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Research Article

Cristina Di Girolami and Francesco Russo*

About classical solutions of the path-dependent heat equation

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Abstract: This paper investigates two existence theorems for the path-dependent heat equation, which is the Kolmogorov equation related to the window Brownian motion, considered as a $C([-T, 0])$-valued process. We concentrate on two general existence results of its classical solutions related to different classes of terminal conditions: the first one is given by a cylindrical not necessarily smooth random variable, the second one is a smooth generic functional.

Keywords: Infinite-dimensional analysis, Kolmogorov type equations, path-dependent heat equation, window Brownian motion

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1 Introduction

The path-dependent heat equation is a natural extension of the classical heat equation to the path-dependent world. If the heat equation constitutes the Kolmogorov equation associated with Brownian motion viewed as a real-valued process, then the path-dependent heat equation is the Kolmogorov equation related to the Wiener process as $C([-T, 0])$-valued process, that we will denominate as *window Brownian motion*. One particularity of $C([-T, 0])$ is that it is a (even non-reflexive) Banach space, and for integrator processes taking values in it, it is not obvious to define a stochastic integral. In the recent past, many works were devoted to various types of path-dependent PDE under different perspectives (for instance, under the perspective of viscosity solutions; see e.g. [2, 5, 14]), using generally approaches close to the functional Itô calculus of [9]. A recent contribution in the study of the path-dependent heat equation (in the spirit of Banach space) was carried on by [10], which considered (not necessarily smooth in time) mild type solutions, involving at the same time a path-dependent drift; see also references therein for related contributions. The problem of finding *classical* or *smooth* solutions has been neglected, especially using the Banach space approach, except for some particular final conditions; see e.g. [4, 6].

In this paper, we focus on classical solutions of the path-dependent heat equation with two types of terminal conditions. In reality, this work updates [6, 7], somehow a pioneering (never published) work of the authors, which formulated similar results in a Hilbert framework.

***Corresponding author: Francesco Russo,** Unité de Mathématiques Appliquées, ENSTA Paris, Institut Polytechnique de Paris, 828, Boulevard des Maréchaux F-91120 Palaiseau, France, e-mail: francesco.russo@ensta-paris.fr
Cristina Di Girolami, Département de mathématiques, UFR Sciences et Techniques, Le Mans Université, Avenue Olivier Messiaen, 72085, Le Mans CEDEX 9, France, e-mail: cristina.di_girolami@univ-lemans.fr. http://orcid.org/0000-0003-4711-8228

Let $H: C([-T,0]) \to \mathbb{R}$ be continuous, and let σ be a real constant. Even though some results can be extended to a more general context, we have preferred for clarity to work with σ being a constant. See Section 4.3 for some results with general $\sigma: [0,T] \times \mathbb{R} \to \mathbb{R}$.

Our path-dependent heat equation can be expressed as

$$\begin{cases} \partial_t u(t,\eta) + \int_{]-t,0]} D^{\perp}_{dx} u(t,\eta)\, d^-\eta(x) + \frac{1}{2}\sigma^2 \langle D^2 u(t,\eta), \mathbb{1}_{\{0\}} \otimes \mathbb{1}_{\{0\}}\rangle = 0 & \text{for } (t,\eta) \in [0,T[\times C([-T,0]), \\ u(T,\eta) = H(\eta) & \text{for } \eta \in C([-T,0]). \end{cases} \tag{1.1}$$

A function $u: [0,T] \times C([-T,0]) \to \mathbb{R}$ will be a classical solution of (1.1) if it belongs to

$$C^{1,2}([0,T[\times C([-T,0])) \cap C^0([0,T] \times C([-T,0]))$$

in the Fréchet sense and if it verifies (1.1). For any given $(t,\eta) \in [0,T] \times C([-T,0])$, $DF(t,\eta)$ denotes the first-order Fréchet derivative with respect to η, $D^{\delta_0}F(t,\eta)$ the component of $DF(t,\eta)$ concentrated on the Dirac zero defined by $D^{\delta_0}F(t,\eta) := DF(t,\eta)(\{0\})$, and $D^{\perp}F(t,\eta)$ denotes the component of $DF(t,\eta)$ *singular* to the Dirac zero component, i.e. the measure defined by $D^{\perp}F(t,\eta) := DF(t,\eta) - DF(t,\eta)(\{0\})\delta_0$. For every $\eta \in C([-\tau,0])$, we observe that $t \mapsto D^{\delta_0}F(t,\eta)$ is a real-valued function. If, for each (t,η), $D^{\perp}F(t,\eta)$ is absolutely continuous with respect to Lebesgue measure on the reals, $D^{ac}F(t,\eta)$ denotes its density, and in particular, it holds that $D^{\perp}_{dx}F(t,\eta) = D^{ac}_x F(t,\eta)\,dx$.

A central object appearing in the path-dependent heat equation PDE (1.1) is the deterministic integrals denoted by

$$\int_{]-t,0]} D^{\perp}_{dx} u(t,\eta)\, d^-\eta(x),$$

where $D^{\perp}u(t,\eta)$ is a measure on $[-T,0]$ and $\eta \in C([-T,0])$. We will give a sense, for $-T \le a \le b \le 0$, to the term $\int_{]a,b]} D^{\perp}u(t,x)\, d^-\eta(x)$ as the *deterministic forward integral* $\lim_{\epsilon\to 0} \int_{]a,b]} D^{\perp}_{dx} u(t,x) \frac{\eta(x+\epsilon)-\eta(x)}{\epsilon}\, dx$; see Definition 2.2. More generally, let μ be a finite Borel measure on $[-T,0]$ and f a càdlàg function; we will give a sense to the integral $\int_{]a,b]} \mu(dx)\, d^-f(x)$. Whenever f has bounded variation and μ is absolutely continuous with respect to the Lebesgue measure, it will coincide with the classical Riemann–Stieltjes integral; see Proposition 2.3.

As we mentioned, we state two existence theorems of the classical solution of (1.1) under two different types of terminal condition given by a function H. In Proposition 3.4, we consider as terminal condition a possibly not smooth function H of a finite numbers of integrals of the type $\int_{-T}^0 \varphi\, d^-\eta$. The reason of validity of that result (when $\sigma \neq 0$) can be understood through the non-degeneracy feature of Brownian motion.

In Theorem 4.11, we suppose the terminal condition function H to be $C^3(C([-T,0]))$. This result generalizes an existence result already established in the unpublished monograph [6, Sections 9.8 and 9.9], where we assumed a Fréchet smooth dependence with respect to $L^2([-T,0])$.

In this paper, we have only concentrated our efforts on the problem of existence of a solution of (1.1), the uniqueness constituting a simpler task which can be obtained as an application of a Banach space valued Itô formula established in [8].

Let $W = (W_t)_{0\le t\le T}$ be a classical real Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$; $(\mathcal{F}_t)$ will denote its canonical filtration. $(W_t(\cdot))$ (or simply $W(\cdot)$) stands for the window Brownian process with values in $C([-T,0])$ defined by $W_t(x) := W_{t+x}$; see Definition 2.1.

An application of our two existence results consists in obtaining a Clark–Ocone type formula for a path-dependent random variable $h := H(X_T(\cdot))$, where X is a finite quadratic variation process with quadratic variation given by $[X]_t = \sigma^2 t$, but X not necessarily a semimartingale. A possible example of such process is given by $X = W + B^H$, i.e. a Brownian motion plus a fractional Brownian motion of parameter $H > \frac{1}{2}$ or the weak k-order Brownian motion of [11].

Let u be the solution of (1.1) provided by Proposition 3.4 or Theorem 4.11. By the Itô formula (see e.g. [8, Theorem 5.2]), if u verifies some more technical conditions, then

$$h = u(0, X_0(\cdot)) + \int_0^T \mathcal{L}u(t, X_t(\cdot))\, dt + \int_0^T D^{\delta_0} u(t, X_t(\cdot))\, d^-X_t, \tag{1.2}$$

where $\mathcal{L}$ denotes differential operator for $u \in C^{1,2}([0,T[\times C([-T,0]))$ defined by

$$\mathcal{L}u(t,\eta) := \partial_t u(t,\eta) + \int_{]-t,0]} D^{\perp}_{dx} u(t,\eta)\, d^-\eta(x) + \frac{1}{2}\sigma^2 \langle D^2 u(t,\eta), \mathbb{1}_{\{0\}} \otimes \mathbb{1}_{\{0\}} \rangle$$

for $(t,\eta) \in [0,T] \times C([-T,0])$. Now, by (1.2),

$$h = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(t, X_t(\cdot))\, d^- X_t, \tag{1.3}$$

where we remind that $\int_0^t Y\, d^- X$ is the forward integral via regularization defined first in [15, 16] for X (resp. Y) a continuous (resp. locally integrable) real process; see also [17] for a survey. Whenever $X = W$, the forward real-valued integral equals the classical Itô integral; see [16, Proposition 1.1]. In particular, if $h \in \mathbb{D}^{1,2}$, it holds that the representation stated in (1.3) coincides with the classical Clark–Ocone formula $h = \mathbb{E}[h] + \int_0^T \mathbb{E}[D^{\mathrm{m}}_t h | \mathcal{F}_t]\, dW_t$, i.e. $u(0, W_0(\cdot)) = \mathbb{E}[h]$ and $D^{\delta_0} u(t, W_t(\cdot)) = D^{\mathrm{m}}_t (h|\mathcal{F}_t)$, D^{m} denoting the Malliavin derivative. This follows by the uniqueness of decomposition of square integrable random variables with respect to the Brownian filtration. We remark that our representation (1.3) can be proved in some cases, where $h \notin \mathbb{D}^{1,2}$; see e.g. Section 3.

The paper is organized as follows. After this introduction, in Section 2, we recall some preliminaries: basic notions of calculus via regularization in finite and infinite dimension, Fréchet derivatives of functionals and the important Subsection 2.2 about deterministic calculus via regularization. In Section 3, we show the existence of a classical solution of the Kolmogorov PDE for a cylindrical H. Finally, in Section 4, we show the existence for H being general but smooth.

2 Preliminaries

2.1 General notations

Let A and B be two general sets such that $A \subset B$; $\mathbb{1}_A : B \to \{0,1\}$ will denote the indicator function of the set A, so $\mathbb{1}_A(x) = 1$ if $x \in A$, and $\mathbb{1}_A(x) = 0$ if $x \notin A$. Let $k \in \mathbb{N} \cup \{+\infty\}$; $C^k(\mathbb{R}^n)$ indicates the set of all functions $g\colon \mathbb{R}^n \to \mathbb{R}$ which admits all partial derivatives of order $0 \le p \le k$ and is continuous. If I is a real interval and g is a function from $I \times \mathbb{R}^n$ to $\mathbb{R}$ which belongs to $C^{1,2}(I \times \mathbb{R}^n)$, the symbols $\partial_t g(t,x)$, $\partial_i g(t,x)$ and $\partial^2_{ij} g(t,x)$ will denote respectively the partial derivative with respect to variable I, the partial derivative with respect to the i-th component and the second-order mixed derivative with respect to j-th and i-th component evaluated in $(t,x) \in I \times \mathbb{R}^n$.

Let $a < b$ be two real numbers; $C([a,b])$ will denote the Banach linear space of real continuous functions equipped with the uniform norm denoted by $\|\cdot\|_\infty$. Let B be a Banach space over the scalar field $\mathbb{R}$. The space of bounded linear mappings from B to E will be denoted by $L(B;E)$, and we will write $L(B)$ instead of $L(B;B)$. The topological dual space of B, i.e. $L(B;\mathbb{R})$, will be denoted by B^*. If ϕ is a linear functional on B, we shall denote the value of ϕ at an element $b \in B$ either by $\phi(b)$ or $\langle \phi, b \rangle$ or even ${}_{B^*}\langle \phi, b \rangle_B$. Let K be a compact space; $\mathcal{M}(K)$ will denote the dual space $C(K)^*$, i.e. the so-called set of all real-valued finite signed measures on K. In the article, if not specified, the mention *absolutely continuous* for a real-valued measure will always refer to the Lebesgue measure.

Let E, F, G be Banach spaces; we shall denote the space of G-valued bounded bilinear forms on the product $E \times F$ by $\mathcal{B}(E \times F; G)$ with the norm given by $\|\phi\|_{\mathcal{B}} = \sup\{\|\phi(e,f)\|_G : \|e\|_E \le 1;\ \|f\|_F \le 1\}$. If $G = \mathbb{R}$, we simply denote it by $\mathcal{B}(E \times F)$. We recall that $\mathcal{B}(B \times B)$ is identified with $(B \hat{\otimes}_\pi B)^*$; see [12, 18] for more details.

We recall some notions about differential calculus in Banach spaces; for more details, the reader can refer to [1]. Let B be a Banach space. A function $F\colon [0,T] \times B \to \mathbb{R}$ is said to be $C^{1,2}([0,T] \times B)$ (Fréchet), or $C^{1,2}$ (Fréchet), if the following three properties are fulfilled: (1) F is once continuously differentiable; the partial derivative with respect to t will be denoted by $\partial_t F : [0,T] \times B \to \mathbb{R}$; (2) for any $t \in [0,T]$, $x \mapsto DF(t,x)$ is of class C^1, where $DF\colon [0,T] \times B \to B^*$ denotes the derivative with respect to the second argument; (3) the second-order derivative with respect to the second argument $D^2F\colon [0,T] \times B \to \mathcal{B}(B \times B)$ is continuous.

If $B = C([-T,0])$, we remark that DF defined on $[0,T] \times B$ takes values in $B^* \cong \mathcal{M}([-T,0])$. For all $(t,\eta) \in [0,T] \times C([-T,0])$, we will denote by $D_{dx}F(t,\eta)$ the measure such that, for all $h \in C([-T,0])$,

$$ {}_{\mathcal{M}([-T,0])}\langle DF(t,\eta), h\rangle_{C([-T,0])} = DF(t,\eta)(h) = \int_{[-T,0]} h(x)\, D_{dx}F(t,\eta). $$

Whenever $B = E = F = C([-T,0])$, then the space of finite signed Borel measures on $[-T,0]^2$ is included in the space $\mathcal{B}(B \times B)$ in the following way:

$$ {}_{\mathcal{M}([-T,0]^2)}\langle \mu, \eta\rangle_{C([-T,0]^2)} = \int_{[-T,0]^2} \eta(x,y)\mu(dx,dy) = \int_{[-T,0]^2} \eta_1(x)\eta_2(y)\mu(dx,dy). $$

We convene that the continuous functions (and real processes) defined on $[0,T]$ or $[-T,0]$ are extended by continuity to the real line.

Definition 2.1. Given a real continuous process $X = (X_t)_{t\in[0,T]}$, we will call *window process* and denote by $X(\cdot)$ the $C([-T,0])$-valued process

$$ X(\cdot) = (X_t(\cdot))_{t\in[0,T]} = \{X_t(x) := X_{t+x};\ x \in [-T,0],\ t \in [0,T]\}. $$

$X(\cdot)$ will be understood, sometimes without explicit mention, as $C([-T,0])$-valued. In this paper, B will be often taken to be $C([-T,0])$.

We recall now the integration by parts in Wiener space. Let δ be the *Skorohod integral* or the adjoint operator of Malliavin derivative D^{m} as defined in [13, Definition 1.3.1]. If u belongs to $\operatorname{Dom}\delta$, then $\delta(u)$ is an element of $L^2(\Omega)$ characterized, for any $F \in \mathbb{D}^{1,2}$, by

$$ \mathbb{E}[F\,\delta(u)] = \mathbb{E}\Big[\int_0^T D^{\mathrm{m}}_t F\, u_t\, dt\Big]. \tag{2.1} $$

2.2 Deterministic calculus via regularization

Let $-T \le a \le b \le 0$; we will essentially concentrate on the definite integral on an interval $J =]a,b]$ and $\bar{J} = [a,b]$, where $a < b$ are two real numbers. Typically, in our applications, we will consider $a = -T$ or $a = -t$ and $b = 0$. That integral will be a real number.

