthermore, $n-1$ of such first integrals are functionally independent (Theorem 2). The Jacobi last multiplier (6) is used in Section 4 to compute a remaining first integral, which constitutes, together with the $n-1$ previously determined, a complete set of first integrals of $P_n = 0$. As a consequence, we obtain explicitly the general solution of all the equations in any of the chains of the family. The results are applied to study the Riccati and Abel chains in sections 5 and 6, respectively. Additionally, we prove that for each equation in the Riccati chain it is possible to derive an additional generalized symmetry. As a consequence, a complete set of first integral for the nth-order equation in the Riccati chain is obtained without any kind of integration.

Remarkably, the unified procedure presented in this paper is valid for all the chains generated by $g(u) = ku^m,$, providing a common expression, depending on m, for the exact solutions of all the equations in any chain. This greatly improves the procedure presented in [13], which required an additional integration to derive such solutions from the solutions of the higher Riccati equations. Moreover, our results are valid for real values of the parameter m (Section 7), enlarging significantly the classes of ODEs which can be completely solved by this new procedure, without any kind of integration.

2 Derivation of n generalized symmetries by using a $\mathcal{C}^\infty-$symmetry

In this section we address the problem of determining n generalized symmetries of the nth-order equation:

$$P_n := (\mathbf{D}_t + ku^m)^n(u) = 0, \qquad k \in \mathbb{R}, \quad m \in \mathbb{Z}, \tag{7}$$

by using the corresponding $\mathcal{C}^\infty-$symmetry (4), which for $g(u) = ku^m$ becomes

$$\mathbf{v} = \partial_u \quad \text{and} \quad \lambda = \frac{u_1}{u} - k\,m\,u^m. \tag{8}$$

We recall that the concept of $\mathcal{C}^\infty-$symmetry arises naturally from the concept of Lie point symmetry when $\lambda-$prolongations are considered instead of standard prolongations: for any vector field $\mathbf{v} = \xi(t,u)\partial_t + \eta(t,u)\partial_u$ and a smooth function $\lambda = \lambda(t,u,u_1)$, the jth$-$order $\lambda-$prolongation of $\mathbf{v}$ is denoted by $\mathbf{v}^{[\lambda,(j)]}$ and defined as

$$\mathbf{v}^{[\lambda,(j)]} := \mathbf{v} + \eta^{[\lambda,(1)]}\big(t,u^{(1)}\big)\partial_{u_1} + \cdots + \eta^{[\lambda,(j)]}\big(t,u^{(j)}\big)\partial_{u_j}, \tag{9}$$

where

$$\eta^{[\lambda,(0)]} = \eta, \qquad \eta^{[\lambda,(i)]} = (\mathbf{D}_t + \lambda)\big(\eta^{[\lambda,(i-1)]}\big) - u_i(\mathbf{D}_t + \lambda)\big(\xi\big), \tag{10}$$

for $i = 1,\ldots,j$. The pair $(\mathbf{v},\lambda)$ defines a $\mathcal{C}^\infty-$symmetry (or $\mathbf{v}$ is a $\lambda-$symmetry) of equation (7) if and only if [10]

$$\mathbf{v}^{[\lambda,(n)]}\big(P_n\big) = 0 \quad \text{when} \quad P_n = 0. \tag{11}$$

Equivalently, $(\mathbf{v}, \lambda)$ is a $\mathcal{C}^\infty-$symmetry of (7) if and only if [10]

$$[\mathbf{v}^{[\lambda,(n-1)]}, \mathbf{A}_n] = \lambda \mathbf{v}^{[\lambda,(n-1)]} - (\mathbf{A}_n + \lambda)(\xi)\mathbf{A}_n, \tag{12}$$

where now, and henceforth, $\mathbf{A}_n$ denotes the restriction of $\mathbf{D}_t$ to the manifold defined by equation (7).

This primitive concept of $\mathcal{C}^\infty-$symmetry was later extended to permit the function λ or the infinitesimals ξ, η of $\mathbf{v}$ belong to the space of smooth functions on t, u and the derivatives of u with respect to t up to some finite but unspecified order [11, Def. 2.1], [12, Sect. 2.3]. In these cases the pair $(\mathbf{v}, \lambda)$ is called a generalized $\lambda-$symmetry (or a generalized $\mathcal{C}^\infty-$symmetry). Clearly, when $\lambda = 0$, (9) is the standard prolongation of $\mathbf{v}$ and therefore the pair $(\mathbf{v}, 0)$ is a (generalized) $\mathcal{C}^\infty-$symmetry if and only if $\mathbf{v}$ is a (generalized) Lie symmetry.

A remarkable property of $\mathcal{C}^\infty-$symmetries for the purpose of this section is the following [11]: if $(\mathbf{v}, \lambda)$ is a $\mathcal{C}^\infty-$symmetry of equation (7) and $\rho = \rho(t, u^{(i-1)})$, $i \le n$, is any given smooth function, then the pair $(\rho\,\mathbf{v}, \bar{\lambda})$ is a generalized $\mathcal{C}^\infty-$symmetry of equation (7) for the function

$$\bar{\lambda} = \lambda - \frac{\mathbf{A}_n(\rho)}{\rho}.$$

In particular, when $\bar{\lambda} = 0$, i.e., if the function ρ verifies $\dfrac{\mathbf{A}_n(\rho)}{\rho} = \lambda$, then $\rho\mathbf{v}$ is a generalized symmetry.

