The Boussinesq approximation for buoyant flows

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Abstract

The aim of this paper is to present a simplified, yet rigorous, deduction of the Boussinesq approximated governing equations for buoyant flows. In order to carry out the core deduction procedure, a simplified version of the manifold asymptotic analyses available in the literature is discussed. The method adopted in this study is focussed on the local balance equations valid for a general, not necessarily Newtonian, fluid. The analysis is carried out by demonstrating the leading order terms in the governing equations for the asymptotic limit which characterises the approximation. The role played by the effect of viscous dissipation is also taken into account.

Keywords — Fluid flow; Convection; Buoyancy force; Boussinesq approximation; Pressure; Viscous dissipation.

1   Introduction

There are several instances in fluid dynamics where an asymptotic analysis of the governing equations may lead to a simplified framework which offers an easier scheme for analytical or numerical solutions. Generally speaking, such asymptotic analyses lead to an approximation of the governing equations for the fluid flow. One of the most well-known asymptotic analyses is that leading to the boundary layer approximation widely employed for the study of external fluid flows and for the evaluation of the fluid-solid viscous interaction at the interface [1]. Another example is the Boussinesq approximation for buoyancy-induced flows [2–7].

The Boussinesq approximation is the basis for the largest part of theoretical works on thermal convection published so far. The reliability of this approximation is mainly due to the extremely good agreement with experimental data. On the other hand, its foundation as an asymptotic theory has been laid out by several authors in quite complicated ways where the Boussinesq approximation turns out to be the result of a limit with multiple parameters tending to zero.

This paper has not the aim of providing a comprehensive review of the really huge literature available on the topic. In fact, there are several excellent and comprehensive surveys in the literature [8–17]. The objective to be pursued in this paper is sketching a simplified, but rigorous, logical path that leads one from the general form of the local mass, momentum and energy balance equations to their approximate form. Strictly speaking, this aim is not completely original. In fact, the idea of the Boussinesq framework as the result of a limiting condition is already present in the literature. An example is the paper by Gray and Giorgini [10], where the limit is carried out starting from a Newtonian fluid with density, specific heat, viscosity, coefficient of thermal expansion and thermal conductivity dependent both on the temperature and on the pressure. The Boussinesq approximation is retrieved by defining eleven dimensionless ϵ-factors which are considered as small. The complication in this approach, common also to other studies published before and after the paper by Gray and Giorgini [10], is in the lack of a precise and explicit definition of the physical assumptions at the bulk of the approximation scheme. Furthermore, the complicated mathematical framework sometimes tends to shade the physics underlying the analysis.

In this paper, the main purpose is capturing the chain of evidence supporting the Boussinesq approximation, by keeping the mathematical procedure as simple as possible and by declaring from the beginning the physical assumptions strictly needed to achieve the approximate governing equations. Despite the relatively simple structure of the presentation, no specific hypothesis is made for the Newtonian or non-Newtonian rheology of the fluid, or regarding the variability of

fluid properties such as the viscosity or the thermal conductivity wherever not necessary. The density, as well as the pressure, the velocity and the temperature, is treated as a dynamic variable subject to specific assumptions. A brief discussion about the possible role of the viscous dissipation term in the local energy balance is also provided.

An important point to be emphasised is that the limiting procedure leading to the Boussinesq approximation of the local balance equations for the fluid flow is an ad-hoc procedure. More precisely, the limit is taken in a specific way that leads exactly to those terms recognised a-priori as qualifying the approximation. This is neither questionable nor unexpected as every approximation is focussed on showing up that some specific terms in the balance equations turn out to be dominant over other terms, while further terms share the same order of magnitude. This is particularly evident if one thinks, for instance, to the boundary layer approximation for external flows [1] where the ad-hoc assumption of a scale for the coordinate perpendicular to the bounding wall much smaller than the scale of the coordinate parallel to the wall is the basis to lay out the approximation.

2 Governing equations

Non-isothermal fluid flows are governed by the local mass, momentum and energy balance equations. We limit our analysis to cases where the buoyancy force typical of convection is due to the temperature differences. Mass diffusion and convection induced by the concentration differences of a solute can be treated in a completely similar manner and will not be considered here for the sake of simplicity.