We start with a convention. If $f\colon [a,b] \to \mathbb{R}$ is a càdlàg function, we extend it naturally to two possible càdlàg functions f_J and $f_{\bar{J}}$ in real line setting

$$ f_J(x) = \begin{cases} f(b), & x > b,\\ f(x), & x \in [a,b],\\ f(a), & x < a, \end{cases} \qquad \text{and} \qquad f_{\bar{J}}(x) = \begin{cases} f(b), & x > b,\\ f(x), & x \in [a,b],\\ 0, & x < a. \end{cases} $$

Definition 2.2. Let μ be a finite Borel measure on $[0,T]$, $\mu \in \mathcal{M}([-T,0])$ and $f\colon [a,b] \to \mathbb{R}$ a càdlàg function. We define the *deterministic forward integral* on $J =]a,b]$ and on $\bar{J} = [a,b]$ denoted by

$$ \int_{]a,b]} \mu(dx)\, d^- f(x) \quad \Big(\text{or simply } \int_{]a,b]} \mu\, d^- f\Big) \qquad \text{and} \qquad \int_{[a,b]} \mu(dx)\, d^- f(x) \quad \Big(\text{or simply } \int_{[a,b]} \mu\, d^- f\Big) $$

as the limit of

$$ I^-(]a,b],f,\epsilon) = \int_{]a,b]} \frac{f_J(x+\epsilon) - f_J(x)}{\epsilon}\mu(dx) \quad \text{and} \quad I^-([a,b],f,\epsilon) = \int_{[a,b]} \frac{f_{\bar{J}}(x+\epsilon) - f_{\bar{J}}(x)}{\epsilon}\mu(dx) $$

when $\epsilon \downarrow 0$, provided it exists.

If μ is absolutely continuous, we denote by μ^{ac} the density with respect to the Lebesgue measure. In this case, we set

$$\int_{]a,b]} \mu \, d^{-}f := \int_{]a,b]} \mu^{\mathrm{ac}} \, d^{-}f, \qquad \int_{[a,b]} \mu \, d^{-}f := \int_{[a,b]} \mu^{\mathrm{ac}} \, d^{-}f. \tag{2.2}$$

The first integral on $]a, b]$ appears in the path-dependent PDE (1.1); the second one on the closed interval $[a, b]$ is fundamental in Section 3. The proposition below discusses the case when f or μ is absolutely continuous.

Proposition 2.3. *Let $\mu(dx) = \mu^{\mathrm{ac}}(x)\,dx$, i.e. μ be absolutely continuous with Radon–Nikodym derivative density denoted by μ^{ac}. By default, the bounded variation functions will be considered as càdlàg.*
(1) *If f has bounded variation, then*

$$\int_{]a,b]} \mu^{\mathrm{ac}}(x) \, d^{-}f(x) = \int_{]a,b]} \mu^{\mathrm{ac}}(x-) \, df(x) \quad \textit{(classical Lebesgue–Stieltjes integral)}.$$

In particular, whenever $\mu^{\mathrm{ac}} \equiv 1$, $\int_{]a,b]} \mu^{\mathrm{ac}}(x) \, d^{-}f(x) = f(b) - f(a)$.
(2) *If the function $\mu^{\mathrm{ac}} : [a, b] \to \mathbb{R}$ is càdlàg with bounded variation, then*
(a)

$$\int_{[a,b]} \mu^{\mathrm{ac}}(x) \, d^{-}f(x) = \mu^{\mathrm{ac}}(b)f(b) - \int_{]a,b]} f(x) \, d\mu^{\mathrm{ac}}(x), \tag{2.3}$$

(b)

$$\int_{]a,b]} \mu^{\mathrm{ac}}(x) \, d^{-}f(x) = \mu^{\mathrm{ac}}(b)f(b) - \mu^{\mathrm{ac}}(a)f(a) - \int_a^b f(x) \, d\mu^{\mathrm{ac}}(x). \tag{2.4}$$

Proof. The statements follow directly from the definition. Concerning the case when the integration interval is $[a, b]$, we remark that our definition is compatible with [3, Definitions 4, 18]; see also [3, Proposition 8]. By [3, Proposition 4], we get item (2) (a). The other items can be established by similar considerations and are left to the reader. □

3 The existence result for cylindrical terminal condition

The central object of this section is Proposition 3.4 which gives an existence result of the solution of the path-dependent heat equation (1.1) when the terminal condition H depends on a finite number of integrals, but it is not necessarily smooth. As we mentioned, here the idea is to exploit the non-degeneracy aspect of the Brownian motion in the sense that the covariance matrix of every finite-dimensional distribution is invertible. In this section, the standard deviation parameter σ will be supposed to be different from 0. This in opposition to the case of Section 4 where H is Fréchet smooth, but not necessarily cylindrical; there σ is allowed even to vanish.

We introduce now the functional H. For all $i = 1, \dots, n$, let $\varphi_i : [0, T] \to \mathbb{R}$ be $C^2([0, T]; \mathbb{R})$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable and with linear growth. We consider the functional $H : C([-T, 0]) \to \mathbb{R}$ defined by

$$H(\eta) = f\bigg(\int_{[-T,0]} \varphi_1(u + T) \, d^{-}\eta(u), \dots, \int_{[-T,0]} \varphi_n(u + T) \, d^{-}\eta(u) \bigg). \tag{3.1}$$

We recall that, for smooth φ_i, $i \in \{1, \dots, n\}$, the deterministic integral $\int_{[-T,0]} \varphi_i(u + T) \, d^{-}\eta(u)$ exists pointwise, according to Definition 2.2. That integral exists since, by (2.3) in Proposition 2.3, we have

$$\int_{[-T,0]} \varphi_i(u + T) \, d^{-}\eta(u) = \varphi_i(T)\eta(0) - \int_0^T \eta(s - T) \, d\varphi_i(s). \tag{3.2}$$

So, replacing η by the random path $\sigma W_T(\cdot)$ in (3.1), we get

$$\begin{aligned} h = H(W_T(\cdot)) &= f\Big(\sigma \int_{[-T,0]} \varphi_1(u+T)\, d^- W_T(u), \dots, \sigma \int_{[-T,0]} \varphi_n(u+T)\, d^- W_T(u)\Big) \\ &= f\Big(\sigma \int_0^T \varphi_1(s)\, d^- W_s, \dots, \sigma \int_0^T \varphi_n(s)\, d^- W_s\Big) \\ &= f\Big(\sigma \int_0^T \varphi_1(s)\, dW_s, \dots, \sigma \int_0^T \varphi_n(s)\, dW_s\Big). \end{aligned} \tag{3.3}$$

We stress that, in the first line of (3.3), the integrands are deterministic forward integrals; those integrals exist pathwise; however, in the second line of (3.3), there appear stochastic forward integrals. The second equality is justified because the convergence for every realization ω implies of course the convergence in probability, which characterizes the stochastic forward integral. The latter equality holds because Itô integrals with Brownian motion are also forward integrals; see [16, Proposition 1.1]. On the other hand, for every $i \in \{1, \dots, n\}$, since φ_i are of class C^2, then Proposition 2.3 and in particular (2.3) gives

$$\int_0^t \varphi_i(s)\, d^- W_s = \int_{[-t,0]} \varphi_i(u+t)\, d^- W_t(u) = \varphi_i(t) W_t - \int_0^t W_s\, d\varphi_i(s), \tag{3.4}$$

where the first equality holds by similar reasons as for the first equality in (3.3). The second equality holds by (2.3).

We formulate the following *non-degeneracy* assumption.

Assumption 1. For $t \in [0, T]$, we denote by Σ_t the matrix in $\mathbb{M}_{n\times n}(\mathbb{R})$ defined by

$$(\Sigma_t)_{1\le i,j\le n} = \Big(\int_t^T \varphi_i(s)\varphi_j(s)\, ds\Big)_{1\le i,j\le n}.$$

We suppose $\det(\Sigma_t) > 0$ for all $t \in [0, T[$.

Remark 3.1. (1) We observe that, by continuity of function $t \mapsto \det(\Sigma_t)$, there is always $\tau > 0$ such that $\det(\Sigma_t) \neq 0$ for all $t \in [0, \tau[$.

(2) It is not restrictive to consider $\det(\Sigma_0) \neq 0$ since it is always possible to orthogonalize $(\varphi_i)_{i=1,\dots,n}$ in $L^2([0, T])$ via a Gram–Schmidt procedure.

We remember that W is a classical Wiener process equipped with its canonical filtration $(\mathcal{F}_t)$. We set $h = H(W_T(\cdot))$, and we evaluate the conditional expectation $\mathbb{E}[h|\mathcal{F}_t]$. It gives

$$\begin{aligned} \mathbb{E}[h|\mathcal{F}_t] &= \mathbb{E}\Big[f\Big(\sigma \int_0^T \varphi_i(s)\, dW_s, \dots, \sigma \int_0^T \varphi_n(s)\, dW_s\Big)|\mathcal{F}_t\Big] \\ &= \Psi\Big(t, \sigma \int_0^t \varphi_1(s)\, dW_s, \dots, \sigma \int_0^t \varphi_n(s)\, dW_s\Big) \\ &= \Psi\Big(t, \int_{[-t,0]} \varphi_1(u+t)\, d^- \sigma W_t(u), \dots, \int_{[-t,0]} \varphi_n(u+t)\, d^- \sigma W_t(u)\Big) \\ &= \Psi\Big(t, \int_{[-T,0]} \varphi_1(u+t)\, d^- \sigma W_t(u), \dots, \int_{[-T,0]} \varphi_n(u+t)\, d^- \sigma W_t(u)\Big), \end{aligned} \tag{3.5}$$

where the function $\Psi\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\Psi(t, y_1, \dots, y_n) = \mathbb{E}\Big[f\Big(y_1 + \sigma \int_t^T \varphi_1(s)\, dW_s, \dots, y_n + \sigma \int_t^T \varphi_n(s)\, dW_s\Big)\Big] \tag{3.6}$$

for any $t \in [0, T]$, $y_1, \dots, y_n \in \mathbb{R}$. In particular, $\Psi(T, y_1, \dots, y_n) = f(y_1, \dots, y_n)$. The second equality in (3.5) holds because, for every $1 \le i \le n$,

$$\int_0^t \varphi_n(s)\sigma\, dW_s = \int_{-t}^0 \varphi_n(u+t)\, d^-\sigma W_t(u)$$

for the same reasons as in (3.4). We evaluate expression (3.6) introducing the density function p of the Gaussian vector

$$\Big(\int_t^T \varphi_1(s)\, dW_s, \dots, \int_t^T \varphi_n(s)\, dW_s\Big),$$

whose covariance matrix equals Σ_t. The function $p\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is characterized by

$$p(t, z_1, \dots, z_n) = \sqrt{\frac{1}{(2\pi)^n \det(\Sigma_t)}} \exp\Big\{-\frac{(z_1, \dots, z_n)\Sigma_t^{-1}(z_1, \dots, z_n)^*}{2}\Big\},$$

and function Ψ becomes

$$\Psi(t, y_1, \dots, y_n) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} f(\tilde{z}_1, \dots, \tilde{z}_n)\, p\Big(t, \frac{\tilde{z}_1 - y_1}{\sigma}, \dots, \frac{\tilde{z}_n - y_n}{\sigma}\Big)\, d\tilde{z}_1 \cdots d\tilde{z}_n & \text{if } t \in [0, T[, \\ f(y_1, \dots, y_n) & \text{if } t = T. \end{cases} \tag{3.7}$$

Remark 3.2. (1) If f is not continuous, we remark that, at time $t = T$, $\Psi(T, \cdot)$ is a function which strictly depends on the representative of f and not only on its Lebesgue a.e. representative. So Ψ, as a class, does not admit a restriction to $t = T$.

(2) The function p is a $C^{3,\infty}([0, T[\times \mathbb{R}^n)$ solution of

$$\partial_t p(t, z_1, \dots, z_n) = -\frac{1}{2} \sum_{i,j=1}^n \varphi_i(t)\varphi_j(t)\partial_{ij}^2 p(t, z_1, \dots, z_n).$$

Therefore, the function Ψ is $C^{1,2}([0, T[\times \mathbb{R}^n)$ and solves

$$\partial_t \Psi(t, z_1, \dots, z_n) = -\frac{\sigma^2}{2} \sum_{i,j=1}^n \varphi_i(t)\varphi_j(t)\partial_{ij}^2 \Psi(t, z_1, \dots, z_n). \tag{3.8}$$

We define now a function $u\colon [0, T] \times C([-T, 0]) \to \mathbb{R}$ by

$$u(t, \eta) = \Psi\Big(t, \int_{[-t,0]} \varphi_1(s+t)\, d^-\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t)\, d^-\eta(s)\Big), \tag{3.9}$$

where $\Psi(t, y_1, \dots, y_n)$ is defined by (3.7).

By the fact that, for every i, the functions φ_i are C^2, so in particular with bounded variation, similarly to (3.2), we can write

$$\int_{[-t,0]} \varphi_i(s+t)\, d^-\eta(s) = \eta(0)\varphi_i(t) - \int_0^t \eta(s-t)\dot{\varphi}_i(s)\, ds; \tag{3.10}$$

see (2.3).

Remark 3.3. By construction, we have $u(t, \sigma W_t(\cdot)) = \mathbb{E}[h|\mathcal{F}_t]$ and in particular $u(0, W_0(\cdot)) = \mathbb{E}[h]$.

We state now the main proposition of this section.

Proposition 3.4. *Let $H\colon C([-T, 0]) \to \mathbb{R}$ be defined by* (3.1) *and $u\colon [0, T] \times C([-T, 0]) \to \mathbb{R}$ by* (3.9).
(1) *The function u belongs to $C^{1,2}([0, T[\times C([-T, 0]))$, and it is a classical solution of* (1.1).
(2) *If f is continuous, then we have moreover $u \in C^0([0, T] \times C([-T, 0]))$.*

Proof. We will see that $D^{\perp}u(t,\eta)$ is absolutely continuous with density that we will denote $x \mapsto D^{ac}_x$, so (1.1) simplifies in

$$\begin{cases} \mathcal{L}u(t,\eta) = \partial_t u(t,\eta) + \int_{]-t,0]} D^{ac}_x u(t,\eta)\, d^-\eta(x) + \frac{1}{2} D^2 u(t,\eta)(\{0,0\}) = 0, \\ u(T,\eta) = H(\eta). \end{cases}$$

We first evaluate the derivative $\partial_t u(t,\eta)$, for a given $(t,\eta) \in [0,T] \times C([-T,0])$:

$$\begin{aligned} \partial_t u(t,\eta) &= \partial_t \Psi\Big(t, \int_{[-t,0]} \varphi_1(s+t)\, d^-\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t)\, d^-\eta(s)\Big) \\ &\quad + \sum_{i=1}^n \Big(\partial_i \Psi\Big(t, \int_{[-t,0]} \varphi_1(s+t)\, d^-\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t)\, d^-\eta(s)\Big) \\ &\qquad \cdot \Big(\partial_t \int_{[-t,0]} \varphi_i(s+t)\, d^-\eta(s)\Big)\Big) \\ &= \partial_t \Psi\Big(t, \int_{[-t,0]} \varphi_1(s+t)\, d^-\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t)\, d^-\eta(s)\Big) \\ &\quad + \sum_{i=1}^n \Big(\partial_i \Psi\Big(t, \int_{[-t,0]} \varphi_1(s+t)\, d^-\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t)\, d^-\eta(s)\Big) \cdot I_i\Big), \end{aligned} \tag{3.11}$$

where

$$I_i := \Big(\int_{]-t,0]} \dot\varphi_i(s+t)\, d^-\eta(s)\Big).$$

Indeed, by usual theorems of Lebesgue integration theory and by Proposition 2.3, (2.3) and (2.4), for every $1 \le i \le n$, we obtain

$$\begin{aligned} \partial_t\Big(\int_{[-t,0]} \varphi_i(s+t)\, d^-\eta(s)\Big) &= \partial_t\Big(\eta(0)\varphi_i(t) - \int_{-t}^0 \eta(s)\dot\varphi_i(s+t)\, ds\Big) \\ &= \eta(0)\dot\varphi_i(t) - \eta(-t)\dot\varphi_i(0^+) - \int_{-t}^0 \eta(s)\ddot\varphi_i(s+t)\, ds = I_i. \end{aligned}$$

In order to evaluate the derivatives of u with respect to η, we observe that, by (3.9) and (3.10), we get

$$u(t,\eta) = \Psi\Big(t, \eta(0)\varphi_1(t) - \int_0^t \eta(s-t)\dot\varphi_1(s)\, ds, \dots, \eta(0)\varphi_n(t) - \int_0^t \eta(s-t)\dot\varphi_n(s)\, ds\Big).$$

For every $t \in [0,T]$, $\eta \in C([-T,0])$, the first derivative Du evaluated at (t,η) is the measure on $[-T,0]$ defined by

$$D_{dx}u(t,\eta) = D^{ac}_x u(t,\eta)\, dx + D^{\delta_0}u(t,\eta)\, \delta_0(dx)$$

with

$$\begin{aligned} D^{ac}_x u(t,\eta) &= -\sum_{i=1}^n \Big(\partial_i \Psi\Big(t, \int_{[-t,0]} \varphi_1(s+t)\, d^-\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t)\, d^-\eta(s)\Big)\Big) \cdot (\mathbb{1}_{[-t,0]}(x)\dot\varphi_i(x+t)), \\ D^{\delta_0} u(t,\eta) &= \sum_{i=1}^n \Big(\partial_i \Psi\Big(t, \int_{[-t,0]} \varphi_1(s+t)\, d^-\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t)\, d^-\eta(s)\Big)\Big) \cdot \varphi_i(t). \end{aligned}$$

As anticipated, we observe that $x \mapsto D^{ac}_x u(t,\eta)$ has bounded variation.

