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BOUNDED COHOMOLOGY OF FINITELY PRESENTED GROUPS: VANISHING, NON-VANISHING, AND COMPUTABILITY

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Abstract. We provide new computations in bounded cohomology:

A group is boundedly acyclic if its bounded cohomology with trivial real coefficients is zero in all positive degrees. We show that there exists a continuum of finitely generated non-amenable boundedly acyclic groups and construct a finitely presented non-amenable boundedly acyclic group.

On the other hand, we construct a continuum of finitely generated groups, whose bounded cohomology has uncountable dimension in all degrees greater than or equal to 2, and a concrete finitely presented one.

Countable non-amenable groups with these two extreme properties were previously known to exist, but these constitute the first finitely generated/finitely presented examples.

Finally, we show that various algorithmic problems on bounded cohomology are undecidable.

1. Introduction

Bounded cohomology of groups is defined via the topological dual of the simplicial resolution. This rich theory has applications to the geometry of manifolds [33], dynamics [27], rigidity theory [13, 64], quasimorphisms [9, 31] and stable commutator length [14]. However, beyond the case of amenable groups, computing the bounded cohomology of a group is a very hard task, which can typically only be done in low degrees. We provide new computations in bounded cohomology of finitely generated and finitely presented groups in arbitrarily large degrees.

1.1. Finitely generated non-amenable boundedly acyclic groups.

Boundedly acyclic groups are those groups whose bounded cohomology with trivial real coefficients vanishes in all positive degrees:

Definition 1 (Boundedly acyclic groups)**.** *A group Γ is* boundedly acyclic *if $\mathrm{H}_b^n(\Gamma;\mathbb{R}) \cong 0$ for all $n \geq 1$. Here, $\mathbb{R}$ denotes the real coefficients endowed with the trivial Γ-action.*

We write BAc *for the class of boundedly acyclic groups.*

The main examples of boundedly acyclic groups are amenable groups, as proved by Johnson [46]. Matsumoto and Morita showed that the class

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of boundedly acyclic groups also contains non-amenable groups, by proving that the group of homeomorphisms of $\mathbb{R}^n$ with compact support has this property [54]. Similar techniques show that there exist non-amenable boundedly acyclic groups that are countable: all mitotic groups are boundedly acyclic [52]. However, also these examples are not finitely generated.

Recently, the interest in finding finitely presented bounded acyclic groups has increased significantly because of the following applications to spaces: The version of Gromov's Vanishing Theorem for boundedly acyclic covers by Ivanov [44] and the extended version of Gromov's Mapping Theorem [65].

Combining mitoses and suitable HNN-extensions, we show:

Theorem 2 (Finitely generated non-amenable boundedly acyclic groups; Theorem 4.1)**.** *There exists a functor* $\mu\colon$ **Groups** $\to$ **Groups** *associating to each group* Γ *a boundedly acyclic group* $\mu(\Gamma)$ *into which* Γ *embeds. The group* $\mu(\Gamma)$ *has the following properties:*

(1) Torsion elements of $\mu(\Gamma)$ *are conjugate to elements of* Γ*.*
(2) If Γ *is infinite, then* $\mu(\Gamma)$ *has the same cardinality as* Γ*, otherwise* $\mu(\Gamma)$ *is countably infinite.*
(3) The group $\mu(\Gamma)$ *contains a non-abelian free subgroup.*
(4) If Γ *is* n*-generated, then* $\mu(\Gamma)$ *is* $(n+3)$*-generated. In particular, if* Γ *is finitely generated, then* $\mu(\Gamma)$ *is finitely generated.*

Thanks to parts (1) and (4) of the theorem, in the same spirit as in classical embedding results [37], we deduce:

Corollary 3 (Corollary 4.4)**.** *There exist continuum many non-isomorphic* 5*-generated non-amenable boundedly acyclic groups.*

Furthermore, we construct a finitely presented non-amenable boundedly acyclic group:

Theorem 4 (A finitely presented non-amenable boundedly acyclic group; Corollary 5.2)**.** *There exists a finitely presented non-amenable boundedly acyclic group.*

This group is non-amenable in a very strong sense, since it contains an isomorphic copy of every finitely presented group.

Our proofs are based on constructions with mitotic groups by Baumslag–Dyer–Heller [2] and Baumslag–Dyer–Miller [3]. As mitotic groups are far from being finitely generated, we proceed as follows:

- Apply HNN-extensions to obtain finitely generated or finitely presented groups from mitotic groups;
- Preserve bounded acyclicity along the construction.

This can be achieved by combining Monod–Popa's result on ascending HNN-extensions [63] with appropriate algebraic constructions.

A note from the future. After the first version of this article was posted, new computations of bounded cohomology have emerged. It is now known that the bounded cohomology of $\mathrm{Homeo}_+(S^1)$ is a polynomial ring, generated by the Euler class [62]; similarly, the bounded cohomology of Thompson's group T is a polynomial ring, generated by the Euler class [21], provided that Thompson's group F is boundedly acyclic.

A proof of bounded acyclicity of F was recently given by Monod [61]. Along the way, he also shows that wreath products of the form $\Gamma \wr \mathbb{Z}$ are boundedly acyclic, for every group Γ. This provides an alternative proof of the fact that every finitely generated group embeds into a finitely generated boundedly acyclic group.

1.2. **Finitely generated groups with large bounded cohomology.** Conversely, it is also interesting to construct finitely generated groups *with large bounded cohomology*:

Definition 5 (Groups with large bounded cohomology)**.** *A group* Γ *has* large bounded cohomology *if* $\dim_{\mathbb{R}} \mathrm{H}_b^n(\Gamma;\mathbb{R}) \geq |\mathbb{R}|$ *for all* $n \geq 2$.

Countable examples with large bounded cohomology can be constructed through product constructions [52]. Recently, Nitsche proved that certain groups of homeomorphisms have similar properties [66].

Until now, no finitely generated examples with large bounded cohomology were known [59, 34, 24]. As recently remarked by Heuer [35], "It is notoriously hard to explicitly compute bounded cohomology, even for most basic groups: There is no finitely generated group G for which the full bounded cohomology $(\mathrm{H}_b^n(G;\mathbb{R}))_{n\in\mathbb{N}}$ with real coefficients is known except where it is known to vanish in all degrees".

Theorem 6 (Finitely generated groups with large bounded cohomology (Corollary 6.2))**.** *There exist continuum many non-isomorphic* 8*-generated groups with large bounded cohomology.*

The main challenge in the proof of Theorem 6 is to find a finitely generated group whose bounded cohomology is large in infinitely many degrees (e.g., all even degrees): the rest can be done by taking appropriate cross products. Our construction starts with a finitely generated group Γ introduced by Meier [55] with the striking property of being isomorphic to its direct square. We then introduce a sufficient condition for groups with this property to have large bounded cohomology in all even degrees (Theorem 6.7) and we show that Γ satisfies that hypothesis.

It is not clear whether this construction can produce finitely presented examples (Scholium 6.15). But a concrete finitely presented example can be obtained through Thompson's group T:

Theorem 7 (A finitely presented group with large bounded cohomology (Theorem 7.3))**.** *If* Λ *is the fundamental group of an oriented closed connected hyperbolic* 3*-manifold, then* $T \times \Lambda$ *has large bounded cohomology.*

1.3. **Non-computability.** We consider the problem of algorithmic computability of bounded cohomology. Despite of the Hopf formula for group homology, the algorithmic problem

> Given a finite presentation $\langle S \mid R\rangle$, decide whether $\mathrm{H}_2(\langle S \mid R\rangle\,;\mathbb{Z})$ is trivial or not.

is undecidable, as one can show by a variation of the Adian–Rabin constructions [29, Theorem 4]. The same method also shows that the algorithmic problem

> Given a finite presentation $\langle S \mid R\rangle$, decide whether $\mathrm{H}^2(\langle S \mid R\rangle\,;\mathbb{R})$ is trivial or not.

is undecidable. Similarly, we obtain for bounded cohomology:

Theorem 8 (Non-computability; Theorem 8.1)**.** *Let $d \in \mathbb{N}_{\geq 2}$. The following algorithmic problems are undecidable: Given a finite presentation $\langle S \mid R\rangle$, decide whether*

(1) $\mathrm{H}_b^d(\langle S \mid R\rangle\,;\mathbb{R}) \cong 0$ *or not;*
(2) $\dim_{\mathbb{R}} \mathrm{H}_b^d(\langle S \mid R\rangle\,;\mathbb{R}) = |\mathbb{R}|$ *or not;*
(3) $\langle S \mid R\rangle$ *is boundedly acyclic or not.*

Further statements of this type are contained in Theorem 8.1. The witness constructions used in the proof of Theorem 8 also apply to show non-computability results for L^2-Betti numbers and cost (Remark 8.11).

Ordinary cohomology of finite simplicial complexes with coefficients in $\mathbb{R}$ or $\mathbb{Z}$ is computable through elementary algorithms from linear algebra. In contrast, Gromov's Mapping Theorem lets us deduce from Theorem 8 that the corresponding property does *not* hold for bounded cohomology:

Theorem 9 (Non-computability for spaces; Corollary 8.2)**.** *Let $d \in \mathbb{N}_{\geq 2}$. The following algorithmic problems are undecidable: Given a finite simplicial complex X, decide whether*

(1) $\mathrm{H}_b^d(X;\mathbb{R}) \cong 0$ *or not;*
(2) $\dim_{\mathbb{R}} \mathrm{H}_b^d(X;\mathbb{R}) = |\mathbb{R}|$ *or not;*
(3) X is boundedly acyclic or not.

Undecidability of vanishing in degrees ≥ 5 could also be deduced from Weinberger's non-computability result for simplicial volume [71, Chapter 2.6]. Our proof is based on similar witness constructions.

We conclude by asking:

Question 10. *Which sequences of semi-normed vector spaces can be realised as bounded cohomology $\mathrm{H}_b^*(\Gamma;\mathbb{R})$ of finitely generated/finitely presented groups Γ?*

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Organisation of this article. In Section 2, we collect background material: We recall the basics of bounded cohomology and ℓ^1-homology with some of their properties. In Section 2.2 we list classical embedding theorems that we will need in the sequel. In Section 2.3, we define mitotic groups and discuss some classical constructions.

Section 3 is devoted to the study of some closure properties of the class of boundedly acyclic groups. In Section 4, we construct finitely generated boundedly acyclic groups; in particular, we prove Theorem 2. In Section 5, we construct a finitely presented non-amenable boundedly acyclic group, proving Theorem 4. Section 6 contains the construction of groups with large bounded cohomology and the proof of Theorem 6. The proof of Theorem 7 is given in Section 7. Finally, we prove the non-computability results Theorem 8 and Theorem 9 in Section 8. The appendix contains basics on

(co)homological dimension in the setting of bounded cohomology, which are used in Section 8.

2. Basic definitions

Unless explicitly stated, all groups considered are discrete.

2.1. ℓ^1-Homology and bounded cohomology.

