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The empirics of economic growth over time and across nations: a unified growth perspective

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Abstract
This research develops an expanded unified growth theory that incorporates the endogenous accumulation of physical capital, population, human capital, and technology. The model incorporates a complementarity between physical capital and human capital and can be extended to a multi-country setting with international technology diffusion. The analytical characterization of the mechanisms behind the observed patterns of long-run growth and comparative development delivers a consistent explanation for a large set of seemingly unrelated empirical facts. A quantitative multi-country version of the model matches various empirical regularities of long-run growth dynamics and comparative development patterns that have previously been studied in isolation. The findings also shed new light on the role of the demographic transition for convergence patterns, the specification of cross-country growth regressions, technology spillovers, and the secular stagnation debate.

Keywords Unified growth · Long-run development · Demographic transition · Secular stagnation

JEL Classification $O47 \cdot O11 \cdot O15 \cdot E24$

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1 Introduction

Over the past decades, the literature on economic growth and economic history has documented a considerable body of empirical patterns of long-run growth that relate to the dynamics of income, demographic variables, production factors, and factor returns. The most evident of these regularities is the secular acceleration of economic well-being over human history. While traditional models of balanced economic growth had limited success in accounting for this acceleration, the development of unified growth theory has shifted the focus to the mechanics behind the transition from Malthusian stagnation to sustained economic growth and the deep roots of comparative development (see Galor, 2011, 2022). While the last two decades have seen remarkable progress in the understanding of the theoretical mechanisms behind this transition, the applicability of quantitative unified growth models to empirical patterns of long-run growth dynamics and comparative development constitutes a relatively underexplored area. This is related, on the one hand, to the discrepancy in the emphasis given to physical capital accumulation in empirical studies of growth, which is not shared by most unified growth models with their focus on education and fertility. On the other hand, studying comparative development requires an explicit consideration of cross-country heterogeneity and interdependencies, in particular, of the diffusion of technology across countries.

This paper develops an expanded unified growth framework that makes progress in several dimensions. We present an analytically tractable unified growth model that extends the focus beyond endogenous fertility and education by accounting explicitly for savings and physical capital accumulation. In particular, this allows incorporating complementarities between physical capital and human capital, an aspect that has received relatively little attention in existing work on long-run growth. We then show that a quantitative version of the model can go a long way in matching the empirical patterns of long-run growth, including the behavior of the state variables physical capital, human capital, population, and technology based on an up-to-date account of the facts of long-run growth observed in historical time series data. An extension of the model to the multi-country context accounts for interdependencies in the diffusion of technology. The comparison of the simulated data of this extension to actual cross-country panel data shows that the quantitative unified growth model can match a variety of non-targeted data moments. In addition, the findings provide new insights for standard cross-country growth empirics and the patterns of technology diffusion.

Section 2 presents a tractable unified growth model that combines the standard components of neo-classical growth models with the mechanisms and insights highlighted by unified growth theory. In particular, we consider a two-sector framework that features complementarities between human capital and physical capital and that delivers endogenous dynamics of population, physical capital, and human capital. The model delivers analytical predictions about the demographic transition as the critical turning point in the dynamics of all state variables and about non-linear dynamics in all core dimensions that are consistent with the empirical facts. In particular, technological change induces the accumulation of physical capital, with the consequence of moderate growth in income per capita even prior to the demographic transition, when the economy is still characterized by high fertility, low education, and Malthusian dynamics. The accumulation of physical capital in combination with a capital-skill complementarity eventually induces an expansion of human capital that is associated with a reduction in fertility. These transition dynamics have implications for the dynamics of skill premia, factor income shares, and factor returns.

Section 3 investigates the consistency of a quantitative version of the model with the empirical facts of long-run growth, closing a gap between theoretical models and quantitative empirical facts in the literature. In this section, we focus on the development trajectory of a single economy and calibrate the model to data moments for England, paralleling the usual approach in unified growth and historical studies. The results document the ability of the model to replicate the facts of long-run growth, including the secular acceleration of income growth, which coincides with the demographic transition that marks the beginning of the education expansion. The model also replicates several empirical patterns that have been documented only recently, including the temporary overshooting of growth rates above the rate of sustained modern growth, the secular stagnation during the later phase of the transition, the corresponding non-monotonic dynamics of skill premia and factor income shares, and the relatively stable dynamics of the returns to physical capital. In particular, the non-monotonic dynamics of capital intensity per worker and of the skill premium, both of which tend to decline during the early phases of development but increase during the later stages of development, can be rationalized by the interactions between human capital and physical capital, without requiring specific assumptions about skill-biased technological change or inter-temporal spillovers. Moreover, the joint rapid expansion of human capital and physical capital after the onset of the demographic transition leads to a temporary overshooting of income growth above the growth rate along the balanced growth path. The analysis also shows that the simulated model closely matches actual time series in various dimensions that have not been targeted.

Section 4 extends the model to a multi-country setting and explicitly incorporates the process of international technology diffusion. The only source of heterogeneity across countries is variation in a single initial condition that governs the timing of the demographic transition, otherwise the calibrated model is identical to the baseline calibration for England. The endogenous improvement of the technological frontier displays a sharp acceleration when the forerunner country undergoes the economic and demographic transition. Follower countries benefit from this acceleration, but only after the onset of their own demographic transition that corresponds to the expansion in human capital and thereby enables the adoption of technologies from the frontier. As a result, productivity initially displays a rapid divergence across countries as the frontier accelerates while the spillovers to follower countries are still limited. Eventually, follower countries achieve even faster income growth than forerunners once they undergo their demographic transition. The analysis documents that the simulated data deliver comparative development patterns that are consistent with those observed in the data, including the variation in income and productivity across countries, cross-country differences in capital intensity and capital returns, and in the variability of growth rates relative to the technology frontier.

Section 5 makes further progress by exploring the similarity between simulated and actual data patterns through the lens of standard growth empirics. In particular, we document that a synthetic panel data set produced by a simulation of a multi-country version of the quantitative model delivers reduced-form estimation results that resemble those obtained from reduced-form empirical growth regressions. Extending the model to human capital-dependent technology adoption from the endogenously evolving world frontier allows us to explore the implications of the mechanism for the facts on technology diffusion. Regression results deliver new evidence on the drivers of cross-country growth and show that the demographic transition marks a critical turning point for technology adoption both at the extensive and at the intensive margin.

Taken together, the analytical, quantitative, and empirical results provide a proof of concept for both the quantitative and the empirical validity of the generalized unified paradigm that is based on the close interplay between the economic and demographic transition. The expanded unified growth framework sheds new light on the mechanisms that give rise to the patterns of comparative development, provides a link to the results of empirical growth regressions, and documents that the proposed model can go a long way in providing a consistent, unified explanation for a large variety of seemingly unrelated facts of long-run growth.

Contribution to the literature This research contributes to several strands of the literature. In terms of theory, our model incorporates prototypical elements and trade-offs from the literature on neo-classical growth and unified growth within a single, coherent framework. The model incorporates endogenous fertility choice and a demographic transition (as in, e.g., Galor and Weil 2000, Strulik and Weisdorf 2008) in a two-sector setting where sectors differ in their use of unskilled and skilled labor (as in, e.g., Doepke, 2004 or Cervellati and Sunde, 2005). While most of the unified growth literature uses a quantity-quality trade-off to illustrate that fertility reductions are ultimately the result of an increasing demand for human capital, we consider a trade-off between individuals' own education and their fertility (as in, e.g., Cervellati and Sunde, 2015). Both approaches deliver similar predictions regarding the nexus of education and fertility, but our approach provides a more direct and parsimonious representation that facilitates the incorporation of savings. Malthusian dynamics emerge by relating production in the unskilled sector to a fixed factor (as in, e.g., Ashraf and Galor 2011). Production in the skilled sector involves physical capital (as in, e.g., Hansen and Prescott 2002, Bar and Leukhina 2010, or Strulik et al. 2013). Physical capital complements skilled human capital in aggregate production (as in, e.g., Galor and Moav, 2006, or Galor et al. 2009), while resource constraints imply a trade-off between investment in education, fertility, and savings at the household level. The model complements the existing theoretical literature by presenting a mechanism that links the demographic transition and physical capital dynamics. In particular, by considering a capital-skill complementarity, the endogenous physical capital dynamics generate an increasing demand for skilled workers, which allows reconciling the predictions of unified growth theory with a variety of facts of long-run growth without relying on specific forms of inter-generational externalities.

The consideration of a unified growth framework with endogenous savings and education investments illustrates that the end of the Malthusian period not only constitutes the onset of a demographic transition with an expansion of human capital and a reduction in fertility, but also marks a crucial turning point in the accumulation of physical capital. As a result, skill premia do not exhibit a persistent decline in spite of the expansion of human capital. This provides a foundation for non-decreasing returns to human capital that have been assumed in reduced form by, e.g., Becker et al. (1990), and parallels arguments by Kaldor (1957) or Zeira (2009). Likewise, capital returns do not exhibit a pronounced decline in spite of the expansion of physical capital. The results also contribute insights to the convergence debate. Conventionally, unified growth theories predict income growth to accelerate in the context of the demographic transition and ultimately converge to the balanced growth path "from below" (see, e.g., Galor and Weil, 2000, Cervellati and Sunde, 2005). In contrast, neo-classical models of economic growth and augmented neo-classical models that account for the historical development patterns predict convergence patterns with growth rates converging to the balanced growth path "from above" (see, e.g., Mankiw et al. 1992, Dalgaard and Strulik, 2013). Rather than convergence "from below" or "from above", the transition dynamics of our model presented below imply an acceleration of growth during the transition from slow, quasi-stagnant growth dynamics to sustained growth, with growth rates during this transition temporarily exceeding the long-run growth rate as consequence of factor accumulation. This sheds new light on the mechanisms behind empirically observed patterns of convergence (see Johnson and Papageorgiou, 2020, for a recent survey).

Our analysis also contributes to the literature of quantitative models of long-run economic and demographic dynamics that target historical data. Most of this literature focuses on fertility dynamics along the transition to modern growth and either abstracts from human capital or physical capital accumulation (see, e.g., Greenwood and Sheshadri, 2002, Kalemli-Ozcan, 2003, Doepke, 2004, Bar and Leukhina, 2010) or focuses on the long-run structural change at the level of families (Greenwood et al., 2021). Methodologically, our approach departs from this literature by providing a dynamic simulation that accounts for the different phases of development dynamics in all state variables. Technically, we proceed by calibrating the structural parameters and simulating the endogenous dynamics for the entire development path of a country given initial conditions. All state variables and factor prices evolve endogenously over time and their trajectories are compared to the evolution of the time series of the different variables, most of which are not targeted in the calibration.

The quantitative analysis goes beyond the time series perspective by considering the implications for comparative development patterns, following a similar approach as in Cervellati and Sunde (2015). This also complements recent work by Delventhal et al. (2021), who investigate the timing of the demographic transition in different countries and the role of access to technology while abstracting from endogenous take-offs and Malthusian dynamics. Compared to these works, we apply a very parsimonious methodology that focuses exclusively on the timing of the demographic transition by changing one single initial condition in otherwise identically parameterized economies. In the extended version of the model, countries are linked through an endogenous process of technology diffusion. The resulting cross-country patterns effectively represent non-targeted moments that can be used to assess the external validity of the underlying mechanism. After simulating the multi-country version of the model, we extend the scope of the quantitative analyses in the existing literature by comparing simulated and actual cross-country panel data through the lens of empirical growth regressions as a novel way to study the empirical relevance of the proposed generalized unified growth framework.

