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ON THE CHVÁTAL-JANSON CONJECTURE

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Abstract. Let $q_m = P(X \leq m)$, where m is a positive integer and X a binomial random variable with parameters n and m/n. Vašek Chvátal conjectured that, for fixed $n \geq 2$, q_m attains its minimum when m is the integer closest to $2n/3$. As shown by Svante Janson, this conjecture is true for large n. Here, we prove that the conjecture is actually true for every $n \geq 2$.

1. Introduction

Denoting by $B(n,p)$ a binomial random variable with parameters n and p, Janson [4] investigates the following conjecture suggested by Chvátal in a personal communication.

Conjecture 1 (Chvátal). *For any fixed $n \geq 2$, as m ranges over $\{0,\dots,n\}$,*

$$q_m := P(B(n,m/n) \leq m)$$

is smallest when $m = [\![2n/3]\!]$ where $[\![\cdot]\!]$ represents the nearest integer function.

In addition to be intriguing, Conjecture 1 may have useful applications, since the probability that a binomial random variable exceeds its expected value plays a role in the machine learning framework; see e.g. [1], [3], [9] and references therein. Such a probability is also connected to an equation given by Ramanujan, as emphasized by [5]. See also [6] and [8] for further results on this topic.

For large n, Conjecture 1 is actually true and the q_m have a unique minimum.

Theorem 1 (Janson, [4]). *There exists an integer n_0 such that, for each $n \geq n_0$: i) q_m is minimum for $m = [\![2n/3]\!]$ and ii) $q_m > q_{m+1}$ or $q_m < q_{m+1}$ according to whether $m + \frac{1}{2} < 2n/3$ or $m + \frac{1}{2} > 2n/3$.*

As noted in [4, Remark 1.5], in principle, the value of n_0 could be computed and then Conjecture 1 could be proved (or disproved) by considering all $n < n_0$. Even if potentially possible, however, this strategy looks not practically feasible and Janson wishes for a general proof of Conjecture 1.

The only purpose of this note is to prove that Conjecture 1 is actually true.

Theorem 2. *For each $n \geq 2$, one obtains $q_m > q_{m+1}$ or $q_m < q_{m+1}$ according to whether $m + \frac{1}{2} < 2n/3$ or $m + \frac{1}{2} > 2n/3$. Hence, if $m_0 = [\![2n/3]\!]$, then $q_{m_0} < q_m$ for each $m \neq m_0$.*

Key words and phrases. Binomial distribution, Binomial tail probability, Bernoulli inequality.

Our proof of Theorem 2 is quite plain and relies on completely different arguments with respect to [4]. In fact, [4] exploits the version for integer-valued random variables of the asymptotic Edgeworth expansion for probabilities in the central limit theorem - as proposed by Esseen [2]. Instead, our proof is closer to the approach introduced by [7, Appendix B] for showing that $q_m \geq q_{m+1}$ for $0 \leq m < n/2$ and $n \geq 2$.

2. Two preliminary lemmas

Let $U_1, \ldots, U_n$ be n independent copies of a uniform random variable on $[0,1]$ and $U_{(1)} \leq \ldots \leq U_{(n)}$ the corresponding order statistics. For $m < n$, since $U_{(m+1)}$ has a beta distribution with parameters $m+1$ and $n-m$, one obtains

$$q_m = P\Big(\sum_{i=1}^{n} I_{\{U_i \leq m/n\}} \leq m\Big) = P(U_{(m+1)} > m/n)$$

$$= (m+1)\binom{n}{m+1}\int_{m/n}^{1} x^m(1-x)^{n-m-1}\,dx.$$

Lemma 1. *Let $n \geq 2$ and $m \leq n-2$. Then, $q_m \geq q_{m+1}$ if and only if*

$$\int_0^1 (1-\frac{v}{m+1})^m(1+\frac{v}{n-m-1})^{n-m-1}\,dv \geq 1. \tag{2}$$

Proof. First note that $q_m \geq q_{m+1}$ is equivalent to

$$\frac{m+1}{n-m-1}\int_{m/n}^{1} x^m(1-x)^{n-m-1}\,dx \geq \int_{(m+1)/n}^{1} x^{m+1}(1-x)^{n-m-2}\,dx.$$

Integrating the left-hand side by parts, this inequality becomes

$$\int_{m/n}^{(m+1)/n} x^{m+1}(1-x)^{n-m-2}\,dx \geq \frac{(m/n)^{m+1}(1-m/n)^{n-m-1}}{n-m-1}.$$