We recall the notions of *ℓ^1-homology* and *bounded cohomology* of groups. Bounded cohomology of groups was introduced by Johnson [46] and Trauber and then extended to spaces by Gromov [33]. We recall the definition of bounded cohomology of groups. For convenience, we introduce bounded cohomology as a dual construction to ℓ^1-homology [51].

2.1.1. *ℓ^1-Homology.* For a group Γ, let $\mathrm{C}_\bullet(\Gamma)$ denote the simplicial resolution of Γ over $\mathbb{R}$. For $n \in \mathbb{N} = \{0, 1, \dots\}$, we have $\mathrm{C}_n(\Gamma) := \bigoplus_{g \in \Gamma^{n+1}} \mathbb{R} \cdot g$ and

$$\begin{aligned} \partial_n \colon \mathrm{C}_n(\Gamma) &\to \mathrm{C}_{n-1}(\Gamma) \\ (g_0, \dots, g_n) &\mapsto \sum_{j=0}^{n} (-1)^n \cdot (g_0, \dots, \widehat{g_j}, \dots, g_n). \end{aligned}$$

We endow $\mathrm{C}_\bullet(\Gamma; \mathbb{R})$ with the *ℓ^1-norm*:

$$\left| \sum_{g \in \Gamma^{\bullet+1}} a_g \cdot g \right|_1 := \sum_{g \in \Gamma^{\bullet+1}} |a_g|.$$

The boundary operator $\partial_\bullet$ is bounded in each degree; thus, we can define the *ℓ^1-resolution* $\mathrm{C}_\bullet^{\ell^1}(\Gamma)$ of Γ as the completion of $\mathrm{C}_\bullet(\Gamma)$ with respect to the ℓ^1-norm.

Definition 2.1 (ℓ^1-Homology of groups)**.** Let Γ be a group and let V be a Banach Γ-module (e.g., $\mathbb{R}$ with the trivial Γ-action). We set

$$\mathrm{C}_\bullet^{\ell^1}(\Gamma; V) := \mathrm{C}_\bullet^{\ell^1}(\Gamma) \mathbin{\overline{\otimes}}_\Gamma V,$$

where $\overline{\otimes}$ denotes the projective tensor product. Then the *ℓ^1-homology* of Γ with coefficients in V is defined by

$$\mathrm{H}_\bullet^{\ell^1}(\Gamma; V) := \mathrm{H}_\bullet\big(\mathrm{C}_\bullet^{\ell^1}(\Gamma; V)\big).$$

We will mainly be interested in the case of trivial $\mathbb{R}$-coefficients. The construction of $\mathrm{H}_\bullet^{\ell^1}(\,\cdot\,; \mathbb{R})$ is functorial with respect to group homomorphisms.

Remark 2.2. We recall that ℓ^1-homology groups are endowed with an ℓ^1-seminorm induced by the ℓ^1-norm on $\mathrm{C}_\bullet^{\ell^1}(\Gamma; \mathbb{R})$. In the sequel we will also make use of the *reduced ℓ^1-homology* $\overline{\mathrm{H}}_\bullet^{\ell^1}(\Gamma; \mathbb{R})$:

$$\overline{\mathrm{H}}_\bullet^{\ell^1}(\Gamma; \mathbb{R}) := \frac{\ker\big(\partial_\bullet \colon \mathrm{C}_\bullet^{\ell^1}(\Gamma; \mathbb{R}) \to \mathrm{C}_{\bullet-1}^{\ell^1}(\Gamma; \mathbb{R})\big)}{|\cdot|_1\text{-closure of } \partial_{\bullet+1}\, \mathrm{C}_{\bullet+1}^{\ell^1}(\Gamma; \mathbb{R})}.$$

2.1.2. *Bounded cohomology.* The construction of ℓ^1-homology allows us to define bounded cohomology as follows [50]:

Definition 2.3 (Bounded cohomology)**.** Let Γ be a group and let V be a Banach Γ-module. The *bounded cochain complex* of Γ with coefficients in V is defined as the Γ-invariants of the topological dual:

$$\mathrm{C}_b^\bullet(\Gamma; V) := B\big(\mathrm{C}_\bullet^{\ell^1}(\Gamma), V\big)^\Gamma.$$

The *bounded cohomology of* Γ *with coefficients in* V is then defined as

$$\mathrm{H}_b^\bullet(\Gamma; V) := \mathrm{H}^\bullet\big(\mathrm{C}_b^\bullet(\Gamma; V)\big).$$

The construction of $\mathrm{H}_b^\bullet(\,\cdot\,;\mathbb{R})$ is functorial with respect to group homomorphisms.

Remark 2.4. Bounded cohomology with trivial real coefficients is well known in degrees 0 and 1: For every group Γ we have $\mathrm{H}_b^0(\Gamma;\mathbb{R}) \cong \mathbb{R}$ and $\mathrm{H}_b^1(\Gamma;\mathbb{R}) \cong 0$ [33, 22]. For this reason, we did not specify low degrees in Definitions 1 and 5.

2.1.3. *Duality with ℓ^1-homology.* We recall how to use reduced ℓ^1-homology and the interaction through the evaluation map $\mathrm{H}_b^\bullet \otimes_{\mathbb{R}} \overline{\mathrm{H}}_\bullet^{\ell^1} \to \mathbb{R}$ to show non-vanishing of bounded cohomology:

Definition 2.5. Let Γ be a group and let $k \in \mathbb{N}$. We set

$$\bar{b}_k^{\ell^1}(\Gamma) := \dim_{\mathbb{R}} \overline{\mathrm{H}}_k^{\ell^1}(\Gamma;\mathbb{R}).$$

Proposition 2.6 ([52, Proposition 3.2, Proposition 3.3])**.** *Let Γ and Λ be groups and let $k, m \in \mathbb{N}$. Then, we have:*

(1) $\dim_{\mathbb{R}} \mathrm{H}_b^k(\Gamma;\mathbb{R}) \geq \bar{b}_k^{\ell^1}(\Gamma)$*;*
(2) $\bar{b}_2^{\ell^1}(\Gamma) \geq \dim_{\mathbb{R}} \mathrm{H}_b^2(\Gamma;\mathbb{R})$ [54, Theorem 2.3, Corollary 2.7]*;*
(3) $\bar{b}_{k+m}^{\ell^1}(\Gamma \times \Lambda) \geq \bar{b}_k^{\ell^1}(\Gamma) \cdot \bar{b}_m^{\ell^1}(\Lambda)$*;*

Remark 2.7. The previous proposition has the following special situations:

- If $\Gamma_d = \Lambda \times \Lambda_2^d$ for some group Λ_2, then
$$\dim_{\mathbb{R}} \mathrm{H}_b^{2d}(\Gamma_d;\mathbb{R}) \geq \bar{b}_{2d}^{\ell^1}(\Gamma_d) \geq \bar{b}_2^{\ell^1}(\Lambda_2)^d.$$
- For every group Λ_3, the above property in degree $2d$ still holds for $\Gamma_d \times \Lambda_3$, and moreover
$$\dim_{\mathbb{R}} \mathrm{H}_b^{2d+3}(\Gamma_d \times \Lambda_3;\mathbb{R}) \geq \bar{b}_{2d+3}^{\ell^1}(\Gamma_d \times \Lambda_3) \geq \bar{b}_{2d}^{\ell^1}(\Gamma_d) \cdot \bar{b}_3^{\ell^1}(\Lambda_3).$$

For $k \in \{2,3\}$, choosing Λ_k with $\bar{b}_k^{\ell^1}(\Lambda_k) > 0$, we thus obtain lower bounds on the dimension of the bounded cohomology spaces. This will be used in Sections 6 and 7 to construct groups with large bounded cohomology, and in Section 8 to extend results in degree 2 and 3 to higher degrees.

Example 2.8. If Γ is the fundamental group of an oriented closed connected hyperbolic n-manifold M, then $\bar{b}_n^{\ell^1}(\Gamma) > 0$, because the simplicial volume of M is non-zero [33, 70] and hence the fundamental class of M yields a non-trivial element of $\overline{\mathrm{H}}_n^{\ell^1}(M;\mathbb{R}) \cong \overline{\mathrm{H}}_n^{\ell^1}(\Gamma;\mathbb{R})$.

2.2. **Embedding theorems.** We collect some classical embedding theorems, which we will need in the sequel (Sections 4, 5 and 6). We will adopt the following notation for group presentations: Given a group presentation $H = \langle S \mid R \rangle$ and a disjoint set of generators S', we will simply write

$$\langle H; S' \mid R' \rangle := \langle S \cup S' \mid R \cup R' \rangle,$$

where R' is a new set of relations (possibly) involving both elements in S and S'. Moreover, it will always be clear from the context whether $\langle S \mid R \rangle$ denotes a presentation or the group given by this presentation.

Definition 2.9 (Recursively presented group)**.** A group presentation $\langle S \mid R \rangle$ is *recursively enumerable* if the generating set S is countable and the set of relations R is a recursively enumerable subset of the free group over S. A group is *recursively presented* if it has a recursively enumerable presentation.

The importance of recursively presented groups is elucidated by the following well-known result by Higman:

Theorem 2.10 ([36, Theorem 1])**.** *A group is recursively presented if and only if it embeds into a finitely presented group.*

Moreover, Higman also showed that there exists a universal finitely presented group in the following sense:

Theorem 2.11 ([36, p. 456])**.** *There exists a universal finitely presented group, that is, a finitely presented group that contains an isomorphic copy of every finitely presented group.*

To move from one to continuum many examples in Corollary 3 and Theorem 6, we make use of the following:

Theorem 2.12 ([37, Section 4])**.** *Every countable group Γ embeds into a 2-generator group K with the following property: For every prime p the group K contains p-torsion if and only if Γ does.*

Corollary 2.13. *There exist continuum many 2-generator groups, which are pairwise distinguished by their torsion.*

Proof. For each (possibly infinite) set P of prime numbers, it is sufficient to apply Thorem 2.12 to $\bigoplus_{p \in P} \mathbb{Z}/p\mathbb{Z}$. □

Remark 2.14. More explicit constructions allow to show that there exist continuum many pairwise non-isomorphic 2-generator groups, which are moreover torsion-free [15].

Remark 2.15. Notice that all these results make essential use of HNN-extensions. Indeed, every group Γ embeds into every HNN-extension of the form $\Gamma *_\varphi$ [37]. We will use this fact repeatedly in the sequel.

2.3. **Mitotic groups.** *Mitotic groups* are acyclic groups, first introduced by Baumslag, Dyer and Heller [2] as building blocks in order to prove new results about functorial embeddings of groups into acyclic groups. Mitotic groups are based on *mitoses*:

Definition 2.16 (Mitosis)**.** Let H be a subgroup of a group Γ. We say that Γ is a *mitosis of H* if there exist two elements $s, d \in \Gamma$ such that the following hold

(1) Γ is generated by H, s and d;
(2) For all $h, h' \in H$, we have $[h', s^{-1}hs] = 1$.
(3) For all $h \in H$, we have $d^{-1}hd = hs^{-1}hs$;

We use the commutator convention $[x, y] := x^{-1}y^{-1}xy$.