Applying this result to the $\mathcal{C}^\infty-$symmetry (8) of equation (7), we have that if a function $\rho = \rho(t, u^{(n-1)})$ verifies

$$\frac{\mathbf{A}_n(\rho)}{\rho} = \lambda = \frac{u_1}{u} - k\,m\,u^m, \tag{13}$$

then the generalized vector field $\rho(x, u^{(n-1)})\,\partial_u$ becomes a generalized symmetry of the equation (7).

In the next theorem we provide n functions satisfying (13) which are used to construct n generalized symmetries of equation (7):

Theorem 1. Let $P_0 := u$ and $P_n := (\mathbf{D}_t + ku^m)(P_{n-1})$ for $n \ge 1$. For $1 \le i \le n$, the vector fields

$$\mathbf{w}_{n,i} := u(P_{n-1})^{m-1}\left(\sum_{j=1}^{i}\frac{(-1)^{j+1}}{(i-j)!}t^{i-j}P_{n-j}\right)\partial_u, \qquad i = 1, \ldots, n \tag{14}$$

define n generalized symmetries of the nth-order equation $P_n = 0$.

Proof. For $i = 1, 2, \cdots, n$, let $\rho_{n,i}$ be the functions defined by

$$\rho_{n,i} := u(P_{n-1})^{m-1}H_i, \quad \text{where} \quad H_i = \sum_{j=1}^{i}\frac{(-1)^{j+1}}{(i-j)!}t^{i-j}P_{n-j}. \tag{15}$$

By using the identities

$$\mathbf{A}_n(P_{i-1}) = P_i - ku^m P_{i-1} \quad \text{for} \quad i = 1, 2, \ldots, n-1,$$
$$\mathbf{A}_n(P_{n-1}) = -ku^m P_{n-1},$$

(16)

which follow immediately from (5), a straightforward calculation leads to

$$\frac{\mathbf{A}_n(\rho_{n,i})}{\rho_{n,i}} = \frac{\mathbf{A}_n(u)}{u} + \frac{\mathbf{A}_n\big((P_{n-1})^{m-1}\big)}{(P_{n-1})^{m-1}} + \frac{\mathbf{A}_n(H_i)}{H_i}$$
$$= \frac{u_1}{u} - k(m-1)u^m - ku^m = \lambda.$$

(17)

This shows that the functions given in (15) satisfy (13). The theorem follows immediately from the discussion included at the beginning of this section. ∎

Remark 1. It can be easily checked that the functions given in (15) satisfy $\partial_t(\rho_{n,i}) = \rho_{n,i-1}$, for $1 < i \leq n$. Therefore, the $n-1$ generalized symmetries $\mathbf{w}_{n,i} = \rho_{n,i}\partial_u$, for $1 \leq i \leq n-1$, can be directly determined from $\mathbf{w}_{n,n} = \rho_{n,n}\partial_u$ by using successive derivations with respect to t, because $\partial_t^i(\rho_{n,n}) = \rho_{n,n-i}$.

3 First integrals derived from the generalized symmetries (14)

Let ρ_1 and ρ_2 be any functions satisfying (13). Then

$$\frac{\mathbf{A}_n\left(\rho_1/\rho_2\right)}{\rho_1/\rho_2} = \frac{\mathbf{A}_n(\rho_1)}{\rho_1} - \frac{\mathbf{A}_n(\rho_2)}{\rho_2} = 0.$$

(18)

This implies that ρ_1/ρ_2 is a first integral for the nth-order equation (7). As a consequence of Theorem 1, it follows that the functions

$$I_{(n;i,j)} := \frac{\rho_{n,i}}{\rho_{n,j}} = \frac{H_i}{H_j}, \quad i, j \in \{1, 2, \ldots, n\}$$

(19)

are first integrals of equation (7). In order to construct a complete set of first integrals, we can consider, for instance, the $n-1$ first integrals:

$$I_{(n;i)} := \frac{\rho_{n,i}}{\rho_{n,1}} = \sum_{r=1}^{i} \frac{(-1)^{r+1}}{(i-r)!} t^{i-r} Q_{n-r} \quad (2 \leq i \leq n),$$

(20)

where $Q_j := P_j/P_{n-1}$, for $0 \leq j \leq n-1$.

We observe that the functions (20) satisfy the relation $\partial_t(I_{(n;i)}) = I_{(n;i-1)}$, for $2 < i \leq n$; therefore the first integrals $I_{(n,2)}, \ldots, I_{(n,n-1)}$ can be generated by successive derivations with respect to t of $I_{(n;n)}$ as follows:

$$I_{(n;j)} = \partial_t^{n-j}(I_{(n;n)}) \quad (2 \leq j \leq n-1).$$

(21)

In the next theorem we prove that the $n-1$ first integrals (20) are functionally independent:

Theorem 2. For $n \geq 2$, the functions defined by (20) are $n-1$ functionally independent first integrals of the nth-order equation (7).

