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Density Estimation in Randomly Distributed Wireless Networks

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***Abstract*—Networks of randomly distributed nodes appear in various fields, including forestry and wireless communications, and can often be modeled, using stochastic geometry theory, as Poisson point processes (PPPs). In these contexts, estimation of nodes density is important for monitoring and optimizing the network. Originally, this problem has been addressed in forestry where the trees are the nodes and, assuming these are distributed according to an infinite two-dimensional homogeneous PPP, the spatial density can be estimated by measuring the distances from one reference tree to its neighbors. However, in many other scenarios, nodes could result invisible with some probability, for example depending on distance. In this paper, we derive the Cramér-Rao bounds and new estimators for the node spatial density, taking into account a limited capability in sensing neighbors. As an example, we provide estimators of the spatial density of transmitting devices in wireless networks with links affected by thermal noise, path loss, and shadowing.**

***Index Terms*—Spatial density estimation, Poisson point processes, Cramér-Rao bounds, maximum likelihood estimation, stochastic geometry, wireless networks.**

I. Introduction

THE estimation of the spatial density of entities is a well-known problem that appears in different disciplines. In forestry, for example, it is crucial to estimate the density of the trees, both for census purposes and for evaluating the dynamic changes in the environment [1]–[3]. In geographical epidemiology, the density of individuals afflicted by a specified disease can be used to produce mapped information on disease incidence [4], [5]. In zoology, density estimators have been used to examine different foraging strategies and density of group-living animals [6], [7]. In wireless communication networks, the knowledge of the spatial density of the nodes allows to evaluate and optimize the performance in terms of throughput and delay [8]–[16]. Moreover, the applicability of novel spectrum reuse concepts, enabling the coexistence of heterogeneous networks, is based on the capability of accurately controlling the interference that each network generates to the others [17]–[20]. In scenarios where the use of the radio channel is uncoordinated, network interference is determined by the number of active nodes transmitting on the same radio channel in a finite region. More specifically, density estimation for wireless networks will find application in:

- 6G mobile radio systems and internet of things, where the ultradense network scenario imposes new energy-efficient and spectrum-efficient protocols [21]–[29];
- wireless ad hoc and sensor networks, where access protocols can be optimized based on the expected traffic [30]–[40];
- cognitive radio, where the presence of transmitting nodes must be estimated to understand if a given frequency band is congested [41]–[44].

Spatial density estimators have been developed originally in the context of forestry, assuming trees distributed according to an infinite two-dimensional Poisson point process (PPP) [45], [46]. When the trees spatial location follows a homogeneous PPP, a straightforward procedure to compute the density is to count the total number of trees in a given region and divide by the area. Pollard proposed an estimator based on the distances to neighbors, which are easier to measure than the absolute positions of trees and thus simplifies the measurement procedure [46]. The same methods can be adopted in other stochastic geometry scenarios described by homogeneous PPPs. However, in some situations, the available measurements could make the observed PPP inhomogeneous. For example, in wireless networks with measurements based on radio signals, propagation impairments may obscure some nodes or limit the maximum sensed distance, leading to a modified statistical description of the PPP observed by a receiver, which impacts on the aggregate interference. In fact, the PPP model has been extensively used to study wireless networks [8]–[15], [17]–[20], [27], [41], [47], [48], always assuming the knowledge of the nodes density.

We here consider the spatial density estimation of a wireless network where detection of nodes may occasionally fail due to path-loss and shadowing. With reference to the PPP observed by a node, thinned by detection failures, the key contributions of the paper can be summarized as follows:

- we derive the distribution of the number of sensed nodes and sensed distances, taking into account the statistical characterization of the thinned spatial process;
- we derive the Cramér-Rao bounds (CRBs) for the spatial density estimation, considering different sensing capabilities;
- we derive maximum likelihood (ML) density estimators for different sensing capabilities;
- we apply the methodology to wireless networks in the presence of thermal noise, path-loss, and shadowing, providing some easy-to-implement spatial density estimators.

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The results, specified with reference to two-dimensional PPPs, have been generalized to the n-dimensional case in Appendix A.

The remainder of the paper is organized as follows. In Section II we review the estimators for the homogeneous PPP networks. Section III presents the model for thinned PPP and the novel estimators, assuming both a single agent and cooperative agents estimation. In Section IV we specialize the analysis to a wireless scenario where propagation impairments are considered. In Section V we report a numerical comparison among different estimators, followed by concluding remarks in Section VI.

Throughout the paper, we distinguish between the random variable (r.v.) X or x with its realization x, boldface letters denote vectors, $\mathbb{E}\{\cdot\}$ stands for expectation, $\mathbb{V}\{\cdot\}$ stands for variance, and $\mathbb{P}\{\cdot\}$ denotes the probability of an event.

II. Density Estimation with Perfect Visibility

We consider the problem of estimating the spatial density, defined as entities per unit area, in two-dimensional stochastic geometry. More precisely, we assume a network with nodes distributed according to a homogeneous 2D Poisson point process having an unknown density $\lambda\,[\mathrm{nodes/m^2}]$. To avoid edge effects, which are negligible for large networks, it is assumed that the process is extended infinitely in all directions. For a homogeneous PPP with density λ, the number of nodes n in a region $\mathcal{A}$ of area $|\mathcal{A}|$ follows a Poisson distribution, with [49]

$$\mathbb{P}\{\mathsf{n}=n\} = \frac{(\lambda|\mathcal{A}|)^n}{n!}e^{-\lambda|\mathcal{A}|}\,. \tag{1}$$

In the following we use "agent" to denote special nodes which have the role of estimating the network density. We start by describing the two approaches that are commonly adopted to estimate the density of a PPP network. The first assumes the knowledge of node locations, and the second is based on ranging measurements [46]. More precisely, we consider the case where the agents can measure locations or distances with all neighboring nodes (so, all nodes are visible). In Section III the possibility that some nodes are not visible to the agents, for example, due to poor propagation conditions, will be taken into account.

Through the paper, we use $\mathsf{R}_j^{(k)}$ to describe the r.v. distance between the agent j and its k-th neighbor node. When it is clear what neighbor we are referring, the notation is simplified as R_j.