The second condition says that Γ contains two conjugate, commuting copies of H (which are allowed to intersect non-trivially). This implies that there exists a third, diagonal copy, and the third condition says that this is also conjugate to the first copy. This is better illustrated by the next example: Following Baumslag, Dyer and Heller [2, Section 5], we recall how to embed every group into its *standard mitosis*:

Example 2.17 (Standard mitosis)**.** Let Γ be a group. We begin by embedding Γ into $\Gamma \times \Gamma$ via the identification $g \mapsto (g, 1)$. Then, we construct a mitotis for Γ by applying two HNN-extensions as follows: We add an element s that conjugates $(g, 1)$ to $(1, g)$, and then an element d that conjugates $(g, 1)$ to (g, g). This leads to a group

$$\begin{aligned} m(\Gamma) &:= \langle \Gamma \times \Gamma; s, d \mid s^{-1}(g,1)s = (1,g), d^{-1}(g,1)d = (g,g) \rangle \\ &= \langle \Gamma; s, d \mid d^{-1}gd = gs^{-1}gs, [g, s^{-1}hs] = 1 : g, h \in \Gamma \rangle, \end{aligned}$$

which is called the *standard mitosis* of Γ. It is immediate to check that Γ is embedded into $m(\Gamma)$ (Remark 2.15).

This construction is functorial in the following sense [2, Lemma 5.2]: Every homomorphism $\varphi\colon \Gamma_1 \to \Gamma_2$ induces a homomorphism $m(\varphi)\colon m(\Gamma_1) \to m(\Gamma_2)$ just by sending $g \to \varphi(g)$ for every $g \in \Gamma_1$ and $s_1 \mapsto s_2$, $d_1 \mapsto d_2$. Moreover, the functor $m\colon$ **Groups** $\to$ **Groups** preserves monomorphisms.

To prove non-amenability of the boundedly acyclic groups constructed in Theorem 2, the following will be useful:

Remark 2.18. The subgroup of $m(\Gamma)$ generated by s and d is free of rank 2: Indeed, the map $m(\Gamma) \to F_2$ sending s and d to the two generators and annihilating Γ is a surjective homomorphism, as can be easily seen from the latter presentation of $m(\Gamma)$.

Definition 2.19 (Mitotic groups)**.** A group Γ is *mitotic* if every finitely generated subgroup of Γ admits a mitotis in Γ.

Using the construction of Example 2.17, it is easy to show that every group embeds into a mitotic group [2, Lemma 5.4]:

Example 2.20 (Standard mitotic embedding)**.** Given a group Γ we already know that it can be embedded into its standard mitosis $m(\Gamma)$ (Example 2.17). Moreover, since the functor m preserves monomorphisms, we can construct the direct union (colimit) $m^\infty(\Gamma) = \bigcup_{i \geq 0} m^i(\Gamma)$, which is mitotic.

Mitotic groups are known to be acyclic [2]. Our interest in them is motivated by the following result (see Definition 1):

Theorem 2.21 ([52, Theorem 1.2])**.** *All mitotic groups are boundedly acyclic.*

3. Basic properties of boundedly acyclic groups

The class BAc of *boundedly acyclic groups* consists of those groups with vanishing bounded cohomology in all positive degrees and trivial real coefficients $\mathbb{R}$ (Definition 1). They appeared in the literature also under the name of groups with zero bounded cohomological dimension [52], but we prefer to stick with the name boundedly acyclic because of the recent characterisation of boundedly acyclic maps [65] and Ivanov's work on boundedly acyclic open covers [44]. Moreover, this choice also avoids any confusion with the bounded cohomological dimension cd_b that we will introduce and discuss later (Sections 8 and Appendix A). In this section we discuss some hereditary properties of boundedly acyclic groups.

3.1. **Boundedly acyclic groups and HNN extensions.** We recall the definition of *co-amenability* and how this property provides information about BAc and HHN-extensions.

Proposition 3.1 ([65, Corollary 4.2.2])**.** *Let* $1 \to H \to \Gamma \to Q \to 1$ *be a short exact sequence of groups. Suppose* $H \in \mathrm{BAc}$*. Then* $\Gamma \in \mathrm{BAc}$ *if and only if* $Q \in \mathrm{BAc}$*.*

In the situation of Proposition 3.1, if $H \in \mathrm{BAc}$ and Q is amenable, then $\Gamma \in \mathrm{BAc}$. In fact, following a result by Monod and Popa [63], one can strengthen this statement via co-amenability:

Definition 3.2 (Co-amenable subgroups)**.** Let Γ be a group and let $H \leq \Gamma$ be a subgroup. We say that H is *co-amenable* in Γ if there exists a Γ-invariant mean on the space $\ell^\infty(\Gamma/H)$ of bounded functions on Γ/H.

Example 3.3. The following are examples of co-amenable subgroups:

(1) Suppose that H is normal in Γ. Then H is co-amenable in Γ if and only if the quotient Γ/H is amenable.
(2) Let H be a group and let $\varphi\colon H \to H$ a monomorphism. Let $\Gamma = H*_\varphi$ be the corresponding HNN-extension: Following most of the literature, we will call such HNN-extensions *ascending*. Then, H is co-amenable in Γ [63, Proposition 2].
(3) Given a chain $K < H < \Gamma$ of groups, we have: If K is co-amenable in H and H is co-amenable in Γ, then K is also co-amenable in Γ [63].
(4) Given a chain $K < H < \Gamma$ of groups, we have: If K is co-amenable in Γ, then H is co-amenable in Γ [63].
(5) On the other hand, it is not true in general that the co-amenability of K in Γ implies that K is co-amenable in H [63].

The importance of co-amenability in our setting is evident from the following result by Monod and Popa:

Proposition 3.4 ([63, Proposition 3], [58, 8.6])**.** *Let* $H \leq \Gamma$ *be a co-amenable subgroup. Then, the inclusion map* $i\colon H \to \Gamma$ *induces an injective map in bounded cohomology:*

$$\mathrm{H}_b^\bullet(i)\colon \mathrm{H}_b^\bullet(\Gamma;\mathbb{R}) \hookrightarrow \mathrm{H}_b^\bullet(H;\mathbb{R})$$

In particular, if $H \in \mathrm{BAc}$*, then* $\Gamma \in \mathrm{BAc}$*.*

We will see in Corollary 5.3 that $\mathrm{H}_b^\bullet(i)$ is, in general, far from being surjective.

Corollary 3.5. *Ascending HNN-extensions of groups in* BAc *are in* BAc.

Proof. We combine Proposition 3.4 and Example 3.3.2. □

The following example shows that Corollary 3.5 does not hold for general HNN-extensions.

Example 3.6. The group $\mathrm{BS}(2,3)$ is an HNN-extension of $\mathbb{Z}$ along finite-index subgroups isomorphic to $\mathbb{Z}$. However, $\mathrm{BS}(2,3)$ is not a boundedly acyclic group. Indeed, it admits non-trivial quasimorphisms [32, 25], whence it has non-trivial second bounded cohomology group.

3.2. **Quotients and Lex groups.** One of the most remarkable properties of mitotic groups is that they are closed under quotients [6, Appendix B]. Since the same property is also true for amenable groups [22, Proposition 3.4], a natural question is whether the quotient of a boundedly acyclic group is still boundedly acyclic.

This problem is related to the problem of showing existence of a non-Lex group, i.e., of groups that are not left-exact in the following sense:

Definition 3.7 (Lex groups [8])**.** We say that a group Γ lies in the family Lex if it satisfies the following left-exactness property: For every group Λ and every epimorphism $\psi\colon \Lambda \to \Gamma$, the induced map $\mathrm{H}_b^\bullet(\psi)$ in bounded cohomology with trivial real coefficients is injective in all degrees.

It is an open problem to find examples of groups that do not lie in Lex. On the other hand, in degree 2 the situation is understood: Every group epimorphism induces an injective map between the second bounded cohomology groups [7] (in fact the induced map is even isometric [41, Theorem 2.14] and the result also holds for a larger family of coefficients [65, Example 4.1.2]).

Of course amenable groups and, more generally, boundedly acyclic groups lie in Lex. Some more interesting examples are for instance:

- Free groups;
- Fuchsian groups [8, Corollaire 3.9];
- Fundamental groups of geometric 3-manifolds [8, Corollaire 3.13 and p. 267] (together with Agol's proof of Thurston's Virtual Fibering Conjecture [1]).
- The class Lex is closed under the following constructions: Quotients by amenable subgroups, extensions of an amenable group by an element in Lex [8, Proposition 3.16] and free products of amenable groups amalgamated over a common normal subgroup [8, Corollaire 3.17].

Remark 3.8. Following Bouarich's proof [8, Proposition 3.16], one can in fact deduce the corresponding statement for boundedly acyclic groups. Namely, Lex is also closed under the following constructions:

- Quotients by boundedly acyclic groups;
- Extensions of a boundedly acyclic group by an element in Lex.

The proof follows *verbatim* the one by Bouarich with the additional fact that epimorphisms with boundedly acyclic kernels induce isomorphism in all bounded cohomology groups with trivial real coefficients [65, Theorem 4.1.1]

The connection between quotients of BAc groups and Lex groups is given in the following proposition:

Proposition 3.9. *The family* BAc *is closed under quotients if and only if all quotients of boundedly acyclic groups lie in* Lex.

Proof. Since boundedly acyclic groups lie in Lex, one implication trivially holds. Vice versa, if we assume that BAc is not closed under quotients, there exists an epimorphism $\psi\colon \Gamma \to \Lambda$ with $\Gamma \in$ BAc and $\Lambda \notin$ BAc. Then, ψ cannot induce an injective map in bounded cohomology for all degrees. This shows that $\Lambda \notin$ Lex, whence the thesis. □

This proposition provides a strategy to find a non-Lex group: namely, it would suffice to exhibit a quotient of a BAc group that is not in BAc. Note that by the discussion above, every quotient of a BAc group has vanishing second bounded cohomology with trivial real coefficients. Several groups of geometric nature are then natural candidates for a counterexample:

Example 3.10. Let X be an n-dimensional irreducible symmetric space of non-compact type, and G the associated Lie group. Let $\Gamma < G$ be a torsion-free cocompact lattice. Then $\|\Gamma\backslash X\| > 0$ [48], and so by Gromov's Duality Principle [33] it follows that $\mathrm{H}_b^n(\Gamma;\mathbb{R}) \ncong 0$ (in fact something can be said about lower degrees as well [49]).

On the other hand, if X is not Hermitian symmetric and has real rank at least 3, then $\mathrm{H}_b^2(\Gamma;\mathbb{R}) \cong 0$ [12].

4. Finitely generated boundedly acyclic groups

In this section, we show that each finitely generated group embeds into a finitely generated boundedly acyclic group. The latter will always contain a non-abelian free group, providing the first examples of (an infinite family of) non-amenable finitely generated boundedly acyclic groups.