The governing equations of fluid flow can be expressed as [5]

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\, \mathbf{v}) = 0, \\
&\rho\, \frac{\mathcal{D}\mathbf{v}}{\mathcal{D}t} = -\boldsymbol{\nabla}P + \rho\, \mathbf{g} + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}, \\
&\rho\, \frac{\mathcal{D}u}{\mathcal{D}t} = -\boldsymbol{\nabla} \cdot \mathbf{q} + \mathbf{D} : \boldsymbol{\tau} - P\, \boldsymbol{\nabla} \cdot \mathbf{v},
\end{aligned} \tag{1}
$$

with the first Eq. (1) yielding the local mass balance, the second the momentum balance and the third the energy balance.

Here, ρ is the density, $\mathbf{v}$ is the velocity, t is time, P is the pressure, $\mathbf{g}$ is the constant gravitational acceleration, $\boldsymbol{\tau}$ is the viscous stress tensor, u is the internal energy per unit mass, $\mathbf{q}$ is the heat flux

density and $\mathbf{D}$ is the strain tensor whose ij component is given by

$$
D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{2}
$$

with x_i the ith Cartesian coordinate.

Furthermore, the substantial derivative, $\mathcal{D}/\mathcal{D}t$, and the double scalar product between tensors, $\mathbf{D} : \boldsymbol{\tau}$, are defined as

$$
\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}, \qquad \mathbf{D} : \boldsymbol{\tau} = D_{ij}\, \tau_{ij}, \tag{3}
$$

where repeated indices ij are implicitly summed over. The term $\mathbf{D} : \boldsymbol{\tau}$ expresses the physical effect of viscous dissipation.

As a consequence of the first Eq. (1), the third Eq. (1) can be rewritten as

$$
\rho\left[\frac{\mathcal{D}u}{\mathcal{D}t} + P\, \frac{\mathcal{D}}{\mathcal{D}t}\left(\frac{1}{\rho}\right)\right] = -\boldsymbol{\nabla} \cdot \mathbf{q} + \mathbf{D} : \boldsymbol{\tau}. \tag{4}
$$

By employing the thermodynamic equation for stable equilibrium states,

$$
\mathrm{d}u = T\, \mathrm{d}s - P\, \mathrm{d}\left(\frac{1}{\rho}\right), \tag{5}
$$

where s is the entropy per unit mass and T is the thermodynamic temperature, we can write

$$
\frac{\mathcal{D}u}{\mathcal{D}t} = T\, \frac{\mathcal{D}s}{\mathcal{D}t} - P\, \frac{\mathcal{D}}{\mathcal{D}t}\left(\frac{1}{\rho}\right). \tag{6}
$$

Here, $1/\rho$ is the volume per unit mass. Thus, Eq. (4) reads

$$
\rho\, T\, \frac{\mathcal{D}s}{\mathcal{D}t} = -\boldsymbol{\nabla} \cdot \mathbf{q} + \mathbf{D} : \boldsymbol{\tau}. \tag{7}
$$

3 The Boussinesq scheme

The Boussinesq approximation can be formulated as an asymptotic theory based on five physical assumptions:

1. The fluid density depends only on the local temperature or, equivalently, pressure changes yield negligible density changes;

2. The local difference between the temperature, T, and its reference constant value, T_0, is very small;

3. The acceleration experienced by the fluid elements in the flow domain is very small;

4. The viscous stress is very small;

5. The local difference between the pressure gradient and the hydrostatic pressure gradient is very small.

We will introduce and comment each of these assumptions, step by step, pointing out their effects.

3.1 Small density variations

We assume that the temperature varies very weakly both in time and in space. Hence, we can write

$$T - T_0 = \epsilon\, T', \qquad (8)$$

where ϵ is a small, dimensionless and constant perturbation parameter. Here and in the following, the primed quantities are implicitly assumed as being $\mathcal{O}(1)$ when, eventually, the limit of a vanishing ϵ is taken.