Our analysis thereby establishes a link between unified growth theory and the empirical growth literature (e.g., Mankiw et al. 1992). In particular, we estimate reduced-form panel regressions using synthetic panel data generated from the multi-country version of our unified growth model. The estimation results closely resemble those obtained with cross-country panel data, without relying (explicitly or implicitly) on the notion of balanced growth or convergence. The findings complement earlier empirical work on the role of the fertility transition for cross-country income growth (see also Galor, 2012), the role of the demographic transition as the starting point of a neo-classical convergence process (Dalgaard and Strulik, 2013), and evidence for different regimes of growth before and after the demographic transition (Cervellati et al., 2019). The findings also provide a rationale for the weak evidence on the role of human capital for development and growth differences, by showing that human capital becomes important mainly after the onset of the transition and by illustrating that human capital affects development through different channels (see, e.g., Sunde and Vischer, 2015). The results complement previous evidence for the existence of important non-linearities in the growth process depending on the stage of the demographic transition (see, e.g., Cervellati and Sunde, 2011a, b) by accounting for the distinct dynamics of income and population during different phases of demographic development.

By documenting a temporary overshooting of growth due to the expansion of human capital and physical capital during the early phases of the take-off, the analysis also sheds new light on the "new secular stagnation" debate and the factors that are responsible for the slow-down in growth observed in developed economies during the past decades. While this debate focused on arguments related to a delayed recovery from cyclical fluctuations (see, e.g., Teulings and Baldwin, 2014, and OECD, 2015), some scholars have emphasized the possibility of a decline in growth potential due to unfavorable dynamics in demographic composition, education, and globalisation (see, e.g., Gordon, 2012, 2014, 2016). Our results consider the dynamics of the entire transition phase and suggest that part of the observed growth stagnation in developed economies can be rationalized by the predicted growth slow-down during the later phases of the transition from stagnation to growth.

Finally, the analysis contributes to the recent empirical literature on the patterns of comparative development and technology diffusion. With only few notable exceptions such as Galor and Mountford (2008), most of the existing unified growth literature has focused on a single-country setting. Our modeling complements recent work that focused on technology spillovers from a world frontier that grows at exogenous and constant rates, see, e.g., Benhabib and Spiegel (2005) or Stokey (2015, 2021). Our model also relates to work by Lindner and Strulik (2020) that considers an endogenous world technology frontier that emerges from a global network of R&D activities and provides a rationale for growth rates of follower economies that temporarily exceed those on the balanced growth path. In contrast, our focus is on the unexplored quantitative and empirical implications of the demographic transition and the associated human capital expansion for the diffusion of technology and abstracts from an explicit modeling of R&D-based growth. Combining a simple framework of technology adoption along the lines of classic work by Nelson and Phelps (1966) with a unified growth mechanism allows us to illustrate the role of the demographic transition for the endogenous acceleration in technology dynamics in the frontier economy and for the timing and intensity of the adoption of new technologies in less developed economies. To explore the empirical relevance of this mechanism, we extend the empirical investigation of the patterns of adoption lags and intensity of use by Comin and Mestieri (2018) and explicitly account for different phases of demographic development. The results provide novel evidence that the demographic transition indeed plays a significant role for explaining the empirical patterns of technology diffusion in terms of adoption lags and intensity of use that has not been documented previously.

The remainder of the paper is structured as follows. Section 2 presents an expanded unified growth model that incorporates endogenous savings and a capital-skill complementarity and that can be extended to account for human capital externalities and to a multi-country setting. Section 3 analyzes the quantitative model dynamics and their consistency with the dynamics of long-run growth. Section 4 explores the consistency of the expanded unified growth framework with patterns of comparative development. Section 5 investigates the implications for cross-country growth empirics and technology diffusion. Section 6 concludes.

2 A generalized unified growth model

In this section, we develop a tractable unified growth model that accounts for the dynamic interactions between fertility, education, physical capital accumulation, and technology. While kept deliberately simple and stylized, the model incorporates the state variables that are central to the long-run dynamics of economic and demographic variables and can be used for quantitative analysis. In particular, the model combines an OLG framework that features optimal decisions about human capital, savings, and fertility with a two-sector production economy. The dynamic general equilibrium path exhibits Malthusian dynamics and an endogenous growth take-off. This take-off is associated with a demographic transition that entails an acceleration of investments in human capital and physical capital which is paralleled by a decline in fertility. After this take-off, the economy enters a regime of modern balanced growth.

2.1 Production

Total output of the economy in period t is produced in two sectors. The sectors employ unskilled and skilled labor inputs; sectors are denoted by U and S, respectively

$$Y_t = Y_t^U + Y_t^S \tag{1}$$

Production in the unskilled sector combines unskilled labor L_t and a fixed amount of land X, using a technology

$$Y_t^U = L_t^{\alpha}\left(A_t^U X\right)^{1-\alpha} \quad \text{with } \alpha \in (0,1) \tag{2}$$

where A_t^U denotes the level of (land-augmenting) productivity. Production in the skilled sector uses skilled labor, denoted by human capital H_t, and physical capital K_t, using a neoclassical technology

$$Y_t^S = H_t^{\beta}\left(A_t^S K_t\right)^{1-\beta} \quad \text{with } \beta \in (0,1) \tag{3}$$

with (capital-augmenting) productivity denoted by A_t^S.

To streamline the exposition, and for transparency of the mechanism and the main results, we make some simplifying assumptions. First, the homogeneous output good can be used for consumption and investment. Moreover, we assume that physical capital fully depreciates after one generation, which simplifies the law of motion of aggregate capital without altering the main insights.[1] Second, we assume that property rights are defined and enforced over capital and labor, while no property rights are defined over land (as in Galor and Weil, 2000). Third, we assume that factor returns are determined on competitive markets. This implies that returns to human capital and physical capital in the skilled sector are given by the respective marginal products, whereas returns to unskilled labor are set equal to the average product, such that all income produced in the unskilled sector is distributed to the unskilled workers. This allows us to abstract from having to make assumptions about land markets and bequests of land in an economy with changing cohort sizes. Factor rents are thus given by

[1] This assumption is standard in the literature, see, e.g., Strulik et al. (2013). The consideration of an extended model with incomplete depreciation is analytically more involved but delivers similar qualitative results; details are available upon request. Moreover, with a realistic depreciation rate of 10% or more per year and a generation length of 20 years, the assumption that physical capital fully depreciates after one generation is quantitatively reasonable.

$$w_t^L = \frac{Y_t^U}{L_t} = \left(\frac{A_t^U X}{L_t}\right)^{1-\alpha} \tag{4}$$

$$w_t^H = \frac{\partial Y_t^S}{\partial H_t} = \beta\left(\frac{A_t^S K_t}{H_t}\right)^{1-\beta} \tag{5}$$

$$1 + r_t = R_t = \frac{\partial Y_t^S}{\partial K_t} = (1-\beta)\left(\frac{H_t}{K_t}\right)^{\beta}\left(A_t^S\right)^{1-\beta} \tag{6}$$

2.2 Optimal education, fertility, and savings

Households are modeled as in a standard overlapping generations framework. In period t, a new generation of size L_t enters the economy. This generation consists of identical individuals that live for two periods and leave the economy after period $t+1$.

2.2.1 Budget constraints and preferences

In the first period of their lives, individuals are unskilled and work in the unskilled sector. In this period of life, individuals make decisions about education, fertility, and savings. Investments in education are denoted by $e_{t+1} \in [0,1]$ and increase the level of human capital individuals can supply to the skilled sector in the second period of their lives according to a human capital production function $h\left(e_{t+1}\right)$ with $h(0) > 0$, $h'\left(e_{t+1}\right) > 0$, $h''\left(e_{t+1}\right) \leq 0$, and $h'(0) < \infty$.

For simplicity, we restrict attention to consumption during the second period of life. Hence, in the first period of life, individuals decide about how to split their labor income, net of the investment in education e_{t+1}, between giving birth to children, n_t, and savings, s_t. The budget constraint for the first period of life is therefore given by

$$(1 - e_{t+1})w_t^L = n_t + s_t \tag{7}$$

Education imposes (opportunity) costs and the essential trade-off for individuals during their first period of life is that both fertility and education investments imply lower savings.[2]

In the second period of their lives, individuals work in the skilled sector.[3] Consumption is financed from the returns to human capital and the returns to the invested savings from the first period of life, such that the budget constraint in period $t+1$ reads

[2] The investment e_{t+1} could alternatively be interpreted as the share of time spent on education, which reduces effective labor supply to $1 - e_{t+1}$. Interpreting the cost of education as resource cost (in terms of resources that are foregone for consumption) is technically convenient as it simplifies the aggregation of labor supply and the analytical derivation of the general equilibrium of the economy, as discussed below.

[3] This is consistent with a mild assumption about old individuals having experience that enables them to work in the skilled sector even without spending time on education, and endogenous labor market sorting in equilibrium (as result of a skill premium that is strictly larger than one). This setting is also equivalent to adopting a vintage human capital perspective where different generations acquire human capital that is specific to operating certain vintages of technology, see, e.g., Cervellati and Sunde (2005).

$$(1 + r_{t+1})s_t + w^H_{t+1} h\left(e_{t+1}\right) = c_{t+1} \tag{8}$$

These individual budget constraints hold in every period of life. This implies individuals do not have access to means of transferring resources across periods of life other than education and non-negative savings. In particular, individuals cannot borrow against their future income to finance education or fertility. Combining the period budget constraints (7) and (8) gives the consolidated lifetime budget constraint

$$c_{t+1} = (1 + r_{t+1})\left[(1 - e_{t+1})w^L_t - n_t\right] + w^H_{t+1} h\left(e_{t+1}\right) \tag{9}$$

For simplicity, we assume quasi-linear preferences as in Strulik and Weisdorf (2008)

$$U = c_{t+1} + \gamma \ln n_t \quad \text{with } \gamma > 0 \tag{10}$$

where c_{t+1} denotes consumption and n_t is the number of children.

2.2.2 Optimization

Substituting (9) into (10) allows expressing the lifetime utility maximization as an optimization problem over n_t and e_{t+1}, subject to non-negativity constraints. The optimal individual choices are characterized by the following system of first order conditions

$$1 + r_{t+1} = \frac{\gamma}{n_t} \tag{11}$$

$$(1 + r_{t+1})w^L_t \geq w^H_{t+1} h'\left(e_{t+1}\right) \tag{12}$$

Notice that an interior solution for fertility is ensured by log-utility. The optimal education choice implies a corner with $e_{t+1} = 0$ if the return to education is lower than the return on savings, $(1 + r_{t+1})w^L_t > w^H_{t+1} h'\left(e_{t+1}\right)$. In the following we will refer to the corner case of $e_{t+1} = 0$ as pre-transitional and to the case of $e_{t+1} > 0$ as post-transitional.