Proof. Let

$$\mathcal{M} = \frac{\partial(I_{(n;2)}, \ldots, I_{(n;n)})}{\partial(t, u, u_1, \ldots, u_{n-1})}. \tag{22}$$

denote the Jacobian matrix associated to the functions (20). Let $\bar{\mathcal{M}}$ be the square sub-matrix of $\mathcal{M}$ formed by its last $n-1$ columns. Our goal is to check that $\det(\bar{\mathcal{M}}) \neq 0$, in order to prove that the rank of the Jacobian matrix $\mathcal{M}$ is $n-1$.

The elements of the last row R_{n-1} of $\bar{\mathcal{M}}$ can be written in the form $\partial_{u_j}(I_{(n;n)}) = a_{j;n-2}t^{n-2} + \cdots + a_{j;0}$, for $1 \leq j \leq n-1$, where

$$a_{j;n-l} = \frac{(-1)^{l-1}}{(n-l)!} \partial_{u_j}(Q_{n-l}), \quad \text{for} \quad 2 \leq l \leq n.$$

According to (21), the elements of the row R_{n-2} are obtained by deriving the elements of R_{n-1} with respect to t and so on. Consequently, for $1 \leq i \leq n-1$, the elements of the row R_i of $\bar{\mathcal{M}}$ are polynomials on t of degree $i-1$, and their respective constant terms, which will be used later, become

$$a_{j;n-i-1} = (-1)^i \partial_{u_j}(Q_{n-i-1}), \quad \text{for} \quad 1 \leq j \leq n-1. \tag{23}$$

In order to cancel out the leading terms of the polynomials of the row R_i of $\bar{\mathcal{M}}$ for $2 \leq i \leq n-1$, we replace R_i by

$$R_i^{(1)} = R_i - \frac{t}{i-1} R_{i-1}, \tag{24}$$

and denote the resulting matrix by $\bar{\mathcal{M}}^{(1)}$. Clearly $\det(\bar{\mathcal{M}}^{(1)}) = \det(\bar{\mathcal{M}})$, and it is easy to check that all the elements of the new row $R_i^{(1)}$, for $i \geq 2$, are polynomials on t of degree $i-2$ and whose respective constant terms remain as in (23).

This process can be successively repeated until getting a matrix $\bar{\mathcal{M}}^{(n-2)}$ whose elements do not depend on t, and are given by (23):

$$\bar{\mathcal{M}}^{(n-2)} = \left((-1)^i \partial_{u_j}(Q_{n-i-1}) \right), \quad \text{for} \quad 1 \leq i, j \leq n-1. \tag{25}$$

Let C_j denote the columns of the matrix $\left(\partial_{u_j}(Q_{n-i-1}) \right)$, for $1 \leq i, j \leq n-1$. By replacing the column C_j by C_j' where

$$\begin{aligned} C_j' &= P_{n-1}\left(C_j - \partial_{u_j}(P_{n-1})C_{n-1} \right), \quad (1 \leq j < n-1, n > 2), \\ C_{n-1}' &= -P_{n-1}^2 C_{n-1} \end{aligned} \tag{26}$$

it can be checked that matrix $\left(\partial_{u_j}(Q_{n-i-1}) \right)$ is transformed into

$$\begin{pmatrix} \partial_{u_1}(P_{n-2}) & \cdots & 1 & P_{n-2} \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & P_1 \\ 0 & \cdots & 0 & P_0 \end{pmatrix}. \tag{27}$$

The determinant of matrix (27) is P_0 for $n = 2, 3$, and $(-1)^{k+1}P_0$ where $k > 1$ is such that $n = 2k$ or $n = 2k + 1$. By taking (25) and (26) into account we finally deduce that

$$\delta_n := \det(\bar{\mathcal{M}}) = \frac{P_0}{(P_{n-1})^n} \neq 0. \tag{28}$$

Therefore the rank of matrix $\mathcal{M}$ is $n - 1$, which proves the theorem. ∎

4   Complete integrability by using Jacobi last multipliers

In Theorem 2, $n - 1$ functionally independent first integrals of the nth-order equation $P_n = 0$ have been obtained. It is clear that any other first integral of the form (19) is functionally dependent of the mentioned $n-1$ first integrals, because $I_{(n;k,l)} = I_{(n;k)}/I_{(n;l)}$, for $1 \leq l, k \leq n$. In order to integrate completely equation (7), in this section we discuss how the Jacobi last multiplier (6) can be used to find a remaining first integral. With this objective we follow the next steps:

1. As a consequence of Theorem 2 the set

$$\Delta = \{(t, u^{(n-1)}) \in M^{(n-1)} : I_{(n;j)}(t, u^{(n-1)}) = c_{(n;j)}, 2 \leq j \leq n\}, \tag{29}$$

where the $c_{(n;j)}$ are constants, is a submanifold of $M^{(n-1)}$ of dimension 2. It follows from (28) and the Implicit Function Theorem that Δ can be locally parametrized by (t, u) as follows:

$$\{(t, u^{(n-1)}) \in M^{(n-1)} : u_j = G_{(n;j)}(t, u, c_{(n;2)}, \ldots, c_{(n;n)}), 2 \leq j \leq n - 2\},$$

for some smooth functions $G_{(n;j)}$. The first order ODE

$$u_1 = G_{(n;1)}(t, u, c_{(n;2)}, \ldots, c_{(n;n)}) \tag{30}$$

will be called the auxiliary equation. Next we deduce the explicit expression of this auxiliary equation.