Theorem 4.1. *There exists a functor* $\mu\colon$ **Groups** $\to$ **Groups** *associating to each group* Γ *a boundedly acyclic group* $\mu(\Gamma)$ *into which* Γ *embeds. The group* $\mu(\Gamma)$ *has the following properties:*

1. *Torsion elements of* $\mu(\Gamma)$ *are conjugate to elements of* Γ.
2. *If* Γ *is infinite, then* $\mu(\Gamma)$ *has the same cardinality as* Γ*, otherwise* $\mu(\Gamma)$ *is countably infinite.*
3. *The group* $\mu(\Gamma)$ *contains a non-abelian free subgroup.*
4. *If* Γ *is* n*-generated, then* $\mu(\Gamma)$ *is* $(n+3)$*-generated. In particular, if* Γ *is finitely generated, then* $\mu(\Gamma)$ *is finitely generated.*

Proof. We construct our functor starting with the standard mitosis of Γ (Example 2.17): Every group Γ embeds into its standard mitosis

$$m(\Gamma) = \langle \Gamma; s, d \mid d^{-1}gd = gs^{-1}gs, [g, s^{-1}hs] = 1 : g, h \in \Gamma \rangle.$$

To make the notation more transparent, let us denote s by s_1 and d by d_1. Then, we iterate the process as in the standard mitotic embedding

(Example 2.20): Denoting by s_i and t_i the new generators of $m^i(\Gamma)$, we obtain the directed union (colimit) $m^\infty(\Gamma)$, which is generated by Γ, together with $\{s_1, d_1, \dots\}$. There exists a self-monomorphism φ of $m^\infty(\Gamma)$ given by $g \mapsto g, s_i \mapsto s_{i+1}, d_i \mapsto d_{i+1}$ [2, p. 20]. We now set $\mu(\Gamma)$ to be the ascending HNN-extension $m^\infty(\Gamma)*_\varphi$:

$$\mu(\Gamma) \coloneqq \langle m^\infty(\Gamma); t \mid t^{-1}xt = \varphi(x) : x \in m^\infty(\Gamma)\rangle.$$

Notice that μ is in fact a functor, because it is constructed via iterated standard mitoses (Example 2.17) followed by an HNN-extension. Moreover, $\mu(\Gamma)$ is a boundedly acyclic group because it is an ascending HNN-extension of the mitotic group $m^\infty(\Gamma)$ (Theorem 2.21 and Corollary 3.5). Finally, by construction, Γ embeds into $\mu(\Gamma)$ (Remark 2.15).

We are left to check that $\mu(\Gamma)$ satisfies the properties *(1)–(4)*.

Ad 1. The statement on torsion follows from the fact that all torsion in an HNN-extension is conjugate into the base group [37].

Ad 2. The statement on the cardinality follows from the fact that HNN-extensions of infinite groups preserve the cardinality, while HNN-extensions of finite groups are countably infinite.

Ad 3. By Remark 2.18, the subgroup of $m(\Gamma)$ generated by s_1 and d_1 is free of rank 2. Since $m(\Gamma)$ also embeds into $\mu(\Gamma)$ (Remark 2.15), we also have that $\mu(\Gamma)$ contains a non-abelian free group.

Ad 4. It is immediate to check that the generators of Γ, together with s_1, d_1 and t, suffice to generate $\mu(\Gamma)$. This shows that if Γ is n-generated, then $\mu(\Gamma)$ is $(n+3)$-generated, whence the claim. □

Remark 4.2. Notice that $\mu(\Gamma)$ is not acyclic. Indeed, the presentation of $\mu(\Gamma)$ shows that its abelianization is an infinite cyclic group, whence $\mathrm{H}_1(\mu(\Gamma);\mathbb{Z}) \cong \mathbb{Z}$ [2, p. 20]. Nevertheless, it is worth mentioning that $\mathrm{H}_n(\mu(\Gamma);\mathbb{Z}) \cong 0$ for all $n > 1$, because $m^\infty(\Gamma)$ is acyclic [2, Theorem 4.2; p. 20].

Corollary 4.3. *Every countable group Γ embeds into a 5-generated non-amenable boundedly acyclic group that has p-torsion if and only if Γ does.*

Proof. This is the combination of parts (1), (3) and (4) of Theorem 4.1 together with Theorem 2.12. □

Similar embedding results for *mixing* coefficients have been obtained by Monod [60, Proposition 6.5].

Using Corollary 4.3, the same proof as Corollary 2.13 gives:

Corollary 4.4. *There exist continuum many 5-generated non-amenable boundedly acyclic groups, which are pairwise distinguished by their torsion.*

Remark 4.5. In fact, there exist continuum many *torsion-free* 5-generated non-amenable boundedly acyclic groups, which are pairwise non-isomorphic. Indeed, there exist continuum many pairwise non-isomorphic 2-generated torsion-free groups $(\Gamma_i)_{i\in I}$, by Remark 2.14. Now each group $\mu(\Gamma_i)$ is finitely generated, in particular it is countable, and so has only countably many finitely generated subgroups. Therefore there must be continuum many distinct isomorphism types in the collection $(\mu(\Gamma_i))_{i\in I}$ as well. Finally, they are torsion-free by Theorem 4.1.

One natural question is whether the same construction leads to finitely *presented* boundedly acyclic groups. Unfortunately, as in the classical case of acyclic groups, this is never the case [2, Theorem 5.6]. Let us recall the following construction by Baumslag, Dyer and Heller [2, Section 5]: Let A be a finitely presented torsion-free acyclic group with generators a, b, c (one such example can be found in [2, Section 3]), and let $A(\Gamma) := \mu(\Gamma) *_{t=a} A$. Then, $A(\Gamma)$ has the following properties:

(P1) The group $A(\Gamma)$ is never finitely presented [2, Theorem 5.6];
(P2) If $A(\Gamma)$ is finitely generated, then so is Γ [2, Lemma 5.7];
(P3) If Γ is recursively presented, then so is $A(\Gamma)$ [3, p. 38].

Using this group we can deduce the following additional properties of $\mu(\Gamma)$:

Proposition 4.6. *The functor* $\mu\colon$ **Groups** $\to$ **Groups** *associating to each group* Γ *the boundedly acyclic group* $\mu(\Gamma)$ *also satisfies the following properties:*

(1) The group $\mu(\Gamma)$ *is finitely generated if and only if* Γ *is;*
(2) The group $\mu(\Gamma)$ *is never finitely presented;*
(3) If Γ *is recursively presented, then so is* $\mu(\Gamma)$.

Proof. We can easily deduce all the properties from the ones of $A(\Gamma)$:

Ad 1. We have already shown in Theorem 4.1.4 that if Γ is finitely generated, then also $\mu(\Gamma)$ is finitely generated. Vice versa, if $\mu(\Gamma)$ is finitely generated, then so is $A(\Gamma)$. This shows that Γ is also finitely generated by (P2) of $A(\Gamma)$.

Ad 2. By contradiction, assume that $\mu(\Gamma)$ is finitely presented. Then also $A(\Gamma)$ is finitely presented being a free product of finitely presented groups amalgamated along a finitely generated subgroup. This leads to a contradiction ((P1) of $A(\Gamma)$).

Ad 3 If Γ is recursively presented, we already know that $A(\Gamma)$ is too ((P3) of $A(\Gamma)$). This implies that $\mu(\Gamma)$ is also recursively presented, as subgroups of recursively presented groups are recursively presented [36]. □

The previous result has an important consequence, which will be a building block in our construction of a finitely presented non-amenable boundedly acyclic group (Theorem 4). Let U denote a universal finitely presented group (Theorem 2.11). Then we have the following:

Corollary 4.7. *Let* Γ *be a finitely presented group. Then,* $\mu(\Gamma)$ *embeds into* U.

Proof. By definition of U we know that every finitely presented group embeds into U. So it is sufficient to show that we can embed $\mu(\Gamma)$ into a finitely presented group. Since Γ is finitely presented (whence recursively presented), Proposition 4.6.3 shows that $\mu(\Gamma)$ is also recursively presented. Hence, by Theorem 2.10 $\mu(\Gamma)$ embeds into a finitely presented group, which in turn embeds into U (Theorem 2.11). □

5. A finitely presented non-amenable boundedly acyclic group

The aim of this section is to make use of Theorem 4.1 in order to construct a finitely presented boundedly acyclic group that contains all finitely

presented groups. In particular this provides the first example of a finitely presented non-amenable boundedly acyclic group. The fundamental tool in the process is the following result, which is based on the techniques by Baumslag, Dyer and Miller [3, Section 4]:

Theorem 5.1. *Let Γ be a group such that $\mu(\Gamma)$ embeds into Γ. Then Γ has a boundedly acyclic ascending HNN-extension. In particular, a universal finitely presented group has a boundedly acyclic ascending HNN-extension.*

Proof. Let Γ' be an isomorphic copy of Γ with $\mu(\Gamma) \leq \Gamma'$. Let $f\colon \Gamma' \to \Gamma$ be such an isomorphism. Then there exists a monomorphism $\varphi\colon \Gamma' \to \Gamma \leq \Gamma'$ obtained by composing f with the inclusion of Γ into $\mu(\Gamma) \leq \Gamma'$ (Theorem 4.1). We claim that the corresponding ascending HNN-extension $\Gamma'*_\varphi$ is boundedly acyclic. Indeed, since $\varphi(\Gamma') = t^{-1}\Gamma' t$ inside $\Gamma'*_\varphi$, we have the following chain of inclusions:

$$\Gamma' = t\Gamma t^{-1} < t\mu(\Gamma)t^{-1} < t\Gamma' t^{-1} < \Gamma' *_\varphi .$$

Moreover, Γ' is co-amenable in $\Gamma'*_\varphi$ (Example 3.3.2) and so by Example 3.3.4 we also know that $t\mu(\Gamma)t^{-1}$ is co-amenable in $\Gamma'*_\varphi$. By using the fact that $t\mu(\Gamma)t^{-1}$ is isomorphic to $\mu(\Gamma)$, whence boundedly acyclic, we conclude that $\Gamma'*_\varphi$ is boundedly acyclic as claimed (Proposition 3.4).