2.3 Intra-generational equilibrium

2.3.1 Dynamics of macroeconomic variables

Notice that the aggregate stocks of unskilled labor, human capital, and physical capital depend on the fertility, education, and savings decisions of the respective generations. The state variables evolve according to the following laws of motion

$$L_{t+1} = n_t L_t \tag{13}$$

$$H_{t+1} = h\left(e_{t+1}\right) L_t \tag{14}$$

$$K_{t+1} = s_t L_t \tag{15}$$

Note that at each point in time the population consists of two generations that are alive, so that the total population size in period $t+1$ is given by

$$N_{t+1} = L_t + L_{t+1} = (1+n_t)L_t \tag{16}$$

2.3.2 Equilibrium

The general equilibrium requires mutual consistency between optimal individual choices in terms of education, fertility, savings, and the resulting aggregate allocation with the corresponding prices. In the pre-transitional environment with no investment in education (i.e. $e_{t+1} = 0$) optimal fertility is given by the first order condition in (11), that is

$$n_t = \frac{\gamma}{1+r_{t+1}} = \frac{\gamma}{1-\beta} \frac{s_t^{\beta}}{h(0)^{\beta}\left(A_{t+1}^S\right)^{1-\beta}} \equiv n\left(s_t, A_{t+1}^S\right) \tag{17}$$

where the last equality follows from (6), (14), and (15). Note that fertility is an increasing, strictly concave function of savings and a decreasing, strictly convex function of the level of skilled productivity. As there are no investments in education and hence (12) does not bind, the level of savings is implicitly given by combining the budget constraint (7) with (17)

$$w_t^L = n\left(s_t, A_{t+1}^S\right) + s_t \tag{18}$$

Lemma 1 *For given, positive levels of w_t^L and A_{t+1}^S, there exists a unique level of savings $s_t = s\left(w_t^L, A_{t+1}^S\right)$ with savings being an increasing function of both unskilled wages and skilled productivity.*

Proof Note that the right-hand side of (18) is strictly monotonically increasing and concave in s_t, takes value 0 if $s_t \to 0$ and converges to infinity for $s_t \to \infty$. Hence, the first result follows from the Intermediate Value Theorem. The second result follows from the Implicit Function Theorem as an increase in w_t^L leads to a monotonic increase in the left-hand side of (18), whereas an increase in A_{t+1}^S leads to a monotonic decrease in the right hand side. Maintaining equality thus requires s_t to increase in both cases. □

In the post-transitional environment, with positive investment in education, (12) binds with equality. Substituting for factor rents and solving (12) for s_t gives

$$s_t = \frac{1-\beta}{\beta} \frac{h\left(e_{t+1}\right)}{h'\left(e_{t+1}\right)} w_t^L \tag{19}$$

This implies that, for any given positive level of education, savings are proportional to wages in the unskilled sector since increasing labor income during the first period of life facilitates savings via an income effect. It is worth noting, however, that once accounting for factor rents and solving for the general equilibrium, savings are increasing in education for any non-convex human capital production function.[4]

Combining the budget constraint (7) and optimal savings (19) yields

$$n_t = \left[(1 - e_{t+1}) - \frac{1-\beta}{\beta} \frac{h(e_{t+1})}{h'(e_{t+1})} \right] w_t^L \tag{20}$$

Optimal fertility is increasing in the level of unskilled wages w_t^L due to an income effect. Likewise, fertility is decreasing in education for any non-convex human capital production function due to a substitution effect since education imposes an opportunity cost on fertility.

Finally, optimal education is implicitly characterized by combining (11), (12), and inserting (20)

$$\beta(1 - e_{t+1})h'(e_{t+1}) - (1-\beta)h(e_{t+1}) - \frac{\beta\gamma}{w_{t+1}^H} = 0 \tag{21}$$

Note that this implies an upper bound on education $e_{max} < 1$ as the left-hand side of (21) becomes negative for $e_{t+1} = 1$. Assuming that investments in education are sufficiently effective in producing human capital, such that

$$\beta h'(0) - (1-\beta)h(0) > 0 \tag{A1}$$

ensures that for some level of $w_{t+1}^H > 0$ the optimal level of education is larger than zero (i.e. investment in education is profitable in a sufficiently advanced environment). This leads to the following result.

Lemma 2 *If Assumption* (A1) *is satisfied, then there exists a level* $\tilde{w}^H > 0$ *such that the optimal level of education is given by*

$$e_{t+1} = \begin{cases} 0 & \text{if } w_{t+1}^H \leq \tilde{w}^H \\ \in (0,\ e_{max}) & \text{if } w_{t+1}^H > \tilde{w}^H \end{cases}$$

and $e_{t+1} > 0$ *is an increasing function of* w_{t+1}^H *for any* $w_{t+1}^H \in (\tilde{w}^H, \infty)$.

Proof Note that (21) is monotonically increasing in w_{t+1}^H. Moreover, there exists an upper bound e_{max} such that for $e_{t+1} > e_{max}$ (21) cannot hold with equality.[5] Hence, if (A1) holds, it follows from the Intermediate Value Theorem that there exits a $\tilde{w}^H$ such that if w_{t+1}^H exceeds this value, (21) will hold with $e_{t+1} > 0$. As (21) is monotonically increasing in w_{t+1}^H and monotonically decreasing in e_{t+1} for any admissible mapping of education into human capital it follows that e_{t+1} is an increasing function of w_{t+1}^H if $w_{t+1}^H \in (\tilde{w}^H, \infty)$. □

[4] Non-convexity of the human capital production function is a sufficient condition. In fact, it can be easily verified that savings are increasing in education as long as the human capital production function is not too convex, i.e. as long as $h'(e_{t+1})h'(e_{t+1}) - h''(e_{t+1})h(e_{t+1}) > 0$.

[5] It is straightforward to show that, as $w_{t+1}^H \to \infty$, the upper bound is characterized implicitly by $\frac{\beta}{1-\beta} \frac{(1-e_{max})h'(e_{max})}{h(e_{max})} = 1$.

2.4 The dynamics of long-run growth

2.4.1 Dynamics of income and productivity

To derive the evolution of income per capita over time, consider the dynamics of the different components. The growth of production in the unskilled sector, $g_{t+1}^{Y_U}$, depends on the dynamics of the stock of unskilled workers and on the growth of productivity in the unskilled sector

$$1 + g_{t+1}^{Y_U} = n_t^{\alpha}(1 + g_{t+1}^{A_U})^{1-\alpha} \tag{22}$$

where $g_{t+1}^{A_U}$ denotes the growth rate of productivity in the unskilled sector.[6]

Analogously, the growth of production in the skilled sector, $g_{t+1}^{Y_S}$, depends on the growth of the stock of individual human capital, g_{t+1}^{h}, the growth of productivity in the skilled sector, $g_{t+1}^{A_S}$, and on the accumulation of physical capital, which depends on the growth of savings, g_t^s. Substituting the respective terms gives

$$1 + g_{t+1}^{Y_S} = n_{t-1}(1 + g_{t+1}^{h})^{\beta}(1 + g_{t+1}^{A_S})^{1-\beta}(1 + g_t^{s})^{1-\beta} \tag{23}$$

Finally, population growth is given by

$$1 + g_{t+1}^{N} = \frac{n_{t-1}(1 + n_t)}{1 + n_{t-1}} \tag{24}$$

The dynamics of income per capita can be derived by combining these elements. Let θ_t denote the share of the skilled sector in total production, $\theta_t \equiv \frac{Y_t^S}{Y_t} \in (0, 1)$. Then using (1) and (24), the growth rate of income per capita can be expressed as

$$1 + g_{t+1}^{y} = \frac{(1 + g_{t+1}^{Y_U})(1 - \theta_t) + (1 + g_{t+1}^{Y_S})\theta_t}{1 + g_{t+1}^{N}} \tag{25}$$

where $g_{t+1}^{Y_U}$, $g_{t+1}^{Y_S}$ denote the growth rates of total production in the unskilled sector and the skilled sector, respectively.

The evolution of technology is reflected by the productivity dynamics in the two sectors that affect the state variables A_t^U and A_t^S. The qualitative predictions of the model do not depend on any specific functional form. To illustrate the working of the model with a simple and transparent benchmark, we derive the analytical results while restricting attention to the case of exogenous productivity growth. In particular, assume

$$A_{t+1}^U = \phi_U A_t^U \tag{26}$$

$$A_{t+1}^S = \phi_S A_t^S \tag{27}$$

with $\{\phi_U;\phi_S\} > 1$.

In order to account for the endogenous acceleration of productivity growth and match the model quantitatively to the data, we consider a more realistic formulation that reflects

[6] See the Appendix for the derivation of this expression as well as of the expressions that follow.

human capital-driven endogenous growth mechanisms in reduced form in the quantitative implementation of the model in Sect. 3 below.

2.4.2 The phases of development

The dynamic evolution of the economy exhibits different phases. Initially, the economy is characterized by a Malthusian phase, which is followed by an endogenous phase transition that is associated with positive and increasing investments in human capital, physical capital, a corresponding fertility transition, and eventually a phase of sustained growth. In the following, we provide an analytical characterization of these phases and of the dynamic evolution of the main variables of interest.

2.4.3 Malthusian phase

We refer to the Malthusian phase as an environment that is characterized by a setting in which education investments are not profitable and hence $e_{t+1} = 0$, as described in Lemma 2. This setting arises as an equilibrium in a technologically underdeveloped economy, in which the level of skilled wages is sufficiently low. Fertility and savings are then characterized by

$$n_t = n\left(s_t, A_{t+1}^S\right) \quad \text{and} \quad s_t = s\left(w_t^L, A_{t+1}^S\right)$$

as implied by expressions (17) and (18). Considering fertility in $t+1$, using (6), (14), (15), and re-arranging yields the dynamics of fertility as

$$\frac{n_{t+1}}{n_t} = \left(\frac{s_{t+1}}{s_t}\right)^{\beta} \frac{1}{(1+g_{t+2}^{A_S})^{1-\beta}} \tag{28}$$

This implies faster fertility growth in phases of faster savings growth and vice versa, but with savings growth exceeding fertility growth. Suppose fertility is above replacement but constant, such that $\frac{n_{t+1}}{n_t} = 1$. Since $1 + g_{t+2}^{A_S} = \phi_S > 1$, this implies that $s_{t+1} > s_t$ so that savings increase over time. Moreover, from (18), it follows that in this case the dynamics of savings depend on the dynamics of the unskilled wage. Due to constant growth in productivity in the unskilled sector, $\phi_U > 1$, wage dynamics (and hence savings dynamics) depend on the relative sizes of fertility and productivity growth.[7] As an immediate result, the size of the population and the aggregate physical capital stock grow during the Malthusian phase. A noteworthy feature of the model is related to the evolution of income per capita. Although individual human capital remains constant in this phase, productivity grows exogenously and the aggregate stock of capital grows as consequence of individual savings and population growth. This implies that growth of income per capita can be positive during this phase. In the Appendix, we derive the lower bound for fertility in the Malthusian phase as

[7] See "Appendix 7.2" for a formal derivation and for derivations of the following expressions.

$$n_t > \bar{n}_M = \frac{\phi_U}{\phi_S^{\frac{1-\beta}{\beta(1-\alpha)}}} \tag{29}$$

Likewise, we show that the upper bound for output per capita growth in the Malthusian phase is given by

$$1 + g_{t+1}^{y} < 1 + g_{t+1}^{y_M} = \phi_S^{\frac{1-\beta}{\beta}} \tag{30}$$

2.4.4 Demographic transition and growth take-off.

The increase in the returns to skills that is due to the growth of productivity and savings implies that the skilled wage w_{t+1}^H increases monotonically and eventually becomes larger than $\tilde{w}^H$. Lemma 2 implies that from this point on, individuals invest in education, such that $e_{t+1} > 0$. Since a positive investment in education reduces disposable income, this change in education behavior leads to a reduction in fertility, which constitutes the starting point of the demographic transition. Note that education and savings are complements while education (as well as its complement savings) and fertility are substitutes as shown by (19) and (20). That is, fertility decreases with further expansion of education while savings increase.

