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Direct and converse applications: two sides of the same coin?

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Abstract

In this paper I present two cases, taken from the history of science, in which mathematics and physics successfully interplay. These cases provide, respectively, an example of the successful application of mathematics in astronomy and an example of the successful application of mechanics in mathematics. I claim that an illustration of these cases has a twofold value in the context of the applicability debate. First, it enriches the debate with an historical perspective which is largely omitted in the contemporary discussion. Second, it reveals a component of the applicability problem that has received little attention. This component concerns the successful application of physical principles in mathematical practice. With the help of the two examples, in the final part of the paper I address the following question: are successful applications of mathematics to physics (direct applications) and successful applications of physics to mathematics (converse applications) two sides of the same problem?

Keywords: Applicability of mathematics; mathematical practice; physical principles; Euclidean geometry; mechanics; Archimedes; Aristarchus.

1 Introduction

Since the publication of Eugene Wigner's paper "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" (1960), the philosophical literature on the applicability of mathematics in science has rapidly increased and several accounts have been proposed.[1] The goal of the present paper is to

[1] See, for instance, the accounts given in Steiner 1998, Pincock 2004, Bueno and Colyvan 2011, Rizza 2013, Bueno and French 2018 and McCullough-Benner 2019. In recent years, the most influential picture of applied mathematics has been the so-called 'mapping account' view, in which an explanation of the applicability of mathematics in science is given in terms of mappings (like homomorphisms, epimorphisms, and monomorphisms) that are established between mathematics and the empirical systems studied (e.g., see Bueno and Colyvan 2011 and Bueno and French 2018).

contribute to the applicability debate along two separate, although interconnected, directions of investigation: one more historical, which draws on the examples presented in sections 2 and 3, and the other of more philosophical nature, examined in the remaining sections.

First, in section 2, I will illustrate an example of successful application of mathematics that is taken from the history of science: Aristarchus' application of Euclidean geometry to estimate the distance of the Earth from the Sun in terms of the distance of the Earth from the Moon. Such example provides a clear and fascinating case for the success of Euclidean geometry in astronomy. Next, in section 3, I shall present another case of application: Archimedes' application of the law of the lever in mathematics, to find the area of a parabolic segment. Both these examples, which have not been discussed before in the context of the applicability problem, involve a successful interplay between mathematics and physics. Nevertheless, there is an essential difference between them. Aristarchus' case is a case of application of mathematics to physics. Archimedes' case, on the other hand, provides a clear example of how physics (in the form of principles and methods) is sometimes applied with success to mathematics.

One motivation for presenting the two cases is that they may serve as a step towards a more broader, historically informed, picture of the applicability problem. Indeed, examples as those presented here bring into the debate more material for case-study analysis and also an historical perspective which, I believe, is largely ignored in the contemporary discussion. But there is also another motivation for my choice. As I shall maintain in sections 4 and 5, the significance of these examples stretches far beyond their historical value and reaches the philosophical debate.

Until now, discussions on the applicability problem have focused on the successful application of mathematics in science. Call this the 'direct applicability problem', and 'direct applications' the successful applications of mathematics in science. The case presented in section 2 provides an exemplar of a direct application. Nevertheless, there is an aspect of the applicability problem that has has received very little attention. This aspect has to do with the successful use of physical considerations in mathematics. Call 'converse applications' these successful applications (of physics to mathematics) and 'converse applicability problem' the philosophical problem that stems from them, namely the problem of accounting for the effectiveness of (methods and ideas that are proper to) physics in mathematics.[2]

Although the direct applicability problem is generally seen as *the* appli-

[2]The converse applicability problem is acknowledged in Ginammi 2018. Ginammi considers different kinds of applicability involving mathematics and physics. Among these, he explicitly addresses applications of physics to mathematics, which he calls 'physics-to-

cability problem, many examples of converse applications can be found in scientific practice. Archimedes' use of the law of the lever in mathematics is one of these. Such examples clearly show how the converse applicability problem is an essential, though less known and less studied, component of the philosophical analysis of the successful interplay between science and mathematics. Hence, while in section 4 I shall discuss two distinct criteria to evaluate the success of direct and converse applications, in section 5 I shall contrast the two examples introduced in sections 2 and 3 to address the following question: are converse and direct applications two sides of the very same problem? I won't propose a clear-cut answer to this question but I shall show how the study of direct applications can benefit from an investigation of the converse problem. Finally, in the Conclusions, I will resume my analysis and point to some aspects of the applicability debate that I left out of my analysis.

2 An early example of direct application

The example of applied mathematics illustrated here is taken from the work of the mathematician Aristarchus of Samos. Aristarchus lived in the 3rd century BCE, during the early Hellenistic period, and he is often remembered for having been the first to put forward the heliocentric hypothesis. Nevertheless, historians of science and mathematics agree that Aristarchus' major contributions come from his only extant treatise, *On the Sizes and Distances of the Sun and the Moon* (henceforth *On Sizes*), which is built on the classical geocentric assumption that the Sun and the Moon move in circles round the Earth as center. Here I will present one proposition from such treatise, together with the relative demonstration.[3]

Before addressing the proposition, let's consider the assumptions that Aristarchus uses to prove it. At the beginning of the treatise, in purely Euclidean style, Aristarchus lists six hypothesis that are used to prove all the propositions of *On Sizes*. The hypothesis can be divided into two types: geometric and computational (Berggren and Sidoli 2007). The geometric hypothesis make assumptions about the celestial world that allow the mathematician to construct a geometric diagram. The computational hypothesis make assumptions about the physical world which allow the application of numerical parameters to the geometric models and are then used to derive

math applications'. In sections 4 and 5, I shall bring up Ginammi's analysis and discuss some of its aspects in the context of the present work.

[3]In my presentation of Aristarchus' proof I will follow Thomas Heath's edition of *On Sizes* (Heath 2004).

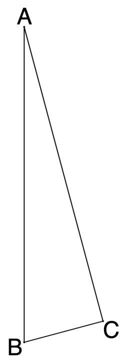

Figure 1

numerical solutions to the problems at hand. Here we are concerned with four of these hypothesis (H1-H3, which are geometric, and H4, which is computational):

- H1 The Moon receives its light from the Sun.
- H2 We can consider the Earth as a point and as the center of the lunar orbit.
- H3 When the Moon is at quadrature the circle that divides the dark side from the bright side lies in the same plane as our eye.
- H4 When the Moon is at quadrature the angle between the Moon and the Sun viewed from the Earth is 87°.