Deriving again in a similar way, for every $t \in [0, T]$, $\eta \in C([-T, 0])$, the second-order derivative $D^2 u$ evaluated at (t, η) gives

$$D^2_{dx,dy} u(t,\eta) = \sum_{i,j=1}^{n} \Big(\partial^2_{i,j}\Psi\Big(t, \int_{[-t,0]} \varphi_1(s+t)\,d^-\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t)\,d^-\eta(s)\Big)\Big)$$
$$\cdot \big(\varphi_i(t)\varphi_j(t)\,\delta_0(dx)\,\delta_0(dy) - \varphi_i(t)\,\mathbb{1}_{[-t,0]}(x)\ddot{\varphi}_j(x+t)\,\delta_0(dy)$$
$$- \varphi_j(t)\,\mathbb{1}_{[-t,0]}(y)\ddot{\varphi}_i(y+t)\,\delta_0(dx) + \mathbb{1}_{[-t,0]}(x)\,\mathbb{1}_{[-t,0]}(y)\ddot{\varphi}_i(x+t)\ddot{\varphi}_j(y+t)\big). \tag{3.12}$$

We get

$$\int_{]-t,0]} D^{ac}_x u(t,\eta)\,d^-\eta(x) = \sum_{i=1}^{n} \Big(\partial_i\Psi\Big(t, \int_{[-t,0]} \varphi_1(s+t)\,d^-\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t)\,d^-\eta(s)\Big)\Big) \cdot I_i. \tag{3.13}$$

Using (3.8), (3.11), (3.13) and (3.12), we obtain that $\mathcal{L}u(t,\eta) = 0$. Condition $u(T,\eta) = H(\eta)$ is trivially verified by definition. This concludes the proof of point (1).

Point (2) is immediate. □

Remark 3.5. In this section, we have often used the concept of deterministic forward integral on a closed interval $[-t, 0]$, given in Definition 2.2,

$$\int_{[-t,0]} \varphi_i(s+t)\,d^-\eta(s), \quad \text{instead of} \quad \int_{]-t,0]} \varphi_i(s+t)\,d^-\eta(s). \tag{3.14}$$

Since $W_0 = 0$, the two integrals are the same when we replace $\eta = W_t(\cdot)$, so

$$\int_{[-t,0]} \varphi_i(s+t)\,d^-\eta(s)|_{\eta=W_t(\cdot)} = \int_{]-t,0]} \varphi_i(s+t)\,d^-\eta(s)|_{\eta=W_t(\cdot)}.$$

The choice of the left expression in (3.14), which is compatible with the fact of considering

$$\int_{]-t,0]} D^{ac}_x u(t,\eta)\,d^-\eta(x)$$

in (1.1), is justified since

$$t \mapsto \int_{]-t,0]} \varphi_i(s+t)\,d^-\eta(s)$$

is not differentiable.

4 The existence result for smooth Fréchet terminal condition

4.1 Preliminary considerations

In this section, we will prove an existence theorem for classical solutions of (1.1) under smooth Fréchet terminal condition. In order to define explicitly the solution of the PDE, we need to introduce two central objects for this section: the Brownian stochastic flow which is a real-valued stochastic flow denoted by $(X^{s,x}_t)_{0\le s\le t\le T,\, x\in\mathbb{R}}$ and the functional Brownian stochastic flow which is a $C([-T, 0])$-valued stochastic flow denoted by $(Y^{s,\eta}_t)_{0\le s\le t\le T,\, \eta\in C([-T,0])}$.

Definition 4.1. Let $\Delta := \{(s,t) : 0 \le s \le t \le T\}$ and $\eta \in C([-T, 0])$. We define the flows that will appear in this section.

(1) We denote by $(X^{s,x}_t)_{0\le s\le t\le T,\, x\in\mathbb{R}}$ the real-valued random field defined by

$$(s,t,x) \mapsto X^{s,x}_t = x + \sigma(W_t - W_s). \tag{4.1}$$

This will be called *Brownian stochastic flow*.

(2) We denote by $(Y_t^{s,\eta})_{0\le s\le t\le T,\,\eta\in C([-T,0])}$ the $C([-T,0])$-valued random field defined by

$$(s,t,\eta)\mapsto Y_t^{s,\eta}(x)=\begin{cases}\eta(x+t-s), & x\in[-T,s-t[,\\ \eta(0)+\sigma(W_t(x)-W_s), & x\in[s-t,0].\end{cases} \tag{4.2}$$

This will be called *functional Brownian stochastic flow*.

Let $H\colon C([-T,0])\to\mathbb{R}$ be the functional appearing in equation (1.1) and a path-dependent random variable $h:=H(\sigma W_T(\cdot))$. We define the functional $u\colon[0,T]\times C([-T,0])\to\mathbb{R}$ by

$$u(t,\eta)=\mathbb{E}[H(Y_T^{t,\eta})]. \tag{4.3}$$

Since $\sigma W_T(\cdot)=Y_T^{t,\sigma W_t(\cdot)}$, we have

$$\mathbb{E}(h|\mathcal{F}_t)=\mathbb{E}[H(\sigma W_T(\cdot))|\mathcal{F}_t]=\mathbb{E}[H(Y_T^{t,\sigma W_t(\cdot)})|\mathcal{F}_t]=u(t,\sigma W_t(\cdot)).$$

For this reason, u defined in (4.3) is a natural candidate to be a solution of (1.1). In Theorem 4.11, we will show, under smooth regularity of H, that such a u is sufficiently smooth to be a classical solution of the path-dependent heat equation (1.1).

We dedicate the next two subsections to investigate some properties of $Y_T^{t,\eta}$ that we will use in the proof of the main theorem. Section 4.2 below contains the general results for the flows introduced in Definition 4.1. In Section 4.3, we will introduce the Markovian stochastic flow for a general $\sigma\colon[0,T]\times\mathbb{R}\to\mathbb{R}$, which coincides with the Brownian stochastic flow when σ is constant. We will derive some properties for this flow that we need in the theorem. We recall that, given X and Y two random elements taking values in the same space, we write $X\sim Y$ if they have the same law. From now on, a realization $\omega\in\Omega$ will be often fixed.

4.2 Some properties of the Brownian (resp. functional Brownian) flow

First of all, we observe that the functional Brownian stochastic flow is time-homogeneous in law.

Proposition 4.2. *$Y_t^{s,\eta}$ and $Y_{t-s}^{0,\eta}$ have the same law as $C([-T,0])$-valued random variables. In particular, for every $x\in[-T,0]$, we have $Y_{t-s}^{0,\eta}(x)\sim Y_t^{s,\eta}(x)$.*

Proof. It follows from the two following arguments. For $x\in[-T,s-t]$, $Y_t^{s,\eta}(x)$ and $Y_{t-s}^{0,\eta}(x)$ are deterministic and are equal to $\eta(x+t-s)$. For $x\in[s-t,0]$, the real-valued processes $Y_t^{s,\eta}(x)=\eta(0)+\sigma(W_t(x)-W_s)$ and $Y_{t-s}^{0,\eta}(x)=\eta(0)+\sigma(W_{t-s}(x)-W_0)$ have the same law by well-known properties of Brownian motion. □

The next proposition concerns the continuity of the field $Y_t^{s,\eta}$ with respect to its three variables.

Proposition 4.3. $(Y_t^{s,\eta})_{0\le s\le t\le T,\,\eta\in C([-T,0])}$ *is a continuous random field.*

Proof. As usual in this section, $\omega\in\Omega$ is fixed, and ϖ_η (resp. $\varpi_{W(\omega)}$) is the modulus of continuity of η (resp. the Brownian path $W(\omega)$).

Let (s,t,η) be such that $0\le s\le t\le T$, $\eta\in C([-T,0])$, and let a sequence (s_n,t_n,η_n) be also such that $0\le s_n\le t_n\le T$, $\eta_n\in C([-T,0])$ with

$$\lim_{n\to\infty}(|s-s_n|+|t-t_n|+\|\eta-\eta_n\|_\infty)=0.$$

We have to show that $Y_{t_n}^{s_n,\eta_n}\to Y_t^{s,\eta}$ in $C([0,T]$ when $n\to\infty$ i.e. uniformly. For $x\in[0,T]$, we evaluate

$$|Y_{t_n}^{s_n,\eta_n}-Y_t^{s,\eta}|(x)\le(I_1(n)+I_2(n)+I_3(n))(x),$$

where

$$\begin{aligned}I_1(n)(x)&=|Y_{t_n}^{s_n,\eta_n}-Y_{t_n}^{s_n,\eta}|(x),\\ I_2(n)(x)&=|Y_{t_n}^{s,\eta}-Y_t^{s,\eta}|(x),\\ I_3(n)(x)&=|Y_{t_n}^{s_n,\eta}-Y_{t_n}^{s,\eta}|(x).\end{aligned}$$

By Definition 4.1, it is easy to see that $\|I_1(n)\|_\infty \le \|\eta - \eta_n\|_\infty + |\eta_n(0) - \eta(0)| \le 2\|\eta - \eta_n\|_\infty$. Since $I_3(n)$ behaves similarly to $I_2(n)$, we only show that $\lim_{n\to\infty} I_2(n) = 0$. Without restriction of generality, we will suppose that $t_n \le t$ for any n since the case when the sequence (t_n) is greater or equal than t could be treated analogously. We observe that the following equality holds:

$$\begin{aligned}(Y^{s,\eta}_{t_n} - Y^{s,\eta}_t)(x) &= \eta(x + t_n - s)\,\mathbb{1}_{[-T,s-t_n]}(x) - \eta(x + t - s)\,\mathbb{1}_{[-T,s-t]}(x)\\ &\quad + (\eta(0) + \sigma W_{t_n}(x) - \sigma W_s)\,\mathbb{1}_{[s-t_n,0]}(x)\\ &\quad - (\eta(0) + \sigma W_t(x) - \sigma W_s)\,\mathbb{1}_{[s-t,0]}(x)\\ &= (\eta(x + t_n - s) - \eta(x + t - s))\,\mathbb{1}_{[-T,s-t]}(x)\\ &\quad + (\eta(x + t_n - s) - \eta(0) - \sigma W_t(x) + \sigma W_s)\,\mathbb{1}_{[s-t,s-t_n]}(x)\\ &\quad + (\sigma W_{t_n}(x) - \sigma W_t(x))\,\mathbb{1}_{[s-t_n,0]}(x).\end{aligned} \tag{4.4}$$

Using (4.4) to evaluate $\|I_2(n)\|_\infty$, we obtain

$$\begin{aligned}\sup_{x\in[-T,0]} |Y^{s,\eta}_{t_n}(x) - Y^{s,\eta}_t(x)| &\le \sup_{x\in[-T,0]} |\eta(x + t_n - s) - \eta(x + t - s)|\\ &\quad + \sup_{x\in[s-t,s-t_n]} |\eta(x + t_n - s) - \eta(0)|\\ &\quad + \sup_{x\in[s-t,s-t_n]} \sigma|W_t(x) - W_s|\\ &\quad + \sup_{x\in[-T,0]} \sigma|W_{t_n}(x) - W_t(x)|\\ &\le 2\varpi_\eta(|t_n - t|) + 2\sigma\varpi_{W(\omega)}(|t_n - t|) \xrightarrow[n\to+\infty]{} 0.\end{aligned}$$

Since η and $W(\omega)$ are uniformly continuous on the compact set $[0, T]$, both moduli of continuity converge to zero when $t_n \to t_0$. □

At this point, we make some simple observations that will be often used in the sequel.

Remark 4.4. (1) There are universal constants C_1, C_2, C_3 and C_4 such that, for every $t \in [0, T]$, $\epsilon > 0$ with $t + \epsilon \in [0, T]$, it holds

$$\begin{aligned}\|Y^{t,\eta}_T\|_\infty &\le C_1(1 + \|\eta\|_\infty + \sup_{t\in[0,T]} \sigma|W_t|),\\ \|Y^{t+\epsilon,\eta}_T\|_\infty &\le C_2(1 + \|\eta\|_\infty + \sigma \sup_{t\in[0,T]} |W_t|)\end{aligned} \tag{4.5}$$

and

$$\|Y^{T-t,\eta}_0\|_\infty \le C_3(1 + \|\eta\|_\infty + \sigma \sup_{t\in[0,T]} |W_t|). \tag{4.6}$$

Further, (4.5) implies that, for any $\alpha \in [0, 1]$, $t \in [0, T]$, ϵ with $t + \epsilon \in [0, T]$, it holds

$$\|\alpha Y^{t+\epsilon,\eta}_T + (1 - \alpha) Y^{t+\epsilon, Y^{t,\eta}_{t+\epsilon}}_T\|_\infty \le C_4(1 + \|\eta\|_\infty + \sigma \sup_{t\in[0,T]} |W_t|). \tag{4.7}$$

(2) For any $\alpha \in [0, 1]$, $t \in [0, T]$, it holds

$$\alpha Y^{t+\epsilon,\eta}_T + (1 - \alpha) Y^{t+\epsilon, Y^{t,\eta}_{t+\epsilon}}_T \xrightarrow[\epsilon\to 0]{C([-T,0])} Y^{t,\eta}_T. \tag{4.8}$$

In fact, expanding the term $Y^{t+\epsilon, Y^{t,\eta}_{t+\epsilon}}_T$, which equals $Y^{t,\eta}_T$, we obtain

$$\|\alpha Y^{t+\epsilon,\eta}_T + (1 - \alpha) Y^{t+\epsilon, Y^{t,\eta}_{t+\epsilon}}_T - Y^{t,\eta}_T\|_\infty = \alpha\|Y^{t+\epsilon,\eta}_T - Y^{t,\eta}_T\|_\infty.$$

The right-hand side converges to zero because of Proposition 4.3.

(3) In the sequel, we will make explicit use of the expression

$$(Y^{t+\epsilon,\eta}_T - Y^{t,\eta}_T)(x) = \begin{cases} \eta(x + T - t + \epsilon) - \eta(x + T - t), & x \in [-T, t - T],\\ \eta(x + T - t + \epsilon) - \eta(0) - \sigma W_T(x) + \sigma W_t, & x \in [t - T, t - T + \epsilon],\\ \sigma W_t - \sigma W_{t+\epsilon}, & x \in [t - T + \epsilon, 0].\end{cases}$$

4.3 About Markovian stochastic flow and functional Markovian stochastic flow

The Brownian (resp. functional Brownian) stochastic flow can be generalized considering $\sigma\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ Lipschitz with linear growth, i.e. not necessarily constant. We introduce the Markovian flow, and we show some properties.