The density changes are induced only by temperature changes, while pressure variations yield only negligible effects on the density. An alternative way to formulate this assumption is: "any convective velocity is much smaller than the speed of sound in the fluid" [18]. Thus, we can write

$$\rho = \rho_0\, (1 - \epsilon\, \theta). \qquad (9)$$

In Eq. (9), ρ_0 is a constant expressing the reference fluid density and θ is a dimensionless function of T' modelling the slight density changes in the fluid. More precisely, ρ_0 is the fluid density evaluated at the reference temperature T_0. The simplest and most common situation is when $\theta(T')$ is given by a linear function,

$$\theta(T) = \beta\, T', \qquad (10)$$

where β is the thermal expansion coefficient. When Eq. (10) is employed, ϵ has a direct physical meaning expressed as

$$\epsilon = \beta\, \Delta T_r, \qquad (11)$$

where ΔT_r is a constant reference temperature difference depending on the temperature boundary conditions imposed for the flow. A nonlinear expression of $\theta(T')$ may be needed to get a proper model of a density maximum, as it happens for water in a temperature range around $4\,^\circ$C.

The role of ϵ is a qualifying point for the Boussinesq approximation, which can be formulated as an asymptotic regime achieved in the limit $\epsilon \to 0$.

3.2 Local mass balance

We rewrite the first Eq. (1) by employing Eq. (9),

$$-\rho_0\, \epsilon\, \frac{\partial \theta}{\partial t} + \rho_0\, \boldsymbol{\nabla} \cdot \mathbf{v} - \rho_0\, \epsilon\, \boldsymbol{\nabla} \cdot (\theta\, \mathbf{v}) = 0. \quad (12)$$

By letting $\epsilon \to 0$, we obtain the usual condition of a solenoidal velocity field, namely

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0. \qquad (13)$$

3.3 Small acceleration and viscosity

Another qualifying point of the Boussinesq approximation is the small acceleration undergone in a buoyant flow, which means that the acceleration field

$$\mathbf{a} = \frac{\mathcal{D}\mathbf{v}}{\mathcal{D}t}, \qquad (14)$$

can be scaled through the parameter ϵ. Therefore, we can write

$$\mathbf{a} = \epsilon\, \mathbf{a}'. \qquad (15)$$

In fact, Normand et al. [18] explicitly state: "any accelerations in the fluid are much less than g", where g is the modulus of $\mathbf{g}$.

Due to the definition of substantial derivative, Eq. (3), the assumption of small acceleration can be mapped into one of large time and small velocity, through the scaling

$$t = \epsilon^{-1/2}\, t', \qquad \mathbf{v} = \epsilon^{1/2}\, \mathbf{v}'. \qquad (16)$$

By using Eq. (16), Eqs. (3), (14) and (15) in fact yield

$$\mathbf{a}' = \frac{\partial \mathbf{v}'}{\partial t'} + (\mathbf{v}' \cdot \boldsymbol{\nabla})\mathbf{v}'. \qquad (17)$$

On account of Eq. (16), Eq. (13) can be easily rewritten in terms of $\mathbf{v}'$,

$$\boldsymbol{\nabla} \cdot \mathbf{v}' = 0. \qquad (18)$$

The hypothesis of small viscous stress can be formulated as

$$\boldsymbol{\tau} = \epsilon\, \boldsymbol{\tau}'. \qquad (19)$$

The idea of small viscous stress as a characteristic feature of buoyant flows has a direct foundation in the argument usually set up to justify the onset of convection cells in a fluid layer heated from below [18]. In fact, the onset of convection emerges as a phenomenon where the viscous stress competes with the buoyancy force by damping out the initiation of the flow. Steady convection cells

may arise only when the buoyancy force turns out to prevail over the viscous stress [18]. These considerations are valid also in a mixed convection regime when an external forcing cooperates with buoyancy in setting up the flow without dominating the buoyancy contribution. Hence, in mixed convection, the viscous stress is to be considered as small as both the buoyancy and the external forcing.