As consequence of these behavioral changes, the demographic transition marks a crucial turning point for the dynamics of all state variables of the model. First, the drop in fertility increases the growth rate of per capita income in the unskilled sector. As a result, wages in the unskilled sector increase. Second, this and the increasing education investments have a positive effect on savings. The reason for this is the complementarity between human capital and physical capital in aggregate production, which implies that investments in human capital also increase the returns to savings and vice versa. Together, these effects imply a reallocation of individual expenditures from raising children to investments in human capital and physical capital during the first period of life. A third, more subtle implication is that increases in productivity in the unskilled sector, and hence unskilled wages, reinforce the accumulation of human capital through a positive effect on the skilled wage. This can be seen from substituting (14), (15), and (19) into (5), which yields

$$w_{t+1}^H = \beta \left(\frac{A_{t+1}^S \frac{1-\beta}{\beta} w_t^L}{h'\left(e_{t+1}\right)} \right)^{1-\beta} \tag{31}$$

Due to the capital-skill complementarity in the skilled sector, the wage in the skilled sector w_{t+1}^H increases with physical capital. The supply of capital, in turn, increases with the level of unskilled wages from (19). This feature of the model is also quantitatively relevant since it implies that even with constant exogenous technological change in both sectors, skilled wages and investment in education are fuelled by productivity gains in the unskilled sector through the accumulation of physical capital. This effect receives additional momentum after the onset of the demographic transition, with the consequence of a growth acceleration due to factor accumulation.

Subsequently, the demographic transition leads to an acceleration of income growth via the decline in population growth and the associated acceleration in the accumulation of human capital and physical capital. To see this, note that the growth rate in income per capita is still given by (25). However, after the end of the Malthusian phase and with declining fertility, productivity growth now further outweighs the growth in population and hence in the unskilled labor force, that is

$$\frac{1+g^{Y_U}_{t+1}}{1+g^N_{t+1}}=\frac{n_t^{\alpha}(1+g^{A_U}_{t+1})^{1-\alpha}}{1+g^N_{t+1}}>\phi_S^{\frac{1-\beta}{\beta}} \tag{32}$$

implying that the unskilled sector exhibits faster growth in per capita terms. Likewise, the accumulation of human capital (i.e., $g^h_{t+1}>0$), and the decline in population growth (i.e., a falling g^N_{t+1}) imply an acceleration of growth in per capita terms in the skilled sector. This also implies faster growth in per capita terms than during the Malthusian phase as

$$\frac{1+g^{Y_S}_{t+1}}{1+g^N_{t+1}}=\frac{n_{t-1}^{1-(1-\alpha)(1-\beta)}\frac{1+g^h_{t+1}}{(1+g^{h'}_{t+1})^{1-\beta}}(1+g^{A_S}_{t+1})^{1-\beta}(1+g^{A_U}_t)^{(1-\alpha)(1-\beta)}}{1+g^N_{t+1}}>\phi_S^{\frac{1-\beta}{\beta}} \tag{33}$$

Growth in income per capita is given by the sum of (32) and (33), weighted by the respective sector shares as in (25). Note that the share of the skilled sector in total production increases over time as the skilled sector grows at a faster rate than the unskilled sector due to factor accumulation.

Another insight from (31) is that the wages in the skilled sector are non-decreasing in the aggregate level of human capital for any non-convex human capital production. This provides a rationale for the observation that a massive increase in human capital does not necessarily imply a reduction in the skill premium, even without skill-biased technical change.

2.4.5 Modern growth and the balanced growth path

The demographic transition triggers an acceleration of economic development. Ultimately, the economy converges back to a balanced growth path that is based on sustained productivity improvements. To see this, notice that in the long run, education approaches its upper bound, with $e_{t+1}\to e_{max}$ as $t\to\infty$. The dynamic system admits a balanced growth path if population grows at a constant rate. Inserting (21) into (20) gives the following condition for a balanced growth path with constant population growth,

$$\frac{n_{t+1}}{n_t}=1=\left(\frac{1}{1+g^{A_S}_{t+2}}\right)^{1-\beta}\left(\frac{1+g^{A_U}_{t+1}}{n_t}\right)^{\beta(1-\alpha)}(1+g^{h'}_{t+2})^{\beta} \tag{34}$$

Inserting the respective growth rates of productivity in the unskilled sector and in the skilled sector, and solving for n_t given $\{e_{t+1};e_{t+2}\}\to e_{max}$ yields (asymptotic) fertility along the balanced growth path as

$$\bar{n}_{BGP}\to\frac{1+g^{A_U}_{t+1}}{(1+g^{A_S}_{t+2})^{\frac{1-\beta}{\beta(1-\alpha)}}}=\frac{\phi_U}{\phi_S^{\frac{1-\beta}{\beta(1-\alpha)}}} \tag{35}$$

This balanced growth path exhibits a stable population (see, e.g., Preston et al., 2001, ch. 7) that is characterized by constant birth rates, constant death rates, constant population growth, and a constant age composition. Under a stable population along the balanced growth path, the (asymptotic) growth rate of income per capita is given by evaluating (25) at $1+g^h_{t+1} \to 1$, $1+g^{h'}_{t+1} \to 1$, $n_{t-1} = n_t = \bar{n}_{BGP} \to \frac{\phi_U}{\phi_S^{\frac{1-\beta}{\beta(1-\alpha)}}}$, $1+g^{A_U}_t = 1+g^{A_U}_{t+1} = \phi_U$, and $1+g^{A_S}_{t+1} = \phi_S$, which yields

$$1+g^{y_{BGP}}_{t+1} \to \phi_S^{\frac{1-\beta}{\beta}} \tag{36}$$

The growth rate of income per capita on the balanced growth path is lower than during the transition, but higher than during the Malthusian phase. This result is the consequence of the accumulation dynamics of population and human capital, which fade out in the long run. The following proposition provides a summary of the model dynamics.

Proposition 1 *The development path of the economy exhibits three distinct regimes.*

1. *A Malthusian phase as long as $w^H_{t+1} < \tilde{w}^H$, in which the economy is characterized by no investment in education, as well as positive growth of the population, of the aggregate capital stock, and of output per capita.*
2. *An economic and demographic transition, which is characterised by a rapid accumulation of human capital and physical capital, a decline in fertility and population growth, and an acceleration of income per capita growth.*
3. *A phase of modern growth with a balanced growth path that exhibits constant fertility, a stable population with constant population growth, positive human capital investments, and a positive constant growth rate of capital per capita and income per capita. Income per capita grows at a faster rate than during the Malthusian phase, but at a lower rate than during the transition.*

Proof The statements follow immediately from Lemma 2, (17)–(21), as well as (30), (32), (33), (35), and (36). □

2.5 Extensions

The model developed so far describes the dynamics of a single country in autarky with exogenous productivity growth in both sectors. The purpose of the model is to illustrate the role of the demographic transition for the development dynamics of a country and the resulting empirical implications. The assumptions that productivity growth is purely exogenous and that all countries undergo the same dynamic process are restrictive and counterfactual. In the quantitative analysis of the model, we relax both assumptions, one at a time, to obtain a more realistic representation.

2.5.1 Human capital externalities

While the baseline model with purely exogenous productivity growth is able to account for the qualitative patterns of income per capita, in particular for a temporary acceleration of growth in the context of the economic and demographic transition, this model is unable to match the quantitative acceleration in productivity and income per capita growth after the transition. It is well established that factor accumulation alone cannot explain the divergence in incomes (see, e.g., Hsieh and Klenow, 2010, and Sunde et al., 2022). In order to match the quantitative patterns of income per capita, we consider an extension of the model with endogenous technological change. In particular, we consider a reduced-form representation that allows for human capital externalities on productivity growth in the form of a "level effect", according to which a higher level of human capital leads to higher rates of productivity growth, and in the form of a "growth effect" of human capital on productivity growth, according to which a faster expansion of human capital implies a faster rate of productivity growth.

2.5.2 Technology diffusion

Some of the facts of long-run growth explicitly refer to comparative development patterns and cross-country comparisons. To account for these patterns, we consider a multi-country version of the model with international diffusion of technology, which allows us to account for the possibility of cross-country technology spill-overs and assess their relevance in accounting for the facts. Specifically, we consider an extension of the model with technology diffusion in the spirit of Nelson and Phelps (1966). This extension does not affect the timing of the demographic transition, since technology imports are not possible without education. However, it clearly affects the equilibrium dynamics after the onset of the demographic transition by accelerating technological development and, as consequence, subsequent economic and demographic development. At the same time, this helps concentrating on the central role of the demographic transition by allowing only one channel through which international spillovers affect the development trajectory of a country.[8] As will be shown in more detail below, this extension provides novel empirical implications. In particular, countries that undergo the economic and demographic transition later have access to technologies developed in more developed countries that experienced an earlier transition.

2.6 Parametrization

We end the discussion by laying out a quantitative version of the baseline version of the model. In the following, we calibrate the model to match empirical data moments for England. In the next section, we will compare the simulated data to actual statistical data and investigate whether the model can serve as a data generating process that accounts for the empirical patterns of long-run growth.

The simulation makes use of the fact that, given a set of initial values for L_t, H_t, K_t, A_t^U, and A_t^S, the productivity levels A_{t+1}^U and A_{t+1}^S are determined. This allows computing the equilibrium allocation in terms of n_t, s_t, and e_{t+1}, and thus L_{t+1}, K_{t+1}, and H_{t+1}, yielding the state variables that are needed to simulate the entire development path forward.

[8] Alternative dimensions for international spillovers are trade in goods and international capital mobility. As argued by Comin and Mestieri (2018), models of technology diffusion are more suitable for studying income divergence than trade-based theories, which are inconsistent with the divergence in incomes across countries that was observed between 1913 and the 1970s despite the collapse in world trade between those points in time. Similarly, international capital mobility was historically subject to various degrees of restrictions, adding complications that go beyond the purpose of this paper.

2.6.1 Data

The data moments that serve as targets for the model calibration are data for England from the Maddison data base (Bolt and van Zanden, 2020). We also make use of data for the timing of the demographic transition by Reher (2004), data for education by Lee and Lee (2016), and data for comparative development from the Penn World Tables (Feenstra et al., 2015, version 9.1).

2.6.2 Calibration

We set the length of a generation to be 20 years and simulate the model for a sequence of 35 generations beginning with the year 1700, which illustrates the working of the model over the long run. The choice of the simulation window is without consequence for the main results and serves the purpose of illustration. Below we will restrict the simulated data to a time frame that is comparable to the observed panel data in some of the empirical analysis. It is important to note, however, that all simulations are based on the same long simulation window.[9]

For simplicity, we consider a parameter-free, linear production function of human capital with $h\left(e_{t+1}\right) = 1 + e_{t+1}$. This implies an optimal education level of $e_{t+1} = \max\left(2\beta - 1 - \frac{\beta\gamma}{w^H_{t+1}}, 0\right)$ as well as $h'\left(e_{t+1}\right) = 1$, $h(0) = 1$, $h'(0) = 1$, and $e_{max} = 2\beta - 1$.

The parameters of the model are set to match data moments for England (United Kingdom) in the Maddison database by Bolt and van Zanden (2020). The parameter X (land size) plays no crucial role except for determining initial values and therefore can be normalized to one. We set the capital income share in the skilled sector to $\frac{1}{3}$, which implies setting $\beta = \frac{2}{3}$, in line with standard calibrations of growth or business cycle models. The labor share in the unskilled sector is set to $\frac{1}{2}$, which implies a value of $\alpha = 0.5$, similar to values used by Hansen and Prescott (2002) and Doepke (2004).