Let $\sigma, b\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ be Lipschitz functions with linear growth. Let, for every $s\in[0,T[$, $x\in\mathbb{R}$, $X = X^{s,x}$ be the solution of the SDE

$$X_t = x + \int_s^t \sigma(u, X_u)\,dW_u + \int_s^t b(u, X_u)\,du, \quad t\in[s,T].$$

Let again $\Delta := \{(s,t) : 0\le s\le t\le T\}$. It is well known that the real-valued random field $(s,t,x)\mapsto X_t^{s,x}$ defined over $\Delta\times\mathbb{R}\to\mathbb{R}$ admits a continuous modification.

Definition 4.5 (Stochastic flows).
(1) The random field $(s,t,x)\mapsto X_t^{s,x}$ will be called *Markovian stochastic flow*.
(2) We denote by $(Y_t^{s,\eta})_{0\le s\le t\le T,\eta\in C([-T,0])}$ the random field defined over $\Delta\times C([-T,0])\to C([-T,0])$ by

$$(s,t,\eta)\mapsto Y_t^{s,\eta}(x) = \begin{cases}\eta(x+t-s), & x\in[-T,s-t[,\\ X_t^{s,\eta(0)}, & x\in[s-t,0].\end{cases}$$

This will be called *functional Markovian stochastic flow*.

Remark 4.6. (1) The Brownian flow $(X_t^{s,x})$ introduced in Definition 4.1 is a particular case of the Markovian flow when $\sigma(t,x)=\sigma$, σ a constant. We could have formulated this chapter in this more general framework, but for simplicity of exposition, we have restricted us to the case σ constant.

(2) The Markovian stochastic flow verifies the flow property for $0\le s\le t\le r\le T$,

$$X_r^{s,x} = X_r^{t,X_t^{s,x}}. \tag{4.9}$$

We set

$$Y_t^{s,\eta}(x) = \begin{cases}\eta(x+t-s), & x\in[-T,s-t],\\ X_{t+x}^{s,\eta(0)}, & x\in[s-t,0].\end{cases} \tag{4.10}$$

The functional flow $(Y_t^{s,\eta})$ coincides of course with (4.2) when $(X_t^{s,x})$ is given by (4.1).

The following lemma shows a "flow property" for the functional flow.

Lemma 4.7. *Let $\eta\in C([-T,0])$ for $0\le s\le t\le r\le T$. Then*

$$Y_r^{s,\eta} = Y_r^{t,Y_t^{s,\eta}}. \tag{4.11}$$

Proof. It follows from the flow property (4.9) for the Markovian stochastic flow. For fixed $\omega\in\Omega$, we inject $\tilde\eta = Y_s^{t,\eta}$ into $Y_r^{t,\tilde\eta}$ obtaining

$$Y_r^{t,Y_t^{s,\eta}}(x) = \left.\begin{cases}\eta(x+r-s), & x\in[-T,s-r],\\ X_{r+x}^{s,\eta(0)}, & x\in[s-r,t-r],\\ X_{r+x}^{t,\tilde\eta(0)} = X_{r+x}^{t,X_t^{s,\eta(0)}} = X_{r+x}^{s,\eta(0)}, & x\in[t-r,0],\end{cases}\right\} = Y_r^{s,\eta}(x),$$

which concludes the proof of the lemma. □

We concentrate now on the derivatives of the functional Markovian stochastic flow. Let $t\in[0,T[$.

By (4.10), we remind that

$$Y_T^{t,\eta}(\rho) = \begin{cases}\eta(\rho+T-t), & \rho\in[-T,t-T[,\\ X_{T+\rho}^{t,\eta(0)}, & \rho\in[t-T,0].\end{cases}$$

It is possible to calculate formally the first and second derivatives of $Y_T^{t,\eta}(\rho)$ for $\rho\in[-T,0]$.

Remark 4.8. For $\rho \in [-T, 0]$, then

$$Y_T^{t,\cdot}(\rho)\colon C([-T,0]) \times \Omega \to \mathbb{R},$$
$$DY_T^{t,\cdot}(\rho)\colon C([-T,0]) \times \Omega \to (C([-T,0]))^* = \mathcal{M}([-T,0]).$$

In particular, if $f \in C([-T, 0])$,

$$_{\mathcal{M}([-T,0])}\langle DY_T^{t,\eta}(\rho), f\rangle_{C([-T,0])} = \int_{[-T,0]} f(x)\, D_{dx} Y_T^{t,\eta}(\rho,\omega).$$

In particular, we have

$$D_{dx} Y_T^{t,\eta}(\rho) = \begin{cases} \delta_{\rho+T-t}(dx), & \rho \in [-T, t-T[,\\ \delta_0(dx)\partial_\xi X_{T+\rho}^{t,\eta(0)}, & \rho \in [t-T, 0], \end{cases} \tag{4.12}$$

and

$$D^2_{dy\,dx} Y_T^{t,\eta}(\rho) = \begin{cases} 0, & \rho \in [-T, t-T[,\\ \delta_0(dx)\,\delta_0(dy)\partial^2_{\xi\xi} X_{T+\rho}^{t,\eta(0)}, & \rho \in [t-T, 0]. \end{cases}$$

Avoiding some technicalities, it is possible to evaluate the first and second derivatives of the functional flow itself. In the sequel, η will always be a generic element in $C([-T, 0])$. Let $(X_t^{s,x})$ be the real stochastic flow as in (4.1) and the associated functional stochastic flow $(Y_t^{s,\eta})$ as in Definition 4.1.

Lemma 4.9. *Let $t \in [0, T[$.*
(1) *The map $Y_T^{t,\cdot}\colon C([-T,0]) \times \Omega \to C([-T,0])$ acting as $\eta \mapsto Y_T^{t,\eta}$ is of class $C^2(C([-T,0]); C([-T,0]))$ a.s.*
(2) *The derivatives*

$$DY_T^{t,\cdot}\colon C([-T,0]) \times \Omega \to \mathcal{L}(C([-T,0]); C([-T,0])),$$
$$D^2 Y_T^{t,\cdot}\colon C([-T,0]) \times \Omega \to \mathcal{B}(C([-T,0]) \times C([-T,0]); C([-T,0]))$$

are characterized as follows. For $f, g \in C([-T, 0])$, we have

$$\begin{aligned} \rho &\mapsto \int_{[-T,0]} D_{dx} Y_T^{t,\eta}(\rho) f(x) = \begin{cases} f(\rho+T-t), & \rho \in [-T, t-T[,\\ f(0)\partial_\xi X_{T+\rho}^{t,\eta(0)}, & \rho \in [t-T, 0], \end{cases}\\ \rho &\mapsto \int_{[-T,0]^2} D^2_{dy\,dx} Y_T^{t,\eta}(\rho) f(x) g(y) = \begin{cases} 0, & \rho \in [-T, t-T[,\\ f(0)g(0)\partial^2_{\xi\xi} X_{T+\rho}^{t,\eta(0)}, & \rho \in [t-T, 0]. \end{cases} \end{aligned} \tag{4.13}$$

In the remark below, we express Lemma 4.9 in the case of the functional Brownian flow.

Remark 4.10. When $\sigma(t, x) \equiv \sigma$ is a constant, by (4.1), the following holds.
(1)

$$\partial_\xi X_t^{s,\xi} = 1 \quad \text{and} \quad \partial^2_{\xi\xi} X_s^{t,\xi} = 0. \tag{4.14}$$

(2) By (4.14), the derivatives given by (4.13) for the functional Brownian flow reduce to

$$\rho \mapsto \int_{[-T,0]} D_{dx} Y_T^{t,\eta}(\rho) f(x) = \begin{cases} f(\rho+T-t), & \rho \in [-T, t-T[,\\ f(0), & \rho \in [t-T, 0], \end{cases}$$
$$\rho \mapsto \int_{[-T,0]^2} D^2_{dy\,dx} Y_T^{t,\eta}(\rho) f(x) g(y) = 0, \quad \rho \in [-T, 0].$$

4.4 The existence result for smooth Fréchet terminal condition

In this section, Theorem 4.11 states the existence result and Fréchet regularity of the solution of the infinite-dimensional PDE (1.1) when σ is constant and H is $C^3(C([-T, 0]))$. In particular, we will give conditions on the function H such that u defined in (4.3) solves the PDE stated on (1.1). Those conditions are reasonable, but they are however not optimal.

Theorem 4.11. *Let $H \in C^3(C([-T,0]))$ such that D^3H has polynomial growth (for instance bounded). Let u be defined by $u(t,\eta) = \mathbb{E}[H(Y_T^{t,\eta})]$, $t \in [0,T]$, $\eta \in C([-T,0])$.*

(1) *Then $u \in C^{0,2}([0,T] \times C([-T,0]))$.*

(2) *Suppose moreover the following for every $\eta \in C([-T,0])$:*

(i) *The measure $D_{dx}H(\eta)$ is Lebesgue absolutely continuous. We will denote by $x \mapsto D_xH(\eta)$ its density, and we suppose that $DH(\eta) \in H^1([-T,0])$, i.e. the function $x \mapsto D_xH(\eta)$ is in $H^1([-T,0])$.*

(ii) *DH has polynomial growth in $H^1([-T,0])$, i.e. there is $p \geq 1$ such that*

$$\eta \mapsto \|DH(\eta)\|_{H^1} \leq \text{const}\,(\|\eta\|_\infty^p + 1). \tag{4.15}$$

In particular,

$$\sup_{t\in[-T,0]} |D_xH(\eta)| \leq \text{const}\,(\|\eta\|_\infty^p + 1) \leq \text{const}\,(\|\eta\|_\infty^p + 1).$$

(iii) *The map*

$$\eta \mapsto DH(\eta) \text{ considered as } C([-T,0]) \to H^1([-T,0]) \text{ is continuous.} \tag{4.16}$$

Then $u \in C^{1,2}([0,T] \times C([-T,0]))$ and u is a classical solution of (1.1) in $C([-T,0])$, i.e. u solves

$$\begin{cases} \partial_t u(t,\eta) + \int_{]-t,0]} D_{dx}^{\perp} u(t,\eta)\, d^-\eta(x) + \frac{1}{2}\sigma^2 \langle D^2u(t,\eta), \mathbb{1}_{\{0\}} \otimes^2 \rangle = 0, \\ u(T,\eta) = H(\eta). \end{cases}$$

Remark 4.12. Contrarily to the (non-degenerate) situation of Section 3, Theorem 4.11 holds even when $\sigma = 0$. In that case, one gets a first-order equation; the regularity on H could be relaxed, but we are not specifically interested in this refinement.

Remark 4.13. (1) Assumption (4.15) implies in particular that DH has polynomial growth in $C([-T,0])$, i.e. there is $p \geq 1$ such that

$$\eta \mapsto \sup_{x\in[-T,0]} |D_xH(\eta)| = \|DH(\eta)\|_\infty \leq \text{const}\,(\|\eta\|_\infty^p + 1). \tag{4.17}$$

Indeed, it is well known that $H^1([-T,0]) \hookrightarrow C([-T,0])$ and for a function $f \in H^1$ it holds $\|f\|_\infty \leq \text{const}\,\|f\|_{H^1}$.

(2) By a Taylor's expansion, given for instance by [1, Theorem 5.6.1], the fact that D^3H has polynomial growth implies that H, DH and D^2H have also polynomial growth in $C([-T,0])$.

(3) $Du(t,\eta)$, $D^2u(t,\eta)$ and $\partial_t u(t,\eta)$ will be explicitly expressed in term of H at (4.21), (4.23) and (4.51).

Proof. By expression (4.3), it is obvious that $u(T,\eta) = H(\eta)$.

Proof of (1). *Continuity of function u with respect to time t.* We consider a sequence

$$(t_n) \text{ in } [0,T] \quad \text{such that} \quad t_n \xrightarrow[n\to\infty]{} t_0.$$

By assumption, $H \in C^0(C([-T,0]))$. Consequently, by Proposition 4.3,

$$H(Y_{T-t_n}^{0,\eta}) \xrightarrow[n\to\infty]{\text{a.s.}} H(Y_{T-t_0}^{0,\eta}). \tag{4.18}$$

By Remark 4.13 (1), H has also polynomial growth; therefore, there is $p \geq 1$ such that

$$|H(\zeta)| \leq \text{const}\,(1 + \sup_{x\in[-T,0]} |\zeta(x)|^p) \quad \text{for all } \zeta \in C([-T,0]).$$

By (4.6), we observe that

$$\begin{aligned} |H(Y_{T-t}^{0,\eta})| &\leq \text{const}\,(1 + \|Y_{T-t}^{0,\eta}\|_\infty^p) \\ &\leq \text{const}\,(1 + \sup_{x\in[-T,0]} |\eta(x)|^p + \sigma^p \sup_{t\leq T} |W_t|^p). \end{aligned}$$

By the Lebesgue dominated convergence theorem, the fact that $\sup_{t\le T}|W_t|^p$ is integrable and (4.18), it follows that

$$u(t_n,\eta)=\mathbb{E}[H(Y^{0,\eta}_{T-t_n})]\xrightarrow[n\to\infty]{}\mathbb{E}[H(Y^{0,\eta}_{T-t_0})]=u(t_0,\eta).$$

First-order Fréchet derivative. We express now the derivatives of u with respect to the derivatives of H. We start with $Du\colon[0,T]\times C([-T,0])\to\mathcal{M}([-T,0])$. Omitting some details, by integration theory for every $t\in[0,T]$, $u(t,\cdot)$ is of class $C^1(C([-T,0]))$. By usual derivation rules for composition, we have

$$D_{dx}H(Y^{t,\eta}_T)=\int_{[-T,0]}D_{d\rho}H(Y^{t,\eta}_T)\,D_{dx}Y^{t,\eta}_T(\rho),$$

and

$$D_{dx}u(t,\eta)=\mathbb{E}[D_{dx}H(Y^{t,\eta}_T)]=\mathbb{E}\bigg[\int_{[-T,0]}D_{d\rho}H(Y^{t,\eta}_T)\,D_{dx}Y^{t,\eta}_T(\rho)\bigg].\tag{4.19}$$

We compute explicitly (4.19) using expression (4.12). Integrating with respect to ρ (for a fixed x), we obtain the following:

$$\begin{aligned}D_{dx}u(t,\eta)&=\mathbb{E}\bigg[\int_{[-T,t-T[}D_{d\rho}H(Y^{t,\eta}_T)\,D_{dx}Y^{t,\eta}_T(\rho)\bigg]+\mathbb{E}\bigg[\int_{[t-T,0]}D_{d\rho}H(Y^{t,\eta}_T)\,D_{dx}Y^{t,\eta}_T(\rho)\bigg]\\&=\mathbb{E}\bigg[\int_{[-T,t-T[}D_{d\rho}H(Y^{t,\eta}_T)\,\delta_{\rho+T-t}(dx)\bigg]+\mathbb{E}\bigg[\int_{[t-T,0]}D_{d\rho}H(Y^{t,\eta}_T)\bigg]\,\delta_0(dx).\end{aligned}\tag{4.20}$$

Consequently,

$$D_{dx}u(t,\eta)=D^{\perp}_{dx}u(t,\eta)+D^{\delta_0}u(t,\eta)\,\delta_0(dx),\tag{4.21}$$

where

$$D^{\perp}_{dx}u(t,\eta)=\mathbb{E}[D_{dx-T+t}H(Y^{t,\eta}_T)]\,\mathbb{1}_{[-t,0[}(x),\tag{4.22}$$

and

$$D^{\delta_0}u(t,\eta)=\mathbb{E}\bigg[\int_{[t-T,0]}D_{d\rho}H(Y^{t,\eta}_T)\bigg].$$

Indeed, the first addend $D^{\perp}_{dx}u(t,\eta)$ of (4.21), i.e. expression (4.22), comes from (4.20), using the fact that $\delta_{\rho+T-t}(dx)=\delta_{dx-T+t}(d\rho)$ and integrating with respect to ρ. The continuity of $(t,\eta)\mapsto D_{dx}u(t,\eta)$ in (4.21) can be justified since the functions

$$\begin{aligned}&[0,T]\times C([-T,0])\to\mathbb{R}, &&(t,\eta)\mapsto D^{\delta_0}u(t,\eta),\\&[0,T]\times C([-T,0])\to\mathcal{M}([-T,0]), &&(t,\eta)\mapsto D^{\perp}u(t,\eta)\end{aligned}$$

are both continuous. The latter fact follows from the fact that $H\in C^1(C([-T,0]))$, DH with polynomial growth, (4.6), (4.5), the fact that, for any given Brownian motion $\bar W$, $\sup_{x\le T}|\bar W_x|$ has all moments and finally the Lebesgue dominated convergence theorem.