A universal finitely presented group (Theorem 2.11) satisfies the hypothesis of the theorem. Indeed by Theorem 4.1 and Corollary 4.7, we have embeddings $U < \mu(U) < U$. Therefore U admits a boundedly acyclic ascending HNN-extension. □

Corollary 5.2. *There exists a finitely presented boundedly acyclic group that contains an isomorphic copy of every finitely presented group.*

Proof. Let U be a universal finitely presented group. It admits a boundedly acyclic ascending HNN-extension, by the last statement of Theorem 5.1. This is finitely presented, being an ascending HNN-extension of a finitely presented group, and it contains U, thus all finitely presented groups. □

By Proposition 3.4, if Λ is an ascending HNN-extension of Γ, then Γ is co-amenable in Λ and so the embedding $\Gamma \to \Lambda$ induces an injection in bounded cohomology. The techniques from this section allow us to show that, in general, this injection is very far from being an isomorphism:

Corollary 5.3. *For all $d \geq 2$ there exists a finitely presented group Γ with $\mathrm{H}^d_b(\Gamma;\mathbb{R}) \ncong 0$ that admits a boundedly acyclic ascending HNN-extension.*

Proof. For every $d \geq 2$, let Λ_d be the fundamental group of an oriented closed connected hyperbolic d-manifold. Then, we know that $\mathrm{H}^d_b(\Lambda_d;\mathbb{R}) \ncong 0$ (Proposition 2.6 and Example 2.8)). Since Λ_d is finitely presented, we can set $\Gamma \coloneqq U \times \Lambda_d$, where U is a universal finitely presented group. As Γ retracts onto Λ_d, we have $\mathrm{H}^d_b(\Gamma;\mathbb{R}) \ncong 0$. On the other hand, Γ is itself a universal finitely presented group, so by the last statement of Theorem 5.1 it has a boundedly acyclic ascending HNN-extension. □

6. Finitely generated groups with large bounded cohomology

In this section, we turn to groups with large bounded cohomology, namely whose bounded cohomology is at least continuum-dimensional in every degree at least 2 (Definition 5). Countable examples have been constructed before [52], but no finitely generated example was known. Here, we provide the first recipe for finitely generated examples:

Theorem 6.1. *There exists a* 6*-generated group with large bounded cohomology. Moreover, this group can be chosen to be torsion-free.*

The proof of Theorem 6.1 will be completed in Section 6.3.

Combining this theorem with Theorem 2.12 [37], we obtain:

Corollary 6.2. *There exist continuum many non-isomorphic* 8*-generated groups with large bounded cohomology.*

Proof. Let Γ be a 6-generated torsion-free group with large bounded cohomology (Theorem 6.1). Then, for every group Λ the product $\Lambda \times \Gamma$ also has large bounded cohomology, because the product retracts onto Γ. Now by Corollary 2.13 there exist continuum many 2-generated groups pairwise distinguished by their torsion. Taking Λ to be in this family, since Γ is torsion-free, we obtain continuum many non-isomorphic 8-generated groups with large bounded cohomology. □

Remark 6.3. Again, we can also choose these groups to be torsion-free using Remark 4.5.

6.1. **Constructing groups with large bounded cohomology.** We provide a criterion to construct finitely generated groups with large bounded cohomology, by starting with a group that is isomorphic to a proper direct factor of itself. We begin by introducing a local version of Definition 5:

Definition 6.4. Let $n \geq 2$. A group Γ has *large* n*-th bounded cohomology* if $\dim_{\mathbb{R}} \mathrm{H}_b^n(\Gamma;\mathbb{R}) \geq |\mathbb{R}|$.

Remark 6.5. Let Γ be a group. The following inequality always holds:

$$\dim_{\mathbb{R}} \mathrm{H}_b^n(\Gamma;\mathbb{R}) \leq |\,\mathrm{C}_b^n(\Gamma;\mathbb{R})| \leq |\mathbb{R}|^{\Gamma^{n+1}}$$

In particular, if Γ is countable, then $\dim_{\mathbb{R}} \mathrm{H}_b^n(\Gamma;\mathbb{R}) \leq |\mathbb{R}|$. This shows that countable groups with large n-th bounded cohomology have n-th bounded cohomology of dimension equal to $|\mathbb{R}|$. For arbitrary groups, larger cardinalities are possible [20].

Example 6.6. The following examples will be useful in the rest of this section:

(1) Let $G *_C H$ be an amalgamated product with $|C\backslash G/C| \geq 3$ and $C \neq H$. Then $G *_C H$ has large second bounded cohomology [32, 25].
(2) If Δ has large second bounded cohomology and Γ surjects onto Δ, then Γ has large second bounded cohomology (Section 3.2).
(3) If H is a retract of Γ, and H has large n-th bounded cohomology, then so does Γ, since the epimorphism $\Gamma \to H$ induces an injection in bounded cohomology. A special case of this is when H is a direct factor of Λ, which was used in the proof of Corollary 6.2.

(4) Acylindrically hyperbolic groups have large second [42] and third [24] bounded cohomology. We will only need the case of fundamental groups of oriented closed connected hyperbolic 3-manifolds for the proof of Theorem 6.1. This case was known earlier [9, 19, 69].

A further example is contained in Lemma 8.10.

The main tool in the proof of Theorem 6.1 is the following:

Theorem 6.7. *Let Γ and Σ be groups such that $\Gamma \cong \Gamma \times \Sigma$. Suppose that Σ has large second bounded cohomology. Then Γ has large bounded cohomology in all even degrees.*

Moreover, if Λ is the fundamental group of an oriented closed connected hyperbolic 3-manifold, then $\Gamma \times \Lambda$ has large bounded cohomology.

Proof. For every $d \geq 1$ we can write Γ as $\Gamma \times \Sigma^d$. Now it follows at once from Remark 2.7 that Γ has large bounded cohomology in all even degrees. Next, since $\overline{b}_3^{\ell^1}(\Lambda) > 0$ (Example 2.8), again it follows from Remark 2.7 that $\Gamma \times \Lambda$ has large bounded cohomology in all even degrees, and all degrees of the form $2d+3$ for $d \geq 1$, which includes all integers $n \geq 2$, except for $n = 3$. Finally, $\Gamma \times \Lambda$ retracts onto Λ, and so has large third bounded cohomology by Examples 6.6.3 and 6.6.4. □

Remark 6.8. The last statement of Theorem 6.7 also holds if we only assume that $\mathrm{H}_b^2(\Sigma; \mathbb{R}) \neq 0$. Indeed, it is still true that $\Gamma \times \Lambda$ has large second and third bounded cohomology, and that $\overline{b}_2^{\ell^1}(\Gamma \times \Lambda) \geq |\mathbb{R}|$. We can write all greater even integers as $2d + 2$ (Remark 2.7), thus obtaining largeness in all even degrees. For the larger odd degrees we proceed as in the proof of Theorem 6.7.

Remark 6.9. Notice that we might replace Λ in the product above by an arbitrary acylindrically hyperbolic group A, if we knew that $\overline{b}_3^{\ell^1}(A) > 0$. This, however, seems to be an open problem, even for non-abelian free groups.

The remaing part of this section is devoted to the construction of a finitely generated group Γ satisfying the assumptions of Theorem 6.7.

6.2. **Meier's finitely generated group.** Finitely generated groups Γ with the property that $\Gamma \cong \Gamma \times \Sigma$ for some $\Sigma \ncong 1$ were first constructed by Jones [47]. Further constructions were given by Meier [55], Rhemtulla [67] and Hirshon [38]. The last paper provides examples with Σ finitely presented. Among the constructions, particular attention has been devoted to the case in which $\Gamma = \Sigma$. We will show that Meier's group satisfies the assumption of Theorem 6.7.

Definition 6.10 (Meier's finitely generated group)**.** Let

$$B \coloneqq \langle a, t \mid t^{-1}a^2t = a^3 \rangle$$

be the Baumslag–Solitar group $\mathrm{BS}(2,3)$ and let $\overline{B}$ be another copy of B with generators $\overline{a}, \overline{t}$. Let

$$L \coloneqq \langle t, [a, t^{-1}at] \rangle \leq B,$$

which is a free subgroup of rank 2, and let $\overline{L}$ be its copy inside $\overline{B}$. We then set Δ to be the free product of B and $\overline{B}$ amalgamated over $L \cong \overline{L}$ by switching the generators; namely

$$\Delta := B *_{L \cong \overline{L}} \overline{B} = \langle B, \overline{B} \mid t = [\overline{a}, \overline{t}^{-1}\overline{a}\overline{t}], \overline{t} = [a, t^{-1}at] \rangle.$$

Notice that Δ is torsion-free, being an amalgamated product of two torsion-free groups, and it is generated by $a, \overline{a}$ and t. Let Γ be the subgroup of $\Delta^{\mathbb{N}}$ generated by the diagonal elements $(a, a, \ldots), (\overline{a}, \overline{a}, \ldots), (t, t, \ldots)$ together with the element $(1, a, a^2, \ldots)$. Thus Γ is a four-generated subgroup of $\Delta^{\mathbb{N}}$ containing the diagonal. We call Γ *Meier's finitely generated group*.

Theorem 6.11 ([55, Proposition 7])**.** *Meier's finitely generated group* Γ *satisfies* $\Gamma \cong \Gamma \times \Gamma$.

Remark 6.12. The projection $\Delta^{\mathbb{N}} \to \Delta$ onto the first coordinate restricts to an epimorphism $\psi \colon \Gamma \to \Delta$, because Γ contains the diagonal of $\Delta^{\mathbb{N}}$. Hence, by Example 6.6.2, if Δ has large second bounded cohomology then so does Γ. Thus, in this case, Γ satisfies the hypotheses of Theorem 6.7.

The previous remark shows that we reduced the problem of computing the second bounded cohomology of Meier's finitely generated group Γ to the one of computing $\mathrm{H}^2_b(\Delta; \mathbb{R})$. Recall from Example 6.6.1 that an amalgamated free product $G *_C H$ has large second bounded cohomology provided $|C \backslash G / C| \geq 3$ and $C \neq H$. We will spend the last part of this section by proving that T satisfies the previous condition.

Before proving that $|L \backslash B / L|$ has the desired cardinality, it is useful to recall the proof that B is non-Hopfian [5]: This amounts in constructing a non-injective self-epimorphism φ. The desired $\varphi \colon B \to B$ is defined on the standard generators of B by $\varphi(a) = a^2, \varphi(t) = t$. Since the image of φ contains both the generators $a = \varphi([t, a^{-1}])$ and $t = \varphi(t)$, the homomorphism φ is surjective. On the other hand, φ is non-injective because $\varphi([a, t^{-1}at]) = 1$, and this element is easily seen to be non-trivial using Britton's Lemma [68, Theorem 11.81].

We are now ready to show that the cardinality $|L \backslash B / L|$ is infinite, by using the previous homomorphism φ.

Lemma 6.13. *Let* $\varphi \colon B \to B$ *be the non-injective self-epimorphism defined above. Then, there exists a strictly nested sequence*

$$L < \varphi^{-1}(L) < \varphi^{-2}(L) < \cdots .$$

In particular $|L \backslash B / L| = \infty$.

Proof. Since $[a, t^{-1}at] \in \ker(\varphi)$, we know that

$$\varphi(L) = \langle \varphi(t), \varphi([a, t^{-1}at]) \rangle = \langle t \rangle.$$

In particular L is strictly contained in the group $\varphi^{-1}(L)$. Applying φ^{-1} repeatedly to both sides we obtain the desired sequence.