The equations $I_{(n;i)}(t, u^{(n-1)}) = c_{(n;i)}$, for $2 \leq i \leq n - 1$, can be written in matrix form as follows

$$\mathcal{N}_{n-1}\begin{pmatrix} P_{n-1} \\ P_{n-2} \\ \vdots \\ P_2 \\ P_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (-1)^n P_0 \end{pmatrix} \tag{31}$$

where

$$\mathcal{N}_{n-1} = \begin{pmatrix} t - c_{(n;2)} & -1 & \cdots & 0 \\ \dfrac{t^2}{2!} - c_{(n;3)} & -t & \cdots & 0 \\ \dfrac{t^3}{3!} - c_{(n;4)} & -\dfrac{t^2}{2!} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{t^{n-1}}{(n-1)!} - c_{(n;n)} & -\dfrac{t^{n-2}}{(n-2)!} & \cdots & (-1)^n t \end{pmatrix}, (n \geq 2).$$

It can be checked that $\det(\mathcal{N}_{n-1})$ is a polynomial of t of degree $n-1$:

$$S_{n-1}(t) := \det(\mathcal{N}_{n-1}) = \sigma_n\Big(\frac{t^{n-1}}{(n-1)!} + \sum_{r=2}^{n}\frac{(-1)^{r+1}}{(n-r)!}t^{n-r}c_{(n;r)}\Big), \tag{32}$$

where

$$\sigma_n = \begin{cases} (-1)^{k+1} & \text{for} \quad n = 2k, \\ (-1)^{k} & \text{for} \quad n = 2k+1. \end{cases} \tag{33}$$

The following relation between the determinants of $\mathcal{N}_{n-1}$ and $\mathcal{N}_{n-2}$ can be easily proved:

$$S'_{n-1}(t) = (-1)^n S_{n-2}(t). \tag{34}$$

It follows from (31) and (34) that

$$P_1\Big|_\Delta = (-1)^n P_0\frac{\det(\mathcal{N}_{n-2})}{\det(\mathcal{N}_{n-1})} = P_0\frac{S'_{n-1}(t)}{S_{n-1}(t)}.$$

Therefore the auxiliary equation (30) becomes

$$u_1 = -ku^{m+1} + u\frac{S'_{n-1}(t)}{S_{n-1}(t)}. \tag{35}$$

This is a Bernoulli equation and it can be solved by a well-known procedure. Nevertheless, in what follows, we provide explicitly its general solution by determining an integrating factor and an associated first integral.

2. According to [14, Corollary 6], an integrating factor of the auxiliary equation (30) is the function $\mu = \mu(t, u, c_{(n;2)}, \cdots, c_{(n;n)})$ defined by

$$\mu := \frac{(P_{n-1})^{-(n+m)}}{\delta_n}\Big|_\Delta = \frac{(P_{n-1})^{-m}}{P_0}\Big|_\Delta, \tag{36}$$

where δ_n has been defined in (28). The restriction of P_{n-1} to Δ can be calculated from (31) as

$$P_{n-1}\Big|_\Delta = \frac{P_0}{\det(\mathcal{N}_{n-1})} = \frac{P_0}{S_{n-1}(t)}.$$

Therefore, the integrating factor (36) of equation (35) becomes

$$\mu = u^{-(m+1)}\big(S_{n-1}(t)\big)^m.$$

3. A corresponding first integral $H = H(t, u, c_{(n;2)}, \cdots, c_{(n;n)})$ of the exact first order equation $\mu(u_1 - G_{(n;1)}) = 0$ can be calculated by a single quadrature and becomes:

$$H := u^{-m}\big(S_{n-1}(t)\big)^m - km\int \big(S_{n-1}(t)\big)^m\, dt. \tag{37}$$

Consequently, the function $I_{(n;1)} = I_{(n;1)}(t, u^{(n-1)})$ defined by

$$I_{(n;1)} = H(t, u, I_{(n;2)}, \ldots, I_{(n;n)}) \tag{38}$$

is a first integral of the equation $P_n = 0$. Furthermore, the functions in $\{I_{(n;j)}\}_{j=1}^n$ are n functionally independent first integrals of the equation $P_n = 0$.