We have not specified whether the fluid is Newtonian or not. Should the fluid be Newtonian, one can turn the hypothesis of a small viscous stress into one of small viscosity coefficient. Indeed, for a Newtonian fluid, we have

$$\tau_{ij} = 2\,\mu\,D_{ij}, \tag{20}$$

where Eq. (13) has been used and μ is the dynamic viscosity. Then, Eqs. (2), (16) and (19) require that one must consistently assume

$$\mu = \epsilon^{1/2}\,\mu', \tag{21}$$

so that one may write

$$\tau'_{ij} = 2\,\mu'\,D'_{ij}, \tag{22}$$

where $\mathbf{D}'$ is the tensor whose ij component is given by

$$D'_{ij} = \frac{1}{2}\left(\frac{\partial v'_i}{\partial x_j} + \frac{\partial v'_j}{\partial x_i}\right). \tag{23}$$

3.4 Hydrostatic pressure gradient

The local momentum balance equation expressed by the second Eq. (1) contains the gravitational body force $\rho\,\mathbf{g}$ which, on account of Eq. (9), can be written as

$$\rho\,\mathbf{g} = \rho_0\,\mathbf{g} - \rho_0\,\epsilon\,\theta\,\mathbf{g}. \tag{24}$$

Hence, the difference between the pressure gradient, $\boldsymbol{\nabla}P$, and the gravitational body force, $\rho\,\mathbf{g}$, is given by

$$\boldsymbol{\nabla}P - \rho\,\mathbf{g} = [\boldsymbol{\nabla}P - \boldsymbol{\nabla}(\rho_0\,\mathbf{g}\cdot\mathbf{x})]$$
$$+\rho_0\,\epsilon\,\theta\,\mathbf{g}, \tag{25}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ is the position vector. In Eq. (25), the term $\boldsymbol{\nabla}(\rho_0\,\mathbf{g}\cdot\mathbf{x})$ is the hydrostatic pressure gradient.

The last hypothesis that defines the basis of the Boussinesq asymptotic regime is relative to the difference between the pressure gradient and the hydrostatic pressure gradient. Such a difference is to be considered as extremely small everywhere. In mathematical terms, this hypothesis is

formulated as

$$\boldsymbol{\nabla}P - \boldsymbol{\nabla}(\rho_0\,\mathbf{g}\cdot\mathbf{x}) = \boldsymbol{\nabla}p,$$
$$\boldsymbol{\nabla}p = \epsilon\,\boldsymbol{\nabla}p'. \tag{26}$$

In Eq. (26), $\boldsymbol{\nabla}p$ is nothing but a shorthand notation for the difference between the pressure gradient and the hydrostatic pressure gradient. Equation (26) has very important physical consequences, because the hydrostatic pressure gradient is large or, at the very least, not negligible. Then, also the pressure gradient, $\boldsymbol{\nabla}P$, is a large or non-negligible vector quantity. This means that the pressure is markedly non-uniform in the vertical direction, despite the negligibly small compressibility effects on the density distribution.

3.5 Local momentum balance

We now employ the assumptions laid out in Sections 3.1, 3.3 and 3.4 to rewrite the second Eq. (1),

$$\rho_0\,\epsilon\left[\frac{\partial\mathbf{v}'}{\partial t'} + (\mathbf{v}'\cdot\boldsymbol{\nabla})\mathbf{v}'\right]$$
$$-\rho_0\,\epsilon^2\theta\left[\frac{\partial\mathbf{v}'}{\partial t'} + (\mathbf{v}'\cdot\boldsymbol{\nabla})\mathbf{v}'\right]$$
$$= -\epsilon\,\boldsymbol{\nabla}p' - \rho_0\,\epsilon\,\theta\,\mathbf{g} + \epsilon\,\boldsymbol{\nabla}\cdot\boldsymbol{\tau}'. \tag{27}$$

We divide Eq. (27) by ϵ, we simplify and then we let $\epsilon \to 0$. Thus, we obtain

$$\rho_0\left[\frac{\partial\mathbf{v}'}{\partial t'} + (\mathbf{v}'\cdot\boldsymbol{\nabla})\mathbf{v}'\right]$$
$$= -\boldsymbol{\nabla}p' - \rho_0\,\theta\,\mathbf{g} + \boldsymbol{\nabla}\cdot\boldsymbol{\tau}'. \tag{28}$$