Letting $x = (m+t)/n$ in the integral, one obtains

$$\int_0^1 (1+t/m)^{m+1}(1-t/(n-m))^{n-m-2}\,dt \geq \frac{n-m}{n-m-1}.$$

Integrating again the left-hand side by parts, such inequality turns into

$$\int_0^1 (1+\frac{t}{m})^m(1-\frac{t}{n-m})^{n-m-1}\,dt \geq (1-\frac{1}{n-m})^{n-m-1}(1+\frac{1}{m})^m$$

or equivalently

$$\int_0^1 (\frac{t+m}{1+m})^m(\frac{n-m-t}{n-m-1})^{n-m-1}\,dt \geq 1.$$

Now, inequality (2) follows from the transformation $t = 1-v$. □

Lemma 2. *Fix $n \geq 3$ and define*

$$g_v(x) = \left(1 - \frac{v}{x+1}\right)^x \left(1 + \frac{v}{n-x-1}\right)^{n-x-1}$$

for all $v \in (0,1]$ and $x \in [1, n-2]$. Then, $x \mapsto g_v(x)$ is strictly decreasing for each fixed v. In particular, if

$$h(m) = \int_0^1 g_v(m)\, dv = \int_0^1 \left(1 - \frac{v}{m+1}\right)^m \left(1 + \frac{v}{n-m-1}\right)^{n-m-1} dv$$

for $m \in \{1, \dots, n-2\}$, the function h is strictly decreasing.

Proof. Fix $(v, x) \in (0,1] \times [1, n-2]$, and note that

$$g'_v(x) = g_v(x)\,\big[\log(1 - \frac{v}{x+1}) + \frac{\frac{vx}{(x+1)^2}}{1 - \frac{v}{x+1}} - \log(1 + \frac{v}{n-x-1}) + \frac{\frac{v}{n-x-1}}{1 + \frac{v}{n-x-1}}\big].$$

Therefore,

$$g'_v(x) < 0 \quad \Longleftrightarrow \quad \frac{\frac{vx}{(x+1)^2}}{1 - \frac{v}{x+1}} + \frac{\frac{v}{n-x-1}}{1 + \frac{v}{n-x-1}} < \log\left(\frac{1 + \frac{v}{n-x-1}}{1 - \frac{v}{x+1}}\right).$$

In addition,

$$\begin{aligned}\log\left(\frac{1 + \frac{v}{n-x-1}}{1 - \frac{v}{x+1}}\right) &= \log\big[\big(1 + \frac{v}{n-x-1}\big)\big(1 + \frac{\frac{v}{x+1}}{1 - \frac{v}{x+1}}\big)\big] \\ &= \log\big(1 + \frac{v}{n-x-1}\big) + \log\big(1 + \frac{\frac{v}{x+1}}{1 - \frac{v}{x+1}}\big).\end{aligned}$$

Hence, in order to prove $g'_v(x) < 0$, it suffices to show that

$$\frac{\frac{v}{n-x-1}}{1 + \frac{v}{n-x-1}} < \log\big(1 + \frac{v}{n-x-1}\big) \tag{3}$$

and

$$\frac{\frac{vx}{(x+1)^2}}{1 - \frac{v}{x+1}} < \log\big(1 + \frac{\frac{v}{x+1}}{1 - \frac{v}{x+1}}\big). \tag{4}$$

To prove (3)-(4), first note that $\log(1+c) > c/(1+c)$ for each $c > 0$. Therefore, (3) holds with $c = v/(n-x-1)$. Similarly, letting $c = v/(x+1-v)$, inequality (4) reduces to

$$\log(1+c) - c + \frac{c^2}{v(c+1)} > 0.$$

Finally, the above inequality is true, since

$$\log(1+c) - c + \frac{c^2}{v(c+1)} \geq \log(1+c) - c + \frac{c^2}{c+1} = \log(1+c) - \frac{c}{c+1} > 0.$$

□

3. A proof of the Chvátal-Janson conjecture

We are now ready to attack Theorem 2. By a direct computation, Theorem 2 holds true for $n \leq 5$. Hence, it can be assumed $n = 3s + r$ where $s \geq 2$ and $r \in \{0, 1, 2\}$. In this case, because of Lemmas 1-2, it suffices to prove that

$$\int_0^1 (1 - \frac{v}{2s})^{2s-1} (1 + \frac{v}{s})^s \, dv > 1 \tag{5}$$

and

$$\int_0^1 (1 - \frac{v}{2s+1})^{2s} (1 + \frac{v}{s-1})^{s-1} \, dv < 1 \tag{6}$$

if $r = 0$, while

$$\int_0^1 (1 - \frac{v}{2s+1})^{2s} (1 + \frac{v}{s+r-1})^{s+r-1} \, dv > 1 \tag{7}$$

and

$$\int_0^1 (1 - \frac{v}{2s+2})^{2s+1} (1 + \frac{v}{s+r-2})^{s+r-2} \, dv < 1 \tag{8}$$

if $r \in \{1, 2\}$.

We point out that, since all the previous inequalities are strict, one obtains $q_m \neq q_{m+1}$ for all m provided such inequalities are true.