Now let $x \in \varphi^{-1}(L) \setminus L$. Then the double cosets L and LxL are distinct and contained in $\varphi^{-1}(L)$. We repeat this argument inductively on the sequence, and obtain that $\varphi^{-k}(L)$ contains at least $(k+1)$ distinct double cosets of L. Thus $|L \backslash B / L| = \infty$. □

The previous discussion leads to the following:

Proposition 6.14. *Meier's finitely generated group* Γ *(Definition 6.10) has large second bounded cohomology.*

Proof. By Remark 6.12, we know that if Δ has large second bounded cohomology, then so does Γ. On the other hand, Lemma 6.13 and Example 6.6.1 immediately imply that Δ has large second bounded cohomology. □

6.3. **Proof of Theorem 6.1.** We now have all the tools for giving a short proof of Theorem 6.1:

Proof of Theorem 6.1. Let Γ be Meier's finitely generated group, which is four-generated and torsion-free. We proved in Proposition 6.14 that Γ has large second bounded cohomology and so it satisfies the hypothesis of Theorem 6.7.

Let Λ be the fundamental group of an oriented closed connected hyperbolic 3-manifold. We can choose Λ to be 2-generated by proceeding as follows. Start with the figure-eight knot complement M, whose fundamental group is 2-generated [17, Example VI.4.3]. Then perform a suitable Dehn Filling on M to obtain an oriented closed connected hyperbolic 3-manifold N [70, Chapter 4]. The fundamental group Λ of N is obtained from the one of M by adding relations. In particular it is still 2-generated.

The thesis now follows by taking $\Gamma \times \Lambda$, which is a torsion-free 6-generated group, and applying Theorem 6.7. □

Scholium 6.15 (On finitely presented examples)**.** The criterion in Theorem 6.7 does not seem to help to construct finitely presented groups with large bounded cohomology. Indeed, as remarked by Hirshon in 1994: "The question whether or not there exists a finitely presented group which is isomorphic to a proper direct factor of itself is an unsolved problem which has been around for a long time" [39].

The most promising constructions in this direction are given by finitely presented groups that allow epimorphisms onto their direct square. These were first constructed by Baumslag and Miller [4]. Further examples were given by Hirshon and Meier [40], see also [39] for a list of striking properties of these groups.

The group H constructed by Baumslag and Miller [4] surjects onto the group Δ appearing in Definition 6.10. Since Δ has large second bounded cohomology (Lemma 6.13 and Example 6.6.1), we then have $\dim_{\mathbb{R}} \mathrm{H}^2_b(H;\mathbb{R}) \geq |\mathbb{R}|$. However, the description of the full bounded cohomology of H is strongly related to a better understanding of Lex groups (Section 3.2). Indeed, since H surjects onto H^d, we can produce quotients of H with large bounded cohomology in every given even degree, but we cannot obtain any information about the bounded cohomology of H itself, if we do not know whether direct powers of H are Lex.

In Section 7, we give an ad-hoc example of a finitely presented group with large bounded cohomology.

Scholium 6.16 (On the bounded cohomology of the free group)**.** The connection between finitely generated groups with large bounded cohomology and Lex groups goes even further. Indeed, if Γ is an n-generated group

with large bounded cohomology, and Γ is moreover Lex, then the free group F_n has large bounded cohomology. In particular, if the group from Theorem 6.1 is Lex, then F_6 has large bounded cohomology. As mentioned in Example 6.6.4, it is known that non-abelian free groups have large second and third bounded cohomology. However, it is not known whether the bounded cohomology of non-abelian free groups vanish or not in degrees 4 and above [11, Question 18.3].

Note that proving that F_6 has large bounded cohomology would have many strong consequences. Indeed, every acylindrically hyperbolic group admits a hyperbolically embedded subgroup of the form $F_n \times K$, where K is finite, for all $n \geq 1$ [18, Theorems 6.8 and 6.14]. It then follows that if some non-abelian free group had large bounded cohomology, then the same would be true of all acylindrically hyperbolic groups [24, Theorem 1.1] .

7. Finitely presented groups with large bounded cohomology

In Section 6, we gave a general recipe for producing finitely generated groups with large bounded cohomology. As remarked in Scholium 6.15, this does not allow to construct finitely presented examples, given the current limited list of examples of groups isomorphic to their own direct factors.

In this section, we produce finitely presented (in fact, type F_∞) groups with large bounded cohomology, by using an ad-hoc construction, and building on previous work.

The main player is *Thompson's group T*:

Definition 7.1. The Thompson group T is the group of orientation-preserving piecewise linear homeomorphisms f of the circle $\mathbb{R}/\mathbb{Z}$ with the following properties:

(1) f has finitely many breakpoints, all of which lie in $\mathbb{Z}[1/2]/\mathbb{Z}$;
(2) Away from the breakpoints, the slope of f is a power of 2;
(3) f preserves $\mathbb{Z}[1/2]/\mathbb{Z}$.

The group T and its siblings F and V were introduced by Richard Thompson in 1965; they are some of the most important groups in geometric and dynamical group theory. We refer the reader to the literature [16] for a detailed discussion.

The group T is finitely presented, and even of type F_∞. The integral cohomology of T has been computed [28], and this result has interesting consequences for its bounded cohomology, as was first noted by Burger and Monod:

Proposition 7.2 (Ghys–Sergiescu [28], Burger–Monod [13])**.** *For each even integer $n \geq 2$, we have $\mathrm{H}^n_b(T;\mathbb{R}) \neq 0$.*

This result is obtained by noticing that every cup power of the Euler class of T is bounded, and using the fact that these cup powers are non-zero in ordinary cohomology [28].

Theorem 7.3. *Let T be Thompson's group, and let Λ be the fundamental group of an oriented closed connected hyperbolic 3-manifold. Then $T \times \Lambda$ has large bounded cohomology.*

Moreover, $T \times \Lambda$ is finitely presented, in fact type F_∞.

Proof. By Proposition 7.2, we have $\mathrm{H}_b^n(T;\mathbb{R}) \neq 0$ for every even integer $n \geq 2$. Using that Λ has large second and third bounded cohomology, and $\overline{b}_3^{\ell^1}(\Lambda) > 0$, we conclude in the same way as in the proof of Theorem 6.7 (see also Remark 6.8) □

8. Non-computability

In the following, we prove Theorem 8 and Theorem 9, i.e., we show that many decision problems concerning bounded cohomology are *not* algorithmically decidable:

Theorem 8.1 (Non-computability of bounded cohomology)**.** *Let $d \in \mathbb{N}_{\geq 2}$ and let $D \in \mathbb{N}$. Then all of the following algorithmic problems are undecidable: Given a finite presentation $\langle S \mid R\rangle$, decide whether*

(1) $\mathrm{H}_b^d(\langle S \mid R\rangle\,;\mathbb{R}) \cong 0$ *or not;*
(2) $\dim_{\mathbb{R}} \mathrm{H}_b^d(\langle S \mid R\rangle\,;\mathbb{R}) \leq D$ *or not;*
(3) $\mathrm{H}_b^d(\langle S \mid R\rangle\,;\mathbb{R})$ *is large (Definition 6.4) or not;*
(4) $\mathrm{H}_d^{\ell^1}(\langle S \mid R\rangle\,;\mathbb{R}) \cong 0$ *or not;*
(5) $\overline{b}_d^{\ell^1}(\langle S \mid R\rangle) > 0$ *or not;*
(6) $\langle S \mid R\rangle$ *is boundedly acyclic or not;*
(7) $\mathrm{cd}_b \langle S \mid R\rangle \leq D$ *or not.*
(8) $\mathrm{hd}_b \langle S \mid R\rangle \leq D$ *or not.*

Basic terminology and properties of bounded (co)homological dimension are recorded in Appendix A.

Corollary 8.2 (Non-computability of bounded cohomology for spaces)**.** *Let $d \in \mathbb{N}_{\geq 2}$, and let $D \in \mathbb{N}$. Then all of the following algorithmic problems are undecidable: Given a finite simplicial complex X, decide whether*

(1) $\mathrm{H}_b^d(X;\mathbb{R}) \cong 0$ *or not;*
(2) $\dim_{\mathbb{R}} \mathrm{H}_b^d(X;\mathbb{R}) \leq D$ *or not;*
(3) $\mathrm{H}_b^d(X;\mathbb{R})$ *is large or not;*
(4) $\mathrm{H}_d^{\ell^1}(X;\mathbb{R}) \cong 0$ *or not;*
(5) X *is boundedly acyclic or not;*

Here, finite simplicial complexes X are given as the finite set V of their vertices together with the finite subsets of V that constitute simplices of X. For the definition of bounded cohomology and ℓ^1-homology of spaces we refer to the literature [33][22, Chapters 5 and 6].

8.1. **Proof of Theorem 8.1, parts (7), (8).** We use the Adian–Rabin Theorem. For the sake of completeness, we recall the basics; we refer to Rotman's book [68, Chapter 12] for further details and history.

Definition 8.3 (Markov property)**.** A subclass P of the class of all finitely presentable groups is a *Markov property* if the following properties all hold:

- The class P is closed under taking isomorphisms.
- *Positive witness.* The class P is non-empty.
- *Negative witness.* There exists a finitely presentable group that is *not* isomorphic to a subgroup of an element of P.

Theorem 8.4 (Adian–Rabin [68, Theorem 12.32])**.** *Let P be a Markov class of finitely presentable groups. Then the following algorithmic problem is undecidable:*

> *Given a finite presentation $\langle S \mid R\rangle$, decide whether $\langle S \mid R\rangle$ lies in P or not.*

Lemma 8.5. *Let $D \in \mathbb{N}$.*

1. *The class of all finitely presentable groups Γ with $\mathrm{cd}_b \Gamma \leq D$ is a Markov property.*
2. *The class of all finitely presentable groups Γ with $\mathrm{hd}_b \Gamma \leq D$ is a Markov property.*

Proof. Clearly, these classes of groups are closed under isomorphisms.

Positive witness. The trivial group is a positive witness.

Negative witness. In view of the hyperbolic examples in Example A.3, there exists a finitely presentable group Λ with $\mathrm{hd}_b \Lambda > D$ and $\mathrm{cd}_b \Lambda > D$. By the monotonicity of bounded (co)homological dimension (Proposition A.5), it follows that Λ is *not* isomorphic to subgroups of groups Γ with $\mathrm{hd}_b \Gamma \leq D$ or $\mathrm{cd}_b \Gamma \leq D$. Hence, these groups provide the desired negative witnesses. □

Proof of Theorem 8.1.(7), (8). The claims follow by applying the Adian–Rabin Theorem (Theorem 8.4) to the Markov class of finitely presentable groups Γ with $\mathrm{cd}_b \Gamma \leq D$ or $\mathrm{hd}_b \Gamma \leq D$, respectively (Lemma 8.5). □

Remark 8.6 ((non-)Markov properties)**.**

- Amenability of finitely presentable groups is a Markov property: The trivial group is a positive witness; the free group of rank 2 is a negative witness.
- However, bounded acyclicity of finitely presentable groups is *not* a Markov property: In view of Corollary 5.2, there do *not* exist negative witnesses.
- Not being boundedly acyclic is *not* a Markov property of finitely presentable groups. Assume by contradiction that there exists a negative witness Γ. Then Γ is isomorphic to a subgroup of $\Gamma \times F_2$ and $\Gamma \times F_2$ is not boundedly acyclic by Proposition 2.6.
- Similarly, (not) having large bounded cohomology is not a Markov property. One can obtain this via the same reasoning as before, using the group from Theorem 7 instead of the free group.