Let's now focus on Proposition 7, which is seen by historians of mathematics as one of the most important theorems in the book. The proposition provides bounds for the solar distance in terms of multiples of the lunar distance:

Proposition 7. *The distance of the Sun from the Earth is greater than 18 times, but less than 20 times, the distance of the Moon from the Earth.*

Let A be the position of the Sun, B the position of the Earth and C that of the Moon. BA is the distance Earth-Sun and BC is the distance Earth-Moon. The geometric configuration is obtained from hypothesis H1-H4 and is represented by the triangle in Figure 1, where $\angle BCA = 90°$ and $\angle CBA = 87°$ (therefore $\angle BAC = 3°$).

Aristarchus' proof is divided into two parts: first, he proves that the distance of the Sun from the Earth is greater than 18 times the distance of the Moon from the Earth (i.e., $BA > 18\,BC$) ; next, he proves that the distance of the Sun from the Earth is less than 20 times the distance of the Moon from the Earth (i.e., $BA < 20\,BC$). Here I will consider only his first

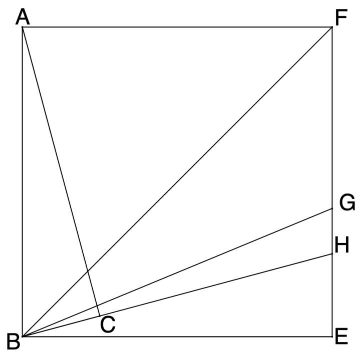

Figure 2

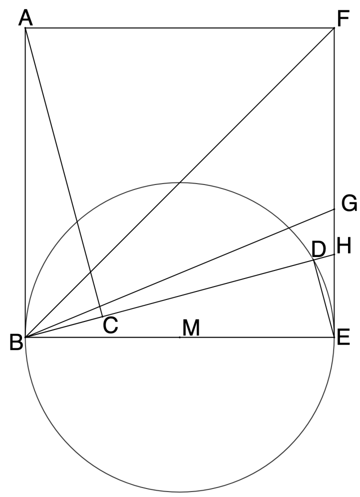

Figure 3

demonstration, which is independent of the second and which provides a case of successful application of mathematics in astronomy.[4]

Draw the square $\square ABEF$ and extend BC to an intersection with EF (Figure 2). Call H the point of intersection. Next, draw the diagonal BF of the square $\square ABEF$. Thus $\angle EBF = 45°$. Let BG the line that bisects the angle $\angle EBF = 45°$ and G the point of intersection of the angle bisector with the side EF of the square. We will therefore have that $\angle EBG = \angle GBF = 22.5°$.

Find M, which is the midpoint of BE, and D, which is the point that results from the intersection of BH with the circle that has radius BM and center M (Figure 3). If we join points D and E, we have that the two triangles $\triangle BAC$ and $\triangle BED$ are congruent because two angles and the included side of $\triangle BAC$ are equal to two angles and the included side of $\triangle BED$: $\angle BAC = \angle EBD$, $\angle ABC = \angle BED$ and $BA = BE$.[5] Therefore, $CA = BD$, $BA = BE$ and $BC = ED$.

So far, we have obtained the following values for angles $\angle EBF$, $\angle EBG$

[4] Aristarchus' first result is correct, since the actual average distance of the Earth to the Sun in terms of its average distance to the Moon is 389. On the other hand, Aristarchus' conclusion that the distance of the Sun from the Earth is less than 20 times the distance of the Moon from the Earth is, although obtained through a correct mathematical proof, false. The source of error lies in what Aristarchus assumes in hypothesis H4, namely that when the Moon is at quadrature the angle between the Moon and the Sun viewed from the Earth is 87° ($\angle CBA$ in Figure 1). The actual value of angle $\angle CBA$ is about $89°51'$.

[5] Angles $\angle BAC$ and $\angle EBD$ are congruent. In fact, $\angle EBD = \angle EBA - \angle CBA$, where $\angle EBA = 90°$ and $\angle CBA = 87°$. Thus, $\angle EBD = 3°$ and it is equal to $\angle BAC$.

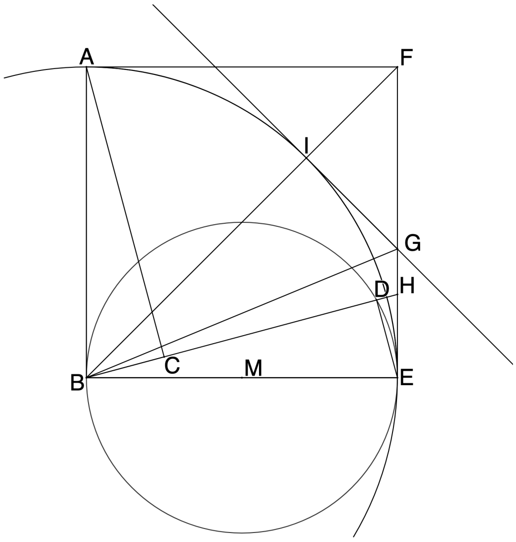

Figure 4

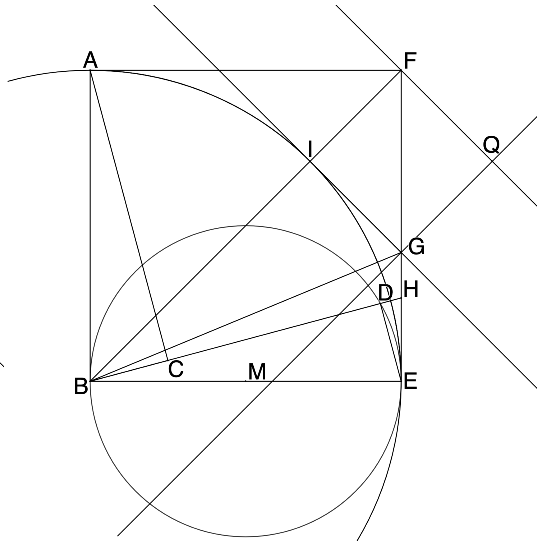

Figure 5

and $\angle EBH$:

$$\angle EBF = 45^\circ \text{ (that is, } \tfrac{1}{2} \text{ of } 90^\circ\text{)}$$

$$\angle EBG = \angle GBF = 22.5^\circ \text{ (that is, } \tfrac{1}{4} \text{ of } 90^\circ\text{)}$$

$$\angle EBH = \angle BAC = 3^\circ \text{ (that is, } \tfrac{1}{30} \text{ of } 90^\circ\text{)}$$

Therefore,

$$\frac{\angle EBG}{\angle EBH} = \frac{\frac{1}{4}}{\frac{1}{30}} = \frac{15}{2}$$

Now, since we know that $GE : HE > \angle EBG : \angle EBH$ (proved by Euclid in his *Optics*) and $\angle EBG : \angle EBH = 15 : 2$, we have that $GE : HE > 15 : 2$.