4. By setting $H = c_{(n;1)}$ in (37), where $c_{(n;1)} \in \mathbb{R}$, the general solution of the equation $P_n = 0$ can be expressed as

$$u(t)^m = \frac{(S_{n-1}(t))^m}{km \int (S_{n-1}(t))^m \, dt + c_{(n;1)}}. \tag{39}$$

Equation (39) provides a formula to solve directly the nth-order equation $P_n = 0$ by using the polynomials (32). These polynomials are the same for all the chains, because they are independent of the constants k and m. In order to simplify the expression of the polynomials (32) we set

$$C_1 := \sigma_n (n-1)! c_{(n;1)},$$
$$C_r := (-1)^{r+1} \sigma_n \frac{(n-1)!}{(n-r)!} c_{(n;r)}, \quad \text{for} \quad 2 \le r \le n,$$

which permits to express the general solution of (7) in terms of arbitrary polynomials of order $n-1$, as it is stated in the next theorem for further reference:

Theorem 3. Let $P_0 := u$ and $P_n := (\mathbf{D}_t + ku^m)(P_{n-1})$ for $n \ge 1$. The general solution of the equation $P_n = 0$ is

$$u(t)^m = \frac{(T_{n-1}(t))^m}{km \int (T_{n-1}(t))^m \, dt + C_1}, \tag{40}$$

where

$$T_{n-1}(t) = t^{n-1} + C_2 t^{n-1} + \cdots + C_{n-1} t + C_n, \tag{41}$$

and $C_i \in \mathbb{R}$ for $1 \le r \le n$.

In the following two sections we apply these results to the Riccati and Abel chains.

5   The Riccati chain

The chain (3) generated by the function g given by $g(u) = ku$, $u \in \mathbb{R}$, is known as the Riccati chain of parameter $k \in \mathbb{R}$. We set

$$R_0 := u \quad \text{and} \quad R_i := (\mathbf{D}_t + ku)(R_{i-1}), \quad i \ge 1. \tag{42}$$

The equations in the Riccati chain are usually known as the higher-order Riccati equations. The first four terms of this sequence define the ODEs displayed in Table 1:

Table 1. Four first equations in the Riccati chain

n	$R_n = 0$
1	$u_1 + ku^2 = 0$
2	$u_2 + k^2 u^3 + 3kuu_1 = 0$
3	$u_3 + k\left(k^2 u^4 + 6ku^2 u_1 + 4uu_2 + 3u_1^2\right) = 0$
4	$u_4 + k\left(k^3 u^5 + 10k^2 u^3 u_1 + 10ku^2 u_2 + 5u\left(3ku_1^2 + u_3\right) + 10u_1 u_2\right) = 0$

5.1 Generalized symmetries and first integrals for the Riccati chain

Theorem 1 can be used to determine n generalized symmetries of the nth-order equation in the considered chain. In this section we derive an additional generalized symmetry for the nth-order equation of the Riccati chain. The term

$$R_{-1} := \frac{1}{k} \tag{43}$$

satisfies

$$(\mathbf{D}_t + ku)(R_{-1}) = R_0, \tag{44}$$

which corresponds to (16) for $i = 0$. This suggests that the term (43) can be used to derive a new generalized symmetry for the Riccati chain:

Theorem 4. For $i \geq -1$, let R_i be the functions given in (42) and (43). The vector field

$$\mathbf{w}_{\mathbf{R}n,n+1} := u\left(\sum_{j=1}^{n+1} \frac{(-1)^{j+1}}{(n+1-j)!} t^{n+1-j} R_{n-j}\right) \partial_u \tag{45}$$

is a generalized symmetry of the nth-order equation $R_n = 0$ of the Riccati chain, for $n \geq 1$.

Proof. By taking into account that $P_n = R_n$ for $m = 1$, the vector field (45) can be written in the form for $\rho_{n,n+1} \partial_u$ where $\rho_{n,n+1}$ is the function defined in (15) but extended to $i = n + 1$:

$$\rho_{n,n+1} := u(P_{n-1})^{m-1} \sum_{j=1}^{n+1} \frac{(-1)^{j+1}}{(n+1-j)!} t^{n+1-j} P_{n-j} \quad (m = 1). \tag{46}$$

By using (44) and the corresponding identities (16):

$$\begin{aligned}
\mathbf{A}_n(R_{i-1}) &= R_i - kuR_{i-1} \quad \text{for} \quad i = 0, 1, 2, \ldots, n-1, \\
\mathbf{A}_n(R_{n-1}) &= -kuR_{n-1},
\end{aligned} \tag{47}$$

it can be proved, as in the proof of Theorem 1, that function (46) satisfies the corresponding condition (17) for $m = 1$:

$$\frac{\mathbf{A}_n(\rho_{n,n+1})}{\rho_{n,n+1}} = \frac{u_1}{u} - ku = \lambda. \tag{48}$$

The result follows immediately from the discussion at the beginning of Section 2. ∎

The additional generalized symmetry (45) for the nth-order equation of the Riccati chain permits to find, without any kind of additional integration, a new first integral, that together with the functionally independent first integrals derived in Theorem 2, constitutes a complete set of first integrals for $R_n = 0$.