Assuming an education level of zero as well as constant fertility n, iterating (16) forward yields population size at time $t > 0$ as

$$N_t = (1 + n)n^{t-1}\left(N_0 - \frac{N_0}{1 + n}\right)$$

where the initial population level N_0 is normalised to one. Assuming the demographic transition to occur in 1880, which corresponds to a take-off in education from 1880 to 1900 (in line with Lee and Lee, 2016), we obtain the population level just before the onset of the demographic transition from the Maddison database and normalise it by the size of the population in the year 1700 (i.e., initial population size). This allows computing the average fertility n that matches the two values. As fertility in the model is increasing during this phase, n constitutes an upper bound and we set the target $\bar{n}_M$ to the arithmetic mean between n and replacement fertility. This yields the initial stocks of aggregate unskilled

[9] The choice of the simulation window is driven by considerations concerning the availability of comparable and reliable data for the period before 1700. The qualitative and quantitative results of the simulation are not sensitive to the size of the simulation window. The model can be simulated for an arbitrarily long period of (quasi-)stagnation by adjusting the respective parameters accordingly.

labor $L_0 = N_0 - \frac{N_0}{1+\bar{n}_M}$ and aggregate human capital $H_0 = h_0 \frac{N_0}{1+\bar{n}_M}$. Moreover, assuming constant growth of GDP per capita gives GDP per capita at time $t > 0$ as

$$y_t = (1+g)^t y_0$$

where again the initial level of GDP per capita y_0 is normalised to one. We retrieve the level of GDP per capita just before the onset of the demographic transition from the Maddison database and normalise it by GDP per capita in the year 1700. Hence we can compute the average growth rate g that matches the two values. As GDP per capita growth varies during this period we set our target g^{Y_M} to the arithmetic mean between g and 0.4 percent per annum, which is consistent with empirical estimates for the 18th and early 19th century (see, e.g., Crafts, 2021). Since (approximate) GDP per capita growth in the Malthusian phase is given by (30) this allows computing $\phi_S \approx 1.189$. Given ϕ_S and (approximate) fertility given by $\bar{n}_M$ as in (29), we can compute $\phi_U \approx 1.2866$.

In order to compute the initial productivity level of the unskilled sector, we set the initial share of income devoted to children ε to approximately two thirds, which implies an initial savings level of $s_0 = (1-\varepsilon)w_0^L$.[10] The savings level, together with initial skilled productivity, determines the initial skilled wage and hence the timing of the demographic transition. The initial level of productivity in the unskilled sector can then be obtained by solving (7) for $e_{t+1} = 0$ and $n_0 = \bar{n}_M$

$$w_0^L = \bar{n}_M + s_0 \iff A_0^U = \frac{L_0}{X}\left(\frac{\bar{n}_M}{\varepsilon}\right)^{\frac{1}{1-\alpha}} \approx 1.3277$$

As consequence, this yields the initial capital stock as $K_0 = \frac{s_0}{1+g^{Y_M}} \frac{N_0}{1+\bar{n}_M} \approx 0.2274$. The initial condition for productivity in the skilled sector is set to satisfy $w_t^H(1+e_t) \geq w_t^L \; \forall \; t$, i.e., a skill premium that always exceeds one. A sufficient condition for this to hold is given by

$$A_0^S \geq \frac{H_0}{K_0}\left(\frac{w_0^L}{\beta h_0}\right)^{\frac{1}{1-\beta}} \approx 29.1066$$

which also delivers a demographic transition in 1880. The parameter γ is set to ensure $n_0 = \bar{n}_M$. Given β, s_0, h_0, ϕ_S, A_0^S, and $\bar{n}_M$, solving (17) for γ yields $\gamma \approx 1.8265$.

To account for the endogenous acceleration of productivity growth, we consider a formulation that contains an exogenous component and includes an endogenous growth component based on human capital externalities. More specifically, as extension to (26) and (27), we consider

$$A_{t+1}^U = \phi_U\left\{1 + \sigma_U\left[h\left(e_{t+1}\right) - h_{min}\right] + \psi\left[h\left(e_{t+1}\right) - h\left(e_t\right)\right]\right\}A_t^U \tag{37}$$

$$A_{t+1}^S = \phi_S\left\{1 + \sigma_S\left[h\left(e_t\right) - h_{min}\right] + \psi\left[h\left(e_t\right) - h\left(e_{t-1}\right)\right]\right\}A_t^S \tag{38}$$

.

[10] This corresponds to spending approximately 50 percent of family income on children, in line with evidence by Turvey (2010) who points towards a yearly expenditure of around 8£ per child in 1750.

Several aspects of this formulation are worth noting. First, with the linear human capital production function, $h_{min} = h(0) = 1$. This implies that this formulation does not affect the rest of the calibration, particularly the timing of the demographic transition, as the additional parameters σ_U, σ_S, and ψ have no effect when $e_{t+1} = 0$. Second, the terms $\sigma_U\left[h\left(e_{t+1}\right) - h_{min}\right]$ and $\sigma_S\left[h\left(e_t\right) - h_{min}\right]$ account for the "level effect" of human capital on productivity growth, i.e., an expansion of human capital leads to higher rates of productivity growth, ceteris paribus. Analogously, the terms $\psi\left[h\left(e_{t+1}\right) - h\left(e_t\right)\right]$ and $\psi\left[h\left(e_t\right) - h\left(e_{t-1}\right)\right]$ account for the "growth effect" of human capital on productivity growth, that is, productivity growth is more rapid when the stock of human capital expands. Existing evidence suggests that both channels are empirically relevant, such that $\{\sigma_U,\ \sigma_S,\ \psi\} > 0$ (see, e.g., Sunde and Vischer, 2015). Third, the timing of this formulation implies that a generation does not exhibit an externality on itself, as future skilled productivity does not depend on the education choice of the current (unskilled) generation.

The parameter σ_S is calibrated to generate a balanced growth path growth rate of income per capita of one percent per year, which is now given by

$$1 + g_{t+1}^{y_{BGP}} \rightarrow \left[\phi_S(1 + \sigma_S e_{max})\right]^{\frac{1-\beta}{\beta}}$$

Given $e_{max} = 2\beta - 1$ this yields $\sigma_S \approx 0.7565$. The parameter σ_U is calibrated to generate an (asymptotic) fertility level along the balanced growth path at replacement (and hence a stationary population size). Solving (35) for $\bar{n}_{BGP} = 1$ yields $\sigma_U \approx 0.4717$. Lastly, the parameter ψ is calibrated to generate a peak growth rate of income per capita of 2.25 percent per year. Given all other parameter values, solving for the peak growth rate of income per capita to equal 2.25 percent per year gives $\psi \approx 11.6656$. Table 1 summarizes the values and targets of the baseline calibration.

3 Accounting for long-run growth dynamics

We are now in a position to assess whether the model developed in the previous section is able to match the stylized empirical facts of long-run growth. This section considers the implications of the baseline model, focusing on the behavior of a single country in isolation. The next section assesses whether the cross-country version of the model can account for the patterns of comparative development across countries. Our exposition follows the six "New Kaldor Facts" proposed by Jones and Romer (2010) and provides an updated and extended version that considers additional evidence that has become available more recently and that led to a more nuanced view of the facts of long-run growth.

3.1 Simulated long-run development dynamics

We begin the analysis by simulating the quantitative version of the model before contrasting it with empirical patterns in various dimensions. The main purpose of this analysis is to illustrate the main features of the model, including the non-linear long-run dynamics in the core variables, namely output, population, physical capital, and human capital.

Figure 1 Panel (a) plots the long-run evolution of (annualized) growth rates of population and output per capita for the baseline country (England) over the period 1700 to 2100, while Panel (b) plots the evolution of all state variables in addition to output per capita growth. The simulation pinpoints several characteristic features of the unified growth

Table 1 Calibration: baseline model

Parameter	Value	Target
X	1	Normalised
β	$\frac{2}{3}$	Capital share of $\frac{1}{3}$
α	0.5	Labor share of $\frac{1}{2}$
N_0	1	Normalised
ϕ_S	1.189014	(Relative) GDP per capita of England in 1860 (from Maddison)
ϕ_U	1.286587	(Relative) population level of England in 1860 (from Maddison)
A_0^U	1.327654	Initial expenditure share of fertility of approximately $\frac{2}{3}$
A_0^S	29.10659	Skill premium
γ	1.826472	Initial fertility level of $n_0 = \bar{n}_M$
σ_S	0.756549	BGP growth rate of 1% p.a. (from Maddison)
σ_U	0.4716599	Replacement fertility along the BGP
ψ	11.66559	Peak growth rate of 2.25% p.a. (from Maddison)

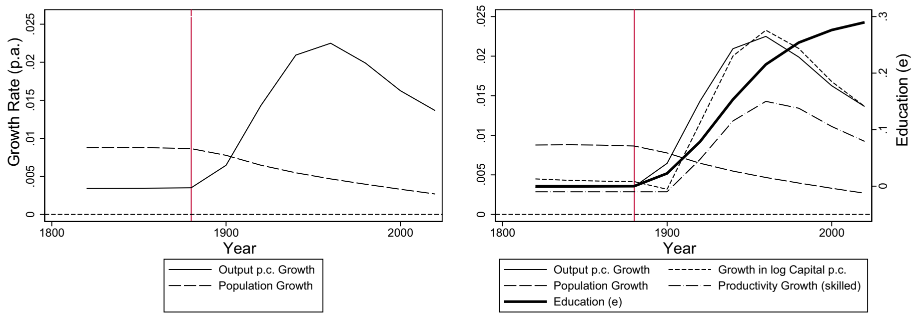
model. As consequence of the choice of parameters and initial conditions, the model exhibits a demographic transition with an onset of the decline in fertility in 1880. Population grows before this point, while the growth rate declines thereafter and converges to a stable population with fertility at replacement.

Panel (b) illustrates the dynamics of the state variables population, physical capital, education, and productivity. The fertility transition marks a turning point in the model dynamics, as declining fertility gives rise to increased education attainment per capita. As consequence of productivity growth, output per capita exhibits a small but positive growth rate even before the demographic transition. It is, however, the demographic transition and the associated co-evolution of human capital accumulation and the intensified accumulation of physical capital that leads to a sharp acceleration in output per capita growth. This acceleration is strengthened by the aggregate complementarity of physical capital and human capital. Growth in output per capita also benefits from the reduction in population growth and from the positive effects of the education expansion on productivity growth. Eventually, however, this process loses momentum and growth peaks in the later phases of the demographic transition before it converges to a lower sustained growth path thereafter. Although output per capita growth remains permanently higher than before the demographic transition, it declines once the transition dynamics lose momentum. As consequence, aggregate output per capita growth stabilises at a growth rate that is higher than during the Malthusian phase, but lower than during the transition when output per capita growth is affected by the one-off dynamics related to the demographic transition.

3.2 Accounting for empirical growth dynamics

3.2.1 The secular acceleration of growth

The most salient empirical fact of long-run growth refers to the secular acceleration of development. Conventionally, the stylized presentation of this fact juxtaposes a phase of stagnant income per capita with a sharp acceleration to sustained income growth in the Western world (see, e.g., data from the Maddison Project, Bolt et al. 2018, 2020). The inconsistency of this observation with models of balanced growth sparked the development



(a) Population and Income Dynamics

(b) Dynamics of State Variables

Fig. 1 Simulated development path of the baseline economy

of theories that explain the income acceleration exclusively by the accumulation of physical capital (Hansen & Prescott, 2002), or by theories that capture also other stylized facts related to the concurrent demographic transition in fertility from a Malthusian regime to a modern regime with fewer but better educated children (see, e.g., Galor and Weil, 2000, Galor, 2011).

As a unified growth framework, the model is consistent with the secular acceleration from a quasi-stagnant environment to sustained growth. The simulation results presented above illustrate that the model exhibits a long-run development path that is initially characterized by a Malthusian phase that is associated with slow (but non-zero) growth in income per capita, high fertility, and no education. The last phase of development is characterized by a balanced growth path with low fertility and a positive level of education. The intermediate phase is given by an economic and demographic transition, during which growth of income per capita accelerates and temporarily exceeds the growth rate of the balanced growth path. Figure 2 documents that the calibrated version of the model closely resembles the long-run growth dynamics of income per capita and population observed for England when abstracting from the fact that stochastic factors that severely affected economic growth, such as World War I, are not accounted for by the model by construction.