Second-order Fréchet derivative. We discuss the second derivative

$$D^2u\colon[0,T]\times C([-T,0])\to\big(C([-T,0])\,\hat\otimes_\pi\,C([-T,0])\big)^*\cong\mathcal{B}\big(C([-T,0]),C([-T,0])\big).$$

For every fixed (t,η), we get

$$\begin{aligned}D^2_{dx,dy}u(t,\eta)=\;&\mathbb{E}[D^2_{dy-T+t,dx-T+t}H(Y^{t,\eta}_T)\,\mathbb{1}_{[-t,0[}(x)\otimes\mathbb{1}_{[-t,0[}(y)]\\&+\mathbb{E}[D_{dx-T+t}\langle DH(Y^{t,\eta}_T),\mathbb{1}_{[t-T,0]}\rangle]\,\mathbb{1}_{[-t,0[}(x)\,\delta_0(dy)\\&+\mathbb{E}[D_{dy-T+t}\langle DH(Y^{t,\eta}_T),\mathbb{1}_{[t-T,0]}\rangle]\,\mathbb{1}_{[-t,0[}(y)\,\delta_0(dx)\\&+\mathbb{E}[\langle D^2H(Y^{t,\eta}_T),\mathbb{1}_{[t-T,0]}\otimes\mathbb{1}_{[t-T,0]}\rangle]\,\delta_0(dx)\,\delta_0(dy).\end{aligned}$$

It is possible to show that all the terms in the first and the second derivative are well defined and continuous using similar techniques used in the first part of the proof. We omit these technicalities for simplicity.

Remark 4.14. For illustration, if D^2H is an absolutely continuous Borel measure on $[-T,0]^2$ with density $D^2_{x,y}H = D_x D_y H$, we obtain the following:

$$\begin{aligned} D^2_{dx,dy}u(t,\eta) &= \mathbb{E}[D_{y-T+t}\,D_{x-T+t}H(Y_T^{t,\eta})]\,\mathbb{1}_{[-t,0[}(x)\,\mathbb{1}_{[-t,0[}(y)\,dx\,dy \\ &\quad + \mathbb{E}\Bigg[\int_{t-T}^{0} D_s\,D_{x-T+t}H(Y_T^{t,\eta})\,ds\Bigg]\,\mathbb{1}_{[-t,0[}(x)\,dx\,\delta_0(dy) \\ &\quad + \mathbb{E}\Bigg[\int_{t-T}^{0} D_{y-T+t}D_sH(Y_T^{t,\eta})\,ds\Bigg]\,\mathbb{1}_{[-t,0[}(y)\,dy\,\delta_0(dx) \\ &\quad + \mathbb{E}\Bigg[\int_{[t-T,0]^2} D_{s_1}D_{s_2}H(Y_T^{t,\eta})\,ds_1\,ds_2\Bigg]\,\delta_0(dx)\,\delta_0(dy). \end{aligned} \tag{4.23}$$

Proof of (2).

Remark 4.15. Under hypothesis (2), we remark the following.

(1) The right-hand side of (4.22) is absolutely continuous in x. In other words, $D^{\perp}_{dx}u(t,\eta) = D^{ac}_x u(t,\eta)\,dx$ and

$$D^{ac}_x u(t,\eta) = \mathbb{E}[D_{x-T+t}H(Y_T^{t,\eta})]\,\mathbb{1}_{[-t,0[}(x) = \begin{cases} 0, & x \in [-T,-t[, \\ \mathbb{E}[D_{x-T+t}H(Y_T^{t,\eta})], & x \in [-t,0[. \end{cases} \tag{4.24}$$

(2) In particular, by item (ii), $x \mapsto D_xH(\eta)$ belongs to H^1, so it has bounded variation. Therefore, the deterministic forward integral in (1.1) exists because of Proposition 2.3, and it can be expressed through (2.4). We will denote by $D'H(\eta)$ the derivative in x of function $x \mapsto D_xH(\eta)$, where $D_xH(\eta)$ is the density of the measure $D_{dx}H(\eta)$ for every fixed η. Since $x \mapsto D_xH(\eta)$ is absolutely continuous then, by (2.2), we have

$$\int_{]-t,0]} D_{dx-T+t}H(Y_T^{t,\eta})\,d^-\eta(x) = \int_{]-t,0]} D_{x-T+t}H(Y_T^{t,\eta})\,d^-\eta(x). \tag{4.25}$$

Previous deterministic integral exists because $x \mapsto D_xH(\eta)$ has bounded variation, and by Proposition 2.3, it equals

$$-D_{-T}H(Y_T^{t,\eta})\eta(-t) + D_{t-T}H(Y_T^{t,\eta})\eta(0) - \int_{-t}^{0} D'_{x-T+t}H(Y_T^{t,\eta})\eta(x)\,dx.$$

Derivability with respect to time t. Let $t \in [0,T]$, $\eta \in C([-T,0)]$. We will show that

$$\partial_t u(t,\eta) = -\mathbb{E}\Bigg[\int_{]-t,0]} D_{x-T+t}H(Y_T^{t,\eta})\,d^-\eta(x) + \frac{\sigma^2}{2}\langle D^2H(Y_T^{t,\eta}), 1_{]t-T,0]}\otimes^2\rangle\Bigg].$$

We need to consider ϵ such that $t+\epsilon \in [0,T]$ and evaluate the limit when $\epsilon \to 0$, if it exists, of

$$\frac{u(t+\epsilon,\eta) - u(t,\eta)}{\epsilon} \tag{4.26}$$

Without restriction of generality, we will suppose here $\epsilon > 0$; the case $\epsilon < 0$ would bring similar calculations.

The flow property (4.11) gives $Y_T^{t,\eta} = Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}$, which allows to write

$$u(t,\eta) = \mathbb{E}[H(Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}})]. \tag{4.27}$$

We go on with the evaluation of the limit of (4.26). By (4.27) and by differentiability of H in $C([-T,0])$, we have

$$\begin{aligned} H(Y_T^{t+\epsilon,\eta}) - H(Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}) &= \langle DH(Y_T^{t,\eta}), Y_T^{t+\epsilon,\eta} - Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}\rangle \\ &\quad + \int_0^1 \langle DH(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}) - DH(Y_T^{t,\eta}), Y_T^{t+\epsilon,\eta} - Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}\rangle\,d\alpha \\ &= \int_{[-T,0]} D_{dx}H(Y_T^{t,\eta})(Y_T^{t+\epsilon,\eta}(x) - Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}(x)) + S(\epsilon,t,\eta), \end{aligned} \tag{4.28}$$

where

$$S(\epsilon,t,\eta)=\int_0^1 \langle DH(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}})-DH(Y_T^{t,\eta}),\,Y_T^{t+\epsilon,\eta}-Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}\rangle\,d\alpha.$$

Setting $\gamma=Y_{t+\epsilon}^{t,\eta}$, we need to evaluate

$$Y_T^{t+\epsilon,\eta}(x)-Y_T^{t+\epsilon,\gamma}(x),\quad x\in[-T,0]. \tag{4.29}$$

Then (4.29) gives

$$Y_T^{t+\epsilon,\eta}(x)-Y_T^{t+\epsilon,\gamma}(x)=\begin{cases}\eta(x+T-t-\epsilon)-\gamma(x+T-t-\epsilon), & x\in[-T,t-T+\epsilon[,\\ \eta(0)-\gamma(0)=-\sigma(W_{t+\epsilon}(0)+W_t), & x\in[t-T+\epsilon,0],\end{cases} \tag{4.30}$$

because $\gamma(0)=Y_{t+\epsilon}^{t,\eta}(0)=\eta(0)+\sigma(W_{t+\epsilon}(0)-W_t)$. Moreover, by (4.10), we have

$$\gamma(x+T-t-\epsilon)=Y_{t+\epsilon}^{t,\eta}(x+T-t-\epsilon)=\begin{cases}\eta(x+T-t), & x\in[-T,t-T[,\\ \eta(0)+\sigma(W_T(x)-W_t), & x\in[t-T,t-T+\epsilon].\end{cases}$$

Finally, we obtain an explicit expression for (4.29); indeed, (4.30) gives

$$Y_T^{t+\epsilon,\eta}(x)-Y_T^{t+\epsilon,\gamma}(x)=\begin{cases}\eta(x+T-t-\epsilon)-\eta(x+T-t), & x\in[-T,t-T[,\\ \eta(x+T-t-\epsilon)-\eta(0)-\sigma(W_T(x)+W_t), & x\in[t-T,t-T+\epsilon[,\\ \sigma(W_t-W_{t+\epsilon}), & x\in[t-T+\epsilon,0].\end{cases} \tag{4.31}$$

Consequently, using (4.27), (4.28) and (4.31), the quotient (4.26) appears to be the sum of four terms.

$$\begin{aligned}\frac{u(t+\epsilon,\eta)-u(t,\eta)}{\epsilon}&=\mathbb{E}\Big[\frac{H(Y_T^{t+\epsilon,\eta})-H(Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}})}{\epsilon}\Big]\\ &=I_1(\epsilon,t,\eta)+I_2(\epsilon,t,\eta)+I_3(\epsilon,t,\eta)+\frac{1}{\epsilon}\,\mathbb{E}[S(\epsilon,t,\eta)],\end{aligned} \tag{4.32}$$

where

$$\begin{aligned}I_1(\epsilon,t,\eta)&=\mathbb{E}\Big[\int_{-T}^{t-T}D_xH(Y_T^{t,\eta})\frac{\eta(x+T-t-\epsilon)-\eta(x+T-t)}{\epsilon}\,dx\Big]\\ &=-\mathbb{E}\Big[\int_{-t}^{0}D_{x-T+t}H(Y_T^{t,\eta})\frac{\eta(x)-\eta(x-\epsilon)}{\epsilon}\,dx\Big],\\ I_2(\epsilon,t,\eta)&=\mathbb{E}\Big[\int_{t-T}^{t-T+\epsilon}D_xH(Y_T^{t,\eta})\frac{\eta(x+T-t-\epsilon)-\eta(0)-\sigma(W_T(x)+W_t)}{\epsilon}\,dx\Big]\\ &\quad-\mathbb{E}\Big[\int_{t-T}^{t-T+\epsilon}D_xH(Y_T^{t,\eta})\frac{W_t-W_{t+\epsilon}}{\epsilon}\,dx\Big]\\ &=\mathbb{E}\Big[\int_{t-T}^{t-T+\epsilon}D_xH(Y_T^{t,\eta})\frac{\eta(x+T-t-\epsilon)-\eta(0)-\sigma(W_T(x)+W_{t+\epsilon})}{\epsilon}\,dx\Big],\\ I_3(\epsilon,t,\eta)&=\mathbb{E}\Big[\int_{t-T}^{0}D_xH(Y_T^{t,\eta})\frac{\sigma(W_t-W_{t+\epsilon})}{\epsilon}\,dx\Big],\end{aligned}$$

and $\mathbb{E}[S(\epsilon,t,\eta)]$ is equal to

$$\int_0^1\mathbb{E}\Big[\int_{-T}^{0}(D_xH(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}})-D_xH(Y_T^{t,\eta}))\cdot(Y_T^{t+\epsilon,\eta}(x)-Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}(x))\,dx\Big]\,d\alpha. \tag{4.33}$$

We will prove that

$$I_1(\epsilon,t,\eta) \xrightarrow[\epsilon\to 0]{} I_1(t,\eta) := I_{11}(t,\eta) + I_{12}(t,\eta) + I_{13}(t,\eta), \tag{4.34}$$

where

$$\begin{aligned} I_{11}(t,\eta) &= \mathbb{E}[D_{-T}H(Y_T^{t,\eta})\eta(-t)],\\ I_{12}(t,\eta) &= \mathbb{E}\Big[\int_{-t}^{0} D'_{x-T+t}H(Y_T^{t,\eta})\eta(x)\,dx\Big],\\ I_{13}(t,\eta) &= -\,\mathbb{E}[D_{t-T}H(Y_T^{t,\eta})\eta(0)].\end{aligned}$$

Admitting (4.34), the additivity and using (4.24) in Remark 4.15, we have

$$I_1(t,\eta) = -\,\mathbb{E}\Big[\int_{]-t,0]} D_{x-T+t}H(Y_T^{t,\eta})\,d^-\eta(x)\Big].$$

It remains to show (4.34). In fact, $I_1(\epsilon,t,\eta)$ can be rewritten as sum of the three terms

$$\begin{aligned} I_{11}(\epsilon,t,\eta) &= \mathbb{E}\Big[\int_{-t}^{-t+\epsilon} D_{x-T+t}H(Y_T^{t,\eta})\frac{\eta(x-\epsilon)}{\epsilon}\,dx\Big],\\ I_{12}(\epsilon,t,\eta) &= \mathbb{E}\Big[\int_{-t}^{0} \frac{D_{x+\epsilon-T+t}H(Y_T^{t,\eta}) - D_{x-T+t}H(Y_T^{t,\eta})}{\epsilon}\eta(x)\,dx\Big],\\ I_{13}(\epsilon,t,\eta) &= -\,\mathbb{E}\Big[\int_{0}^{\epsilon} D_{x-T+t}H(Y_T^{t,\eta})\frac{\eta(x-\epsilon)}{\epsilon}\,dx\Big].\end{aligned}$$

We can apply the dominated convergence theorem. Since $\bar W$, $\sup_{x\le T}|\bar W_x|$ has all moments, and taking into account (4.17) in Remark 4.13, we get that $I_{1i}(\epsilon,t,\eta) \xrightarrow[\epsilon\to 0]{} I_{1i}(t,\eta)$ for $i = 1,2,3$ holds.

$I_2(\epsilon,t,\eta)$ converges to zero when $\epsilon\to 0$. Indeed, the Cauchy–Schwarz inequality yields

$$\begin{aligned}|I_2(\epsilon,t,\eta)|^2 \le \frac{1}{\epsilon}\,&\mathbb{E}\Big[\int_{t-T}^{t-T+\epsilon} D_xH(Y_T^{t,\eta})^2\,dx\Big]\\ &\cdot\frac{1}{\epsilon}\,\mathbb{E}\Big[\int_{t-T}^{t-T+\epsilon} (\eta(x+T-t-\epsilon) - \eta(0) - \sigma(W_T(x) + W_{t+\epsilon}))^2\,dx\Big].\end{aligned}$$

Again, by usual arguments and again because $\sup_{x\le T}|\bar W_x|$ has all moments and taking into account (4.17) in Remark 4.13, it follows that the first integral converges to $\mathbb{E}[D_{t-T}H(Y_T^{t,\eta})^2]$ and the second integral to zero.