A. Estimation based on nodes spatial position

Let us consider M agents used to collect nodes information over disjoint regions $\mathcal{A}_j \subset \mathbb{R}^2$, $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for $i \neq j$, with $i,j = 1,\dots,M$. In the following $|\mathcal{A}_j|$ is denoted as capture area. Also, assume that all nodes in the generic $\mathcal{A}_j$ are aware of their spatial position. Then, if they share this information with the associated agent j, it can compute the number n_j of nodes in its area. From (1) the cooperative maximum likelihood estimator (MLE) of λ is given by

$$\widehat{\lambda} = \frac{\sum_{j=1}^{M} n_j}{\sum_{j=1}^{M}|\mathcal{A}_j|}\,. \tag{2}$$

The estimator (2) is unbiased with variance

$$\mathbb{V}\left\{\widehat{\lambda}\right\} = \frac{\lambda}{\sum_{j=1}^{M}|\mathcal{A}_j|}. \tag{3}$$

Moreover, (2) is an efficient estimator, since its variance coincides with the CRB for all λ.[1]

B. Estimation based on ranging: Pollard's estimator

The estimation of λ assuming the distances can be measured with no error (perfect ranging) has been studied extensively with application to biometrics, ecology, randomly distributed forests, and ad-hoc wireless networks [34], [35], [45], [46]. Let us consider M agents, where each one measures exactly one distance from its own position to the k-th nearest agent. In particular, the j-th agent records the distance to its k_j-th neighbor, where k_j takes some predetermined positive integer value. We assume that the agents have unlimited ranging capabilities, so there are no restrictions on the maximum value of k_j. When the agents of the network are distributed according to a homogeneous PPP, the distance R_j from an agent j to its k_j-th closest neighbor is a r.v. with probability density function (PDF) [46], [50], [51]

$$f_{\mathsf{R}_j}(r;\lambda) = \frac{2(\pi\lambda)^{k_j}}{(k_j-1)!}r^{2k_j-1}e^{-\pi\lambda r^2} \qquad r \geqslant 0 \tag{4}$$

and its square $\mathsf{W}_j = \mathsf{R}_j^2$ follows an Erlang distribution with PDF

$$f_{\mathsf{W}_j}(w;\lambda) = \frac{(\pi\lambda)^{k_j}w^{k_j-1}e^{-\pi\lambda w}}{(k_j-1)!} \qquad w \geqslant 0. \tag{5}$$

The distribution (5) describes also the sum of k_j independent, identically distributed (i.i.d.) exponential r.v.s with mean $\mu = 1/(\pi\lambda)$, so that we can write $\mathsf{R}_j^2 = \mathsf{Z}_1^{(j)} + \cdots + \mathsf{Z}_{k_j}^{(j)}$, where $\mathsf{Z}_i^{(j)}$ are i.i.d. exponentials. In order to derive the ML density estimator based on all measurements, we hence consider the joint distribution of $K \triangleq \sum_{j=1}^{M} k_j$ exponential r.v.s, Z_i, $i = 1,\dots,K$, all with mean $\mu = 1/(\pi\lambda)$

$$f_{\mathbf{Z}}(\boldsymbol{z};\mu) = \prod_{i=1}^{K}\frac{1}{\mu}e^{-\frac{1}{\mu}z_i}. \tag{6}$$

The ML estimate of μ is $\widehat{\mu} = \sum_{i=1}^{K} z_i/K$ which shows that $\sum_{i=1}^{K} z_i$ is a sufficient statistic. Since $\sum_{i=1}^{K}\mathsf{Z}_i = \sum_{j=1}^{M}\mathsf{R}_j^2$ we have

$$\widehat{\mu} = \frac{\sum_{j=1}^{M} r_j^2}{K} \tag{7}$$

and, due to the invariance property of the MLE [52], the ML density estimator is [46]

$$\widehat{\lambda} = \frac{K}{\pi\sum_{j=1}^{M} r_j^2}. \tag{8}$$

[1] Any unbiased estimator $\widehat{\lambda}$ of a deterministic scalar parameter λ, based on measurements distributed according to the PDF $f_{\mathbf{x}}(\boldsymbol{x};\lambda)$, has a variance which satisfies

$$\mathbb{V}\left\{\widehat{\lambda}\right\} \geqslant \mathrm{CRB}(\lambda) \triangleq -1\Big/\mathbb{E}\left\{\frac{\partial^2 \ln f_{\mathbf{x}}(\boldsymbol{x};\lambda)}{\partial\lambda^2}\right\}.$$

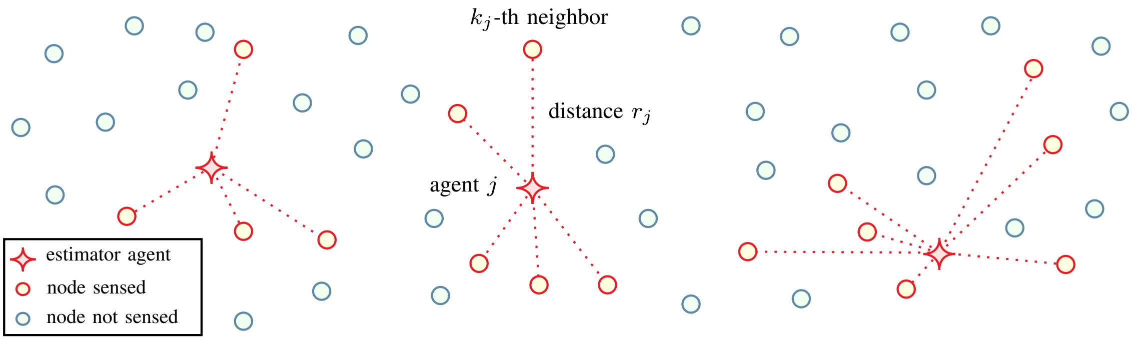

Fig. 1. Density estimation carried out by agents with impaired node visibility.

Using the joint distribution of $\mathbf{R} = (\mathsf{R}_1, \mathsf{R}_2, \ldots, \mathsf{R}_M)$ this estimator has been proved to be biased [46]. An alternative, easier way, to derive the bias is given by the fact that

$$\mathbb{E}\left\{\widehat{\lambda}\right\} = \int_0^\infty \cdots \int_0^\infty \frac{K}{\pi \sum_{j=1}^M r_j^2} f_{\mathbf{R}}(\boldsymbol{r}; \lambda) d\boldsymbol{r}$$
$$= \int_0^\infty \cdots \int_0^\infty \frac{K}{\pi \sum_{i=1}^K z_i} f_{\mathbf{Z}}(\boldsymbol{z}; \lambda) d\boldsymbol{z} \,. \tag{9}$$

After the substitution $u_\ell = \sum_{i=1}^\ell z_i$ it is simple to check that the expectation is $\mathbb{E}\left\{\widehat{\lambda}\right\} = \lambda K/(K-1)$. To have finite expectation, it must be $K > 1$. An unbiased estimator is then given by [46]

$$\widehat{\lambda} = \frac{K-1}{\pi \sum_{j=1}^M r_j^2}. \tag{10}$$

Similarly, we can derive the variance of (10) as

$$\mathbb{V}\left\{\widehat{\lambda}\right\} = \frac{\lambda^2}{K-2} \tag{11}$$

which to be finite requires $K > 2$ and, for large K, approaches the corresponding CRB

$$\mathrm{CRB}(\lambda) = \frac{\lambda^2}{K} \,. \tag{12}$$

From (11) it appears that using the distances of distant neighbors instead of the nearest ones results in a reduced variance [46].