As pointed out in Section 3.1, in most buoyant flows, Eq. (10) holds. Then, Eq. (28) can be rewritten as

$$\rho_0\left[\frac{\partial\mathbf{v}'}{\partial t'} + (\mathbf{v}'\cdot\boldsymbol{\nabla})\mathbf{v}'\right]$$
$$= -\boldsymbol{\nabla}p' - \rho_0\,\beta\,T'\,\mathbf{g} + \boldsymbol{\nabla}\cdot\boldsymbol{\tau}'. \tag{29}$$

The term $-\rho_0\,\beta\,T'\,\mathbf{g}$ is the buoyancy force.

3.6 Local energy balance

The local energy balance equation given by Eq. (7) can be rewritten by employing the results collected in Sections 3.1, 3.3 and 3.4. The first consideration is relative to the term $T\,\mathcal{D}s/\mathcal{D}t$ on the left hand side of Eq. (7). By utilising the thermodynamic definitions of specific heat at constant volume and of specific heat at constant pressure,

$$c_v = T\left(\frac{\partial s}{\partial T}\right)_\rho, \qquad c_P = T\left(\frac{\partial s}{\partial T}\right)_P, \tag{30}$$

one can write

$$T \, \mathrm{d}s = c_v \, \mathrm{d}T, \tag{31}$$

for a constant density process. On the other hand, one has

$$T \, \mathrm{d}s = c_P \, \mathrm{d}T, \tag{32}$$

for a constant pressure process. The question whether the Boussinesq approximation is described effectively by an isobaric process of the elementary fluid element or by an isochoric process can be easily responded by recognising that:

- The density changes are $\mathcal{O}(\epsilon)$ as revealed by Eq. (9);

- The pressure changes in the vertical direction are as large as the hydrostatic pressure changes as revealed by Eq. (26).

Convection cells display a nonvanishing vertical component of velocity which means a sensible pressure variation in the process undergone by the fluid element. Under these conditions, the use of Eq. (31) is the most satisfactory basis for the Boussinesq approximation, which becomes even an exact statement in the case of perfect gases. This can be easily reckoned by inspecting the third Eq. (1) after the substitution of Eq. (13), and by recalling that in the case of a perfect gas we can write $\mathrm{d}u = c_v \, \mathrm{d}T$, whatever is the type of process undergone by the fluid element. If Eq. (31) looks like the most reliable basis for the Boussinesq approximation, it cannot be retained as exact in general. We will thus use the symbol c for the specific heat to be employed in the Boussinesq approximation having in mind that its value is to be established experimentally, though it can be well estimated by equating c with c_v. Then, we rewrite Eq. (7) as

$$\rho \, c \, \frac{\mathcal{D}T}{\mathcal{D}t} = -\boldsymbol{\nabla} \cdot \mathbf{q} + \mathbf{D} : \boldsymbol{\tau}. \tag{33}$$

We now use Eqs. (8) and (16), together with Eq. (19), to conclude that

$$\begin{aligned}
\rho \, c \, \frac{\mathcal{D}T}{\mathcal{D}t} &= \epsilon^{3/2} \, \rho_0 \, c \left(\frac{\partial T'}{\partial t'} + \mathbf{v}' \cdot \boldsymbol{\nabla} T' \right) \\
&\quad - \epsilon^{5/2} \, \rho_0 \, c \, \theta \left(\frac{\partial T'}{\partial t'} + \mathbf{v}' \cdot \boldsymbol{\nabla} T' \right), \\
\mathbf{D} : \boldsymbol{\tau} &= \epsilon^{3/2} \, \mathbf{D}' : \boldsymbol{\tau}'.
\end{aligned} \tag{34}$$

The heat flux density is expressed through Fourier's law, so that

$$\mathbf{q} = -\kappa \, \boldsymbol{\nabla} T, \tag{35}$$

where κ is the thermal conductivity of the fluid.