Corollary 1. The function

$$I_{(n;n+1)} := \frac{\rho_{n,n+1}}{\rho_{n,1}} = \frac{t^n}{n!} - \frac{t^{n-1}}{(n-1)!}\frac{R_{n-2}}{R_{n-1}} + \cdots + (-1)^{n+2}\frac{R_{-1}}{R_{n-1}} \tag{49}$$

is a first integral of the nth-order equation $R_n = 0$ of the Riccati chain. Moreover, (49) and the functions given by (20) (for $m = 1$) constitutes a complete set of first integrals of $R_n = 0$.

Proof. According to (18), the function (49) is a first integral of $R_n = 0$ because $\rho_{n,n+1}$ and $\rho_{n,1}$ satisfy (13) (see Equations (17) and (48)).

In order to prove the functional independence of the first integrals $I_{(n;j)}$, for $1 \leq j \leq n+1$, let

$$\mathcal{M}_R = \frac{\partial(I_{(n;2)}, \cdots, I_{(n;n+1)})}{\partial(t, u, u_1, \cdots, u_{n-1})} \tag{50}$$

denote associated Jacobian matrix. Let $\bar{\mathcal{M}}_R$ be the square submatrix of $\mathcal{M}_R$ formed by its last n columns. As in the proof of Theorem 2, it can be checked that $\det(\bar{\mathcal{M}}_R) = \dfrac{R_{-1}}{(R_{n-1})^{n+1}} \neq 0$, which proves that the rank of the Jacobian matrix $\mathcal{M}_R$ is n. Therefore $\{I_{(n;2)}, \ldots, I_{(n;n+1)}\}$ are n functionally independent first integrals of $R_n = 0$.
∎

Remark 2. It can be easily checked that the functions given in (15) and (46) satisfy $\partial_t(\rho_{n,i}) = \rho_{n,i-1}$ for $1 \leq i \leq n+1$. Therefore the n generalized symmetries of $R_n = 0$ derived in Theorem 1 (for $m = 1$) can be directly determined from (46) by using successive derivations with respect to t. For instance, the corresponding generalized symmetries (14) and (45) for the fourth-order equation in the Riccati chain become

$$\mathbf{w_{R(4,5)}} = u\left(R_3\frac{t^4}{4!} - R_2\frac{t^3}{3!} + R_1\frac{t^2}{2} - R_0 t + R_{-1}\right)\partial_u,$$

$$\mathbf{w_{R(4,4)}} = u\left(R_3\frac{t^3}{3!} - R_2\frac{t^2}{2} + R_1 t - R_0\right)\partial_u,$$

$$\mathbf{w_{R(4,3)}} = u\left(R_3\frac{t^2}{2} - R_2 t + R_1\right)\partial_u, \tag{51}$$

$$\mathbf{w_{R(4,2)}} = u\left(R_3 t - R_2\right)\partial_u,$$

$$\mathbf{w_{R(4,1)}} = u R_3\partial_u.$$

Similarly, it follows from (21) that a complete set of first integrals of $R_n = 0$ can be easily computed from the first integral (49) by successive derivations with respect to t.

For instance, the corresponding first integrals (20) and (49) for the fourth-order equation in the Riccati chain become

$$
\begin{aligned}
I_{(4,5)} &= \frac{t^4}{4!} - \frac{R_2}{R_3}\frac{t^3}{3!} + \frac{R_1}{R_3}\frac{t^2}{2} - \frac{R_0}{R_3}t + \frac{R_{-1}}{R_3}, \\
I_{(4,4)} &= \frac{t^3}{3!} - \frac{R_2}{R_3}\frac{t^2}{2} + \frac{R_1}{R_3}t - \frac{R_0}{R_3}, \\
I_{(4,3)} &= \frac{t^2}{2} - \frac{R_2}{R_3}t + \frac{R_1}{R_3}, \\
I_{(4,2)} &= t - \frac{R_2}{R_3}.
\end{aligned}
\tag{52}
$$

5.2 General solutions to the equations of the Riccati chain

For the Riccati chain (i.e., for $m = 1$), the corresponding formula (40) becomes

$$
u(t) = \frac{T_{n-1}(t)}{k \displaystyle\int T_{n-1}(t)dt + C_1},
\tag{53}
$$

which provides directly the general solutions of $R_n = 0$, by using the polynomials given in (41). In the next table we give the general solutions for the four first equations in the Riccati chain (see Table 1), and where the C_i are arbitrary real constants:

Table 2. General solutions for the four first equations in the Riccati chain

n	General Solution of $R_n = 0$
1	$u(t) = \dfrac{1}{kt + C_1}$
2	$u(t) = \dfrac{2(t + C_2)}{k(t^2 + 2C_2 t) + 2C_1}$
3	$u(t) = \dfrac{3!(t^2 + C_2 t + C_3)}{k(t^3 + 3C_2 t^2 + 6C_3 t) + 6C_1}$
4	$u(t) = \dfrac{4!(t^3 + C_2 t^2 + C_3 t + C_4)}{k(t^4 + 4C_2 t^3 + 6C_3 t^2 + 12C_4 t) + 12C_1}$

6 The Abel chain

The chain (3) generated by the function g given by $g(u) = ku^2$, $u \in \mathbb{R}$, is known as the Abel chain of parameter $k \in \mathbb{R}$. We set

$$
A_0 := u \quad \text{and} \quad A_i := (\mathbf{D}_t + ku^2)(A_{i-1}), \quad i \geq 1.
\tag{54}
$$