Remark 1: It is interesting to note that the estimator (10), based on distance measurements, has a smaller variance than the estimator (2), based on the number of nodes in a given area, $|\mathcal{A}| = \sum_{j=1}^M |\mathcal{A}_j|$, whenever the following condition is satisfied

$$\lambda < \frac{K-2}{|\mathcal{A}|}. \tag{13}$$

Remark 2: It is not always true that the best estimator in terms of mean-square error (MSE) is an unbiased estimator. However, using the information inequality [53], the MSE of any estimator with $\mathbb{E}\left\{\widehat{\lambda}\right\} = \lambda K/(K-1)$ can be bounded as

$$\mathrm{MSE} = \mathbb{E}\left\{\left(\widehat{\lambda} - \lambda\right)^2\right\} \geq \frac{K+1}{(K-1)^2}\lambda^2 \,. \tag{14}$$

When $K \geq 3$, the right side of (14) is always larger than (11). Hence, the unbiased estimator (10) is better than any estimator with expectation equal to $\lambda K/(K-1)$.

III. Density Estimation with Impaired Visibility

In this section, we consider a scenario where, with some probability, nodes could not be visible by the agents. We assume an isotropic case, where the probability that a node is visible, $p(r)$, depends only on the radial distance r from the agent to the node. After the general treatment for arbitrary $p(r)$, we will specialize the analysis to wireless networks, where this probability is related to path-loss and shadowing.

A. Isotropically thinned PPPs

Let us consider the spatial process constituted by the nodes seen by a generic agent. If each node is visible with some probability $p(r)$, this observed spatial process is still a Poisson process, but it is no longer homogeneous. More precisely, the observed is a thinned PPP whose density is [49]

$$\lambda_{\mathrm{t}}(r) = \lambda\, p(r) \,. \tag{15}$$

The probability that in a region $\mathcal{A}$ there are k visible nodes is given by

$$\mathbb{P}\{\mathsf{n}(\mathcal{A}) = k\} = \frac{[\Lambda(\mathcal{A})]^k}{k!}\, e^{-\Lambda(\mathcal{A})} \tag{16}$$

where

$$\Lambda(\mathcal{A}) = \lambda \int_{\mathcal{A}} r\, p(r)\, dr d\theta \tag{17}$$

is the average number of nodes sensed in $\mathcal{A}$. For the whole plane, equation (17) becomes

$$\Lambda = \Lambda(\mathbb{R}^2) = 2\pi\lambda \int_0^\infty r\, p(r) dr \,. \tag{18}$$

For the thinned process, we distinguish two scenarios depending on the convergence of (18). To this aim, we define the function

$$\psi(r) \triangleq \int_0^r \xi\, p(\xi) d\xi \,. \tag{19}$$

In appendix A we show the generalization to the n-dimensional case. Without losing generality, through the paper we assume a 2-dimensional scenario which is the most significant in wireless networks.

Adopting the notation $\psi(\infty) = \lim_{r\to\infty} \psi(r)$, we have the two cases below.

- Infinite number of visible nodes: $\psi(\infty) = \infty$.
  Here, $\Lambda = \infty$ as for the homogeneous PPP, and the average number of nodes sensed from an agent is infinite.

The PDF of the distance R_j between the agent j and its k-th closest neighbor is derived in Appendix B-A as

$$f_{\mathsf{R}_j^{(k)}}(r;\lambda) = \frac{(2\pi\lambda)^k}{(k-1)!}\, r\,p(r)\,[\psi(r)]^{k-1}\, e^{-2\pi\lambda\psi(r)} \quad (20)$$

where $r \geqslant 0$. This can be seen as a generalization of the previously known distance distribution in (4), which is the particular case $p(r) = 1$. Similarly, we can derive the joint ordered distances distribution for the first k closest neighbors of the agent j as (see Appendix B-B)

$$f_{\mathsf{R}_j^{(1)},\mathsf{R}_j^{(2)},\cdots,\mathsf{R}_j^{(k)}}(r_1, r_2, \cdots, r_k;\lambda) = (2\pi\lambda)^k e^{-2\pi\lambda\psi(r_k)} \prod_{\ell=1}^{k} r_\ell\, p(r_\ell)\,. \quad (21)$$

Applying the factorization theorem to (20) or (21), the sufficient statistic for the estimation of λ, assuming the distances to the closest k nodes are known, is reduced to r_k [52, Theorem 6.2.6].

- Finite number of visible nodes: $\psi(\infty) < \infty$.
  In this scenario, typical of wireless networks, the average number of nodes sensed over the whole plane is finite. We can derive the distance distribution of the farthest visible node, jointly to the event that k nodes are sensed, as (see Appendix B-C)

$$f_{\mathsf{R}_j^{(k)},\mathsf{K}}(r,k;\lambda) = \frac{(2\pi\lambda)^k}{(k-1)!} r\,p(r)\,[\psi(r)]^{k-1}\, e^{-2\pi\lambda\psi(\infty)}\,. \quad (22)$$

  Similarly, it is possible to write the joint ordered distances distribution for all the k neighbors of the agent j as (see Appendix B-D)

$$f_{\mathsf{R}_j^{(1)},\mathsf{R}_j^{(2)},\ldots,\mathsf{R}_j^{(k)},\mathsf{K}}(r_1, r_2, \ldots, r_k, k;\lambda) = (2\pi\lambda)^k e^{-2\pi\lambda\psi(\infty)} \prod_{\ell=1}^{k} r_\ell\, p(r_\ell)\,. \quad (23)$$

  In this case, applying the factorization theorem to (23), the sufficient statistic for the estimation of λ, assuming to sense a random number of nodes and corresponding distances, is just their number k.

B. Density estimation with a single agent

Consider the scenario where a single agent is used to estimate the density λ. Its area of interest is $\mathcal{A} = \mathbb{R}^2$, and the sensing capability is therefore limited only by $p(r)$. Below we derive the ML estimators for both infinite and finite number of visible nodes.

1) Infinite visible nodes, distance-based estimator: Let us assume the agent knows the distances to the closest k nodes. Starting from the expression of the PDF in (20), or equivalently from (21), it can be checked that the CRB for estimators of the density is given by

$$\mathrm{CRB}(\lambda) = \frac{\lambda^2}{k}\,. \quad (24)$$

Then, from (20) or (21) we derive the ML estimator in closed form as

$$\widehat{\lambda} = \frac{k}{2\pi\psi(r_k)}\,. \quad (25)$$

As noted before, for a given number k of ordered distances, a sufficient statistic is just that of the farthest node, r_k. So, while the information about the first $k-1$ neighbors could seem not used, actually k takes into account that there are $k-1$ nearest nodes before that at r_k.

For the ML estimator (25), using (20) we have

$$\begin{aligned}\mathbb{E}\left\{\widehat{\lambda}\right\} &= \int_0^\infty \frac{k}{2\pi\psi(r)} f_{\mathsf{R}_j^{(k)}}(r;\lambda)\,dr \\ &= \frac{k\lambda}{k-1}\int_0^\infty \frac{(2\pi\lambda)^{k-1}}{(k-2)!}\, rp(r)\,[\psi(r)]^{k-2}\, e^{-2\pi\lambda\psi(r)}\,dr \\ &= \frac{k}{k-1}\,\lambda\,. \end{aligned} \quad (26)$$

An unbiased estimator is therefore given by

$$\widehat{\lambda} = \frac{k-1}{2\pi\psi(r_k)}\,. \quad (27)$$

The performance of this unbiased estimator in terms of variance are given by

$$\mathbb{V}\left\{\widehat{\lambda}\right\} = \frac{\lambda^2}{k-2} \quad (28)$$

which, for large k, tends to the CRB in (24). To have finite expected value and variance, we thus consider $k > 2$.