Draw the circle of radius BA and center B, and call I the point of intersection of this circle with the diagonal BF (Figure 4). Draw the line IG perpendicular to the diagonal BF, passing through point I. This line intersects the side EF at point G. Since $\triangle(BEG) = \triangle(BGI)$, we have that $IG = GE$.[6]

Now we draw the line perpendicular to the diagonal BF at point F and the line perpendicular to the line IG at G (Figure 5). We get a square, $\square(IGQF)$, of side IG (which is equal to GE) and diagonal FG.

[6] $\triangle(BEG) = \triangle(BGI)$ because two sides and the included angle of $\triangle(BEG)$ are equal to two sides and the included angle of $\triangle(BGI)$: $BE = BI$, BG is common to the two triangles and $\angle EBG = \angle GBI$.

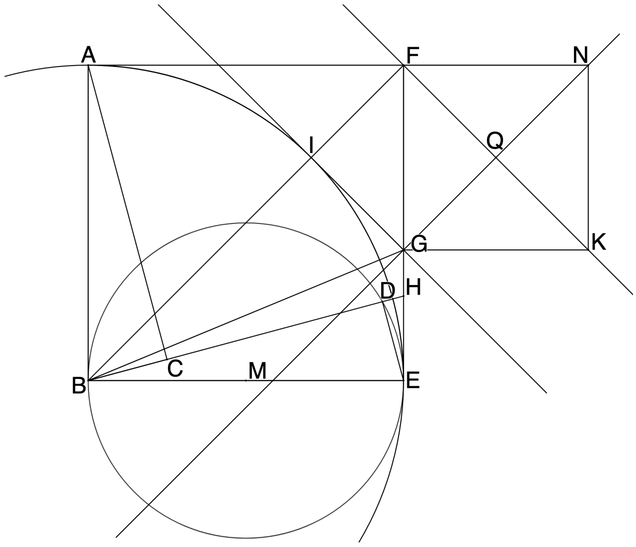

Figure 6

Let the square $\square(FGKN)$ be constructed on FG (Figure 6). Since FG is the diagonal of $\square(IGQF)$, by Pythagoras' theorem we have that $\square(FGKN)$ is twice $\square(IGQF)$.[7] This is equivalent to say that $FG^2 : IG^2 = 2$, and also that $FG^2 : GE^2 = 2$ (since, as we have seen, IG is equal to GE).

At this point we want to evaluate the ratio $FE : GE$. First, we write FE as the composition of GE and FG. Next, we observe that the ratio $FG : GE$ is greater than the ratio $7 : 5$ (i.e., $FG : GE > 7 : 5$).[8] Thus we have the following expression:

$$\frac{FE}{GE} = \frac{GE + FG}{GE} = \frac{GE}{GE} + \frac{FG}{GE} = 1 + \frac{FG}{GE} > 1 + \frac{7}{5}$$

Hence

$$\frac{FE}{GE} > \frac{12}{5}$$

Since $FE : GE > 12 : 5$ and $GE : HE > 15 : 2$, we can evaluate the ratio $FE : HE$ in the following way:

$$\frac{FE}{HE} = \frac{FE}{GE} \cdot \frac{GE}{HE} > \frac{12}{5} \cdot \frac{15}{2}$$

[7]The diagonal FG divides the square $\square(IGQF)$ into two isosceles triangles: $\triangle(GIF)$ and $\triangle(GQF)$. If we apply the Pythagorean theorem to $\triangle(GIF)$, we get that $FG^2 = 2IG^2$, which is equivalent to say that $\square(FGKN)$ is twice $\square(IGQF)$.

[8]Aristarchus remarks that $FG^2 : GE^2 = 2$ and $2 > 49 : 25$. Therefore $FG^2 : GE^2 > 49 : 25$, which gives the inequality $FG : GE > 7 : 5$.

Hence

$$\frac{FE}{HE} > 18$$

Now, since $BH > BE$ and $BE = FE$, we also have that $BH > FE$. Furthermore, $FE : HE > 18$ and $BH > FE$, and therefore $BH : HE > 18$.

Right triangles $\triangle(BAC)$ and $\triangle(BEH)$ are similar because their angles are congruent ($\angle ACB = \angle BEH, \angle BAC = \angle EBH$ and $\angle ABC = \angle BHE$). Thus, the corresponding sides are in proportion: $BA : BH = BC : HE$. If we write the last proportion found as $BH : HE = BA : BC$ and we consider that $BH : HE > 18$, we get our final result:

$$\frac{BA}{BC} > 18 \qquad \square$$

We have proved that $BA > 18\,BC$, namely that the distance of the Sun from the Earth is greater than 18 times the distance of the Moon from the Earth.

3 An early example of converse application

In the previous section we saw an example of direct application of mathematics. More precisely, we saw how Euclidean geometry is applied in astronomy to evaluate the relative distances of the Sun and the Moon from the Earth. In this section I will report an early example of converse application. As the previous case, also the example below involves Euclidean mathematics. Nevertheless, contrary to what we have seen with Aristarchus' Proposition 7, we will have that a mathematical result is reached with the use of physical considerations (in Aristarchus' case we had an opposite scenario: a result concerning the physical world was obtained through a purely mathematical demonstration).

Although many recent textbooks in mathematics contain examples of how physics (mainly mechanics) can be used with success in mathematics (e.g., Uspenskii 1961, Kogan 1974 and Levi 2009), the interest in converse applications (of physics to mathematics) can be traced back to Archimedes and his treatise *The Method of Mechanical Theorems, for Eratosthenes* (henceforth *Method*). In this work, which is a private communication to Eratosthenes, Archimedes shows how the application of some mechanical principles in mathematics has led him to the discovery of several mathematical results.

The first result of the *Method* is found in Proposition 1 and has to do with the area of a parabolic segment (i.e., the region bounded by a parabola and a line): any segment of a parabola is four-thirds of the triangle whose

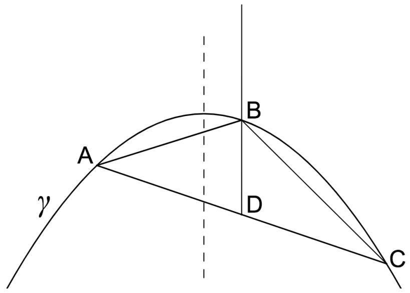

Figure 7

base is the line that bounds the parabola and whose height is the segment that is parallel to the axis of symmetry of the parabola and that joins the middle point of the base to a point on the parabola. A geometric proof of this proposition is given by Archimedes in his treatise *Quadrature of the Parabola*. Nevertheless, in the *Method* he offers an argument that shows how, combining geometric results with the law of the lever, he was led to the mathematical discovery.