Five aspects are noteworthy. First, the growth trajectory exhibits a phase transition from an almost stagnant development path to a sustained growth regime. This transition occurs over less than two centuries and marks the end of Malthusian dynamics and the beginning of a modern growth regime.

Second, the acceleration of income growth eventually fades out and growth rates stabilize at a level below their intermediate peak. The reason for this is that the education expansion eventually ends as education converges to its endogenous upper bound, as discussed in more detail below. As a consequence, the acceleration in growth due to capital-skill complementarities and the acceleration of technological progress due to human capital also loses momentum. This feature is consistent with the observation of a "wave pattern" in productivity growth that occurs in the context of the education expansion (see, e.g., Rangazas, 2002).

Third, the growth acceleration leads to a temporary overshooting of income growth above the endogenous growth rate that is sustainable in the long run. This pattern is consistent with an empirical aspect of long-run growth that has received less attention than the take-off. The slow-down of income dynamics after the growth take-off has implications for the recent debate about "secular stagnation", which mostly focused on the income growth slowdown observed in developed countries over the last decades. In search of the causes for this slowdown, some scholars raised concerns about the dynamism of technological progress and demographic factors that impose increasing impediments for development (see, e.g., Teulings and Baldwin, 2014, OECD, 2015, Gordon, 2012, 2014, 2016). In light of the finding of a hump-shaped pattern of growth also among low-income countries, some have pointed at the potential role of transition dynamics (see, e.g., Gundlach and Paldam, 2020). In fact, evidence by Cervellati et al. (2017) suggests that part of the slowdown might be explained by the acceleration of growth after the demographic transition, which fades out as the temporary dynamics of the sudden accumulation of production factors come to an end. The model provides a rationalization for this hypothesis. The overshooting and subsequent slow-down of growth is related to the additional momentum generated by the sudden accumulation of human capital and physical capital, which are strategic complements from the perspective of individuals. Once this rapid accumulation, which is the direct consequence of the singular, non-recurrent demographic transition, fades out, the only source of growth is sustained productivity improvement, and hence the standard mechanism giving rise to balanced growth. These non-monotonic dynamics, which are consistent with the empirical patterns, differ from previous accounts that considered monotonic convergence dynamics to the balanced growth path.

Fourth, the decline in growth during the approach to the last phase of development has implications for the prospects of future growth on a global scale. This suggests that it might be natural to expect slower growth in the future for countries that underwent the demographic transition in the past as result of the non-recurring take-off dynamics in the context of the economic and demographic transition. The reinforcing interactions between population, human capital, physical capital, and technology, first lead to an acceleration of growth but eventually fade out. In this respect, the model contributes a novel aspect to the secular stagnation debate, which has focused on a slow-down in technology and considered the role of demographic factors primarily from a perspective of aging and public health (Gordon, 2012, 2014, 2016).

Fifth, despite a Malthusian population regime, which is reflected in significant population growth the model exhibits very small but sustained positive growth in income per capita prior to the transition. This is in line with recent empirical findings of moderately positive growth in incomes during the 17th and 18th century that has been viewed as challenging the empirical validity of existing unified growth theories that are based on a Malthusian phase of stagnation (see, e.g., Fouquet and Broadberry, 2015, p. 228) and Broadberry et al., 2015). More recent evidence suggests that earnings trended upwards since the late 16th century, while the increase in incomes was associated with a secular decline in the labor share (Humphries and Weisdorf, 2019). The moderate increase in earnings during this phase of population expansion is consistent with technological progress (see Crafts and Mills, 2022).[11] This modest increase in economic activity prior to the take-off in growth has so far only been detected for Europe. This pattern can be viewed as consequence of Malthusian fluctuations, when reproductive success does not keep pace with productivity increases. In the centuries prior to the demographic transition, improvements in

[11] Similar findings of increasing population and steadily increasing production, mostly due to an increasing share of GDP accounted for by non-agricultural production, have been reported for Italy (Malanima, 2011), Holland (Van Zanden and Van Leeuwen, 2012), Spain (Alvarez-Nogal and Prados de la Escosura, 2013), Sweden (Schön and Krantz, 2015), and Portugal (Palma and Reis, 2019).

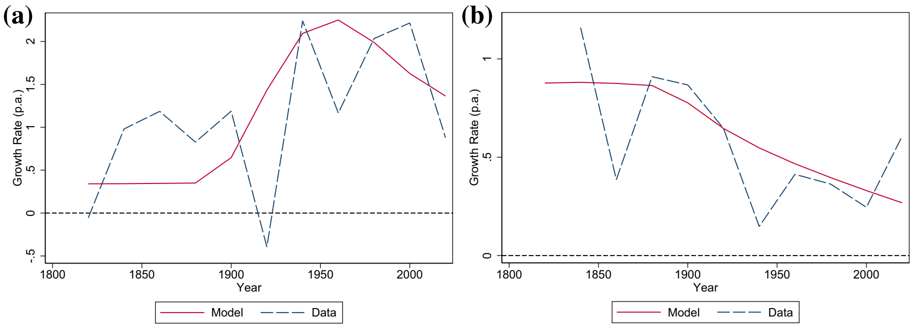


Fig. 2 Growth dynamics—England. Data: Maddison Project, Bolt and van Zanden (2020)

productivity were associated, for instance, with the relaxation of land constraints due to the discovery of the New World, and with gains from trade in the context of increased globalization. Nevertheless, the analysis of the model documents that a unified growth approach is perfectly consistent with evidence of gradually increasing income even before the transition. At the same time, the model analysis also puts this sluggish growth in perspective against the eruptive acceleration of growth during the middle of the 19th century, roughly at the same time when physical capital deepening and human capital deepening experienced a significant acceleration, consistent with recent evidence (see, e.g., Crafts, 2021, Tables 2 and 3). Moreover, the model underscores the important qualitative and quantitative difference between non-zero but slow growth prior to the transition, and the eruption of growth and development across all dimensions of the economy that occurs in the context of the demographic transition.

3.2.2 Population and demographic dynamics

Historically, demographic development was characterized by stable populations and moderate levels of population growth. Eventually, this phase was followed by a demographic transition that brought about a sharp decline in fertility rates. Although this demographic transition occurred simultaneously to the acceleration in income growth this reversal in population growth rates plays no role in early theories explaining the population dynamics (see, e.g., Kremer, 1993) or the acceleration of economic development based on the accumulation of physical capital (see, e.g., Hansen and Prescott, 2002). In contrast, the demographic transition is a central element in explaining the growth acceleration in unified growth models, with the decline in fertility reflecting the onset of investments in education (see, e.g., Galor and Weil, 2000, Galor, 2011). Delays in this transition have been associated with the great divergence in comparative development across the world (see, e.g., Galor, 2011, or Cervellati and Sunde, 2015).

The demographic transition is usually seen as the end of a Malthusian phase of population development. Malthusian dynamics in the pre-industrial era imply that productivity improvements were absorbed by an increasing population (see, e.g., Ashraf and Galor, 2011). The main reason that greater population eroded improvements in living conditions per capita was related to the presence of fixed factors and the associated decreasing returns to labor. However, related to the previous discussion, recent findings suggest that the population increase was indeed associated with a very moderate improvement in income per capita (see, e.g., Bolt and van Zanden, 2020). This suggests the existence of scale effects, which might operate through agglomeration effects, knowledge spillovers, and ultimately improvements in productivity that trigger the demographic transition (as in, e.g., Galor and Weil, 2000) and foster subsequent growth.

The model is consistent with a rapid population expansion prior to the demographic transition and a convergence to a stable population thereafter. Moreover, as illustrated by Fig. 3, the population expansion occurs substantially before the growth acceleration. Again, this figure illustrates the dramatic increase in income per capita over the past two centuries, which dwarfs the economic development in the centuries before.[12]

Several aspects are particularly noteworthy in this context. First, the simulated model stipulates a convergence to a balanced growth path with stationary population size. Empirically, the population dynamics in the aftermath of the onset of the demographic transition are less well understood than the dynamics prior to the transition. Most accounts argue that, based on intuition, population growth will eventually fade out (see, e.g., Jones and Romer, 2010). Unified growth models often impose the convergence to a balanced growth path with stationary population (see, e.g., Galor, 2011, or Cervellati and Sunde, 2015). In other accounts, the asymptotic population dynamics are left unspecified (Strulik et al., 2013). Evidence for population dynamics for the last decades documents that fertility often drops below replacement in the process of the demographic transition. While the majority of post-transitional countries display total fertility rates around or below replacement, total fertility has dropped in some cases to levels below 1.5. There is an ongoing discussion whether this lowest-low fertility scenario is a temporary phenomenon (see, e.g., Bongaarts, 2002, Billari and Kohler, 2004, Goldstein et al., 2009, and Myrskylä et al., 2009). In the model simulation, the stabilization of population in the long run is due to the specific calibration, as discussed above. An alternative calibration could account for sustained positive or negative population growth along the endogenous growth path. In the unified growth literature, this is often implicitly assumed or neglected as an asymptotic behavior of little practical relevance for understanding the transition dynamics.

Second, in the baseline calibration, the model exhibits a monotonic decline in population growth during the demographic transition. In general, however, the model allows for non-monotonic population dynamics. As becomes clear from inspecting (34), fertility dynamics depend on an interplay of income and substitution effects, where the income effect is related to the dynamics of the wage (and hence productivity) in the unskilled sector, whereas the substitution effect is related to the dynamics of productivity in the skilled sector, which provides the incentives for education and savings. In the baseline specification of the productivity dynamics in (37) and (38), productivity in the unskilled sector is affected through intra-generational spillovers of human capital (by considering an influence of $h\left(e_{t+1}\right)$ on productivity A^U_{t+1}), while productivity in the skilled sector is affected through inter-generational spillovers (by considering an influence of $h\left(e_t\right)$ on productivity A^S_{t+1}) as in much of the unified growth literature (see, e.g., Cervellati and Sunde, 2005).

[12] While not accounting for an extension of the market in terms of urbanization or globalization the model is consistent with an association of growth acceleration with scale effects, as suggested by Jones and Romer (2010). Considering the possibility that the unskilled sector reflects the primary (agricultural) sector of production, whereas the skilled sector reflects the secondary (manufacturing) or tertiary (service) sectors, which are predominantly located in urban agglomerations, the model dynamics are consistent with a narrative of urbanization. See also Baudin and Stelter (2022) for recent work on this aspect.