3.6.1 Non-negligible viscous dissipation

From Eq. (35), the diffusion term, $-\boldsymbol{\nabla} \cdot \mathbf{q}$, in Eq. (33) is $\mathcal{O}(\epsilon^{3/2})$, as the other terms expressed by Eq. (34), if

$$\kappa = \epsilon^{1/2} \, \kappa'. \tag{36}$$

Thus, we can write

$$-\boldsymbol{\nabla} \cdot \mathbf{q} = -\epsilon^{3/2} \, \boldsymbol{\nabla} \cdot \mathbf{q}', \tag{37}$$

where

$$\mathbf{q}' = -\kappa' \, \boldsymbol{\nabla} T'. \tag{38}$$

By substituting Eqs. (34) and (37) into Eq. (33), we obtain

$$\begin{aligned}
\epsilon^{3/2} \, \rho_0 \, c &\left(\frac{\partial T'}{\partial t'} + \mathbf{v}' \cdot \boldsymbol{\nabla} T' \right) \\
-\epsilon^{5/2} \, \rho_0 \, c \, \theta &\left(\frac{\partial T'}{\partial t'} + \mathbf{v}' \cdot \boldsymbol{\nabla} T' \right) \\
&= -\epsilon^{3/2} \, \boldsymbol{\nabla} \cdot \mathbf{q}' + \epsilon^{3/2} \, \mathbf{D}' : \boldsymbol{\tau}'.
\end{aligned} \tag{39}$$

We divide Eq. (39) by $\epsilon^{3/2}$ and then we take the limit $\epsilon \to 0$. The resulting local energy balance equation is

$$\begin{aligned}
\rho_0 \, c &\left(\frac{\partial T'}{\partial t'} + \mathbf{v}' \cdot \boldsymbol{\nabla} T' \right) \\
&= -\boldsymbol{\nabla} \cdot \mathbf{q}' + \mathbf{D}' : \boldsymbol{\tau}'.
\end{aligned} \tag{40}$$

Here, there is an implicit assumption that c is $\mathcal{O}(1)$. This assumption, together with that expressed by Eq. (36), may have a possible alternative.

3.6.2 Negligible viscous dissipation

On account of Eq. (35), the diffusion term, $-\boldsymbol{\nabla} \cdot \mathbf{q}$, in Eq. (33) is $\mathcal{O}(\epsilon)$ with

$$-\boldsymbol{\nabla} \cdot \mathbf{q} = -\epsilon \, \boldsymbol{\nabla} \cdot \mathbf{q}', \tag{41}$$

if we introduce the alternative statement

$$\mathbf{q}' = -\kappa \, \boldsymbol{\nabla} T', \tag{42}$$

where κ is considered $\mathcal{O}(1)$. We assume

$$c = \epsilon^{-1/2} \, c', \tag{43}$$

instead, which depicts a situation where the fluid has a large specific heat. We now substitute Eqs. (34), (41) and (43) into Eq. (33), so that we obtain

$$\begin{aligned}
\epsilon \, \rho_0 \, c' &\left(\frac{\partial T'}{\partial t'} + \mathbf{v}' \cdot \boldsymbol{\nabla} T' \right) \\
-\epsilon^2 \, \rho_0 \, c' \, \theta &\left(\frac{\partial T'}{\partial t'} + \mathbf{v}' \cdot \boldsymbol{\nabla} T' \right) \\
&= -\epsilon \, \boldsymbol{\nabla} \cdot \mathbf{q}' + \epsilon^{3/2} \, \mathbf{D}' : \boldsymbol{\tau}'.
\end{aligned} \tag{44}$$

We divide Eq. (44) by ϵ and then we take the limit $\epsilon \to 0$. The resulting local energy balance equation is

$$\rho_0 \, c' \left(\frac{\partial T'}{\partial t'} + \mathbf{v}' \cdot \boldsymbol{\nabla} T' \right) = -\boldsymbol{\nabla} \cdot \mathbf{q}'. \qquad (45)$$

Unlike Eq. (40), Eq. (45) does not include any contribution due to viscous dissipation.