To see how Archimedes reached a mathematical conclusion using physical considerations, let's have a look at his treatment as it appears in the *Method*.[9]

Consider a parabolic segment ABC (henceforth ABC_{ps}), bounded by the straight line AC and the parabola γ (Figure 7). Let AC be bisected in D and DB be drawn parallel to the axis of symmetry, or diameter, of the parabola (the dashed line in Figure 7). Now join AB and BC. Archimedes shows that the area of the parabolic segment ABC_{ps} is equal to four-thirds the area of the triangle $\triangle ABC$ inscribed in the parabola.

From A draw the straight line AKZ parallel to DB (see Figure 8). Line AKZ meets the tangent to the curve at C in Z and the straight line CB in K. Point E is the point of intersection of lines CZ and DB. Extend now line CK to H such that $CK = KH$. Next, consider an arbitrary point O on AC and draw a straight line parallel to DB, which meets the parabola γ in P, CK in N and CZ in M.

Since ABC_{ps} is a parabolic segment, CE its tangent and CD half of the chord AC parallel to the tangent to the parabola at B, then $DB = BE$.[10]

[9]In this paper I am adopting Jan Dijksterhuis' exposition of the *Method* (Dijksterhuis 1987).

[10]This result, concerning the property of the subtangent, is stated in the *Quadrature of the Parabola*, Proposition 2, where Archimedes mentions that the result was obtained by Aristaeus and Euclid in their treatises on conic sections (Heath 2009, p. 235). The chord AC is parallel to the tangent to the parabola at B because D is the middle point of AC and DB is parallel to the axis of the parabola (this result is also stated in the *Quadrature*

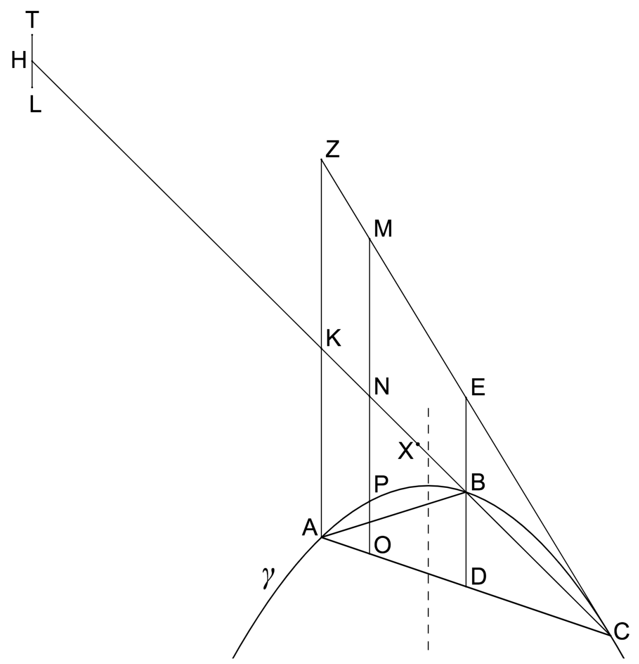

Figure 8

For this reason, and because AZ and OM are both parallel to DE, we also have that $ON = NM$ and $AK = KZ$.[11] Now, since $CA : AO = MO : OP$ (proved in *Quadrature of the Parabola*, Proposition 5), $CA : AO = CK : KN$ (application of Euclid's *Elements*, Proposition 2 from Book VI and Proposition 18 from Book V) and $CK = KH$, we have that $KH : KN = MO : OP$.

After this purely geometrical treatment, Archimedes begins to introduce physical considerations. More precisely, he uses some lemmas on centers of gravity that he mentions at the beginning of the *Method* and that have been proved in his treatise *On the Equilibrium of Planes*. First, he observes that point N is the center of gravity of the straight line OM (the center of gravity of a straight line is its middle point). Therefore, if we take a segment LT equal to OP with H as its center of gravity (so that $LH = HT$), LT will be in equilibrium with OM. Indeed, using the law of the lever (proved by Archimedes in *On the Equilibrium of Planes*, Propositions 6 and 7 of Book I), we can observe that HN is divided into segments (HK and KN) which are inversely proportional to the 'weights' LT and MO, namely in such a

of the Parabola, Proposition 1).

[11]Here Archimedes is implicitly applying three results obtained by Euclid in the *Elements*: Proposition 4 of Book VI and Propositions 11 and 9 of Book V.

way that $HK : KN = MO : LT$, so that K is the center of gravity of the combined weight of the two.[12]

Thus, Archimedes sees the straight line HN as an idealized lever, of arms HK and KN, that remains in equilibrium under the influence of two weights. The two weights in question are the two segments LT and MO, which are imagined as real weights such that the longer segment MO has greater weight than the shorter segment LT.

We can now extend the result obtained for MO and LT, which is equal to OP, to all the segments that are taken in the same way of MO and OP. In fact, all the straight lines that can be drawn in the triangle $\triangle AZC$ parallel to MO will be in equilibrium with their portions cut off from them by the parabola, when transferred to H (as we did for OP). K will be the center of gravity of the combined weight of each straight line and its portion. Now, since the triangle $\triangle AZC$ is made up of all the parallel lines inside it, and since the parabolic segment ABC_{ps} consists of all the parallel lines drawn inside it in the manner of OP, the triangle $\triangle AZC$ will balance about the point K the parabolic segment placed about H as center of gravity, so that K is their common center of gravity.