2) Finite visible nodes, counting estimator: As observed, from (23) the sufficient statistic is the total number k of sensed neighbors, while the extra knowledge of the distances does not add information to the estimation. Therefore, from (16) we can derive the CRB as

$$\mathrm{CRB}(\lambda) = \frac{-1}{\mathbb{E}\left\{\frac{\partial^2 \ln \mathbb{P}\{\mathsf{n}(\mathbb{R}^2)=k;\lambda\}}{\partial\lambda^2}\right\}} = \frac{\lambda}{2\pi\psi(\infty)} \quad (29)$$

and the ML estimator as

$$\widehat{\lambda} = \frac{k}{2\pi\psi(\infty)}\,. \quad (30)$$

We will refer to this as the counting estimator, since the agent needs just to count all its visible neighbors. This estimator is unbiased, as can be checked by computing

$$\mathbb{E}\left\{\widehat{\lambda}\right\} = \sum_{k=0}^{\infty} \frac{k}{2\pi\psi(\infty)} \frac{\Lambda^k}{k!} e^{-\Lambda} = \lambda\,. \quad (31)$$

The variance is

$$\begin{aligned}\mathbb{V}\left\{\widehat{\lambda}\right\} &= \sum_{k=0}^{\infty} \left(\frac{k}{2\pi\psi(\infty)}\right)^2 \frac{\Lambda^k}{k!} e^{-\Lambda} - \lambda^2 \\ &= \frac{\lambda}{2\pi\psi(\infty)}\end{aligned} \quad (32)$$

which, compared to (29), prove that (30) is an efficient estimator.

C. Density estimation with cooperating agents

Now we examine the scenario where $M \geq 2$ agents cooperate to estimate the density of the PPP. Assuming independent measurements from the agents, we derive estimators for all possible types of information the agents could gather from the environment. Again, the sensing capability is limited only by $p(r)$, with two different scenarios characterized by the asymptotic value of $\psi(r)$.

1) Infinite visible nodes, distance-based estimator: In this case, we assume each agent measures one distance. Precisely, the agent j measures the distance r_j to its k_j-th neighbor. Since each distance is a r.v. distributed as (20), the joint PDF for all agent measurements can be expressed as

$$f_{\mathsf{R}_1,\mathsf{R}_2,\cdots,\mathsf{R}_M}(r_1, r_2, \cdots, r_M; \lambda) = \frac{(2\pi\lambda)^K}{\prod_{j=1}^{M}(k_j-1)!} \prod_{j=1}^{M} r_j \; p(r_j) \, [\psi(r_j)]^{k_j-1} \, e^{-2\pi\lambda\psi(r_j)} \,. \tag{33}$$

Comparing (33) with (20) it is simple to derive, similarly to the single agent scenario, the CRB as

$$\mathrm{CRB}(\lambda) = \frac{\lambda^2}{K} \tag{34}$$

and the MLE as

$$\widehat{\lambda} = \frac{K}{2\pi \sum_{j=1}^{M} \psi(r_j)} \tag{35}$$

whose expectation is (see Appendix C)

$$\mathbb{E}\left\{\widehat{\lambda}\right\} = \lambda \, \frac{K}{K-1} \,. \tag{36}$$

From (35) and (36), an unbiased estimator is

$$\widehat{\lambda} = \frac{K-1}{2\pi \sum_{j=1}^{M} \psi(r_j)} \,. \tag{37}$$

Its variance, that can be derived with steps similar to those used to demonstrate (36), is

$$\mathbb{V}\left\{\widehat{\lambda}\right\} = \frac{\lambda^2}{K-2} \tag{38}$$

which, for large K, approaches the CRB.

2) Finite visible nodes, counting estimator: Suppose that each agent can count and estimate the distances from each neighbor so that the joint distribution of distances and number of sensed nodes is given by products of (22). Similarly to the single agent scenario, it can be checked that the sufficient statistic for the estimation of λ is given by the number of neighbors sensed by each agent. Let us indicate by k_j the number of nodes sensed by the j-th agent. Then, from the joint distribution

$$\mathbb{P}\{k_1, \ldots, k_M; \lambda\} = \frac{\Lambda^K}{\prod_{j=1}^{M} k_j!} e^{-M\Lambda} \tag{39}$$

the MLE can be derived as

$$\widehat{\lambda} = \frac{K}{2\pi M \psi(\infty)} \tag{40}$$

which will be indicated as counting estimator. We note that the MLE can be interpreted as the arithmetic mean of the single agent estimations (30). Hence, it is unbiased and efficient, with variance given by

$$\mathbb{V}\left\{\widehat{\lambda}\right\} = \mathrm{CRB}(\lambda) = \frac{\lambda}{2\pi M \psi(\infty)} \,. \tag{41}$$

3) Finite visible nodes, detection-based estimator: Let us now consider the particular case of simple, low-complexity nodes, capable of detecting the presence of neighbors but not able to estimate their number or distances. So, an agent can only sense the absence/presence of neighbors, and we can write the probability of n_0 agents sensing no one as

$$\mathbb{P}\{\mathsf{n}_0 = n_0; \lambda\} = \binom{M}{n_0} e^{-n_0\Lambda} \left(1 - e^{-\Lambda}\right)^{M-n_0} \,. \tag{42}$$

From this distribution it can be shown that the CRB is

$$\mathrm{CRB}(\lambda) = \frac{e^{\Lambda} - 1}{M\,(2\pi\psi(\infty))^2} \tag{43}$$

and that the MLE is

$$\widehat{\lambda} = \frac{1}{2\pi\psi(\infty)} \ln\left(\frac{M}{n_0}\right) \tag{44}$$

which will be indicated as detection-based estimator. Since this estimator diverges for $n_0 = 0$, it is applicable only for not too dense networks. Note that, in wireless scenarios, the detection-based estimator can be implemented using a simple energy detector [43].

Remark 3: For small λ, the CRB in (43) tends to the value $\lambda \big/ (2\pi M \psi(\infty))$, which is equal to (41).

IV. Application: Density Estimation in Wireless Networks

In wireless networks, both the agents and the generic entities constituting our PPP are radio-equipped devices. Adopting the models used for wireless links, we will assume that an agent can hear a node if the received power is above a threshold $P_{\rm th}$, which depends on the thermal noise level. Besides thermal noise, we assume wireless links affected by path-loss and log-normal shadowing [41], [54]. Then, a node at distance r can be detected if

$$P \frac{e^{2\sigma \mathsf{G}}}{r^{\alpha}} > P_{\rm th} \tag{45}$$

where P is the received power at a reference distance $d_0 = 1\,\mathrm{m}$, σ is the shadowing parameter, $\alpha > 0$ is the path-loss exponent, $\mathsf{G} \sim \mathcal{N}(0,1)$ is a Gaussian r.v. accounting for shadowing effects.[2] Due to the condition (45) only a fraction of nodes will be detected. Precisely, a node at a distance r can be sensed with probability $p(r)$, which in this specific propagation environment is given by

$$p(r) = \mathbb{P}\left\{P \frac{e^{2\sigma \mathsf{G}}}{r^{\alpha}} > P_{\rm th}\right\} = \mathcal{Q}\left(\frac{\ln \bar{P}_{\rm th} + \alpha \ln r}{2\sigma}\right) \tag{46}$$

where $\mathcal{Q}(x)$ is the $\mathcal{Q}$-function of the standard normal distribution, and $\bar{P}_{\rm th} = P_{\rm th}/P$ is the normalized threshold. This

[2] The path-loss exponent can also be estimated in large wireless networks [55], [56].