Inequalities (5) and (6). Let $r = 0$. Recalling the Bernoulli inequality

$$(1 + c)^s \geq 1 + sc \qquad \text{for all } c > -1,$$

one obtains

$$\begin{aligned}
\int_0^1 (1 - \frac{v}{2s})^{2s-1} (1 + \frac{v}{s})^s \, dv &= \int_0^1 [(1 - \frac{v}{2s})^2 (1 + \frac{v}{s})]^s (1 - \frac{v}{2s})^{-1} \, dv \\
&= \int_0^1 [1 - \frac{3v^2}{4s^2} + \frac{v^3}{4s^3}]^s (1 - \frac{v}{2s})^{-1} \, dv \\
&\geq \int_0^1 (1 - \frac{3v^2}{4s} + \frac{v^3}{4s^2}) (1 - \frac{v}{2s})^{-1} \, dv \\
&> \int_0^1 (1 - \frac{3v^2}{4s} + \frac{v^3}{4s^2}) (1 + \frac{v}{2s} + \frac{v^2}{4s^2}) \, dv \\
&= 1 + \frac{5}{96s^2} - \frac{1}{80s^3} + \frac{1}{96s^4} > 1.
\end{aligned}$$

Here, the first inequality is because of the Bernoulli's one while the second depends on $(1 - c)^{-1} > 1 + c + c^2$ for all $c \in (0, 1)$. Hence, inequality (5) is actually true.

Let us turn to inequality (6). We have to show that $I_s < 1$, where

$$I_s = \int_0^1 (1 - \frac{v}{2s+1})^{2s} (1 + \frac{v}{s-1})^{s-1} \, dv.$$

First note that

$$\begin{aligned}I_s &= \int_0^1 \frac{2s+1}{2s+1-v} \exp\left((2s+1)\log(1-\frac{v}{2s+1})+(s-1)\log(1+\frac{v}{s-1})\right) dv\\ &= \int_0^1 \frac{2s+1}{2s+1-v} \exp\left(\sum_{k=2}^{\infty} \frac{v^k}{k}(\frac{(-1)^{k-1}}{(s-1)^{k-1}} - \frac{1}{(2s+1)^{k-1}})\right) dv\\ &< \int_0^1 \frac{2s+1}{2s+1-v} \exp\left(\sum_{k=2}^{3} \frac{v^k}{k}(\frac{(-1)^{k-1}}{(s-1)^{k-1}} - \frac{1}{(2s+1)^{k-1}})\right) dv\end{aligned}$$

where the last inequality depends on

$$\sum_{k=4}^{\infty} \frac{v^k}{k}(\frac{(-1)^{k-1}}{(s-1)^{k-1}} - \frac{1}{(2s+1)^{k-1}}) < 0.$$

Since $\exp(c) < 1 + c + \frac{c^2}{2}$ for $c < 0$ and

$$\gamma(s,v) := \sum_{k=2}^{3} \frac{v^k}{k}(\frac{(-1)^{k-1}}{(s-1)^{k-1}} - \frac{1}{(2s+1)^{k-1}}) = \frac{-3v^2 s}{2(s-1)(2s+1)} + \frac{v^3 s(s+2)}{(s-1)^2(2s+1)^2} < 0,$$

one also obtains

$$I_s < \int_0^1 \frac{2s+1}{2s+1-v} \exp\big(\gamma(s,v)\big)\, dv < \int_0^1 \frac{2s+1}{2s+1-v}\big(1+\gamma(s,v)+\frac{\gamma(s,v)^2}{2}\big)\, dv.$$

Moreover, since

$$\begin{aligned}\frac{2s+1}{2s+1-v} &= \frac{1}{1-\frac{v}{2s+1}} = 1 + \frac{v}{2s+1} + \frac{v^2}{(2s+1)^2}\frac{1}{1-\frac{v}{2s+1}}\\ &\le 1 + \frac{v}{2s+1} + \frac{5v^2}{4(2s+1)^2},\end{aligned}$$

it follows that

$$I_s < \int_0^1 \big(1+\frac{v}{2s+1}+\frac{5v^2}{4(2s+1)^2}\big)\big(1+\gamma(s,v)+\frac{\gamma(s,v)^2}{2}\big)\, dv.$$

After some (tedious) algebra, the above integral can be evaluated and the previous inequality can be written as

$$I_s < 1 + \frac{1+3s(-11s^3+(s+4)^2)}{(s-1)^4(2s+1)^6} + \frac{s^6(29+7s-15s^2/2)}{(s-1)^4(2s+1)^6}.$$

Both fractions in the previous expression are negative for $s \ge 3$. Hence, $I_s < 1$ for each $s \ge 3$. Finally, $I_2 < 1$ follows from a direct calculation.

This concludes the proof of inequality (6).