After having established the equilibrium (with center of gravity K) between the triangle $\triangle AZC$ and the parabolic segment ABC_{ps} transferred to H, Archimedes finds the center of gravity of triangle $\triangle AZC$. Since CK is a median of $\triangle AZC$, if we take point X on CK such that $CK = 3XK$, then X will be the center of gravity of $\triangle AZC$.[13]

Since there is equilibrium between the triangle $\triangle AZC$ and the parabolic segment ABC_{ps} about H, and their center of gravity is K, we have the following result: the triangle $\triangle AZC$ is to the parabolic segment ABC_{ps} transferred to H as its center of gravity as HK is to KX (that is, $\triangle AZC : ABC_{ps} = HK : KX$). Now, since $HK = KC = 3KX$, then the triangle $\triangle AZC$ is three times the segment of parabola ABC_{ps}. Moreover, $\triangle AZC$ is four times the triangle $\triangle ABC$, since $ZK = KA$ and $AD = DC$.[14] Hence, if

[12] The law of the lever, found by Archimedes in the treatise *On the Equilibrium of Planes*, states that bodies placed on opposite sides of the fulcrum are in equilibrium at distances reciprocally proportional to their weights. For instance, if two bodies of masses m_1 and m_2 placed on the arms of a straight lever of fulcrum K, and if d_1 and d_2 are the distances of the bodies' centers of mass from the fulcrum, then the two bodies will balance just in case $m_1/d_2 = m_2/d_1$.

[13] The point of intersection of the medians of a triangle divides each median into segments with a $2:1$ ratio. Thus, by taking point X on CK such that $CK = 3XK$, we find the point of intersection of the medians. But the point where the three medians of the triangle meet is also the center of gravity of the triangle, as Archimedes proves in his *On the Equilibrium of Planes*, Proposition 14 of Book I (Heath 2009, p. 201).

[14] The result is obtained through an application of Euclid's *Elements*, Proposition 1 of

$\triangle AZC = 3ABC_{ps}$ and $\triangle AZC = 4\triangle ABC$, we can conclude that

$$3\,ABC_{ps} = 4\,\triangle ABC$$

We have thus shown that the parabolic segment ABC is four-thirds of the triangle ABC. □

4 Successful applications

Aristarchus' estimate of the distance of the Sun from the Earth and Archimedes' treatment in Proposition 1 represent two cases of application. The first is a case of application of mathematics to physics (more precisely, astronomy) and the result obtained concerns the concrete entities of the latter discipline (some astronomical objects, together with their relative distances). The second example, on the other hand, is a case of application of a physical law (the law of the lever) in pure mathematics, and the result reached is a purely mathematical one (it is about geometrical entities and relations between them). But now the question arises as to how can we say that these applications are *successful*. In the present section I offer two criteria to evaluate the effectiveness of mathematics and physics in such cases.

Before moving on with a presentation of these criteria, it is important to clarify in what sense Archimedes' case is an example of application of physics in mathematics. What Archimedes does in his treatment is finding a mathematical proposition. Nevertheless, at the end of his discussion of Proposition 1, he remarks that the treatment he has just presented cannot be said to be a *proof* of the result. Rather, he says, it creates a certain impression that the conclusion is true:

> This as not therefore been proved by the above, but a certain impression has been created that the conclusion is true. (Dijksterhuis 1987, p. 318)

Thus, we can consider that, in Archimedes, the application of physics in mathematics amounts to the use of physics to *discover* a mathematical fact. This sense of 'application' well reflects the way in which Archimedes sees the mechanical-mathematical arguments given in the *Method*, which he explicitly considers as tools to "discover" a mathematical fact (Dijksterhuis 1987, p. 315). Moreover, it also mirrors the way in which historians of mathematics look at Archimedes' way of proceeding in the *Method*. For instance, Reviel

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Book VI.

Netz and William Noel consider Archimedes' way of finding mathematical results in the *Method* as an "act of magic no less spectacular [than the act of finding physical results through pure mathematics]. It is matter-over-mind—physics *discovering* a mathematical fact. This is done in the *Method*" (Netz and Noel 2007, p. 148; my emphasis).

More generally, I take the application of physics as the use of physics to discover a mathematical fact. Such process of discovery does not coincide with the demonstrative process that is typically found in mathematical practice. In fact, the method by which mathematicians are convinced of the truth of a result in many cases is quite different from the way in which that result is established. The process through which mathematical results are discovered may involve non-demonstrative tools such as the ability to reason visually on a diagram or the ability to trace similarities between different mathematical theories (see, e.g., Lakatos 1976 and Pólya 1981). Moreover, as it happens in converse applications, the discovery can be found by tracing several analogies between an empirical set up and mathematics (e.g., in Archimedes' example, a line segment is seen as a lever while other line segments as weights).

The idea that the application of physics in mathematics can be spelled out in terms of discovery, and more precisely in terms of analogies between the empirical set up and mathematics, has already been advanced in the philosophical debate on applicability (Ginammi 2018). In the context of mathematical practice, the very same idea is shared by those mathematics who apply physics in their mathematical work. For instance, Mark Levi observes how "the physical argument can be a tool of discovery" (Levi 2009, p. 3) and "physical reasoning was responsible for some fundamental mathematical discoveries, from Archimedes, to Riemann, to Poincaré, and up to the present day" (*Ibid.*, p. 4). In a similar vein, in his *Théorie analytique de la chaleur* (1822), Fourier famously noted that "profound study of nature is the most fertile source of mathematical discoveries".

Let's now move to the criteria to evaluate the effectiveness of mathematics and physics. The claim that a discipline A (mathematics, or physics) is successfully applied to another discipline B (physics, or mathematics) ultimately depends on the fact that the result obtained in B through the use of A is successful. Nevertheless, to claim that we are confronted with a successful result in B, it is reasonable to think that we should provide a criterion, and this criterion should be somehow independent from the application we are considering (otherwise, the reasoning would strike us as circular). In cases of direct applications, such criterion can been identified as follows: the success of mathematics in the empirical sciences is given by the fact that the (empirical) results obtained through mathematics receive successful empirical confirmations or they allow to make successful empirical predictions.

Surely, there might be other criteria that make an application of mathematics in the empirical science successful. For example, mathematics can be considered successful in science because it leads to simpler calculations, or because it sheds light on certain connections between different scientific areas or aspects of different phenomena. Nevertheless, the specific criterion given in the previous paragraph better renders the usage that is often made of 'successful application of mathematics' in many influential studies on the applicability of mathematics (e.g., see Pincock 2004, p. 136).