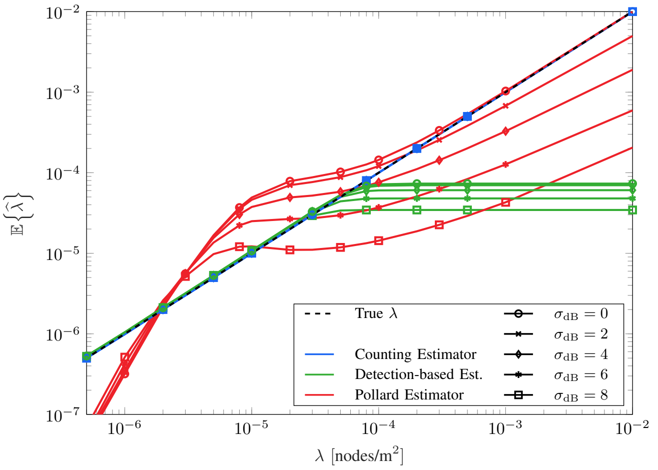

Fig. 2. Expected value of density estimators as a function of the actual density for different shadowing parameter σ_{dB}.

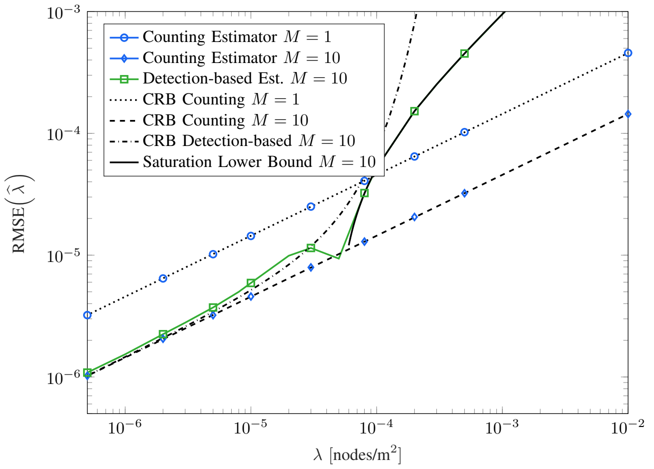

Fig. 3. RMSE of density estimators as a function of the actual density for a different number of agents, M, and $\sigma_{\mathrm{dB}} = 6$.

selection based on the received power is independent for each point and gives rise to the thinning of the PPP. Note that, in the absence of shadowing ($\sigma = 0$) the propagation model can be seen as a disk model, i.e., $p(r) = 1$ for $r < \bar{P}_{\mathrm{th}}^{-1/\alpha}$ and $p(r) = 0$ otherwise.

In order to perform density estimation, we first compute $\psi(r)$ according to (19) as

$$\psi(r) = \int_0^r \xi \mathcal{Q}\left(\frac{\ln \bar{P}_{\mathrm{th}} + \alpha \ln \xi}{2\sigma}\right) d\xi = \\ = \frac{r^2}{2}\mathcal{Q}\Big(\eta\,(r)\Big) + \frac{e^{8\sigma^2/\alpha^2}}{2\,\bar{P}_{\mathrm{th}}^{2/\alpha}}\,\mathcal{Q}\left(\frac{4\sigma}{\alpha} - \eta\,(r)\right) \tag{47}$$

where

$$\eta\,(r) \triangleq \frac{\ln \bar{P}_{\mathrm{th}} + \alpha \ln r}{2\sigma}\,.$$

Its asymptotic behaviour is

$$\psi(\infty) = \frac{1}{2}\exp\left(\frac{8\sigma^2}{\alpha^2} - \frac{2\ln \bar{P}_{\mathrm{th}}}{\alpha}\right) \tag{48}$$

which is finite since the threshold P_{th} cannot be set to zero due to thermal noise. So, as expected, in the wireless network scenario the nodes visibility is limited, and the average number of nodes detected by an agent is finite.

V. Numerical Results

In this section, we report some numerical results for the scenario introduced in section IV, to verify the effectiveness of the proposed density estimators. The parameters used in the simulations are: normalized threshold $\bar{P}_{\mathrm{th}} = 10^{-6}$, path-loss exponent $\alpha \in [1.5, 4]$, and log-normal shadowing standard deviation $\sigma_{\mathrm{dB}} = 10\log_{10}(\exp(2\sigma)) \in [0, 8]$, according to typical wireless scenarios [54]. When not specified, we assume $\alpha = 3$ and $\sigma_{\mathrm{dB}} = 6$. From (48) we can then compute $\psi(\infty)$ and hence the average number of sensed nodes $2\pi\lambda\psi(\infty)$. For example, with the parameters above, each agent is expected to sense on the average 4.8 nodes when $\lambda = 10^{-4}$ [nodes/m^2]. Since in this scenario $\psi(\infty)$ is finite, the density estimators of interest are that based on counting in (40), and the simple one based on detection given in (44). For comparison, we will also report the performance of Pollard's estimator (10), where we chose k_j according to the farthest visible neighbor. In particular, we impose the estimate to be equal to zero when the total number of visible nodes, considering all cooperating agents, is $K < 3$.

Regarding the detection-based estimator, in order to avoid divergence, we modify the estimator in (44) as

$$\widehat{\lambda} = \begin{cases} \dfrac{1}{2\pi\psi(\infty)} \ln\left(\dfrac{M}{n_0}\right) & n_0 > 0 \\ \dfrac{\ln(M)}{2\pi\psi(\infty)} & n_0 = 0\,. \end{cases} \tag{49}$$

Since the estimator is a constant when $n_0 = 0$, for $\lambda > \ln(M)/(2\pi\psi(\infty))$, the MSE is lower-bounded by

$$\mathbb{E}\Big\{(\widehat{\lambda} - \lambda)^2\Big\} > \left(\lambda - \frac{\ln(M)}{2\pi\psi(\infty)}\right)^2 \tag{50}$$

and the CRB in (43) is useful for small λ. We will refer to the bound in (50) as saturation lower bound.