4 Analysis of the approximation

In Sections 3.6.1 and 3.6.2, we have drawn two different paths leading to the approximate local energy balance expressed by either Eq. (40) or Eq. (45). If viscous dissipation turns out to be non-negligible in Eq. (40), this effect is neglected in Eq. (45). The choice between the two alternatives relies on the identification of the appropriate scenario. We can devise a fluid with a very small thermal conductivity where viscous dissipation may be important and Eq. (40) is the appropriate Boussinesq approximation of the energy balance. Alternatively, we can devise a fluid with a very large specific heat where the effect of viscous dissipation is negligibly small and Eq. (45) is the appropriate energy balance. The physics behind this mathematical result can be focussed on by considering the Gebhart number, a nondimensional parameter often employed for the analysis of viscous dissipation in natural or mixed convection flows,

$$\mathrm{Ge} = \frac{g \, \beta \, L}{c}, \qquad (46)$$

where L is a typical length characterising the flow domain. If we follow the path drawn in Section 3.6.1, c is $\mathcal{O}(1)$ so that also Ge is $\mathcal{O}(1)$. If we follow the path drawn in Section 3.6.2, c is $\mathcal{O}(\epsilon^{-1/2})$ so that also Ge is $\mathcal{O}(\epsilon^{1/2})$ and, hence, Ge $\ll 1$.

These findings are consistent with the usual idea that viscous dissipation is negligible in a buoyant flow when the Gebhart number is extremely small. An interesting aspect of the alternative views described in Sections 3.6.1 and 3.6.2 is that both views imply a thermal diffusivity of the fluid,

$$\alpha = \frac{\kappa}{\rho_0 \, c}, \qquad (47)$$

which turns out to be $\mathcal{O}(\epsilon^{1/2})$,

$$\alpha = \epsilon^{1/2} \, \alpha'. \qquad (48)$$

This is a very important point, as the thermal diffusivity is involved in the definition of Prandtl number,

$$\mathrm{Pr} = \frac{\mu}{\rho_0 \, \alpha}. \qquad (49)$$

Equations (21) and (48) allow one to write

$$\mathrm{Pr} = \frac{\mu'}{\rho_0 \, \alpha'}. \qquad (50)$$

Equation (50) means that a finite Prandtl number corresponds to either choices proposed in Sections 3.6.1 and 3.6.2.

Another sensible dimensionless parameter for convection is the Rayleigh number, which is defined as

$$\mathrm{Ra} = \frac{\rho_0 \, g \, \beta \, \Delta T_r \, L^3}{\mu \, \alpha}. \qquad (51)$$

By employing Eqs. (11), (21) and (48), Eq. (51) can be rewritten as

$$\mathrm{Ra} = \frac{\rho_0 \, g \, \epsilon \, L^3}{\epsilon \, \mu' \, \alpha'} = \frac{\rho_0 \, g \, L^3}{\mu' \, \alpha'}, \qquad (52)$$

which means that the Rayleigh number is $\mathcal{O}(1)$. A similar result is drawn for the Reynolds number, which is an important parameter in a regime of mixed convection. Its definition is

$$\mathrm{Re} = \frac{\rho_0 \, V_r \, L}{\mu}, \qquad (53)$$

where V_r is a constant expressing the reference velocity of the forced flow. Such a constant undergoes the same behaviour of every local value of velocity, expressed by Eq. (16), so that we have

$$V_r = \epsilon^{1/2} \, V_r'. \qquad (54)$$

By using Eqs. (21) and (54), we can rewrite Eq. (53) as

$$\mathrm{Re} = \frac{\rho_0 \, \epsilon^{1/2} \, V_r' \, L}{\epsilon^{1/2} \, \mu'} = \frac{\rho_0 \, V_r' \, L}{\mu'}, \qquad (55)$$

meaning that the Reynolds number is $\mathcal{O}(1)$.