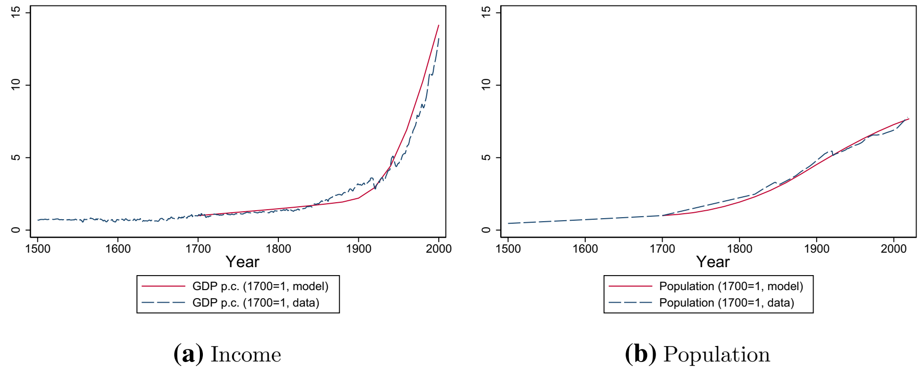


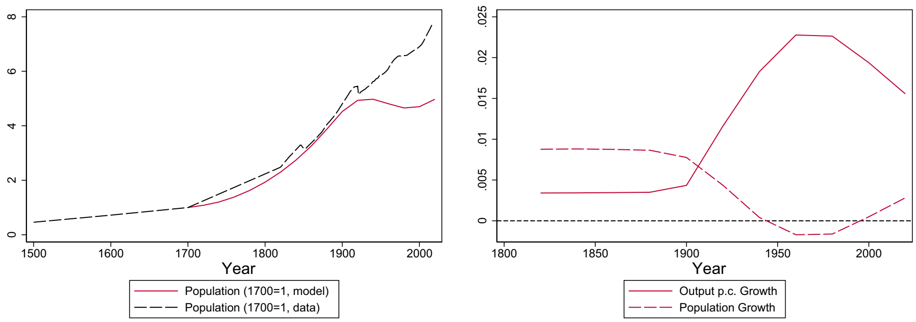
Fig. 3 The expansion of income and population—England. Data: Maddison Project, Bolt and van Zanden (2020)

This implies a delay in the skilled sector and hence a relatively weak substitution effect in the fertility choice. Choosing a symmetric formulation (i.e., where both technologies are affected by inter-generational spill-overs) implies a stronger substitution effect, and hence a sharper decline in fertility during the early phase of the demographic transition. As illustrated in Fig. 4, this leads to a temporary decline in fertility below replacement even if the population stabilizes at a stationary level during the phase of modern growth. This provides a rationalization for the observed drop of fertility below replacement in many countries. The calibration of a balanced growth path with stationary population implies a rebound, which sheds new light on the empirical debate about the reversal in fertility dynamics in many of the "lowest-low" fertility countries.

3.2.3 The expansion of human capital and physical capital

The key mechanism behind the acceleration of growth in income per capita and the slowdown in population growth is related to the expansion of education and the corresponding decline in fertility in the context of the demographic transition. The massive expansion of human capital over the past two centuries represents another stylized fact of long-run growth (Jones and Romer, 2010, Lee and Lee, 2016). In light of converging evidence that human capital plays an important role for economic development (see, e.g., Gennaioli et al., 2012, Sunde and Vischer, 2015, Hanushek et al., 2017), this expansion is viewed as a major driver of the growth acceleration. While the onset of this expansion in Western countries occurred in the temporal context of the demographic transition (see, e.g., Lee and Lee, 2016), there is an ongoing discussion about whether education was a determinant or consequence of the acceleration in growth. This question also relates to the dynamics of wages and the skill premium, which is discussed below.

Figure 5 illustrates the expansion of education in England over the past two centuries and compares the model dynamics to the empirical patterns. The dynamics in education investments (in terms of e_{t+1} or a derived measure of years of schooling) captures the take-off and the overall dynamics in empirical measures of human capital fairly accurately,



(a) Population **(b)** Simulated Long-Run Dynamics

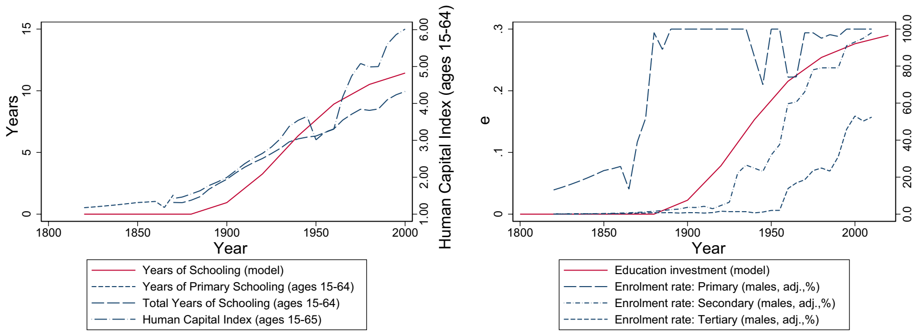
Fig. 4 Alternative scenario: fertility reduction below replacement. Data: Maddison project, Bolt and van Zanden (2020)

regardless of the precise measure used for education.[13] In particular, the take-off in education investments captures the take-off in average years of schooling and a human capital index. Considering enrolment rates, the take-off in education investments occurs with a considerable delay relative to primary enrolment, but closely tracks secondary enrolment. While qualitatively similar, the take-off in tertiary enrolment came later.

A similar expansion has been observed for physical capital. Providing estimates of historical capital stocks has been notoriously difficult, but recent estimates by Broadberry and de Pleijt (2021) indicate that a non-negligible part of output growth during the early phases of the growth acceleration was driven by capital accumulation, which experienced a visible acceleration during the middle of the 19th century. Prior to the acceleration, the capital stock exhibited visible increases as early as during the 17th century, but the evidence suggests that capital intensity did not increase or even decreased in the context of rapid population expansion. The estimates by Broadberry and de Pleijt (2021) for Great Britain also indicate that the importance of (fixed) capital for production increased while the importance of land declined as early as in the 17th century. Their estimates also show that the capital-output ratio declined during this period.

Figure 6 presents the corresponding results for the accumulation of physical capital. Consistent with the discussion of the analytical predictions for the Malthusian phase, individual savings imply an increase in the aggregate capital stock. However, the available resource endowments (in terms of capital per capita and land per capita) slowly decrease as the result of Malthusian dynamics, as illustrated in Panel (a). As soon as the fertility decline begins around 1880, the resource endowments per capita increase in the model. An increase is also visible in the data for the post-transitional period, as illustrated by Panel (b). As consequence of the dynamics of capital endowments per capita and moderate

[13] In the model, education investments are captured by e_{t+1}. As discussed in the introduction, the model is based on a broad notion of skilled human capital, and a conceptually simple and transparent model of skill acquisition as an individual's choice between savings, own education, and fertility. This modeling strategy is consistent with higher levels of education, such as secondary enrolment. A model of parental education choice would likely deliver similar predictions regarding the nexus of education and fertility in the long-run, but would be more suitable for lower levels of education. To construct a measure of years of schooling, we normalize education with e_{max} to represent 14 years of schooling.



(a) Years of Schooling **(b)** Education Investment

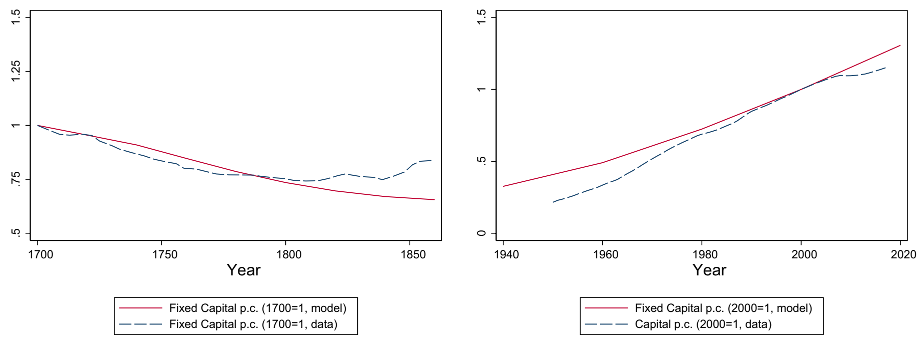
Fig. 5 The expansion of human capital—England. Data: Lee and Lee (2016)

increases in income per capita, the capital output ratio falls during the Malthusian phase. This resembles the decline in the capital-output ratio that has been documented for England for the late 17th to the mid 19th century (see, e.g., Broadberry and de Pleijt, 2021).

3.2.4 The dynamics of the skill premium and labor share

An empirical observation that has generated considerable interest is the development of the skill premium. In particular, despite the massive expansion of human capital, there has been no observation of a marked decline in the skill premium. If anything, the skill premium seems to have widened during the 20th century, even in developing countries (see, e.g., Goldberg and Pavcnik, 2007, and Jones and Romer, 2010). Candidate explanations for this observation are skill-biased technical change (Goldin and Katz, 2007a), or complementarities between human capital and physical capital (as in, e.g., Krusell et al., 2000).

The literature has also pointed out a tension between a human capital-driven acceleration as in unified growth theories and the empirical dynamics of wages and skill premia in the historical context prior and around the time of the growth take-off, mostly for the case of Great Britain. Van Zanden (2009) documented that the skill premium in Western Europe had not increased markedly during the late medieval period and prior to the growth acceleration, while Betran and Pons (2010) show evidence for a decline in the skill premium during the late 19th century. Others, including, e.g., Feinstein (1998) and Allen (2009) argued that real wages stagnated while output per worker expanded during the early phases of the economic transition. The explanation for this was seen in technical change increasing the demand for physical capital, which led to an intermediate increase in the capital income share, which was ultimately followed by an increase in wages. Recent evidence by Humphries and Weisdorf (2019) complements this by showing that the labor share exhibited a temporary minimum during the onset of the growth take-off. These observations have raised doubt about the compatibility of the evidence with unified growth models in which an increase in the demand for skilled labor is typically seen as the major driver behind the onset of the demographic transition and the associated increase in education. At the same time, there is evidence that suggests that education was an important factor during all phases of industrialization, (see, e.g., Becker et al., 2011).



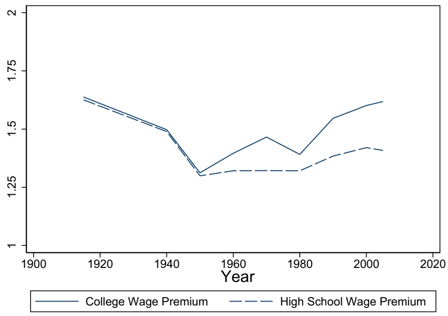
(a) Capital p.c. – Malthusian Phase **(b)** Capital p.c. – Post-Transition

Fig. 6 The expansion of physical capital—England. Data: **a** Broadberry and de Pleijt (2021), Figure 4b (fixed capital excl. dwellings, own digitization). **b** Penn World Tables (Feenstra et al., 2015, version 9.1). Model: fixed capital per capita

As shown before, the model dynamics exhibit a sustained increase in human capital. Due to the complementarity between human capital level and the demand for human capital that results from capital-skill complementarity, the model does not predict a monotonic decline in the skill premium. This is illustrated in Fig. 7, which plots the earnings premium and the wage ratio of the skilled sector relative to the unskilled sector for a re-calibrated version of the model in which the education transition begins in 1920 to have a direct comparison with the college and high school premium for the US reported by Goldin and Katz (2007a). While there is a slight difference in the levels, the overall pattern is strikingly similar. In particular, during the early phases of the transition, the skill premium declines. Despite this, the supply of skilled labor continues to increase, as shown in Fig. 5, and the skill premium even rebounds during the later phases. While in principle, this is consistent with a race between education and technology as discussed by Goldin and Katz (2007a), the model dynamics are also related to the accumulation of physical capital during this period and the associated feedback loop to education.

A related question concerns the dynamics of the income shares of labor and capital. Historically, the capital share exhibited a moderate upward trend. Figure 8 illustrates that the model is consistent with this observation. Panel (a) plots the dynamics of the capital income share for the baseline economy and the corresponding data for England. Underlying the increase in the capital share is a continuing decline in the overall labor income share, which is predominantly due to the decline of the labor income share from the unskilled sector. In fact, the skilled labor share increases despite the decline of the overall labor share. Moreover, this decline exhibits a moderate reversal during the early phase of the demographic transition, consistent with a temporary minimum during the onset of the growth take-off in Britain reported by Humphries and Weisdorf (2019). This is the result of general equilibrium effects. Once the education expansion sets in, the labor income share increases temporarily, leading to a temporary decline of the capital share during the early phase of the transition.