The first three terms of the Abel chain are the following ODEs:

Table 3. Three first equations in the Abel chain

n	$A_n = 0$
1	$u_1 + ku^2 = 0$
2	$u_2 + k^2u^5 + 4ku_1u^2 = 0$
3	$u_3 + k(k^2u^4 + 6ku^2u_1 + 4uu_2 + 3u_1^2) = 0$

For the Abel chain, the parameter m is $m = 2$ and the corresponding formula (40) becomes

$$u(t)^2 = \frac{(T_{n-1}(t))^2}{k \int (T_{n-1}(t))^2 dt + C_1}. \tag{55}$$

In Table 4 we give the general solutions for the three first equations in the Abel chain shown in Table 3:

Table 4.   General solutions for the Abel chain

n	General Solution of $A_n = 0$
1	$u(t)^2 = \dfrac{1}{2kt + C_1}$
2	$u(t)^2 = \dfrac{3(t + C_2)^2}{2k(t + C_2)^3 + 3C_1}$
3	$u(t)^2 = \dfrac{15(t^2 + C_2t + C_3)^2}{k\left(6t^5 + 15C_2t^4 + 10(C_2^2 + 2C_3)t^3 + (30C_3C_2)t^3 + 30C_3^2t\right) + 15C_1}$

These solutions should be compared with those obtained in [13, Theorem 3.2], which were obtained from the solutions of the higher Riccati equations by solving an additional first-order ODE (see Eq. (41) in [13]). The procedure introduced in this paper provides directly the general solutions by means of (55), which does not require either the solutions of the Riccati higher equations or any type of additional integration.

7   Example for $m \notin \mathbb{Z}$

The results obtained in Sections 1-4 have been established for sequences of equations of the form (3) generated by $g(u) = ku^m$, for arbitrary values $k \in \mathbb{R}$ and $m \in \mathbb{Z}$. In this section we present an example in order to show these results are also valid for real values of parameter m, with adequate restrictions on the involved domains.

The third-order ODE

$$\sqrt{u}u_3 + 14u_2u + 5u_1^2 + 72u_1u\sqrt{u} + 64u^3 = 0, \quad u > 0 \tag{56}$$

is the element $P_3 = 0$, of the sequence (3) generated by the function $g(u) = 4\sqrt{u}$, which correspond to $k = 4, m = 1/2$. Therefore, Theorem 3 can be applied to provide directly

its general solution through (40):

$$u(t) = \frac{T_2(t)}{\left((C_3 - C_2^2) \ln \left| C_2 + t + \sqrt{|T_2(t)|} \right| + (t + C_2) \sqrt{|T_2(t)|} + C_1 \right)^2}, \tag{57}$$

where

$$T_2(t) = t^2 + 2C_2 t + C_3, \quad C_i \in \mathbb{R} \quad \text{for} \quad i = 1, 2, 3.$$

Although the Lie symmetry group of equation (56) is two-dimensional (and therefore insufficient to complete its integration by quadratures), Theorem 1 provides three generalized symmetries of the equation,

$$\mathbf{w}_{(3,3)} = u \, (P_2)^{-1/2} \left(P_2 \frac{t^2}{2} - P_1 t + P_0 \right) \partial_u,$$

$$\mathbf{w}_{(3,2)} = u \, (P_2)^{-1/2} \left(P_2 t - P_1 \right) \partial_u, \tag{58}$$

$$\mathbf{w}_{(3,1)} = u \, (P_2)^{1/2} \partial_u,$$

by using the preceding elements in the sequence:

$$P_0 = u, \quad P_1 = u_1 + 4u\sqrt{u}, \quad P_2 = u_2 + 10u_1\sqrt{u} + 16u^2.$$

By Theorem 2, these terms can also be used to obtain, without any integration, the first integrals

$$I_{(3,3)} = \frac{t^2}{2} - \frac{P_1}{P_2} t - \frac{P_0}{P_2}, \quad I_{(3,2)} = t - \frac{P_1}{P_2}. \tag{59}$$

By using (32)

$$S_2(t) = \det(\mathcal{N}_2) = \begin{vmatrix} t - c_{(3;2)} & -1 \\ \dfrac{t^2}{2!} - c_{(3;3)} & -t \end{vmatrix} = -\frac{t^2}{2} + t c_{(3,2)} - c_{(3,3)}$$

a remaining first integral can be calculated from (38), which can be expressed as follows

$$I_{(3,1)} = (\sqrt{u} - t + I_{(3,2)}))J + \sqrt{2} \left(I_{(3,3)} - \frac{(I_{(3,2)})^2}{2} \right) \arctan \left(\frac{t - I_{(3,2)}}{J} \right),$$

where $J = J(t, u^{(2)})$ is the function given by

$$J = \sqrt{\left| -\frac{t^2}{2} + t I_{(3,2)} - I_{(3,3)} \right|}.$$

8   Concluding remarks and extensions

In this paper, we have determined a unified method to study a family of differential sequences, in order to obtain their first integrals, generalized symmetries and exact solutions.