To investigate the bias of the various estimators, in Fig. 2 we report the estimate averages obtained through Monte Carlo simulations, assuming $M = 10$ agents. Let us first discuss Pollard's estimator. This has been designed in the hypothesis of an infinite PPP without thinning, and therefore it is not expected to perform optimally in the investigated wireless scenario. In fact, we can see that the Pollard estimator has a bias that increases with σ_{dB}. Moreover, it is interesting to note that, for large λ and small σ_{dB}, the estimator tends to be unbiased. This is due to the fact that, without shadowing, the model in (45) degenerates to the disk model and, for large λ, the farthest neighbor distance is close to the disk radius. In this case, the Pollard estimator tends to (2), which is the ratio of the number of sensed nodes and the disk area. Let us now discuss the other estimators. We start by noting that, as expected, the counting estimator is unbiased. On the other hand, the detection-based estimator has a slight bias for small λ, while it is not suitable for large λ where, with high probability, all

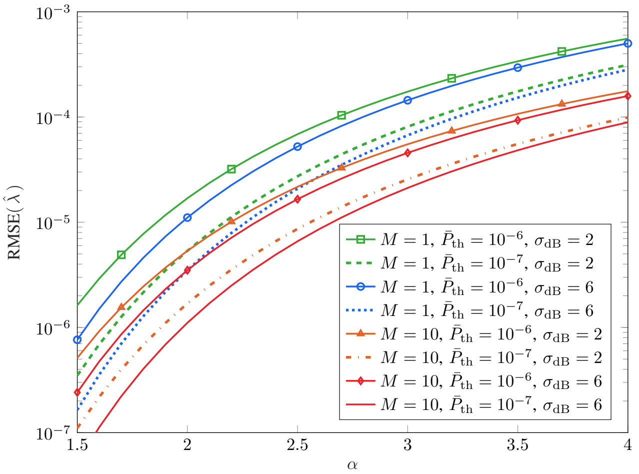

Fig. 4. Counting estimator RMSE as a function of the path-loss exponent for $\lambda = 10^{-3}$ [nodes/m^2].

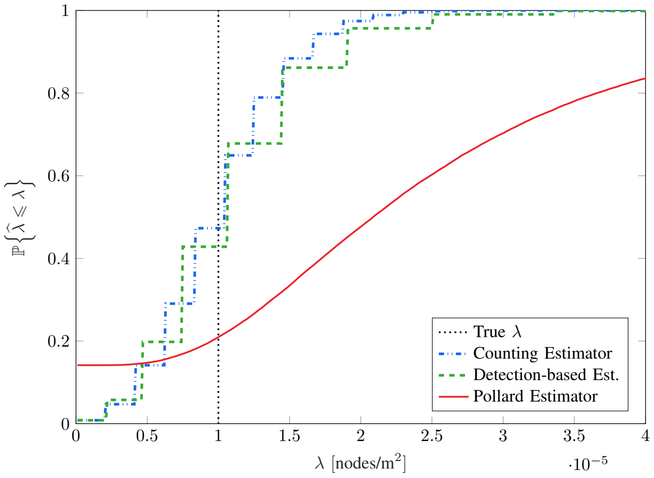

Fig. 5. Cumulative distribution functions of density estimates for $M = 10$, $\sigma_{\text{dB}} = 6$, and $\lambda = 10^{-5}$ [nodes/m^2].

agents will sense at least one node, saturating the estimate to $\ln M / 2\pi\psi(\infty)$.

In Fig. 3 we report the root-mean-square errors (RMSEs) of the two proposed estimators. For the counting estimator, we also show the results when a single agent is in charge of the estimation, $M = 1$. For comparison, we report also the CRB of the analyzed estimators. The counting estimator is efficient and thus, as shown in the figure, its RMSE is equal to the CRB for all λ. Regarding the detection-based estimator, we observe that for small λ the error is comparable with the counting one, and the RMSE is close to (43), since the estimator is almost unbiased (see Fig. 2). As anticipated, for large λ the estimator saturates and its lower bound becomes (50).

In Fig. 4, we focus our attention on the counting estimator. In particular, we show how the performance in terms of RMSE deteriorates when the path-loss exponent α increases, for two different values of the normalized threshold, $\bar{P}_{\text{th}} = 10^{-6}$ and $\bar{P}_{\text{th}} = 10^{-7}$, and $\lambda = 10^{-3}$ [nodes/m^2]. The behavior is due to the fact that increasing α reduces the sensing capability of agents. For example, in a scenario without shadowing, the disk area for each agent decreases.

Finally, in Fig. 5 we report an example of the cumulative distribution of the estimates. As expected from Fig. 3, the counting estimator shows the steepest behavior around the true value of λ. In addition, it is possible to observe the quantized nature of the counting and detection-based estimators. On the other hand, the Pollard's estimator is continuous for all $\lambda > 0$.

VI. Conclusion

This paper addresses the problem of nodes spatial density estimation in wireless networks where the nodes distribution is a two-dimensional PPP. The framework developed accounts for realistic channel models where some nodes could be invisible because of propagation impairments, leveraging on the properties of thinned PPPs. In this scenario, we derived the distribution of the number of sensed nodes and sensed distances from which we found new density estimators that exploit different types of measurements, namely distance-based, counting-based and detection-based estimators. The findings of this paper demonstrate that: i) Pollard's estimator does not provide accurate results in wireless scenarios where channel impairments play a significant role; ii) the counting-based estimator performs the best and is efficient; iii) the very simple detection-based estimator works well when the nodes spatial density is low, where it approaches the CRB.

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Appendix A
Generalization to n-Dimensional Case

The results obtained in this paper are easily generalized to the case where the point process is a n-dimensional homogeneous PPP. The probability that in the n-dimensional space there are k visible nodes considering the radial thinned process described in section III-A is given by

$$\mathbb{P}\{\mathsf{n}(\mathbb{R}^n) = k\} = \frac{[\Lambda(\mathbb{R}^n)]^k}{k!} e^{-\Lambda(\mathbb{R}^n)} . \tag{51}$$

To express $\Lambda(\mathbb{R}^n)$ we use n-dimensional spherical coordinates, i.e., a radial coordinate r, and $n-1$ angular coordinates, where the angles $\theta_1, \theta_2, \ldots, \theta_{n-2}$ range over $[0, \pi]$ and θ_{n-1} ranges over $[0, 2\pi)$. Recalling the expression for the volume element in n-dimensional spherical coordinates $d^n V = r^{n-1} \left(\prod_{i=1}^{n-1} \sin^{n-i-1}(\theta_i)\, d\theta_i \right) dr$, we have

$$\begin{aligned}\Lambda(\mathbb{R}^n) &= \lambda \int_{\mathbb{R}^n} r^{n-1} p(r) \left(\prod_{i=1}^{n-1} \sin^{n-i-1}(\theta_i)\, d\theta_i \right) dr \\ &= \lambda \frac{n\, \pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \int_0^\infty r^{n-1} p(r)\, dr . \end{aligned} \tag{52}$$

By defining the function

$$\Psi(r; n) \triangleq \frac{n\, \pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \int_0^r \xi^{n-1} p(\xi)\, d\xi \tag{53}$$

the considerations and all the estimators derived through the paper are easily generalized to the n-dimensional case. In particular, from (19) we observe that $\psi(r) = \Psi(r;2)/2\pi$.