In the case of converse applications, on the other hand, the criterion cannot be the same and, more importantly, it cannot appeal to empirical considerations. The reason is that successful applications of physics to mathematics are successful *in* mathematics. And therefore the criterion to be adopted should be a wholly mathematical one. This is why I see only one relevant possibility, namely: the success of non-mathematical methods in mathematics is given from the existence of a purely mathematical proof of the result.[15]

It may be observed that the criterion just offered is inadequate, or rather incomplete, when a result is discovered by applying non-mathematical considerations and a proof of that result is not available. In fact, what happens in this scenario is that we cannot use the criterion to establish if the converse application is successful or not. Such observation is surely important because in mathematical practice we have many cases in which a mathematical fact is discovered without there being a proof available. Nevertheless, it is not problematic since the criterion reflects the following intuition: in this scenario, the converse application has heuristic value in fostering mathematical investigation and to claim that such application is successful we have to wait for a proof of the result.[16]

Now we can see whether the two cases are successful applications (according to the two criteria given above). First, consider Aristarchus' application of Euclidean geometry to obtain his astronomical result that the distance of the Sun from the Earth is greater than 18 times the distance of the Moon from the Earth. Although Aristarchus does not consider this result as a prediction, and he does not have empirical means to confirm it, we can see how his estimate is consistent with our current knowledge of the distances between the three bodies. How? It is sufficient to measure (by radar signals) the distance between the Earth and Mercury (or even between the Earth and an artificial solar satellite like the Parker Solar Probe, which is currently

[15]The proof should be formally valid and contain no mistakes. Otherwise it would not count as proof.

[16]I am grateful to a referee for pushing me to clarify this point.

orbiting around the Sun in the ecliptic plane). Since the results confirm that the average distance of this object from the Earth is greater than 18 times the Earth-Moon distance, which can also be measured with high accuracy using radio signals, we also know that the actual average distance of the Earth to the Sun is greater than 18 times its average distance to the Moon. Thus, by using techniques that are independent from the Euclidean geometry used by Aristarchus, we know that the result found in section 2 is correct.[17]

Aristarchus' application of mathematics to astronomy is therefore successful because it receives independent empirical confirmation. But what about Archimedes' case? According to the criterion given above, the converse application is successful if the same result is obtained through a purely mathematical proof. Do we have such a proof?

The answer is given by Archimedes himself. After noting that the treatment of Proposition 1 given in the *Method* cannot be said to be a proof, he observes:

> Since we thus see that the conclusion has not been proved, but we suppose it is true, we shall mention the previously published geometrical proof, which we ourselves have found for it, in its appointed place. (Dijksterhuis 1987, p. 318)

Thus, the passage informs us on the existence of a purely geometrical demonstration of what is stated in Proposition 1. Such demonstration is given by Archimedes in his treatise *Quadrature of the Parabola*, where he obtains the same result using the method of exhaustion, without recurring to non-mathematical considerations (see Katz 2009, pp. 108-109). We have therefore that the result obtained in Proposition 1 through the application of the law of the lever is confirmed on independent grounds, within pure mathematics, and the application should be considered a successful one.[18]

5 Two sides of the same coin?

After having established *that* the two examples illustrated are cases of successful applications, it is now time to see *how* mathematics and physics are

[17] In the radio-signal method, distance are calculated without trigonometry, using only the time-delay of the signal and the speed of light. For a description of the method see Webb 1999.

[18] Archimedes' result is also confirmed by modern calculus. Nevertheless, it is not necessary to resort to modern mathematics, since the geometrical proof given in the *Quadrature of the Parabola* is still regarded by mathematicians as a flawless demonstration.

applied and, next, turn to the more general question of whether direct and converse applications are two sides of the same problem.

Let's first make a point about what is applied to what. One way to present the problem of application is the following: mathematical deals with (abstract) entities that have no spatio-temporal location and lack causality ; the empirical sciences, on the other hand, deal with (concrete) entities that have spatio-temporal location and causal efficacy ; in many cases the entities of the empirical sciences are studied through the entities of mathematics, and this leads to successful results (i.e., successful predictions and successful experimental confirmations) ; how can we account for the successful applicability of mathematics in the empirical sciences?[19]

Now, although this way of putting the problem well renders the conceptual difficulty that derives from the interplay between the abstract realm of mathematics and the concrete realm of the empirical sciences, I think that it does not offer an accurate characterization of the issue. According to the characterization just given, it seems that the (abstract) *entities* of one discipline, mathematics, are applied to the (concrete) entities of an empirical science. Nevertheless, it would be better to talk of application of mathematical *theories* in the empirical sciences. At first glance, the point may seem just a terminological one. However, there is more to it than mere terminology. First, this particular stance well reflects a very intuitive understanding of mathematical objects: mathematical objects are framed within theories and acquire their significance in theories (e.g., a square is a particular object defined in Euclidean geometry, while a group is a particular algebraic structure in group theory). Second, rephrasing the direct applicability problem in terms of mathematical theories applied in the empirical sciences permits to shift the attention from entities to theories and this approach may provide an interesting perspective on the applicability problem.[20]

Take, for instance, Aristarchus' example. We may say that the successful application of mathematics is given by the successful application of the

[19]Mark Steiner calls this the "metaphysical question concerning application: how can facts about numbers [thought of as abstract or nonphysical objects] be relevant to the empirical world?" (Steiner 1998, pp. 1-2). Other philosophers of mathematics present the applicability problem in terms of the abstract/concrete dichotomy, as a metaphysical issue. See, for instance, (Dummett 1991, p. 301) and (Nolan 2015, p. 61).

[20]Similar remarks on the importance to focus on (applied) mathematical theories, rather than (applied) mathematical objects, have been put forward in the debate on the enhanced indispensability argument (see section 3.4 of Panza and Sereni 2016). Moreover, it can be noted that also mapping accounts of mathematical applicability focus on (applied) theories rather than (applied) entities. For instance, Otávio Bueno and Steven French observe how "In applying a mathematical theory to physics, we are often 'bringing in' structure from the mathematical level to the physical" (Bueno and French 2018, p. 56).

abstract entities of Euclidean geometry (i.e., lines, angles, circles and so no). And it is uncontroversial which claims are mathematical and which are not. But do we really mean that the successful application comes from applying 'lines and angles' to the (empirical) objects of astronomy? Such a claim is obscure unless we observe how these entities have mathematical significance within the framework of Euclidean geometry (what is a line in Archimedes' treatment, if not an object of Euclidean geometry?). It would be therefore more correct to talk about the successful applicability of mathematical theories, together with the results obtained in such theories, in the empirical sciences.[21]

A similar point can be made for converse applications. In converse applications we do not apply concrete entities such as levers and tables to mathematics. Rather, we apply physical laws and theories that are about concrete objects. Take Archimedes' case. In his geometrical treatment of the parabolic segment, Archimedes applies his law of the lever, which states that bodies placed on opposite sides of a fulcrum are in equilibrium at distances reciprocally proportional to their weights. Obviously, he does not posses the concept of torque (the intensity with which a force tries to rotate an object it's applied to around a pivot). But we know today that, with levers, the position of the fulcrum determines the distribution of the balanced torques on either side. A lever is in balance if the total left side torque is equal to the total right side torque. Such zero torque condition gives the law of the lever used by Archimedes.[22] Moreover, the zero total torque condition on a system means that the total angular momentum of the system is constant. And therefore the law of the lever essentially depends on a conservation principle, conservation of angular momentum, which is among the most fundamental principles of physics. It is this principle that Archimedes is implicitly applying in his geometrical treatment. And thus in Archimedes' case the converse applicability problem, namely the problem of accounting for the success of physics in mathematics, can be restated in terms of the successful application of a physical principle in mathematics.