For any nth-order equation in each chain, we have obtained a set of n generalized symmetries in evolutionary form, and verified the essential property that the ratio of any two characteristics become a first integral of the equation (Theorem 1). Furthermore, we have demonstrated taht $n-1$ of these first integrals are functionally independent (Theorem 2). It is noteworthy that the generalized symmetries can be easily derived from one another by means of simple derivations with respect to t (see Remark 1 and equation (21)).

In order to obtain a complete set of first integrals, we have exploited the knowledge of a Jacobi last multiplier for each equation of the considered sequences. Thus, a remaining first integral is determined by quadrature only. Finally, we have shown that the complete set of first integrals yields the general solution of each nth-order equation of any sequence and can be expressed through (40) in terms of arbitrary polynomials of order $n-1$ (Theorem 3).

We have applied our general results to the Riccati and Abel chains. In particular, we have derived an additional generalized symmetry in the case of the Riccati chain. Consequently, the $n+1$ determined generalized symmetries yield a complete set of first integrals, without any kind of integration and the use of the Jacobi last multiplier.

The existence of an additional generalized symmetry with similar properties for the other sequences of the family, e.g. the Abel chain, remains as an a open problem. Another issue which needs to be investigated further is how the generalized symmetries obtained in this work can be used to determine the complete symmetry groups of each equation, not only in the Riccati chain (which have been derived in [2] in terms of nonlocal symmetries, and in [17] as Lie point symmetries of the equivalent first-order equations), but for any sequence in the family and, in particular, in the Abel chain.

Acknowledgements

References

[1] Andriopoulos K, Generalised Symmetries and the Connection to Differential Sequences, *Int. J. Pure Appl. Math.*, 2008, V.42, 383–390.

[2] Andriopoulos K, Leach P G L and Maharaj A, On Differential Sequences, *AMIS*, 2011, V. 5, 525–546.

[3] Cariñena J F , Lucas J de and Rañada M F, A Geometric Approach to Integrability of Abel Differential Equations, *Int. J. Theor. Phys.*, 2011, V.50, 2114–2124.

[4] Cariñena J F , Guha P and Rañada M F, A Geometric Approach to Higher-order Riccati Chain: Darboux Polynomials and Constants of the Motion, *Journal of Physics: Conference Series*, 2009, V.175, 012009, 15pp.

[5] Cariñena J F , Guha P and Rañada M F, Higher-order Abel Equations: Lagrangian Formalism, First Integrals and Darboux Polynomials, *Nonlinearity*, 2009, V.22, 2953–2969.

[6] Euler M, Euler N and Leach P G L, The Riccati and Ermakov-Pinney Hierarchies, *J. Nonlinear Math. Phys.* , 2007, V.14, 290–310.

[7] Euler M, Euler N and Petersson N, Linearizable Hierarchies of Evolution Equations in (1+ 1) Dimensions, *Stud. Appl. Math.*, 2003, V.111, 315–337.

[8] Euler M and Euler N, A Tree of Linearizable Second Order Evolution Equations by Generalized Hodograph Transformations, *J. Nonlinear Math. Phys.*, 2001, V.8, 342–362.

[9] Leach P G L and Euler N, A Novel Riccati Sequence, *J. Nonlinear Math. Phys.*, 2009, V.16, 157–164.

[10] Muriel C and Romero J L, New Methods of Reduction for Ordinary Differential Equations, *IMA J. Appl. Math.*, 2001, V.66, 111–125.

[11] Muriel C and Romero J L, C^∞-Symmetries and Reduction of Equations Without Lie Point Symmetries, *J. Lie Theory*, 2003, V.13, 167–188.

[12] Muriel C and Romero J L, Nonlocal Symmetries, Telescopic Vector Fields and $\lambda-$Symmetries of Ordinary Differential Equations, *SIGMA* , 2012, V.8, 106 (21 pp).

[13] Muriel C and Romero J L, λ-Symmetries of Some Chains of Ordinary Differential Equations, *Nonlinear Anal. Real World Appl.*, 2014, V.16, 191–201.

[14] Muriel C and Romero J L, The λ-Symmetry Reduction Method and Jacobi Last Multipliers, *Commun. Nonlinear Sci. Numer. Simulat.*, 2014, V.19, 807–820.

[15] Muriel C and Romero J L, Evolution of the Concept of λ-Symmetry and Main Applications, in Nonlinear Systems and Their Remarkable Mathematical Structures, Editors N Euler and M C Nucci, Chapman and Hall/CRC, Boca Raton, Florida, 2019, V.2, Ch.6.

[16] Nucci M C, Jacobi Last Multiplier and Lie symmetries: a Novel Application of an Old Relationship, *J. Nonlinear Math. Phys.* , 2005, V.12, 284–304.

[17] Nucci M C, Ubiquitous Symmetries, *Theor. Math. Phys.*, 2016, V.188, 1361–1370.

[18] Olver P J, Evolution Equations Possessing Infinitely Many Symmetries, *J. Math. Phys.*, 1977, V.18, 1212, 4pp.