Since $\Psi(r;n)$ is not a function of λ, the maximum likelihood estimator derivations and Cramér-Rao bounds are not affected by this generalization. For example, the counting estimator in (30) easily becomes

$$\widehat{\lambda} = \frac{k}{\Psi(\infty;n)} \,. \tag{54}$$

APPENDIX B
DISTANCE DISTRIBUTION IN AN INHOMOGENEOUS PPP

Let us consider an inhomogeneous 2D PPP with density that depends only on the distance r to the origin, i.e., $\lambda_{\mathrm{t}}(r) = \lambda\, p(r)$. Then, the probability that k nodes can be found in a region $\mathcal{A}$ is given by

$$\mathbb{P}\{\mathsf{n}(\mathcal{A}) = k\} = \frac{[\Lambda(\mathcal{A})]^k}{k!} e^{-\Lambda(\mathcal{A})} \tag{55}$$

where

$$\Lambda(\mathcal{A}) = \lambda \int_{\mathcal{A}} r\, p(r)\, dr d\theta \,. \tag{56}$$

A. Distance distribution of the k-th neighbour

In the following is derived the distance distribution of the k-th neighbor from an agent j, in the case of an inhomogeneous 2D PPP, applying the same ideas previously used in the homogeneous case [50], [51]. In fact, we can write the probability of finding the k-th neighbor at distance r_k as

$$f_{\mathsf{R}_j^{(k)}}(r_k) dr_k = \mathbb{P}\{\mathsf{n}(\mathcal{R}) = 1\}\, \mathbb{P}\{\mathsf{n}(\mathcal{D}) = k-1\} \tag{57}$$

with

$$\begin{aligned} \mathcal{R} &= \{\text{annulus within radii } (r_k, r_k + dr_k)\} \\ \mathcal{D} &= \{\text{disk with radius } (0, r_k)\} \end{aligned}$$

where we applied the independent increment property of PPP along with the fact that $\mathcal{D} \cap \mathcal{R} = \emptyset$ [49]. Now, considering that for a PPP

$$\mathbb{P}\{\mathsf{n}(\mathcal{R}) = 1\} = \Lambda(\mathcal{R}) e^{-\Lambda(\mathcal{R})} \tag{58}$$

$$\mathbb{P}\{\mathsf{n}(\mathcal{D}) = k-1\} = \frac{[\Lambda(\mathcal{D})]^{k-1}}{(k-1)!} e^{-\Lambda(\mathcal{D})} \tag{59}$$

where

$$\Lambda(\mathcal{R}) = \int_{\mathcal{R}} r \lambda\, p(r)\, dr d\theta = 2\pi\lambda\, p(r_k) r_k dr_k \tag{60}$$

$$\Lambda(\mathcal{D}) = \int_{\mathcal{D}} r \lambda\, p(r)\, dr d\theta = 2\pi\lambda \int_0^{r_k} r\, p(r) dr \tag{61}$$

the desired distance distribution becomes

$$f_{\mathsf{R}_j^{(k)}}(r_k) = \frac{(2\pi\lambda)^k}{(k-1)!} r_k\, p(r_k) \left[\psi(r_k)\right]^{k-1} e^{-2\pi\lambda\psi(r_k)} \tag{62}$$

with

$$\psi(r) \triangleq \int_0^r \xi\, p(\xi)\, d\xi \,. \tag{63}$$

B. Joint ordered distances distribution for the first k neighbours

The joint ordered distances distribution for the first k neighbors $\mathsf{R}_j^{(1)} \leqslant \mathsf{R}_j^{(2)} \leqslant \ldots \leqslant \mathsf{R}_j^{(k)}$ from the agent j can be obtained as

$$\begin{aligned} & f_{\mathsf{R}_j^{(1)}, \mathsf{R}_j^{(2)}, \ldots, \mathsf{R}_j^{(k)}}(r_1, r_2, \ldots, r_k) dr_1 dr_2 \ldots dr_k = \\ & \quad = \prod_{\ell=1}^{k} \mathbb{P}\{\mathsf{n}(\mathcal{R}_\ell = 1\}\, \mathbb{P}\{\mathsf{n}(\mathcal{V}_\ell) = 0\} \end{aligned} \tag{64}$$

with

$$\begin{aligned} \mathcal{R}_\ell &= \{\text{annulus within radii } (r_\ell, r_\ell + dr_\ell)\} \\ \mathcal{V}_\ell &= \{\text{annulus within radii } (r_{\ell-1}, r_\ell) \text{ and } r_0 = 0\} \end{aligned}$$

where we applied the independent increment property of PPP along with the fact that $\mathcal{V}_\ell \cap \mathcal{R}_\ell = \emptyset$ for $\ell = 1, \ldots, k$. Now, considering that for a PPP

$$\mathbb{P}\{\mathsf{n}(\mathcal{R}_\ell) = 1\} = \Lambda(\mathcal{R}_\ell) e^{-\Lambda(\mathcal{R}_\ell)} \tag{65}$$

$$\mathbb{P}\{\mathsf{n}(\mathcal{V}_\ell) = 0\} = e^{-\Lambda(\mathcal{V}_\ell)} \tag{66}$$

where

$$\Lambda(\mathcal{R}_\ell) = \lambda \int_{\mathcal{R}_\ell} r\, p(r)\, dr d\theta = 2\pi\lambda\, r_\ell\, p(r_\ell)\, dr_\ell \tag{67}$$

$$\Lambda(\mathcal{V}_\ell) = \lambda \int_{\mathcal{V}_\ell} r\, p(r)\, dr d\theta = 2\pi\lambda \int_{r_{\ell-1}}^{r_\ell} r\, p(r)\, dr \tag{68}$$

the joint distribution becomes

$$\begin{aligned} & f_{\mathsf{R}_j^{(1)}, \mathsf{R}_j^{(2)}, \ldots, \mathsf{R}_j^{(k)}}(r_1, r_2, \cdots, r_k) = \\ & = (2\pi\lambda)^k \prod_{\ell=1}^{k} r_\ell\, p(r_\ell) \exp\left(-2\pi\lambda \int_{r_{\ell-1}}^{r_\ell} r\, p(r)\, dr\right) \\ & = (2\pi\lambda)^k \exp\left(-2\pi\lambda \sum_{\ell=1}^{k} \int_{r_{\ell-1}}^{r_\ell} r\, p(r)\, dr\right) \prod_{\ell=1}^{k} r_\ell\, p(r_\ell) \\ & = (2\pi\lambda)^k e^{-2\pi\psi(r_k)} \prod_{\ell=1}^{k} r_\ell\, p(r_\ell) \end{aligned} \tag{69}$$

with $\psi(r)$ defined in (63).