In view of Remark 8.6, the Adian–Rabin Theorem (Theorem 8.4) cannot be applied directly to establish the remaining parts of Theorem 8.1.

8.2. **Proof of Theorem 8.1, parts (1)–(6).** We use the standard technique of turning the word problem into group presentations.

Construction 8.7. By the Novikov–Boone–Britton Theorem [68, Theorem 12.8], there exists a finitely presented group Λ with unsolvable word problem; let $\langle S \mid R\rangle$ be a finite presentation of Λ with symmetric generating set S. By the proof of the Adian–Rabin Theorem [68, Lemma 12.31], there

exists an algorithm

$$\begin{aligned} \text{words over } S &\to \text{finite presentations} \\ w &\mapsto \langle S_w \mid R_w \rangle \end{aligned}$$

with the following property: For all words w over S, we have

$$w \text{ represents the neutral element of } \Lambda \iff \langle S_w \mid R_w \rangle \cong 1.$$

For a word w over S, we write $\Lambda_w := \langle S_w \mid R_w \rangle$.

In addition, this construction can be refined as follows: We can take Λ to be torsion-free [56, Theorem 12 on p. 88] and we can assume that there is an algorithm

$$\begin{aligned} \text{words over } S &\to \text{words over } S_w \\ w &\mapsto \overline{w} \end{aligned}$$

with the following property [68, (proof of) Lemma 12.31]: If w does not represent the neutral element in Λ, then $\overline{w}$ has infinite order in Λ_w.

Via the groups Λ_w, we can reduce the algorithmic problems in Theorem 8.1.(1)—(6) to the word problem in Λ. As a preliminary stage, we take the free product with $\mathbb{Z}$ (this will be sufficient for the degrees 2 and 3, as well as for the claim on bounded acyclicity):

Construction 8.8. In the situation of Construction 8.7, for words w over S, we consider

$$\Gamma_w := \langle S_w, t \mid R_w \rangle \cong \Lambda_w * \mathbb{Z},$$

where $t \notin S$ is a new generator. By construction, we have:

- If w represents the neutral element of Λ, then
$$\Gamma_w \cong \Lambda_w * \mathbb{Z} \cong 1 * \mathbb{Z} \cong \mathbb{Z}.$$
In particular, Γ_w is amenable.
- If w does *not* represent the neutral element of Λ, then $\Lambda_w \ncong 1$ and so Γ_w is a non-elementary free product, i.e., a free product $A * B$, where A and B are non-trivial and we do not have $A \cong \mathbb{Z}/2 \cong B$. In particular, Γ_w is an acylindrically hyperbolic group [57, Corollary 2.2].

Therefore, in bounded cohomology, we obtain:

- If w represents the neutral element of Λ, then Γ_w is amenable and so it is boundedly acyclic; in particular, Γ_w also has trivial reduced and unreduced ℓ^1-homology [54, Corollary 2.4].
- If w does *not* represent the neutral element of Λ, then $\mathrm{H}^2_b(\Gamma_w; \mathbb{R})$ and $\mathrm{H}^3_b(\Gamma_w; \mathbb{R})$ are large (Example 6.6). Moreover, $\overline{b}^{\ell^1}_2(\Gamma) = |\mathbb{R}|$ (Proposition 2.6).

For the higher degree case of Theorem 8.1, we use the witness construction of Weinberger [71, Chapter 2.6].

Construction 8.9. In the situation of Construction 8.7, we consider the following witness groups: Let $\Gamma = \langle S' \mid R' \rangle$ be a torsion-free finitely presented

group, where $S' = \{s'_1, \dots, s'_k\}$ and where all elements of S' are non-trivial in Γ. For words w over S, we define

$$W(\Gamma, \Lambda, w) \coloneqq \Gamma *_{\mathbb{Z}} \Lambda_w *_{\mathbb{Z}} \Lambda_w *_{\mathbb{Z}} \cdots *_{\mathbb{Z}} \Lambda_w,$$

where the k-fold push-out group is given by the glueings of Γ and Λ_w over the maps $s'_j \hookleftarrow 1 \mapsto \overline{w}$ for $j \in \{1, \dots, k\}$. In other words:

$$W(\Gamma, \Lambda, w) = \big\langle \Gamma; (\Lambda_w)_j : j = 1, \dots, k \;\big|\; s'_j = \overline{w} \in (\Lambda_w)_j : j = 1, \dots, k \big\rangle.$$

The whole construction is algorithmic in the sense that we can also algorithmically give finite presentations of $W(\Gamma, \Lambda, w)$. By construction, we have:

- If w represents the neutral element of Λ, then $W(\Gamma, \Lambda, w) \cong 1$, because all generators of Γ are killed and $\Lambda_w \cong 1$.
- If w does *not* represent the neutral element of Λ, then the construction of $W(\Gamma, \Lambda, w)$ is a proper iterated k-fold amalgamated free product, because s'_j and $\overline{w}$ have infinite order in Γ and Λ_w, respectively. Moreover, the amalgamation is performed over the amenable group $\mathbb{Z}$. In particular, for $d \in \mathbb{N}_{\geq 2}$, we have (Lemma 8.10): If $\mathrm{H}^d_b(\Gamma; \mathbb{R})$ is large, then also $\mathrm{H}^d_b(W(\Gamma, \Lambda, w); \mathbb{R})$ is large. Finally, $\overline{b}^{\ell^1}_d(W(\Gamma, \Lambda, w)) \geq \overline{b}^{\ell^1}_d(\Gamma)$.

Lemma 8.10. *Let Γ and Λ be countable groups, let $\Gamma *_A \Lambda$ be an amalgamated free product over a common amenable subgroup A, and let $d \in \mathbb{N}_{>0}$.*

(1) If $\mathrm{H}^d_b(\Gamma; \mathbb{R})$ is large, then so is $\mathrm{H}^d_b(\Gamma *_A \Lambda; \mathbb{R})$.

(2) We have $\overline{b}^{\ell^1}_d(\Gamma *_A \Lambda) \geq \overline{b}^{\ell^1}_d(\Gamma)$.

Proof. Let $i \colon \Gamma \hookrightarrow \Gamma *_A \Lambda$ denote the canonical inclusion. Then, by [10, Theorem 1], there exists an isometric embedding $\Theta \colon \mathrm{H}^d_b(\Gamma; \mathbb{R}) \to \mathrm{H}^d_b(\Gamma *_A \Lambda; \mathbb{R})$ with

$$\mathrm{H}^d_b(i) \circ \Theta = \mathrm{id}_{\mathrm{H}^d_b(\Gamma; \mathbb{R})}.$$

This immediately gives the first part.

For the second part, it suffices to show that the map $\overline{\mathrm{H}}^{\ell^1}_d(i) \colon \overline{\mathrm{H}}^{\ell^1}_d(\Gamma; \mathbb{R}) \to \overline{\mathrm{H}}^{\ell^1}_d(\Gamma *_A \Lambda; R)$ is injective: Let $\alpha \in \overline{\mathrm{H}}^{\ell^1}_d(\Gamma; \mathbb{R})$ with $\alpha \neq 0$. Then, by duality, there exists a $\varphi \in \mathrm{H}^d_b(\Gamma; \mathbb{R})$ with $\langle \varphi, \alpha \rangle \neq 0$ [54]. Therefore, we obtain

$$\big\langle \Theta(\varphi), \overline{\mathrm{H}}^{\ell^1}_d(i)(\alpha) \big\rangle = \big\langle \mathrm{H}^d_b(i) \circ \Theta(\varphi), \alpha \big\rangle = \langle \varphi, \alpha \rangle \neq 0.$$

In particular, $\overline{\mathrm{H}}^{\ell^1}_d(i)(\alpha) \neq 0$ in $\overline{\mathrm{H}}^{\ell^1}_d(\Gamma *_A \Lambda; \mathbb{R})$. □

Proof of Theorem 8.1.(1)—(6). We consider $w \mapsto \Gamma_w$ as in Construction 8.8.

In degrees 2 and 3 and for the case of bounded acyclicity, we have: If any of the problems (1)–(6) were decidable, then the computations in Construction 8.8 show that also the word problem for Λ would be decidable. This contradicts the choice of Λ.

For the higher degrees, we proceed as follows: Let $d \geq 4$. Let Γ be the fundamental group of an oriented closed connected hyperbolic $(d-2)$-manifold, which is finitely presented and torsion-free. We then consider the

construction

$$\begin{aligned} \text{words over } S &\to \text{finite presentations} \\ w &\mapsto \Pi_w \coloneqq W(F_2, \Lambda, w) \times W(\Gamma, \Lambda, w) \end{aligned}$$

from Construction 8.9 (with the product presentation). Then, we obtain from the computations in Construction 8.9:

- If w represents the neutral element of Λ, then $\Pi_w \cong 1$ is boundedly acyclic and has trivial reduced and unreduced ℓ^1-homology.
- If w does *not* represent the neutral element of Λ, then Remark 2.7, Example 2.8, Example 6.6.4 and Lemma 8.10 show that

$$\begin{aligned} \overline{b}_d^{\ell^1}(\Pi_w) &\geq \overline{b}_2^{\ell^1}\big(W(F_2, \Lambda, w)\big) \cdot \overline{b}_{d-2}^{\ell^1}\big(W(\Gamma, \Lambda, w)\big) \\ &\geq \overline{b}_2^{\ell^1}(F_2) \cdot \overline{b}_{d-2}^{\ell^1}(\Gamma) \geq |\mathbb{R}| \cdot 1. \end{aligned}$$

  In particular, $\mathrm{H}_b^d(\Pi_w; \mathbb{R})$ is large (Proposition 2.6).

Again, we see: If any of the problems (1)–(5) were decidable in degree d, then the word problem for Λ would be decidable, which contradicts the choice of Λ. □

Remark 8.11. The same argument as in the proof of the parts (1)–(6) of Theorem 8.1 also gives simple proofs of the following:

(1) Let $d \in \mathbb{N}$. Then the d-th L^2-Betti number $b_d^{(2)}$ of finitely presentable groups is *not* computable.
    It should be noted that more sophisticated non-computability results for L^2-Betti numbers are already known [30].
(2) The cost of finitely presentable groups is *not* computable.

We refer the reader to [53] for the definition of L^2-Betti numbers, and to [26] for the definition of cost. For the proofs, we use the same notation as above.

Ad 1. If $d = 0$, then computing $b_d^{(2)}$ amounts to computing the cardinality of the group in question [53, Theorem 6.54]; however, it is known that the cardinality of groups is *not* computable from finite presentations [68, Corollary 12.33].