Archimedes' example is not an isolated case of converse application. As

[21]By adopting this stance, I am not providing any argument ruling out the possibility that what is applied are objects (of theories). This possibility should be left open. And this especially because talking in terms of theories seems to leave a lot out (e.g., we may want to be more specific about what is applied to what, and talking about theories does not seem to provide this kind of specificity). I am indebted to a referee for bringing this consideration to my attention.

[22]For a simple lever, if m_1 and m_2 are the masses of the two bodies on the arms and d_1 and d_2 are the distances of the bodies' centers of mass from the fulcrum, the zero torque condition reads $m_1 g \cdot d_1 = m_2 g \cdot d_2$, which is Archimedes' law of the lever $m_1/d_2 = m_2/d_1$.

I observed at the beginning of section 3, although the converse applicability problem has received little attention in the contemporary philosophical debate, we do have many textbooks that illustrate how physics is applied with success in mathematics. For some mathematical results, as for instance the Pythagorean theorem, we also have many different physical arguments (see Levi 2009). Furthermore, a look at these textbooks reveal how applications of physics not only occur in geometry but also in other areas of mathematics (although geometry is the area where such applications have received more attention).[23] These examples show how physics is applied in mathematics under the form of principles and laws, as it happens in Archimedes' case. Moreover, for the most part, these principles and laws draw on considerations about conservation of energy and linear momentum.[24] Thus it seems that there is a cluster of cases of converse applicability which involve the application of some principles, such as conservation of energy and conservation of angular momentum. These principles are neither mathematical, since they are about physical entities and phenomena, nor strictly empirical, as they are about *idealized* concrete entities and state of affairs.[25]

Now, if we have a look at our case of direct application, namely Aristarchus' example, we may ask whether there is a similar involvement of physical principles (similarly to what happens in Archimedes' case and in other cases of converse applications). We saw how Aristarchus begins his treatise with some hypothesis, which are used to derive all the propositions. In these hypothesis we find assumptions about the physical world. Some of these assumptions contain idealizations (e.g., the Earth the Sun and the Moon are considered as points), while others contain numerical values that are useful in the development of the geometrical reasoning (e.g., the angle between the Moon and the Sun viewed from the Earth is 87°). From our modern perspective, however,

[23]For instance, Uspenskii (1961) also focuses on cases where a physical principle is used to establish a purely arithmetical result.

[24]Conservation principles appear as the main ingredients of converse applications, even if sometimes the reference to such principles is not explicit. For instance, Mark Levi discusses a case of application of Kirchhoff's second law (Levi 2009, p. 76). Although not mentioned by Levi, we know that Kirchhoff's second law is a consequence of charge conservation and conservation of energy.

[25]Here I am specifically focusing on the import of principles, and not laws, in converse applications. The distinction between principles and laws is the object of a separate debate in philosophy of science, but for the point at stake here it is sufficient to add that I generally regard principles as more fundamental than laws. For instance, as I show in footnote 22, Archimedes' law of the lever can be obtained from the conservation of angular momentum. And therefore the principle of conservation of angular momentum is more fundamental than the law of the lever. I do not exclude, however, that some physical laws may have such fundamental status. For instance, Newtons' laws are sometimes regarded as fundamental principles of physics (Dilworth 1994).

we know that Aristarchus is also implicitly assuming more than that. In particular, he is assuming that space is Euclidean, which we may consider as another idealization. And, more importantly, that light propagates between two points in space instantaneously and in a straight line. These presuppositions, which are naturally missing from Aristarchus' list of hypothesis, are necessary to the application since they allow to consider the mathematical representation as *the* representation of a specific empirical setting. If we deny one of these assumptions, the geometrical treatment will not be consistent anymore. Take, for instance, the assumption that light travels a straight line path (when it does not interact with the medium, which is homogenous, and does not get bent when it passes near to a massive object). If we drop this premise, or even deny it, we cannot represent the distance between the Earth and the Sun as a Euclidean straight line between two points (where 'straight line between two points' is an entity of geometry, not a physical object). Moreover, this assumption is grounded on a physical principle. Visible light is electromagnetic radiation with a specific wavelength, and the straight-line propagation of electromagnetic (light) waves in a homogeneous medium is a consequence of Fermat's principle of least time, which states the following: light propagates in such a way that the propagation time is minimal. Thus, light travels in a perfectly straight line simply because it is the most economical and efficient way for it to travel. And in a flat geometry the path which requires the shortest time is a straight line. We have therefore that also in Aristarchus' case there is a physical principle that operates and that guarantees the successful interplay between mathematics and physics.

The previous observations suggest that, although direct and converse applications may be discussed separately (the former as successful applications of mathematics to physics, while the latter as successful applications of physics to mathematics), there is also something that they share. This something has to do with the intervention of physical principles in the application process.

Surely, to observe that physical principles are applied in direct and converse applications is not enough to claim that the applicability of mathematics in science and the applicability of science in mathematics are two sides of the very same problem and should be the object of a unified investigation. Nevertheless, I believe that this shortfall can be addressed by adopting a novel stance on the applicability issue. What I have in mind is the following idea: mathematical truths and physical principles successfully interact because they share the same modal status of being metaphysically necessary. The fact that they share the same species of necessity is what makes them mutually interacting (in both directions, namely from mathematics to physics and from physics to mathematics). This proposal would be therefore centered on physical principles and mathematical truths involved in the application, and not on the structural features that can be identified in the mathematical and physical domains (as it happens in the structural approaches to the applicability issue).