The approximate local balance equations (18), (29) and (40), or alternatively (45), are expressed through primed quantities, which means that such equations are upscaled with the respect to their natural units. Such a formulation has been functional to achieving the limit $\epsilon \to 0$ by selecting the dominant terms in each balance equation. Now, one can restore the natural scales of temperature, velocity and pressure by multiplying Eq. (18) by $\epsilon^{1/2}$, Eq. (29) by ϵ, Eq. (40) by $\epsilon^{3/2}$, or Eq. (45) by ϵ. This operation allows one to retrieve the usual physical scales for the fluid dynamics

6

variables, so that the Boussinesq approximation yields

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0,$$

$$\rho_0 \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v} \right]$$
$$= -\boldsymbol{\nabla} p - \rho_0 \, \beta \, (T - T_0) \mathbf{g} + \boldsymbol{\nabla} \cdot \boldsymbol{\tau},$$

$$\rho_0 \, c \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} T \right)$$
$$= -\boldsymbol{\nabla} \cdot \mathbf{q} + \mathbf{D} : \boldsymbol{\tau}, \quad \text{if} \quad \mathrm{Ge} \sim \mathcal{O}(1), \qquad (56)$$

or, as an alternative to the third Eq. (56),

$$\rho_0 \, c \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} T \right)$$
$$= -\boldsymbol{\nabla} \cdot \mathbf{q}, \quad \text{if} \quad \mathrm{Ge} \ll 1. \qquad (57)$$

The value of Ge depends on the fluid and on the typical length scale, L, of the flow domain. Given that $g \approx 9.807 \, \mathrm{m/s^2}$, we may assume for liquids $\beta \approx 10^{-3} \, \mathrm{K^{-1}}$ or lower, while $c \approx 10^3 \, \mathrm{J/(kg\,K)}$. Thus, a typical estimate of Ge is

$$\mathrm{Ge} \approx 10^{-5} \, L. \qquad (58)$$

From Eq. (58), we may easily conclude that a regime $\mathrm{Ge} \sim \mathcal{O}(1)$ can be approached only with L equal to over 10 or even 100 kilometres. This means that the Boussinesq approximation is unlikely to display important effects of viscous dissipation unless we are studying buoyant flows for astrophysical, geophysical or oceanographic applications. We mention that, for stars, the value of g is usually much larger than the standard Earth gravitational acceleration. This argument does not change significantly if we consider gases. In fact, for perfect gases, β is the inverse thermodynamic temperature so that the estimate $\beta \approx 10^{-3} \, \mathrm{K^{-1}}$ is still reasonable, while the specific heat can be smaller than for liquids. An example is $c \approx 10^2 \, \mathrm{J/(kg\,K)}$ for some noble gases. Then, the value of Ge/L can be $10^{-4} \, \mathrm{m^{-1}}$ instead of $10^{-5} \, \mathrm{m^{-1}}$ as predicted by Eq. (58). More detailed sample evaluations of Ge can be found, for instance, in Table 1 of Barletta et al. [19].

The qualifying parametric condition for the asymptotic regime described by the Boussinesq approximation is $\epsilon \ll 1$. By assuming ϵ expressed through Eq. (11) and the values of β for both liquids and gases mentioned above, of the order of $10^{-3} \, \mathrm{K^{-1}}$, one may regard ΔT_r of the order of $10 \, \mathrm{K}$ or smaller to be fairly compatible with the approximation as it yields $\epsilon \approx 10^{-2}$ at its largest.

5 Conclusions

The Boussinesq approximation, widely employed for the theoretical modelling of buoyant flows either in natural convection or in mixed convection has been discussed. The basic physical assumptions leading to this approximate framework have been explicitly declared from the beginning and the limiting procedure has been described step-by-step in a simplified form compared to the previous papers. Unlike in other theoretical studies on the topic, no explicit assumption is made about the fluid rheology or the variability of fluid properties such as the specific heat, the coefficient of thermal expansion or the thermal conductivity. The effect of viscous dissipation can be considered as important when the fluid thermal conductivity is a small parameter. This feature, within the Boussinesq approximation, captures the physics of the viscous dissipation effect in the description of buoyant flows. When the fluid is a poor heat conductor, the internal frictional heating has a chance to influence the temperature distribution in a significant manner. On the other hand, this is an unlikely case if the fluid has a large specific heat.

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