Inequalities (7) and (8). Let $r \in \{1, 2\}$ and

$$J_s^{(r)} = \int_0^1 (1-\frac{v}{2s+1})^{2s}(1+\frac{v}{s+r-1})^{s+r-1}\, dv.$$

Since $(1+\frac{v}{s})^s \leq (1+\frac{v}{s+1})^{s+1}$, one obtains $J_s^{(1)} \leq J_s^{(2)}$. Hence, to prove (7), it suffices to show $J_s^{(1)} > 1$. To this end, we first write

$$\begin{aligned}
J_s^{(1)} &= \int_0^1 (1-\frac{v}{2s+1})^{2s}(1+\frac{v}{s})^s\,dv \\
&= \int_0^1 [(1-\frac{v}{2s+1})^2(1+\frac{v}{s})]^s\,dv \\
&= \int_0^1 [1+\frac{v(1-2v)}{s(2s+1)}+\frac{v^2}{(2s+1)^2}(1+\frac{v}{s})]^s\,dv.
\end{aligned}$$

Hence, the Bernoulli inequality yields

$$\begin{aligned}
J_s^{(1)} &\geq \int_0^1 [1+\frac{v(1-2v)}{2s+1}+\frac{sv^2}{(2s+1)^2}(1+\frac{v}{s})]\,dv \\
&= 1-\frac{1}{6(2s+1)}+\frac{s}{3(2s+1)^2}+\frac{1}{4(2s+1)^2} = 1+\frac{1}{12(2s+1)^2}.
\end{aligned}$$

This proves inequality (7).

Finally, we turn to (8). Let

$$H_s^{(r)} = \int_0^1 (1-\frac{v}{2s+2})^{2s+1}(1+\frac{v}{s+r-2})^{s+r-2}\,dv.$$

Once again, $H_s^{(1)} \leq H_s^{(2)}$. Thus, to prove (8), it suffices to show that $H_s^{(2)} < 1$. To this end, we argue as in the proof of $I_s < 1$. Precisely, we first note that

$$\begin{aligned}
H_s^{(2)} &= \int_0^1 (1-\frac{v}{2s+2})^{2s+1}(1+\frac{v}{s})^s\,dv \\
&= \int_0^1 \frac{2s+2}{2s+2-v}\exp\left((2s+2)\log(1-\frac{v}{2s+2})+s\log(1+\frac{v}{s})\right)dv \\
&= \int_0^1 \frac{2s+2}{2s+2-v}\exp\left(\sum_{k=2}^{\infty}\frac{v^k}{k}(\frac{(-1)^{k-1}}{s^{k-1}}-\frac{1}{(2s+2)^{k-1}})\right)dv \\
&< \int_0^1 \frac{2s+2}{2s+2-v}\exp\left(\sum_{k=2}^{3}\frac{v^k}{k}(\frac{(-1)^{k-1}}{s^{k-1}}-\frac{1}{(2s+2)^{k-1}})\right)dv
\end{aligned}$$

where the last inequality is because

$$\sum_{k=4}^{\infty}\frac{v^k}{k}(\frac{(-1)^{k-1}}{s^{k-1}}-\frac{1}{(2s+2)^{k-1}}) < 0.$$

Moreover,

$$\lambda(s,v) := \sum_{k=2}^{3}\frac{v^k}{k}(\frac{(-1)^{k-1}}{s^{k-1}}-\frac{1}{(2s+2)^{k-1}}) < 0,$$

$$\text{and} \qquad \frac{2s+2}{2s+2-v} \leq 1+\frac{v}{2s+2}+\frac{6v^2}{5(2s+2)^2}.$$

Hence, recalling that $\exp(c) < 1 + c + \frac{c^2}{2}$ for $c < 0$, one obtains

$$\begin{aligned} H_s^{(2)} &< \int_0^1 \frac{2s+2}{2s+2-v} \exp\big(\lambda(s,v)\big)\, dv \\ &< \int_0^1 \big(1 + \frac{v}{2s+2} + \frac{6v^2}{5(2s+2)^2}\big)\big(1 + \lambda(s,v) + \frac{\lambda(s,v)^2}{2}\big)\, dv. \end{aligned}$$

Finally, evaluating the integral, the previous inequality turns into

$$H_s^{(2)} < 1 + \frac{-2s^5 - 9(s^4 + s^3) + 8s^2 + 22s + 18}{64s(s+1)^6} < 1.$$

This proves (8) and concludes the proof of Theorem 2.

Added in proof: After writing this paper, we learned (from an anonymous referee) of the existence of another paper very similar to ours, that is: Ping Sun (2021) Strictly unimodality of the probability that the binomial distribution is more than its expectation, *Discrete Applied Mathematics* 301, 1–5. However, we point out that a preliminary draft of our paper appeared on arXiv previous to Sun's paper; see: arXiv:2104.11971v1 [math.PR]

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