As third step, we prove that

$$I_3(\epsilon,t,\eta) \xrightarrow[\epsilon\to 0]{} -\sigma^2\,\mathbb{E}[\langle D^2H(Y_T^{t,\eta}), \mathbb{1}_{]t-T,0]}\otimes^2\rangle] =: I_3(t,\eta). \tag{4.35}$$

For this, we rewrite $I_3(\epsilon,t,\eta)$ using (A.1), i.e. $W_{t+\epsilon} - W_t = \overline{W}_\epsilon$ and the Skorohod integral to obtain

$$\begin{aligned} I_3(\epsilon,t,\eta) &= -\sigma\,\mathbb{E}\Big[\int_{t-T}^{0} D_xH(Y_T^{t,\eta})\frac{W_{t+\epsilon}-W_t}{\epsilon}\,dx\Big] = -\frac{\sigma}{\epsilon}\,\mathbb{E}\Big[\int_{t-T}^{0} D_xH(Y_T^{t,\eta})\,dx\cdot\overline{W}_\epsilon\Big]\\ &= -\frac{\sigma}{\epsilon}\,\mathbb{E}\Big[\int_{t-T}^{0} D_xH(Y_T^{t,\eta})\,dx\cdot\int_0^\epsilon \delta\overline{W}_r\Big] = -\frac{\sigma}{\epsilon}\,\mathbb{E}\Big[\mathcal{Z}\cdot\int_0^\epsilon \delta\overline{W}_s\Big],\end{aligned} \tag{4.36}$$

where $\mathcal{Z} := \langle DH(Y_T^{t,\eta}), \mathbb{1}_{]t-T,0]}\rangle$. Denoting by the deterministic function $\mathcal{Y} := 1_{]t-T,0]}(x)$, using Proposition A.4 with $n = 1$, it follows that $\mathcal{Z} = \langle DH(Y_T^{t,\eta}), \mathcal{Y}\rangle$ belongs to $\mathbb{D}^{1,2}$ and

$$D_r^{\mathrm{m}}\mathcal{Z} = \sigma\langle D^2H(Y_T^{t,\eta}), \mathbb{1}_{]t-T,0]}(x)\otimes\mathbb{1}_{]r-T+t,0]}(y)\rangle. \tag{4.37}$$

By integration by parts on Wiener space, expression (4.37), Fubini's theorem with respect to r and y, (4.36) gives

$$\begin{aligned} I_3(\epsilon,t,\eta) &= -\frac{\sigma}{\epsilon}\,\mathbb{E}\Big[\int_0^\epsilon D_r^{\mathrm{m}}\mathcal{Z}\,dr\Big] = -\frac{\sigma^2}{\epsilon}\,\mathbb{E}\Big[\int_0^\epsilon \langle D^2H(Y_T^{t,\eta}), \mathbb{1}_{]r-T+t,0]}(x)\otimes \mathbb{1}_{]t-T,0]}(y)\rangle\,dr\Big] \\ &= -\frac{\sigma^2}{\epsilon}\,\mathbb{E}\Big[\Big\langle D^2H(Y_T^{t,\eta}), \int_0^\epsilon \mathbb{1}_{]r-T+t,0]}(x)\,dr \otimes \mathbb{1}_{]t-T,0]}(y)\Big\rangle\Big] \\ &= -\frac{\sigma^2}{\epsilon}\,\mathbb{E}\Big[\Big\langle D^2H(Y_T^{t,\eta}), \int_t^{t+\epsilon} \mathbb{1}_{]z-T,0]}(x)\,dz \otimes \mathbb{1}_{]t-T,0]}(y)\Big\rangle\Big], \end{aligned} \tag{4.38}$$

where the latter equality comes replacing $z := r + t$ in the integral.

Observing that

$$\begin{aligned} \int_t^{t+\epsilon} \mathbb{1}_{]z-T,0]}(x)\,dz &= \int_t^{t+\epsilon} \mathbb{1}_{[0,x+T[}(z)\,dz \\ &= \begin{cases} \int_t^{t+\epsilon} 0\,dz = 0, & x \le t-T \iff x+T \le t, \\ \int_t^{t+\epsilon} \mathbb{1}_{[0,x+T[}(z)\,dz = x-t, & x \in]t-T, t-T+\epsilon] \iff x+T \in]t, t+\epsilon], \\ \int_t^{t+\epsilon} 1\,dz = \epsilon, & x \in]t-T+\epsilon, 0] \iff x+T \in]t+\epsilon, T], \end{cases} \end{aligned} \tag{4.39}$$

we get

$$\frac{1}{\epsilon}\int_t^{t+\epsilon} \mathbb{1}_{]z-T,0]}(x)\,dz = \mathbb{1}_{]t-T+\epsilon,0]}(x) + \frac{(x-t)}{\epsilon}\,\mathbb{1}_{]t-T,t-T+\epsilon]}(x).$$

The previous expression is bounded by 1. Moreover, it converges pointwise to $\mathbb{1}_{]t-T,0]}(x)$ as $\epsilon \downarrow 0$. By Remark 4.13 (1), the fact that D^2H has polynomial growth and that, for any given Brownian motion $\bar{W}$, $\sup_{x\le T}|\bar{W}_x|$ has all moments and finally the Lebesgue dominated convergence theorem, we conclude that (4.38) converges to $I_3(t,\eta)$, i.e.

$$I_3(t,\eta) = -\sigma^2\,\mathbb{E}[\langle D^2H(Y_T^{t,\eta}), \mathbb{1}_{]t-T,0]}(x)\otimes \mathbb{1}_{]t-T,0]}(y)\rangle].$$

So the convergence (4.35) is established.

We study now the term $\frac{1}{\epsilon}\,\mathbb{E}[S(\epsilon,t,\eta)]$ in (4.33). By Lemma 4.7, we get the a.s. equality $Y_T^{t,\eta} = Y_T^{t+\epsilon, Y_{t+\epsilon}^{t,\eta}}$. Using (4.31) and the fact that $H \in C^3(C([-T,0]))$, (4.33) can be rewritten as the sum of the terms

$$\begin{aligned} A_1(\epsilon,t,\eta) &= \int_0^1 \mathbb{E}\Big[\int_{-T}^{t-T} (D_xH(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}) - D_xH(Y_T^{t,\eta})) \\ &\qquad \cdot \frac{\eta(x+T-t-\epsilon)-\eta(x+T-t)}{\epsilon}\,dx\Big]\,d\alpha, \\ A_2(\epsilon,t,\eta) &= \int_0^1 \mathbb{E}\Big[\int_{t-T}^{t-T+\epsilon} (D_xH(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}) - D_xH(Y_T^{t,\eta})) \\ &\qquad \cdot \frac{\eta(x+T-t-\epsilon)-\eta(0)-\sigma W_T(x)+\sigma W_{t+\epsilon}}{\epsilon}\,dx\Big]\,d\alpha, \\ A_3(\epsilon,t,\eta) &= A_{31}(\epsilon,t,\eta) + A_{32}(\epsilon,t,\eta) + A_{33}(\epsilon,t,\eta) + A_{34}(\epsilon,t,\eta), \end{aligned}$$

where

$$\begin{aligned} A_{31}(\epsilon,t,\eta) &= \frac{\sigma^2}{2}\,\mathbb{E}\Big[\langle D^2H(Y_T^{t,\eta}), \mathbb{1}_{]t-T+\epsilon,0]}\otimes \mathbb{1}_{]t-T+\epsilon,0]}\rangle \cdot \frac{(W_t-W_{t+\epsilon})^2}{\epsilon}\Big], \\ A_{32}(\epsilon,t,\eta) &= \sigma^2 \int_0^1 \mathbb{E}\Big[\langle (D^2H(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}) - D^2H(Y_T^{t,\eta})), \mathbb{1}_{]t-T+\epsilon,0]^2}\rangle \cdot \frac{(W_t-W_{t+\epsilon})^2}{\epsilon}\Big]\,d\alpha, \end{aligned}$$

$$A_{33}(\epsilon,t,\eta)=\sigma\int_0^1\mathbb{E}\Big[\Big\langle\big(D^2H(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta})-D^2H(Y_T^{t,\eta})\big),\\ \frac{\eta(y+T-t+\epsilon)-\eta(y+T-t)}{\epsilon}\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{[-T,t-T]}(y)\Big\rangle\cdot(W_t-W_{t+\epsilon})\Big]\,d\alpha,$$

$$A_{34}(\epsilon,t,\eta)=\sigma\int_0^1\mathbb{E}\Big[\Big\langle\big(D^2H(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta})-D^2H(Y_T^{t,\eta})\big),\\ \frac{\eta(y+T-t-\epsilon)-\eta(0)-\sigma(W_T(y)+W_{t+\epsilon})}{\epsilon}\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{]t-T,t-T+\epsilon]}(y)\Big\rangle\\ \cdot(W_t-W_{t+\epsilon})\Big]\,d\alpha.$$

Similarly to $I_1(\epsilon,t,\eta)$, the term $A_1(\epsilon,t,\eta)$ can be decomposed into the sum of terms given below.

$$A_{11}(\epsilon,t,\eta)=\mathbb{E}\Big[\int_0^1\int_{-t}^{-t+\epsilon}D_{x-T+t}H(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta})-D_{x-T+t}H(Y_T^{t,\eta})\frac{\eta(x-\epsilon)}{\epsilon}\,dx\,d\alpha\Big],$$

$$A_{12}(\epsilon,t,\eta)=\mathbb{E}\Big[\int_0^1\int_{-t}^0\frac{D_{x+\epsilon-T+t}H(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta})-D_{x-T+t}H(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta})}{\epsilon}\cdot\eta(x)\,dx\,d\alpha\Big]\\ -\mathbb{E}\Big[\int_0^1\int_{-t}^0\frac{D_{x+\epsilon-T+t}H(Y_T^{t,\eta})-D_{x-T+t}H(Y_T^{t,\eta})}{\epsilon}\eta(x)\,dx\,d\alpha\Big],$$

$$A_{13}(\epsilon,t,\eta)=-\mathbb{E}\Big[\int_0^1\int_{-\epsilon}^0 D_{x-T+t}H(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta})-D_{x-T+t}H(Y_T^{t,\eta})\frac{\eta(x-\epsilon)}{\epsilon}\,dx\,d\alpha\Big].$$

We show now that $A_{11}(\epsilon,t,\eta)$ converges to zero. By the Cauchy–Schwarz inequality, we have

$$[A_{11}(\epsilon,t,\eta)]^2\leq\int_{-t}^{-t+\epsilon}\frac{\eta^2(x-\epsilon)}{\epsilon}\,dx\cdot\mathbb{E}\Big[\int_0^1\int_{-t}^{-t+\epsilon}\frac{1}{\epsilon}[D_{x-T+t}H(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta})-D_{x-T+t}H(Y_T^{t,\eta})]^2\,dx\,d\alpha\Big].$$

The integral $\frac{1}{\epsilon}\int_{-t}^{-t+\epsilon}\eta^2(x-\epsilon)\,dx$ converges to $\eta^2(-t)$ by the finite increments theorem. By hypotheses (4.16) and (4.8), we have

$$\|DH(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta})-DH(Y_T^{t,\eta})\|_{H^1([-T,0])}\xrightarrow[\epsilon\to 0]{\text{a.s.}}0. \tag{4.40}$$

Because of (4.40), it follows that

$$\sup_{x\in[-T,0]}|D_xH(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta})-D_xH(Y_T^{t,\eta})|\xrightarrow[\epsilon\to 0]{\text{a.s.}}0\quad\text{for all } x\in[-T,0]. \tag{4.41}$$

Then (4.41) implies that

$$\int_0^1\int_{-t}^{-t+\epsilon}\frac{1}{\epsilon}[D_{x-T+t}H(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta})-D_{x-T+t}H(Y_T^{t,\eta})]^2\,dx\,d\alpha\xrightarrow[\epsilon\to 0]{\text{a.s.}}0.$$

Using (4.17), (4.7), (4.5) and the fact that, given any Brownian motion $\bar{W}$, $\sup_{x\leq T}|\bar{W}_x|$ has all moments and the Lebesgue dominated convergence theorem, it follows that $A_{11}(\epsilon,t,\eta)$ converges to zero. Using the same technique, we also obtain that $A_{13}(\epsilon,t,\eta)$ converges to zero.

We show that $A_{12}(\epsilon,t,\eta)$ converges to zero. For every fixed continuous function ζ, we can write

$$D_{x-T+t+\epsilon}H(\zeta)-D_{x-T+t}H(\zeta)=\int_{x-T+t}^{x+\epsilon-T+t}D'_uH(\zeta)\,du.$$

It follows that $A_{12}(\epsilon,t,\eta)$ can be rewritten as

$$\mathbb{E}\Big[\int_0^1\int_{-t}^0\frac{1}{\epsilon}\int_{x-T+t}^{x-T+t+\epsilon}[D'_uH(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta})-D'_uH(Y_T^{t,\eta})]\eta(x)\,du\,dx\,d\alpha\Big].$$

Taking the absolute value and considering the fact that $|\eta(x)| \leq \|\eta\|_\infty$, we obtain

$$|A_{12}(\epsilon,t,\eta)| \leq \mathbb{E}\Bigg[\int_0^1\int_{-t}^0 \frac{1}{\epsilon}\int_{x-T+t}^{x-T+t+\epsilon} \big|D'_u H(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}) - D'_u H(Y_T^{t,\eta})\big|\, du\, dx\, d\alpha\Bigg]\|\eta\|_\infty.$$

By Fubini's theorem, it follows

$$|A_{12}(\epsilon,t,\eta)| \leq \mathbb{E}\Bigg[\int_0^1\int_{-T}^{-T+t} \big|D'_u H(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}) - D'_u H(Y_T^{t,\eta})\big|\, du\, d\alpha\Bigg]\|\eta\|_\infty.$$

Now, using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|A_{12}(\epsilon,t,\eta)|^2 &\leq T\,\mathbb{E}\Bigg[\int_0^1\int_{-T}^{-T+t} \big(D'_u H(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}) - D'_u H(Y_T^{t,\eta})\big)^2\, du\, d\alpha\Bigg]\|\eta\|_\infty^2\\ &\leq T\,\mathbb{E}\Bigg[\int_0^1 \big\|D' H(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}) - D' H(Y_T^{t,\eta})\big\|^2_{L^2([-T,0])}\, d\alpha\Bigg]\|\eta\|_\infty^2.\end{aligned}$$

Convergence (4.40) implies in particular

$$\big\|D' H(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}) - D' H(Y_T^{t,\eta})\big\|^2_{L^2([-T,0])} \xrightarrow[\epsilon\to 0]{\text{a.s.}} 0.$$

Again using (4.17), (4.7), (4.5) and the fact that, given any Brownian motion $\bar W$, $\sup_{x\leq T}|\bar W_x|$ has all moments and the Lebesgue dominated convergence theorem, we have that $A_{12}(\epsilon,t,\eta)$ converges to zero. This concludes the proof that $A_1(\epsilon,t,\eta)$ converges to zero.

The term $A_2(\epsilon,t,\eta)$ also converges to zero. In fact, Cauchy–Schwarz implies that

$$\begin{aligned}|A_2(\epsilon,t,\eta)|^2 \leq \int_0^1 \frac{1}{\epsilon}\,\mathbb{E}\Bigg[\int_{t-T}^{t-T+\epsilon} \big(D_x H(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}) - D_x H(Y_T^{t,\eta})\big)^2\, dx\Bigg]\\ \cdot \frac{1}{\epsilon}\,\mathbb{E}\Bigg[\int_{t-T}^{t-T+\epsilon} \big(\eta(x+T-t-\epsilon) - \eta(0) - \sigma W_T(x) + \sigma W_{t+\epsilon}\big)^2\, dx\Bigg]\, d\alpha.\end{aligned}$$

The continuity of DH (see (4.16)), the fact that it has polynomial growth in the sense of Remark 4.13 (1), (4.7) and the Lebesgue dominated convergence theorem imply that the first expectation converges to zero. The second expectation converges to zero by the same arguments together with the fact that $\sup_{x\leq T}|\bar W_x|$ has all moments.

We show now that $A_{31}(\epsilon,t,\eta)$ converges to

$$\frac{\sigma^2}{2}\,\mathbb{E}[\langle D^2 H(Y_T^{t,\eta}), \mathbb{1}_{]t-T,0]^2}\rangle] =: A_{31}(t,\eta). \tag{4.42}$$

At this level, we need two technical results.