We now let $d \geq 1$. For words w over S, we consider

$$\Delta_w \coloneqq \Gamma_w \times (F_2)^{\times(d-1)}$$

(which admits a finite presentation that can be algorithmically constructed from $\langle S_w \mid R_w \rangle$). The standard inheritance properties of L^2-Betti numbers [53, Theorem 6.54] show:

- If w represents the neutral element of Λ, then Δ_w is isomorphic to $\mathbb{Z} \times (F_2)^{\times(d-1)}$ and thus $b_d^{(2)}(\Delta_w) = 0$.
- If w does *not* represent the neutral element of Λ, then

$$\begin{aligned} b_1^{(2)}(\Gamma_w) = b_1^{(2)}(\Lambda_w * \mathbb{Z}) &= b_1^{(2)}(\Lambda_w) + 1 - b_0^{(2)}(\Lambda_w) \\ &> b_1^{(2)}(\Lambda_w) + 1 - 1 \geq 0 \end{aligned}$$

and so

$$\begin{aligned} b_d^{(2)}(\Delta_w) &= b_d^{(2)}(\Gamma_w \times (F_2)^{\times(d-1)}) \\ &\geq b_1^{(2)}(\Gamma_w) \cdot b_{d-1}^{(2)}\big((F_2)^{\times(d-1)}\big) = b_1^{(2)}(\Gamma_w) \cdot 1 \\ &> 0. \end{aligned}$$

Therefore computability of L^2-Betti numbers would imply solvability of the word problem in Λ, which contradicts the choice of Λ.

Ad 2. For words w over S, we again consider the group Γ_w as above.

- If w represents the neutral element of Λ, then $\Gamma_w \cong \mathbb{Z}$; in particular, $\operatorname{cost}\Gamma_w = 1$ [26, Corollaire III.4].
- If w does *not* represent the neutral element of Λ, then $\Lambda_w \ncong 1$. Because finitely presented groups are countable and have finite cost, we obtain [26, Théorème VI.7″]

$$\operatorname{cost}(\Gamma_w) = \operatorname{cost}(\Lambda_w * \mathbb{Z}) \geq \operatorname{cost}(\Lambda_w) + \operatorname{cost}(\mathbb{Z}) > 0 + 1.$$

Therefore computability of cost would imply solvability of the word problem in Λ, which contradicts the choice of Λ.

8.3. **Proof of Corollary 8.2.** We deduce Corollary 8.2 from Theorem 8.1 via Gromov's Mapping Theorem [33, 43, 23]:

Proof of Corollary 8.2. We proceed by contradiction: In view of Theorem 8.1 it suffices to show that if any of the decision problems formulated in Corollary 8.2 were decidable, then the corresponding decision problem for groups in Theorem 8.1 would be decidable.

There is an algorithm

$$P\colon \text{finite presentations} \to \text{finite simplicial complexes}$$

with the following property: For all finite presentations $\langle S \mid R\rangle$, the simplicial complex $P(\langle S \mid R\rangle)$ is connected and

$$\pi_1\big(P(\langle S \mid R\rangle)\big) \cong \langle S \mid R\rangle\,.$$

For example, one such construction is to take the double barycentric subdivision of the presentation cellular complex of the presentation $\langle S \mid R\rangle$.

By Gromov's Mapping Theorem [33, 43, 23], we know that

$$\forall_{d\in\mathbb{N}} \quad \mathrm{H}_b^d\big(P(\langle S \mid R\rangle);\mathbb{R}\big) \cong \mathrm{H}_b^d\big(\langle S \mid R\rangle\,;\mathbb{R}\big);$$

moreover, we have the analogous statement for ℓ^1-homology [51, Corollary 5.2]:

$$\forall_{d\in\mathbb{N}} \quad \mathrm{H}_d^{\ell^1}\big(P(\langle S \mid R\rangle);\mathbb{R}\big) \cong \mathrm{H}_d^{\ell^1}\big(\langle S \mid R\rangle\,;\mathbb{R}\big).$$

Therefore, any algorithm for the problems formulated in Corollary 8.2 would lead to a corresponding algorithm for the bounded cohomology of finitely presented groups. This contradicts Theorem 8.1 and thus completes the proof of Corollary 8.2. □

Appendix A. Bounded (co)homological dimension

We introduce dimension notions of groups in the context of bounded cohomology and ℓ^1-homology following previous works by Johnson [45, 46] and Monod [59, Definition 3.1]. In contrast with the notion [52]

$$\operatorname{bcd}\Gamma := \sup\bigl\{n \in \mathbb{N} \bigm| \mathrm{H}_b^n(\Gamma;\mathbb{R}) \not\cong 0\bigr\} \in \mathbb{N} \cup \{\infty\}$$

these dimensions mimic the usual (co)homological dimension and thus take twisted coefficients into account.

Definition A.1 (bounded (co)homological dimension)**.** Let Γ be a group.

- The *bounded cohomological dimension* of Γ is defined as
$$\operatorname{cd}_b\Gamma := \sup\bigl\{n \in \mathbb{N} \bigm| \exists_{V\in\mathrm{Ban}_\Gamma}\ \mathrm{H}_b^n(\Gamma;V) \not\cong 0\bigr\} \in \mathbb{N} \cup \{\infty\}.$$
- The *bounded homological dimension* of Γ is defined as
$$\operatorname{hd}_b\Gamma := \sup\bigl\{n \in \mathbb{N} \bigm| \exists_{V\in\mathrm{Ban}_\Gamma}\ \mathrm{H}_n^{\ell^1}(\Gamma;V) \not\cong 0\bigr\} \in \mathbb{N} \cup \{\infty\}.$$

Remark A.2. Let Γ be a group. By duality [54, Corollary 2.4.2][51, Corollary 5.3], we have

$$\operatorname{hd}_b\Gamma \leq \operatorname{cd}_b\Gamma.$$

More precisely: Let $n \in \mathbb{N}$ and let V be a Banach Γ-module with $\mathrm{H}_b^k(\Gamma;V') \cong 0$ for all $k \in \mathbb{N}_{>n}$. Then $\mathrm{H}_k^{\ell^1}(\Gamma;V) \cong 0$ for all $k \in \mathbb{N}_{>n}$ [51, Corollary 5.3][50, Remark 3.7].

Example A.3.

- Let Γ be a group. Then $\operatorname{cd}_b\Gamma = 0$ if and only if Γ is finite [22, Theorem 3.12].
- Let Γ be a group. Then $\operatorname{hd}_b\Gamma = 0$ if and only if Γ is amenable [51, Corollary 5.5].
- In particular, $\operatorname{hd}_b\mathbb{Z} = 0 < \operatorname{cd}_b\mathbb{Z}$.
- If Γ is a non-amenable group, then $\operatorname{cd}_b\Gamma \geq 3$. This can be derived from the fact that Γ admits F_2 as a random subgroup [59, Theorem 5.4, Proposition 5.8].
- Let M be an oriented closed connected hyperbolic n-manifold and let $\Gamma := \pi_1(M)$. Then Γ is finitely presentable and $\bar{b}_n^{\ell^1}(\Gamma) \geq 1$ as well as $\mathrm{H}_b^n(\Gamma;\mathbb{R}) \not\cong 0$ (Example 2.8, Proposition 2.6). Thus,
$$\operatorname{hd}_b\Gamma \geq n \quad \text{and} \quad \operatorname{cd}_b\Gamma \geq n.$$
Such examples exist in all dimensions at least 2.
- If Γ is a group with $\operatorname{bcd}\Gamma = \infty$, then $\operatorname{cd}_b\Gamma = \infty$. In particular, this applies to the groups from Section 6.

However, as of now no example seems to be known of a group of finite non-trivial bounded (co)homological dimension.

Proposition A.4 (cohomological dimension as projective dimension)**.** *Let Γ be a group. Then* $\operatorname{cd}_b\Gamma$ *coincides with the* relatively projective dimension *of* Γ

$$\operatorname{projdim}_b\Gamma := \inf\bigl\{n \in \mathbb{N} \bigm| \mathbb{R} \textit{ admits a strong relatively projective } \Gamma\textit{-resolution of length} \leq n\bigr\} \in \mathbb{N} \cup \{\infty\}.$$

Proof. We argue as in the case of classical cohomological dimension:

We have $\mathrm{cd}_b \Gamma \leq \mathrm{projdim}_b \Gamma$, because for all Banach Γ-modules V and all strong relatively projective Γ-resolutions $C_* \to \mathbb{R}$ of $\mathbb{R}$, we have [51, Theorem 3.7]

$$\mathrm{H}_b^*(\Gamma; V) \cong \mathrm{H}^*\big(B(C_*, V)^\Gamma\big).$$

Conversely, we have $\mathrm{projdim}_b \Gamma \leq \mathrm{cd}_b \Gamma$: If $C_* \to \mathbb{R}$ is a strong relatively projective Γ-resolution of $\mathbb{R}$ and if $n \in \mathbb{N}$ satisfies for all Banach Γ-modules V that

$$\mathrm{H}_b^{n+1}(\Gamma; V) \cong 0,$$

then the same argument as in the classical case shows that $\ker \partial_n$ is a relatively projective Γ-module. Therefore, C_* can be truncated at degree n. This shows that $\mathrm{projdim}_b \Gamma \leq n$. □

Proposition A.5 (monotonicity of bounded (co)homological dimension)**.** *Let Γ be a group and let $\Lambda \leq \Gamma$ be a subgroup. Then*

$$\mathrm{hd}_b \Lambda \leq \mathrm{hd}_b \Gamma \quad \textit{and} \quad \mathrm{cd}_b \Lambda \leq \mathrm{cd}_b \Gamma.$$

Proof. This is the usual Shapiro argument: Let V be a Banach Λ-module and let $n \in \mathbb{N}$. In bounded cohomology, we have [58, Proposition 10.1.3]

$$\mathrm{H}_b^n(\Lambda; V) \cong \mathrm{H}_b^n\big(\Gamma; B(\ell^1\Gamma, V)^\Lambda\big).$$

In ℓ^1-homology: There is a natural isomorphism

$$\mathrm{C}_*^{\ell^1}(\Gamma) \mathbin{\overline{\otimes}}_\Gamma (\ell^1\Gamma \mathbin{\overline{\otimes}}_\Lambda V) \cong \mathrm{res}_\Lambda^\Gamma \mathrm{C}_*^{\ell^1}(\Gamma) \mathbin{\overline{\otimes}}_\Lambda V$$

of Banach chain complexes; moreover, $\mathrm{res}_\Lambda^\Gamma \mathrm{C}_*^{\ell^1}(\Gamma)$ is a strong relatively projective Λ-resolution of $\mathbb{R}$. Using the description of ℓ^1-homology via projective resolutions [51, Theorem 3.7], we thus obtain a natural isomorphism

$$\mathrm{H}_n^{\ell^1}(\Lambda; V) \cong \mathrm{H}_n^{\ell^1}(\Gamma; \ell^1\Gamma \mathbin{\overline{\otimes}}_\Lambda V).$$ □

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