The proposal just sketched clearly requires more elaboration, which I am not able to provide at the present stage. Nevertheless, I want to stress two points that provide favorable grounds for pursuing this strategy of analysis. First, the view that some physical principles, as for instance conservation laws, are metaphysically necessary has been defended by various philosophers of science (Swoyer 1982; Leeds 2007; Wolff 2013). Scientific essentialist, for instance, take laws of nature as metaphysically necessary (Ellis 2002, Bird 2007). Thus, if we observe that mathematical truths are usually seen as "paradigmatic metaphysical necessities" (Clarke-Doane 2019, p. 5), we have good grounds to take mathematical truths and physical principles as sharing the same modal status of being metaphysically necessary. Second, an analysis of the interplay between mathematics and physics in terms of the modal status of mathematical facts and physical laws has already been advanced within another debate in philosophy of science. In his recent studies on mathematical explanation, Marc Lange considers that, in some non-causal explanations, mathematical facts act as constraints because they bring into the explanation a particular degree of necessity (mathematical necessity) that is stronger than the necessity of ordinary laws of nature (Lange 2017). Although Lange does not connect his proposal to the applicability debate and he specifically focuses on the modal force of mathematical facts and physical laws, such perspective reinforces the idea that the interplay between mathematics and physics can be analyzed by focusing on the modal status of the mathematical facts and physical principles that are involved in applications.

What exactly is to be gained by pursuing the strategy above is yet to be assessed. Moreover, the sketchy picture just given need to be complemented by the many and multifaceted analysis of physical laws in terms of metaphysical necessity (e.g., those offered in Wolff 2013 and Linnemann 2020). On the other hand, the proposals outlined in the previous lines prompts new questions about applicability and has the virtue of bypassing two major difficulties that arise for the structural mapping account of applications. First, such strategy has the benefit of addressing a metaphysical issue through a purely metaphysical approach, by focusing on the particular species of necessity that is involved in the application process. Why is this a benefit? As I noted at the beginning of the present section, the applicability issue can be considered as a metaphysical one (in both directions: from mathematics to science and from science to mathematics). It is therefore reasonable to think that it should be addressed through considerations of metaphysical nature. Now, if we consider the mapping view of applicability, namely the most discussed picture of applicability in the contemporary debate, we can note that at its core lies the idea that the successful use of mathematics in science can be explained through mathematics (more precisely, through mathematical mappings). But this way of proceeding can be regarded as suspicious, since it essentially uses mathematics to explain the successful interplay between mathematics and science (Why does mathematics work in explaining such interplay? The mapping account is silent on that). Similarly, Tim Räz and Tilman Sauer have observed how the mapping account faces what they call "the circularity objection": "Mappings cannot possibly explain how mathematics can be applied to the world; they can only explain how mathematics can be applied to some other mathematical domain" (Räz and Sauer 2015, p. 60). The approach above clearly bypasses this line of criticism, and the circularity objection that comes with it, since it does not account for the applicability issue in terms of mathematics, but in terms of metaphysical modality. Second, the approach above better deals with physical idealizations. Since idealizations cannot captured through simple mappings like isomorphisms, the mapping account has been considered problematic when taking into account them (Batterman 2010).[26] Now, if we consider the approach outlined above, it seems that this difficulty is easily bypassed too. In fact, the proposal draws on the idea that the interaction between mathematics and physics holds at the level of the necessary status of mathematical truths and physical principles, where the latter refer to *idealized* physical settings. Consequently, physical idealizations are not problematic for this view.

Let me conclude the present section with a short remark. By offering a novel, although sketchy, picture of application I am not ruling out the possibility that a different account of applicability, as for instance the mapping account view, may be able to provide a unified treatment of direct and converse applications.[27] Moreover, there are surely interesting insights to be gained through examining how the accounts of applicability already available deal with direct *and* converse applications. But such a task clearly requires

[26]To overcome the problem with idealizations, some authors have proposed an extension of the mapping account in terms of partial structures (see, e.g., Bueno and French 2018). Note, however, that this refinement does not provide an answer to the first criticism exposed here, since partial structures are still *mathematical* structures.

[27]Interestingly, some positive reasons in favor of such unified perspective in terms of structural similarities and structural representation are given in Ginammi 2018. These considerations are surely important to address direct and converse applications in terms of a structuralist approach. Nevertheless, as observed by Ginammi himself, they still require further elaboration. In the present study I did not follow this route and I preferred to focus on an alternative direction of analysis.

a separate investigation. In this final section, rather than developing this investigation, I focused on an alternative approach. I put forward the main idea behind this proposal and I stressed how some considerations coming from other debates in philosophy of science give us good grounds for believing that the idea can be developed further. Finally, I showed how this proposal may be better suited to handle two difficulties that we can identify in the mapping approach to applications.

6 Conclusions

In this paper I presented two cases of application (of mathematics to physics and physics to mathematics, respectively). These cases enrich the discussion on the applicability problem because they provide an historical perspective on it. Moreover, in sections 4 and 5, I showed how the significance of such examples reaches the philosophical debate. I showed why these cases should be considered as successful applications and how the converse applicability problem, which has been largely ignored in the contemporary discussion, may be approached by focusing on the introduction of physical principles in mathematical practice. In the final part of the paper, I addressed the question of whether direct and converse applications are connected. Although I did not give a clear-cut answer to such question, I showed how in some cases of direct applications, as for instance in Aristarchus', the successful application is also grounded on the successful intervention of physical principles. Furthermore, I sketched a novel proposal that seems to be well suited to exploit this consideration and provide a unified approach to direct and converse applications. So, perhaps, the lesson to be drawn from the present study is that one should not focus solely on one or the other side of the applicability problem, but rather look at both sides of the coin.

I did not advance a full account of application, nor did I address the broader question of how mathematics and the empirical sciences (other than physics) successfully interact. Rather, I focused the successful interplay between mathematics and physics and I traced a potential path which, I believe, is worth exploring. It should also be observed that the problem of the applicability of mathematics has deep ramifications in other debates in philosophy of mathematics and philosophy of science. One of these is the debate over mathematical explanations in science. Articulating a plausible account of the applicability of mathematics in the empirical sciences is particularly important for those philosophers who maintain that mathematics can play an explanatory role in science (see, e.g., Baker 2009, Batterman 2010, Pincock 2015, Lange 2017 and Bangu 2020). And this especially if we consider that successful applicability (of mathematics) is often seen as a necessary, although not sufficient, condition for mathematical explanation of empirical facts (Shapiro 1983, p. 525). Furthermore, some philosophers have proposed the idea that there exist physical explanations of mathematical facts (Skow 2015). Surely, if we agree that physics yields explanatory power in mathematics, it is extremely important to address the question of whether there is an explanatory dimension to applying physical principles within mathematics.

These (as many other) issues connected to the investigation of direct and converse applications cannot be addressed here. I leave them for future work and I hope that the direction of investigation sketched here may have repercussions on such exciting discussions.

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