Lemma 4.16. *The random variable $B(\epsilon) := \frac{(W_{t+\epsilon}-W_t)^2}{\epsilon}$ weakly converges in $L^2(\Omega)$ to 1 when $\epsilon \to 0$.*

Proof. In fact, $\mathbb{E}[B(\epsilon)^2] = 3$ so that $(B(\epsilon))$ is bounded in $L^2(\Omega)$. Therefore, there exists a subsequence (ϵ_n) such that $(B(\epsilon_n))$ converges weakly to some square integrable variable B_0. In order to show that $B_0 = 1$ and to conclude the proof of the lemma, it is enough to prove that $\mathbb{E}[B(\epsilon)\cdot Z] \to \mathbb{E}[Z]$ for any r.v. Z of a dense subset $\mathcal{D}$ of $L^2(\Omega)$. We choose $\mathcal{D}$ and the r.v. Z such that $Z = \mathbb{E}[Z] + \int_0^T \xi_s\, dW_s$, where $(\xi_s)_{s\in[0,T]}$ is a bounded previsible process. We have

$$\mathbb{E}[B(\epsilon)\cdot Z] = \mathbb{E}[B(\epsilon)]\,\mathbb{E}[Z] + \mathbb{E}\Bigg[\frac{(W_{t+\epsilon}-W_t)^2}{\epsilon}\int_0^T \xi_s\, dW_s\Bigg].$$

Since $\mathbb{E}[B(\epsilon)]\,\mathbb{E}[Z]=\mathbb{E}[Z]$, we only need to show that

$$\mathbb{E}\bigg[\frac{(W_{t+\epsilon}-W_t)^2}{\epsilon}\int_0^T \xi_s\,dW_s\bigg]\xrightarrow[\epsilon\to 0]{}0. \tag{4.43}$$

Since $\int_0^T \xi_s\,dW_s$ is a Skorohod integral, integration by parts on Wiener space (2.1) implies that the left-hand side of (4.43) equals

$$\mathbb{E}\bigg[\frac{2}{\epsilon}\int_0^T \xi_s(W_{t+\epsilon}-W_t)\,\mathbb{1}_{[t,t+\epsilon]}(s)\,ds\bigg]=\mathbb{E}\bigg[\frac{1}{\epsilon}\int_t^{t+\epsilon}\xi_s\,ds\,(W_{t+\epsilon}-W_t)\bigg];$$

this converges to zero since ξ is bounded. □

Lemma 4.17. *Let H be a Hilbert space equipped with a product $\langle\cdot,\cdot\rangle$. Let $(Z_n)_n$ and $(Y_n)_n$ be two sequences in H such that Z_n converges strongly to Z and Y_n converges weakly to Y. Then $\langle Z_n, Y_n\rangle$ converges to $\langle Z, Y\rangle$.*

Proof. By the Cauchy–Schwarz inequality, we obtain

$$|\langle Z_n, Y_n\rangle-\langle Z, Y\rangle|=|\langle Z_n-Z, Y_n\rangle+\langle Z, Y_n-Y\rangle|\le\|Z_n-Z\|_H\,\|Y_n\|_H+|\langle Z, Y_n-Y\rangle|\xrightarrow[\epsilon\to 0]{}0$$

since $\|Z_n-Z\|_H$ goes to zero by the strong convergence hypothesis of (Y_n), $\|Y_n\|_H$ is bounded because weakly convergent and $\langle Z, Y_n-Y\rangle$ goes to zero by definition of weak convergence of $(Y_n)_n$ and the fact that $Z\in H$. □

In order to show the convergence of $2A_{31}(\epsilon,t,\eta)=\sigma^2\,\mathbb{E}[\mathcal{Z}(\epsilon)\cdot\frac{(W_{t+\epsilon}-W_t)^2}{\epsilon}]$ to $2A_{31}(t,\eta)$, we use Lemma 4.17 setting the Hilbert space H equal to $L^2(\Omega)$. We only need to show the strong convergence in H of $\mathcal{Z}(\epsilon)$ to $\mathcal{Z}:=\langle D^2H(Y_T^{t,\eta}),\mathbb{1}_{]t-T,0]}\otimes\mathbb{1}_{]t-T,0]}\rangle$. Taking into account $\mathbb{1}_{]t-T+\epsilon,0]}\otimes^2\to\mathbb{1}_{]t-T,0]}\otimes^2$ pointwise and the Lebesgue dominated convergence theorem, it is not difficult to show now that $\mathbb{E}[(\mathcal{Z}(\epsilon)-\mathcal{Z})^2]$ converges to zero, i.e. the strong convergence in $L^2(\Omega)$. Finally, by an immediate application of Lemma 4.16, the term $A_{31}(\epsilon,t,\eta)$ expressed in (4.42) converges to $\frac{\sigma^2}{2}\,\mathbb{E}[\mathcal{Z}]$ which equals $A_{31}(t,\eta)$.

The term $A_{32}(\epsilon,t,\eta)$ converges to zero. In fact, using $\mathbb{1}_{]t-T+\epsilon,0]^2}\le\mathbb{1}_{[t-T,0]^2}$ and then the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\big\langle D^2H(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}})-D^2H(Y_T^{t,\eta}),\mathbb{1}_{]t-T+\epsilon,0]^2}\big\rangle\cdot\frac{(W_{t+\epsilon}-W_t)^2}{\epsilon}\Big]\\
&\quad\le\mathbb{E}\Big[\big\langle D^2H(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}})-D^2H(Y_T^{t,\eta}),\mathbb{1}_{[t-T,0]^2}\big\rangle\cdot\frac{(W_{t+\epsilon}-W_t)^2}{\epsilon}\Big]\\
&\quad\le\sqrt{\mathbb{E}\big[\big|\big\langle D^2H(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}})-D^2H(Y_T^{t,\eta}),\mathbb{1}_{[t-T,0]^2}\big\rangle\big|^2\big]}\cdot\sqrt{3}\\
&\quad\le\sqrt{\mathbb{E}\big[\big\|D^2H(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}})-D^2H(Y_T^{t,\eta})\big\|^2_{(C([-T,0])\hat{\otimes}_\pi^2)^*}\cdot\|\mathbb{1}_{[t-T,0]^2}\|^2\big]}.
\end{aligned}$$

The latter term converges to zero because $D^2H\in C^0(C([-T,0]))$ and D^2H has polynomial growth as we have seen in Remark 4.13 (1).

We show that $A_{33}(\epsilon,t,\eta)$ converges to zero. We rewrite $A_{33}(\epsilon,t,\eta)$ as $\sigma(A_{332}(\epsilon,t,\eta)-A_{331}(\epsilon,t,\eta))$, where

$$\begin{aligned}
A_{331}(\epsilon,t,\eta)&=\mathbb{E}\Big[\Big\langle D^2H(Y_T^{t,\eta}),\frac{\eta(y+T-t+\epsilon)-\eta(y+T-t)}{\epsilon}\,\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{[-T,t-T]}(y)\Big\rangle(W_{t+\epsilon}-W_t)\Big],\\
A_{332}(\epsilon,t,\eta)&=\int_0^1\mathbb{E}\Big[\Big\langle D^2H(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t+\epsilon,Y_{t+\epsilon}^{t,\eta}}),\\
&\qquad\frac{\eta(y+T-t+\epsilon)-\eta(y+T-t)}{\epsilon}\,\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{[-T,t-T]}(y)\Big\rangle(W_{t+\epsilon}-W_t)\Big]\,d\alpha.
\end{aligned}$$

We will show that both $A_{331}(\epsilon,t,\eta)$ and $A_{332}(\epsilon,t,\eta)$ converge to zero. Denoting

$$\mathcal{Z}:=\langle D^2H(Y_T^{t,\eta}),\mathcal{Y}\rangle, \tag{4.44}$$

where

$$\mathcal{Y} := \mathbb{1}_{]t-T+\epsilon,0]}(x) \otimes \mathbb{1}_{[-T,t-T]}(y)[\eta(y+T-t+\epsilon) - \eta(y+T-t)],$$

we rewrite

$$A_{331}(\epsilon,t,\eta) = \frac{1}{\epsilon}\,\mathbb{E}[\mathcal{Z}\cdot(W_{t+\epsilon} - W_t)].$$

Using Proposition A.4 and that $H \in C^3(C([-T,0]))$, with polynomial growth, we get that $\mathcal{Z}$ belongs to $\mathbb{D}^{1,2}$ and

$$\begin{aligned} D_r^{\mathrm{m}}\mathcal{Z} &= \sigma\langle D^3H(Y_T^{t,\eta}), \mathbb{1}_{]r-T+t,0]} \otimes \mathcal{Y}\rangle + \langle D^2H(Y_T^{t,\eta}), D_r^{\mathrm{m}}\mathcal{Y}\rangle \\ &= \sigma\langle D^3H(Y_T^{t,\eta}), \mathbb{1}_{]r-T+t,0]}(z) \otimes \mathbb{1}_{]t-T+\epsilon,0]}(x) \otimes \mathbb{1}_{[-T,t-T]}(y)[\eta(y+T-t+\epsilon) - \eta(y+T-t)]\rangle \end{aligned} \tag{4.45}$$

because $D_r^{\mathrm{m}}\mathcal{Y}$ is zero. Using (4.44), Skorohod integral formulation, integration by parts on Wiener space (2.1), (4.45) and successively Fubini's theorem with respect to the variables r and z and then integrating with respect to r, we obtain

$$\begin{aligned} &A_{331}(\epsilon,t,\eta) \\ &\quad= \frac{1}{\epsilon}\,\mathbb{E}[\mathcal{Z}\cdot(W_{t+\epsilon} - W_t)] = \frac{1}{\epsilon}\,\mathbb{E}[\mathcal{Z}\cdot\overline{W}_\epsilon] = \frac{1}{\epsilon}\,\mathbb{E}\Big[\mathcal{Z}\cdot\int_0^\epsilon \delta\overline{W}_u\Big] = \frac{1}{\epsilon}\,\mathbb{E}\Big[\int_0^\epsilon D_r^{\mathrm{m}}\mathcal{Z}\,dr\Big] \\ &\quad= \frac{\sigma}{\epsilon}\,\mathbb{E}\Big[\int_0^\epsilon \langle D^3H(Y_T^{t,\eta}), \\ &\qquad\qquad \mathbb{1}_{]r-T+t,0]}(z) \otimes \mathbb{1}_{]t-T+\epsilon,0]}(x) \otimes \mathbb{1}_{[-T,t-T]}(y)[\eta(y+T-t+\epsilon) - \eta(y+T-t)]\rangle\,dr\Big] \\ &\quad= \frac{\sigma}{\epsilon}\,\mathbb{E}\Big[\langle D^3H(Y_T^{t,\eta}), \\ &\qquad\qquad \int_0^\epsilon \mathbb{1}_{]r-T+t,0]}(z)\,dr \otimes \mathbb{1}_{]t-T+\epsilon,0]}(x) \otimes \mathbb{1}_{[-T,t-T]}(y)[\eta(y+T-t+\epsilon) - \eta(y+T-t)]\rangle\Big]. \end{aligned} \tag{4.46}$$

Analyzing the term $\int_0^\epsilon \mathbb{1}_{]r-T+t,0]}(z)\,dr$ analogously to (4.38) and (4.39), we can establish the convergence of $A_{331}(\epsilon,t,\eta)$. In fact, the third-order Fréchet derivative of H, denoted by D^3H, is a map from $C([-T,0])$ to the dual of the triple projective tensor product of $C([-T,0])$, i.e. $(C([-T,0])\hat{\otimes}_\pi^3)^*$. We recall that, given a general Banach space E equipped with its norm $\|\cdot\|_E$ and x, y, z three elements of E, then the norm of an elementary element of the tensor product $x \otimes y \otimes z$ which belongs to $E\otimes^3$ is $\|x\|_E \cdot \|y\|_E \cdot \|z\|_E$. We remark that the trilinear form $(\phi,\varphi,\psi) \mapsto \langle D^3H(Y_T^{t,\eta}), \phi\otimes\varphi\otimes\psi\rangle$ extends from $C([-T,0]) \times C([-T,0]) \times C([-T,0])$ to $\phi,\varphi,\psi\colon [-T,0] \to \mathbb{R}$ as a Borel bounded map. Indeed, the application is a measure in each component. Consequently,

$$\begin{aligned} &|\langle D^3H(Y_T^{t,\eta}), \mathbb{1}_{]t-T+\epsilon,0]}(x) \otimes \mathbb{1}_{[-t,0]}(y)[\eta(y+\epsilon) - \eta(y)] \otimes \mathbb{1}_{]r-T+t,0]}(z)\rangle| \\ &\quad\le \sup_{\|\phi\|_\infty\le 1,\, \|\varphi\|_\infty\le 1,\, \|\psi\|_\infty\le 1} |\langle D^3H(Y_T^{t,\eta}), \phi\otimes\varphi\otimes\psi\rangle| \cdot \varpi_\eta(\epsilon) \\ &\quad= \|D^3H(Y_T^{t,\eta})\|_{(C([-T,0])\hat{\otimes}_\pi^3)^*} \cdot \varpi_\eta(\epsilon) \xrightarrow[\epsilon\to 0]{\text{a.s.}} 0 \end{aligned}$$

since $\varpi_\eta(\epsilon)$ is the modulus of continuity of η. By the polynomial growth of D^3H, (4.5), the fact that, for any given Brownian motion $\tilde{W}$, $\sup_{x\le T}|\tilde{W}_x|$ has all moments and finally the Lebesgue dominated convergence theorem, we conclude that (4.46) converges to zero; therefore, $A_{331}(\epsilon,t,\eta)$ converges to zero.

At this point, we should establish the convergence to zero of $A_{332}(\epsilon,t,\eta)$. This can be done using, again as above, integration by parts on Wiener space (2.1). However, there are several technicalities that we have to omit.

We show finally that $A_{34}(\epsilon,t,\eta)$ converges to zero. We rewrite the term $A_{34}(\epsilon,t,\eta)$ as

$$\begin{aligned} &A_{34}(\epsilon,t,\eta) \\ &\quad= \sigma\int_0^1 \mathbb{E}\Big[\Big\langle (D^2H(\alpha Y_T^{t+\epsilon,\eta} + (1-\alpha)Y_T^{t,\eta}) - D^2H(Y_T^{t,\eta})), \\ &\qquad \frac{\eta(y+T-t-\epsilon) - \eta(0) - \sigma(W_T(y) + W_{t+\epsilon})}{\epsilon}\,\mathbb{1}_{]t-T+\epsilon,0]}(x) \otimes \mathbb{1}_{]t-T,t-T+\epsilon]}(y)\Big\rangle \cdot (W_t - W_{t+\epsilon})\Big]\,d\alpha \end{aligned}$$

as $A_{34}(\epsilon,t,\eta)=\sigma(A_{341}(\epsilon,t,\eta)-A_{342}(\epsilon,t,\eta))$, where

$$\begin{aligned}
&A_{341}(\epsilon,t,\eta)\\
&\quad=\int_0^1\mathbb{E}\Big[\Big\langle D^2H(\alpha Y_T^{t+\epsilon,\eta}+(1-\alpha)Y_T^{t,\eta}),\\
&\qquad\frac{\eta(y+T-t-\epsilon)-\eta(0)-\sigma(W_T(y)+W_{t+\epsilon})}{\epsilon}\,\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{]t-T,t-T+\epsilon]}(y)\Big\rangle\cdot(W_t-W_{t+\epsilon})\Big]\,d\alpha,\\
&A_{342}(\epsilon,t,\eta)\\
&\quad=\int_0^1\mathbb{E}\Big[\Big\langle D^2H(Y_T^{t,\eta}),\\
&\qquad\frac{\eta(y+T-t-\epsilon)-\eta(0)-\sigma(W_T(y)+W_{t+\epsilon})}{\epsilon}\,\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{]t-T,t-T+\epsilon]}(y)\Big\rangle\cdot(W_t-W_{t+\epsilon})\Big]\,d\alpha\\
&\quad=\mathbb{E}\Big[\Big\langle D^2H(Y_T^{t,\eta}),\\
&\qquad\frac{\eta(y+T-t-\epsilon)-\eta(0)-\sigma(W_T(y)+W_{t+\epsilon})}{\epsilon}\,\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes\mathbb{1}_{]t-T,t-T+\epsilon]}(y)\Big\rangle\cdot(W_t-W_{t+\epsilon})\Big].
\end{aligned}$$

Firstly, we show that A_{342} converges to zero. It holds in fact

$$A_{342}(\epsilon,t,\eta)=\frac{1}{\epsilon}\,\mathbb{E}[\mathcal{Z}\cdot(W_t-W_{t+\epsilon})]=\frac{1}{\epsilon}\,\mathbb{E}[\mathcal{Z}\cdot\overline{W}_\epsilon]=\frac{1}{\epsilon}\,\mathbb{E}\Big[\mathcal{Z}\cdot\int_0^\epsilon\delta\overline{W}_r\Big],$$

where

$$\mathcal{Z}:=\langle D^2H(Y_T^{t,\eta}),\,\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes[\eta(y+T-t-\epsilon)-\eta(0)-\sigma W_T(y)+\sigma W_{t+\epsilon}]\,\mathbb{1}_{]t-T,t-T+\epsilon]}(y)\rangle.$$