C. Distance distribution of the last neighbor

When the thinning process on the PPP limits the number of sensed nodes to a finite number, we can write the probability to detect exactly k neighbors and that the last one is at a distance r from the agent j as

$$\begin{aligned} f_{\mathsf{R}_\mathsf{j}^{(\mathsf{k})}, \mathsf{K}}(r, k) dr = & \ \mathbb{P}\{\mathsf{n}(\mathcal{E}) = 0\}\, \mathbb{P}\{\mathsf{n}(\mathcal{R}) = 1\} \\ & \times \mathbb{P}\{\mathsf{n}(\mathcal{D}) = k-1\} \end{aligned} \tag{70}$$

with

$$\begin{aligned} \mathcal{E} &= \{\text{annulus with radius } (r, \infty)\} \\ \mathcal{R} &= \{\text{annulus within radii } (r, r + dr)\} \\ \mathcal{D} &= \{\text{disk with radius } (0, r)\} \end{aligned}$$

where we applied the independent increment property of PPP along with the fact that $\mathcal{D} \cap \mathcal{R} = \emptyset$, $\mathcal{D} \cap \mathcal{E} = \emptyset$ and $\mathcal{E} \cap \mathcal{R} = \emptyset$. Now, recalling (58) and (59) and that, for a PPP

$$\mathbb{P}\{\mathsf{n}(\mathcal{E}) = 0\} = e^{-\Lambda(\mathcal{E})} \tag{71}$$

with

$$\Lambda(\mathcal{E}) = \lambda \int_{\mathcal{E}} r\, p(r)\, dr d\theta = 2\pi\lambda \int_r^\infty \xi\, p(\xi)\, d\xi \tag{72}$$

the desired distance distribution assumes the form

$$f_{\mathsf{R}_j^{(k)},\mathsf{K}}(r,k) = \frac{(2\pi\lambda)^k}{(k-1)!} r\, p(r)\, [\psi(r)]^{k-1}\, e^{-2\pi\lambda\psi(\infty)} \tag{73}$$

with $\psi(r)$ defined in (63) and

$$\psi(\infty) \triangleq \int_0^\infty \xi\, p(\xi)\, d\xi\,. \tag{74}$$

D. Joint ordered distances distribution for all the k neighbours

The joint ordered distances distribution for all the k neighbors $\mathsf{R}_j^{(1)} \leqslant \mathsf{R}_j^{(2)} \leqslant \ldots \leqslant \mathsf{R}_j^{(k)}$ of the agent j can be obtained as

$$\begin{aligned} & f_{\mathsf{R}_j^{(1)},\mathsf{R}_j^{(2)},\ldots,\mathsf{R}_j^{(k)},\mathsf{K}}(r_1, r_2, \cdots, r_k, k) dr_1 dr_2 \ldots dr_k = \\ & = \mathbb{P}\{\mathsf{n}(\mathcal{E}) = 0\} \prod_{\ell=1}^{k} \mathbb{P}\{\mathsf{n}(\mathcal{R}_\ell) = 1\}\, \mathbb{P}\{\mathsf{n}(\mathcal{V}_\ell) = 0\} \end{aligned} \tag{75}$$

with

$$\begin{aligned} \mathcal{E} &= \{\text{annulus with radius } (r_k, \infty)\} \\ \mathcal{R}_\ell &= \{\text{annulus within radii } (r_\ell, r_\ell + dr_\ell)\} \\ \mathcal{V}_\ell &= \{\text{annulus within radii } (r_{\ell-1}, r_\ell) \text{ and } r_0 = 0\} \end{aligned}$$

where we applied the independent increment property of PPP along with the fact that $\mathcal{V}_\ell \cap \mathcal{R}_\ell = \emptyset$, $\mathcal{E} \cap \mathcal{R}_\ell = \emptyset$, and $\mathcal{V}_\ell \cap \mathcal{E} = \emptyset$ for $\ell = 1, \ldots, k$. Therefore, considering (65), (66), and (71), the joint distribution becomes

$$\begin{aligned} & f_{\mathsf{R}_j^{(1)},\mathsf{R}_j^{(2)},\ldots,\mathsf{R}_j^{(k)},\mathsf{K}}(r_1, r_2, \cdots, r_k, k) = \\ & = (2\pi\lambda)^k e^{-\Lambda(\mathcal{E})} \prod_{\ell=1}^{k} r_\ell\, p(r_\ell) \exp\left(-2\pi\lambda \int_{r_{\ell-1}}^{r_\ell} r\, p(r)\, dr\right) \\ & = (2\pi\lambda)^k e^{-\Lambda(\mathcal{E})} \exp\left(-2\pi\lambda \sum_{\ell=1}^{k} \int_{r_{\ell-1}}^{r_\ell} r\, p(r)\, dr\right) \prod_{\ell=1}^{k} r_\ell\, p(r_\ell) \\ & = (2\pi\lambda)^k e^{-2\pi\lambda\psi(\infty)} \prod_{\ell=1}^{k} r_\ell\, p(r_\ell) \end{aligned} \tag{76}$$

with $\psi(\infty)$ defined in (74).

APPENDIX C
PROOF OF (36)

Starting from

$$\begin{aligned} f_{\mathbf{R}}(\boldsymbol{r};\lambda) &= (2\pi\lambda)^K \prod_{j=1}^{M} \frac{\psi'(r_j)\, [\psi(r_j)]^{k_j-1}\, e^{-2\pi\lambda\psi(r_j)}}{(k_j-1)!} \\ \mathbb{E}\left\{\widehat{\lambda}\right\} &= \int_0^\infty \!\!\cdots \int_0^\infty \frac{K}{2\pi \sum_{j=1}^{M} \psi(r_j)} f_{\mathbf{R}}(\boldsymbol{r};\lambda) d\boldsymbol{r} \end{aligned} \tag{77}$$

and substituting $\xi_j = \psi(r_j)$ we obtain

$$\int_0^\infty \!\!\cdots \int_0^\infty \frac{K\,(2\pi\lambda)^K}{2\pi \sum_{j=1}^{M} \xi_j} \prod_{j=1}^{M} \frac{\xi_j^{\,k_j-1} e^{-2\pi\lambda\xi_j}}{(k_j-1)!} d\boldsymbol{\xi}\,. \tag{78}$$

Then, using $u_\ell = \sum_{j=1}^{\ell} \xi_j$, we get

$$K_{\mathrm{c}} \int_0^\infty \frac{e^{-2\pi\lambda u_M}}{u_M} \int_0^{u_M} \!\!\cdots \int_0^{u_2} u_1^{k_1-1} \prod_{j=2}^{M} (u_j - u_{j-1})^{k_j-1} d\boldsymbol{u} \tag{79}$$

where

$$K_{\mathrm{c}} \triangleq \frac{K\,(2\pi\lambda)^K}{2\pi \prod_{j=1}^{M} (k_j-1)!}\,. \tag{80}$$

The expression in (79) can be simplified using the Beta function $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$ and its properties. In addition, we define the constant $\eta_\ell \triangleq \sum_{j=1}^{\ell} k_j$ (in particular $\eta_M = K$), obtaining

$$\begin{aligned} \mathbb{E}\left\{\widehat{\lambda}\right\} &= K_{\mathrm{c}} \prod_{j=1}^{M-1} B(\eta_j, k_{j-1}) \int_0^\infty u_M^{K-2} e^{-2\pi\lambda u_M} du_M \\ &= \frac{K\,(2\pi\lambda)^K}{2\pi(K-1)!} \int_0^\infty u_M^{K-2} e^{-2\pi\lambda u_M} du_M \\ &= \lambda \frac{K}{K-1}\,. \end{aligned} \tag{81}$$

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