Since D^2H has polynomial growth and it is of class C^1, by Proposition A.4, $\mathcal{Z}\in\mathbb{D}^{1,2}$. Then the integration by parts on Wiener space gives

$$A_{342}(\epsilon,t,\eta)=\frac{1}{\epsilon}\,\mathbb{E}\Big[\int_0^\epsilon D_r^{\mathrm{m}}\mathcal{Z}\,dr\Big].\tag{4.47}$$

According to Proposition A.4, equation (A.4) for $n=2$ and setting

$$\mathcal{Y}:=\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes[\eta(y+T-t-\epsilon)-\eta(0)-\sigma W_T(y)+\sigma W_{t+\epsilon}]\,\mathbb{1}_{]t-T,t-T+\epsilon]}(y),$$

we get the following expression for the Malliavin derivative of $\mathcal{Z}$ in the Wiener space associated with $(\bar{W}_r)$, for $r\in[0,T-t]$:

$$D_r^{\mathrm{m}}\mathcal{Z}=\langle D^3H(Y_T^{t,\eta}),\,\mathcal{Y}\otimes\mathbb{1}_{]r-T+t,0]}(z)\rangle+\langle D^2H(Y_T^{t,\eta}),\,D_r^{\mathrm{m}}\mathcal{Y}\rangle.\tag{4.48}$$

Replacing (4.48) in (4.47), we get that $A_{342}(\epsilon,t,\eta)$ equals a sum of $A_{3421}(\epsilon,t,\eta)$ and $A_{3422}(\epsilon,t,\eta)$ with

$$\begin{aligned}
A_{3421}(\epsilon,t,\eta)&=\frac{1}{\epsilon}\,\mathbb{E}\Big[\int_0^\epsilon\langle D^3H(Y_T^{t,\eta}),\,\mathcal{Y}\otimes\mathbb{1}_{]r-T+t,0]}(z)\rangle\,dr\Big],\\
A_{3422}(\epsilon,t,\eta)&=\frac{1}{\epsilon}\,\mathbb{E}\Big[\int_0^\epsilon\langle D^2H(Y_T^{t,\eta}),\,D_r^{\mathrm{m}}\mathcal{Y}\rangle\,dr\Big].
\end{aligned}\tag{4.49}$$

The term $A_{3421}(\epsilon,t,\eta)$ converges to zero. In fact, similarly to the method developed in detail in (4.39), we have

$$\begin{aligned}
A_{3421}(\epsilon,t,\eta)&=\frac{1}{\epsilon}\,\mathbb{E}\Big[\int_0^\epsilon\langle D^3H(Y_T^{t,\eta}),\,\mathcal{Y}\otimes\mathbb{1}_{]z-T+t,0]}(r)\rangle\,dr\Big]\\
&=\frac{1}{\epsilon}\,\mathbb{E}\Big[\Big\langle D^3H(Y_T^{t,\eta}),\,\mathcal{Y}\otimes\int_0^\epsilon\mathbb{1}_{]z-T+t,0]}(r)\,dr\Big\rangle\Big]
\end{aligned}$$

and

$$\frac{1}{\epsilon}\int_0^\epsilon \mathbb{1}_{]z-T+t,0]}(r)\rangle\, dr \le \frac{\epsilon \wedge (z+T-t)}{\epsilon} \xrightarrow[\epsilon\to 0]{} 1.$$

By polynomial growth of D^3H, (4.7), the usual property that, given any Brownian motion $\bar{W}$, $\sup_{x\le T}|\bar{W}_x|$ has all moments, the convergence of $\mathcal{Y}$ to zero and through the application of the Lebesgue dominated convergence theorem, we conclude that first term in $A_{3421}(\epsilon, t, \eta)$ converges to zero.

Concerning the term $A_{3422}(\epsilon, t, \eta)$, we firstly need to compute the Malliavin derivative of $\mathcal{Y}$,

$$\begin{aligned} D_r^{\mathrm{m}}\,\mathcal{Y} &= \mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes D_r^{\mathrm{m}}[\eta(y+T-t-\epsilon)-\eta(0)-\sigma W_T(y)+\sigma W_{t+\epsilon}]\,\mathbb{1}_{]t-T,t-T+\epsilon]}(y)\\ &= \sigma\,\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes D_r^{\mathrm{m}}[W_{t+\epsilon}-W_{T+y}]\,\mathbb{1}_{]t-T,t-T+\epsilon]}(y)\\ &= \sigma\,\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes D_r^{\mathrm{m}}[\overline{W}_\epsilon-\overline{W}_{T+y-t}]\,\mathbb{1}_{]t-T,t-T+\epsilon]}(y)\\ &= \sigma\,\mathbb{1}_{]t-T+\epsilon,0]}(x)\otimes \mathbb{1}_{[T+y-t,\epsilon]}(r)\cdot\mathbb{1}_{]t-T,t-T+\epsilon]}(y), \end{aligned} \tag{4.50}$$

since, by the usual property of the Malliavin derivative, $D_r^{\mathrm{m}}[\overline{W}_\epsilon - \overline{W}_{T+y-t}] = \mathbb{1}_{[T+y-t,\epsilon]}(r)$. Now, replacing (4.50) in (4.49), we have, similarly to the method developed in detail in (4.39),

$$A_{3422}(\epsilon,t,\eta) = \frac{1}{\epsilon}\,\mathbb{E}\Big[\langle D^2H(Y_T^{t,\eta}), \int_0^\epsilon D_r^{\mathrm{m}}\mathcal{Y}\,dr\rangle\Big] \quad\text{and}\quad \frac{\sigma}{\epsilon}\int_0^\epsilon D_r^{\mathrm{m}}\mathcal{Y}\,dr \le \sigma\frac{\epsilon\wedge(T+y-t)}{\epsilon} \xrightarrow[\epsilon\to 0]{} 0.$$

We remark that $T+y-t \in [0,\epsilon]$ since $y \in [t-T, t-T+\epsilon]$. Again by polynomial growth of D^2H, (4.7), by the usual property that, for the Brownian motion $\bar{W}$, $\sup_{x\le T}|\bar{W}_x|$ has all moments and applying the Lebesgue dominated convergence theorem, we conclude that the first term in $A_{3422}(\epsilon, t, \eta)$ converges to zero. Finally, $A_{342}(\epsilon, t, \eta)$ converges to zero.

By similar arguments, even though technically a little bit more involved, also $A_{341}(\epsilon, t, \eta)$ converges to zero. This finally proves that

$$A_{34}(\epsilon,t,\eta) \xrightarrow[\epsilon\to 0]{} 0.$$

We are now able to express $\partial_t u\colon [0,T]\times C([-T,0]) \to \mathbb{R}$. For $t \in [0,T]$, $\partial_t u(t,\eta)$ is given by the convergence of term (4.32) to a sum of three terms different from zero

$$\partial_t u(t,\eta) = I_1(t,\eta) + I_3(t,\eta) + A_{31}(t,\eta),$$

i.e.

$$\partial_t u(t,\eta) = -\mathbb{E}\Big[\int_{]-t,0]} D_{x-T+t}H(Y_T^{t,\eta})\,d^-\eta(x) + \frac{\sigma^2}{2}\langle D^2H(Y_T^{t,\eta}), \mathbb{1}_{]t-T,0]}{}^{\otimes 2}\rangle\Big]. \tag{4.51}$$

The path-dependent heat equation. Taking into account (4.25), the second-order Fréchet derivative (4.23) and the time derivative (4.51), it finally follows that u solves (1.1). □

A Appendix: Malliavin and Fréchet derivatives

We need some technical results concerning the link between Fréchet and Malliavin derivatives in a separable Banach space that, for the moment, we set to be equal to $\mathbb{R}$. We need to apply Malliavin calculus related to the Brownian motion. Let $T > 0$ and $t \in [0,T]$ be fixed. We recall that

$$\overline{W}_x := W_{t+x} - W_t, \quad x \in [0, T-t]. \tag{A.1}$$

So the Wiener space will be $C([0,T-t])$ with variable parameter in $[0,T-t]$ and based on $\overline{W}$. We consider the window Brownian element $\overline{W}_{T-t}(\cdot)$ with values in $C([-(T-t),0])$, defined as

$$\overline{W}_{T-t}(x) = W_{t+T-t}(x) - W_t = W_{T+x} - W_t, \quad x \in [-(T-t),0].$$

Lemma A.1. *Let $G\colon C([-(T-t),0])\to\mathbb{R}$ of class C^1 be such that DG has polynomial growth. Let $\mathcal{Y}\in\mathbb{D}^\infty$. Then $G(\sigma\overline{W}_{T-t}(\cdot))$ belongs to $\mathbb{D}^{1,2}$ and*

$$D_r^{\mathrm{m}}(G(\sigma\overline{W}_{T-t}(\cdot))\mathcal{Y})=\sigma\int\limits_{]r-(T-t),0]}(D_{dy}G)(\sigma\overline{W}_{T-t}(\cdot))\mathcal{Y}+G(\sigma\overline{W}_{T-t}(\cdot))\,D_r^{\mathrm{m}}\mathcal{Y},\quad r\in[0,T-t]\ a.e. \tag{A.2}$$

Proof. The proof of this result needs some boring technicalities involving the approximation of a continuous function and its polygonal approximation. Formula (A.2) is stated in a particular case for instance in [13, Example 1.2.1]. □

A consequence of the previous lemma is the possibility of the differentiating

$$h=F(Y_T^{t,\eta}),\quad F\colon C([-T,0])\to\mathbb{R}$$

of class C^1 Fréchet. We remark that $Y_T^{t,\eta}=G_\eta(\sigma\overline{W}_{T-t}(\cdot))$, where $G_\eta\colon C[-(T-t),0]\to C([-T,0])$ is given by

$$G_\eta(\gamma)=\begin{cases}\eta(x+T-t), & x\in[-T,t-T[,\\ \eta(0)+\gamma(T-t+x), & x\in[t-T,0].\end{cases}$$

By Lemma A.1, if $\mathcal{Y}\in\mathbb{D}^\infty$,

$$D_x^{\mathrm{m}}(h\mathcal{Y})=\sigma\int\limits_{]x-T+t,0]}D_{dy}(F\circ G_\eta)(\sigma\overline{W}_{T-t}(\cdot))\mathcal{Y}+F\circ G_\eta(\sigma\overline{W}_{T-t}(\cdot))\,D_x^{\mathrm{m}}\mathcal{Y},\quad x\in[0,T-t]. \tag{A.3}$$

Remark A.2. We remark that, for all $\gamma\in C([-T+t,0])$, $D(F\circ G_\eta)\in\mathcal{M}([-T+t,0])$.

We have, for $\zeta\in C([-T+t,0])$,

$$\int D_{dy}(F\circ G_\eta)(\gamma)\zeta(y)=\int\limits_{[t-T,0]}D_{dy}F(G_\eta(\gamma))\zeta(y).$$

So (A.3) gives, for $x\in[0,T-t]$,

$$\begin{aligned}D_x^{\mathrm{m}}(h\mathcal{Y})&=\sigma\int\limits_{]x-T+t,0]}(D_{dy}F)(G_\eta(\sigma\overline{W}_{T-t}))\mathcal{Y}+(F\circ G_\eta)(\sigma\overline{W}_{T-t}(\cdot))\,D_x^{\mathrm{m}}\mathcal{Y}\\ &=\sigma\int\limits_{]x-T+t,0]}(D_{dy}F)(Y_T^{t,\eta})\mathcal{Y}+F(Y_T^{t,\eta})\,D_x^{\mathrm{m}}\mathcal{Y}.\end{aligned}$$

At this point, we have proved the following.

Proposition A.3. *Let $H\colon C([-T,0])\to\mathbb{R}$ be of class C^1-Fréchet with polynomial growth. Let $\mathcal{Y}\in\mathbb{D}^\infty$. Then $H(Y_T^{t,\eta})\mathcal{Y}$ belongs to $\mathbb{D}^{1,2}$ and*

$$D_r^{\mathrm{m}}(H(Y_T^{t,\eta})\mathcal{Y})=\sigma\int\limits_{]x-T+t,0]}(D_{dy}H)(Y_T^{t,\eta})\mathcal{Y}+F(Y_T^{t,\eta})D_r^{m}\mathcal{Y}.$$

The previous proposition admits a generalization to the case when $H\colon C([-T,0])\to\mathbb{R}$ is replaced by a functional

$$C([-T,0])\to\underbrace{(C([-T,0])\,\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi\,C([-T,0]))}_{n\text{ times}},\quad n\geq 1.$$

Typically, an example will be D^nH. We recall that

$$(\underbrace{C([-T,0])\,\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi\,C([-T,0])}_{n\text{ times}})^*$$

can be isomorphically identified with the space of n-multilinear continuous functionals on $C([-T,0])$. Proposition A.3 can be generalized as follows.

Proposition A.4. *Let $H: C([-T,0]) \to \mathbb{R}$ of class C^{n+1}-Fréchet be such that $D^{n+1}F$ has polynomial growth. Let*

$$\mathcal{Y} \in \mathbb{D}^{\infty}(\underbrace{C([0,T-t]) \hat{\otimes}_{\pi} \cdots \hat{\otimes}_{\pi} C([0,T-t])}_{n \text{ times}}).$$

Then $\langle D^n H(Y_T^{t,\eta}), \mathcal{Y}\rangle$ belongs to $\mathbb{D}^{1,2}$. Moreover, for a.e. $r \in [0,T-t]$, we have

$$D_r^{\mathrm{m}}(\langle D^n H(Y_T^{t,\eta}), \mathcal{Y}\rangle) = \sigma \langle D^{n+1} H(Y_T^{t,\eta}), 1_{]r-T+t,0]} \otimes \mathcal{Y}\rangle + \langle D^n H(Y_T^{t,\eta}), D_r^{\mathrm{m}}\mathcal{Y}\rangle. \tag{A.4}$$

Remark A.5. The function $1_{]r-T+t,0]}$ can be considered as a test function ζ_0. Indeed,

$$\zeta_0 \mapsto D^{n+1} H(Y_T^{t,\eta})(\zeta_0 \otimes \zeta_1 \otimes \cdots \zeta_n)$$

for fixed $\zeta_1, \ldots, \zeta_n \in C([-T,0])$ is a measure.

Proof. Avoiding to state too abstract results, the proof of Proposition A.4 is based on a generalization of Lemma A.1 replacing the value space $\mathbb{R}$ with the separable Banach space B, setting

$$B = C([-T,0]) \hat{\otimes}_{\pi} \cdots \hat{\otimes}_{\pi} C([-T,0]).$$ □

Lemma A.6. *Let B be a separable Banach space. Let $G: C([-T+t,0]) \to B^*$ be of class C^1-Fréchet with polynomial growth. Let $\mathcal{Y} \in \mathbb{D}^{\infty}(B)$. Then $G(\overline{W}_{T-t}(\cdot))\mathcal{Y} \in \mathbb{D}^{1,2}(B)$ and*

$$D_r({}_{B^*}\langle G(\overline{W}_{T-t}(\cdot)), \mathcal{Y}\rangle_B) = {}_{(C([-T,0])\hat{\otimes}_{\pi}B)^*}\langle DG(\overline{W}_{T-t}(\cdot)), 1_{]r-T+t,0]} \otimes \mathcal{Y}\rangle_{C([-T,0])\hat{\otimes}_{\pi}B} + \langle G(\overline{W}_{T-t}(\cdot)), D_r^{\mathrm{m}}\mathcal{Y}\rangle$$

Remark A.7. We remark the following.
(1) $DG: C([-T+t,0]) \to (C([-T,0]) \hat{\otimes}_{\pi} B)^*$.
(2) Proposition A.4 will be used for $n = 1, 2, 3$.
(3) $D_r^{\mathrm{m}}\mathcal{Y} \in B$ for almost all r.

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