Panel (b) of Fig. 8 shows the dynamics of the returns to capital implied by the model. The real return exhibits a moderate decline prior to the onset of the transition, and a modest increase thereafter. When accounting for the variation in productivity (by normalizing the dynamics of the real return relative to income dynamics), the real return exhibits a steady



(a) College/High School Wage Premium (Data, USA)

(b) Skill/Earnings Premium (Model)

Fig. 7 The dynamics of the skill premium. Data: **a** Goldin and Katz (2007b), Figure 1, Table A8.1 (own digitization). **b** Baseline model, recalibrated for an education transition in 1920 (corresponding to the USA)

decline. These dynamics are consistent with the overall pattern found in the data.[14] Again this is the result of general equilibrium effects that lead to capital deepening. In addition, the model also predicts a secular decline in the return of capital relative to the return on (skilled or unskilled) labor as consequence of continued capital accumulation. This complements arguments related to a savings glut that have been made in the recent secular stagnation debate (see, e.g., Teulings and Baldwin, 2014, and OECD, 2015).

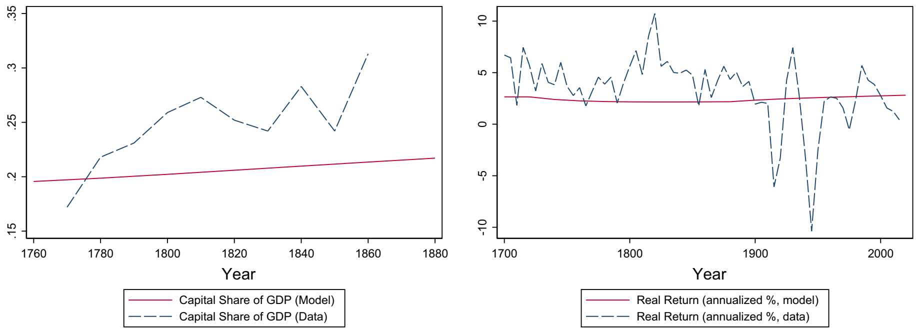
4 Accounting for comparative development

This section extends the baseline model to a multi-country version that allows exploring the implications for patterns of comparative development and cross-country growth studies.

4.1 A multi-country version of the model with technology diffusion

To extend the model, we construct a synthetic panel of countries. In particular, we simulate an artificial cross-country panel data set that consists of 114 countries that are identical in terms of all parameters and initial conditions. The only exception is initial skilled productivity A_0^S, which is assumed to differ for exogenous reasons. A lower A_0^S implies a lower return to production in the skilled sector and, consequently, a later demographic transition, *ceteris paribus*. To discipline the analysis, we calibrate the synthetic panel to match available data on the onset of the demographic transition in each country (following the classification by Reher, 2004). In the baseline parameterization of the model, A_0^S was set to generate a demographic transition in 1880. Figure 9a depicts the observed distribution of the onset of the fertility transition in the data.

[14] Figures. A.1 and A.2 in the Online Appendix show corresponding figures for capital returns when normalized by income levels, and when using data from alternative sources and for different subperiods, respectively.



(a) Capital Income Share **(b)** Capital Returns

Fig. 8 The dynamics of capital shares and real returns data: **a** England, Crafts (2021), Table 10. **b** Schmelzing (2020), Figure IV (global rate, GDP-weighted). Simulated data: (*a*) $(R \cdot K)/(Y^U + Y^S)$. (*b*) $r = (R^{1/20} - 1) \cdot 100$

To simulate additional countries, we solve (21) for A_0^S to match the observed distribution of dates of the onset of the demographic transition in the data for these countries. The baseline multi-country simulation assumes that every country follows the same development path as prescribed by the model. In addition, we conduct a simulation for the extended model that accounts for technology diffusion from the world frontier.

Figure 9b depicts the respective distribution of onset dates of the transition in the simulated sample. By construction, the simulated distribution matches the empirical distribution in Fig. 9a, with minor deviations stemming from the fact that in the simulation the dates are forced to correspond to twenty-year periods.[15]

To account for technology diffusion, we also simulate an extended version of the multi-country model that incorporates an adoption process in the spirit of Nelson and Phelps (1966). Suppose that a country can import the latest technology developed by the technologically most advanced country (corresponding to the world technology frontier in the context of the model), $\bar{A}_t^U$ and $\bar{A}_t^S$, but that the ability to import technology depends on human capital. In particular, consider an extension of the technology dynamics in (37) and (38), where education determines whether the technologies used in the two sectors at time t, A_t^U and A_t^S, are influenced by the world technology frontier $\bar{A}_t^U$ and $\bar{A}_t^S$,

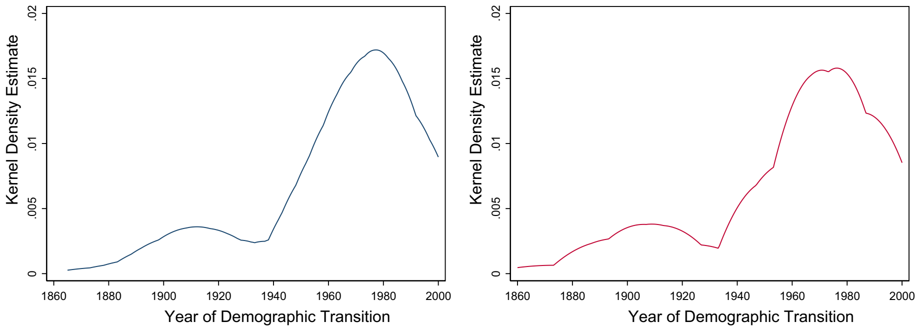
$$A_{t+1}^U = \left(1 + g_{t+1}^U\right)A_t^U + \rho\left[h\left(e_{t+1}\right) - h_{min}\right]\mathcal{F}\left(A_t^U, \bar{A}_t^U, \bar{A}_{t-1}^U, \bar{A}_{t-2}^U\right) \tag{39}$$

$$A_{t+1}^S = \left(1 + g_{t+1}^S\right)A_t^S + \rho\left[h\left(e_t\right) - h_{min}\right]\mathcal{F}\left(A_t^S, \bar{A}_t^S, \bar{A}_{t-1}^S, \bar{A}_{t-2}^U\right) \tag{40}$$

with $\rho > 0$ and $h_{min} = h(0)$.[16] The function $\mathcal{F}\left(A_t^i, \bar{A}_t^i, \bar{A}_{t-1}^i, \bar{A}_{t-2}^i\right)$, given by

[15] The coarseness of the OLG model with a new generation entering every 20 years implies that each of the simulated countries in the synthetic panel of 114 countries effectively belongs to one of eight groups of countries with transition dates between 1860 and 2000. Also notice that the timing of the onset of the demographic transition is identical in the baseline version of the multi-country model and in the extended version with technology diffusion.

[16] The general representation corresponds to widely used diffusion models, see, e.g., Benhabib and Spiegel (2005) and Stokey (2015).



(a) Data **(b)** Simulation

Fig. 9 Distribution of the onset of the transition. **a** Data: Reher (2004)

$$\mathcal{F} = \left[(\bar{A}_t^i - A_t^i) \cdot \frac{\min\left(\bar{A}_{t-1}^i - A_t^i,\ 0\right)}{\bar{A}_{t-1}^i - A_t^i} + \max\left(\bar{A}_{t-1}^i - A_t^i,\ 0\right) \cdot \frac{\min\left(\bar{A}_{t-2}^i - A_t^i,\ 0\right)}{\bar{A}_{t-2}^i - A_t^i} + \max\left(\bar{A}_{t-2}^i - A_t^i,\ 0\right) \right]$$

for $i \in \{U;S\}$ captures the decrease in adoption lags reported by Comin and Mestieri (2018).[17] The parameter ρ determines the level of convergence. In the data the ratio of average GDP per capita in the $G7$ between 1980 and 2000 relative to the GDP per capita of the richest country in the $G7$ in the same time span is around 0.825. In the simulation, we set $\rho = 0.265$, which implies the same convergence for the $G7$ in the simulation.

4.2 The role of the timing of the transition

To illustrate the consequences of heterogeneity in the timing of the transition, consider two economies that are identical to the baseline economy in all dimensions. Both economies only differ with respect to one characteristic, namely the initial productivity in the skilled sector (A_0^S). In particular, the first of these two economies is the baseline economy with an onset of the demographic transition in 1880. The second economy has a relatively lower productivity in skilled production and correspondingly faces a delayed onset of the transition in 1980.

Figure 10 plots the trajectories for population and output per capita growth for these two economies. In the baseline model, both economies exhibit the same dynamics, although on a different time scale. The delay in the demographic transition entails a delay in income per capita growth. This delay is seen also in the extended model that allows for technology spillovers and in which follower countries are able to import technology from the frontier. This import implies steeper productivity gains the larger the gap to the world frontier at the time of the transition. Consequently, the trajectories of population and output per

[17] This specification stipulates that economies that are sufficiently close to the world technology frontier (with productivity exceeding that at the frontier a generation earlier) adopt directly from the frontier, whereas economies further from the frontier first have to catch up by adopting technologies that represented the frontier one or two generations earlier. Note that divergence in intensity of use, as in Comin and Mestieri (2018), would be captured by country-specific factors either affecting the level of education or the parameter ρ.

capita growth differ after the onset of the transition, with the latecomer country exhibiting a sharper growth take-off and higher growth rates.

The shaded area corresponds to the observation window of 1950 – 2010, which covers the usual sample period in empirical growth studies. Observing both economies only during this limited observation period clearly delivers an incomplete and possibly misleading picture of the development path. Moreover, obviously the (implicit or explicit) assumption of a balanced growth path with different growth rates is not warranted during this observation period. Reduced-form estimates are thus likely to be heavily influenced by the observation period for which they are conducted, and the sample on which they are performed. Empirical results obtained with a sample of countries that are mainly observed during the pre-transitional period of their long-run development trajectory are likely to differ from results obtained with a sample of countries that are mainly post-transitional. Notice that this is true even when the maintained assumption about parameter stability across countries is satisfied, since in both the baseline and the extended version of the model the only difference between the two economies is in initial conditions, not structural parameters. We will return to this issue below.

4.3 Accounting for the empirical patterns of comparative development

We are now in a position to assess whether the extended model can account for the empirical patterns of comparative development across countries.

4.3.1 Cross-country differences in income and productivity

The positive association between differences in total factor productivity and income differences across countries constitutes another established empirical fact (Hall and Jones, 1999, Jones and Romer, 2010). In addition, physical capital per worker correlates positively with income per capita while at the same time the returns to physical capital are very similar across countries, despite substantial differences in economic development. This suggests that lower capital ratios in less developed countries are associated with lower levels of complementary factors like human capital (Caselli and Feyrer, 2007). This observation also raised a discussion about the reasons for persistent differences in productivity and the failure of global technology diffusion to foster income convergence. In fact, work by Comin and Hobijn (2010) has shown that a significant part of the cross-country income differences is accounted for by variation in the adoption of technologies. More recently, work by Comin and Mestieri (2018) has documented that a significant part of cross-country income divergence during the 19th century can be traced back to delays in the adoption of new technologies, while the cross-country income divergence during the 20th century is associated with a divergence in the intensity of use of new technologies. Paralleling the arguments explaining persistent differences in capital intensity, a candidate explanation for this pattern is the lack of complementary factors needed for technology adoption, such as human capital (e.g., Nelson and Phelps, 1966, Zeira, 2009).

Large income and productivity differences across countries are a natural feature of the cross-country version of the model. These differences are primarily the result of differences in the timing of the onset of the demographic transition. As a direct implication of the acceleration of productivity growth due to the expansion of human capital and the associated human capital externalities, the gap in income and productivity across countries

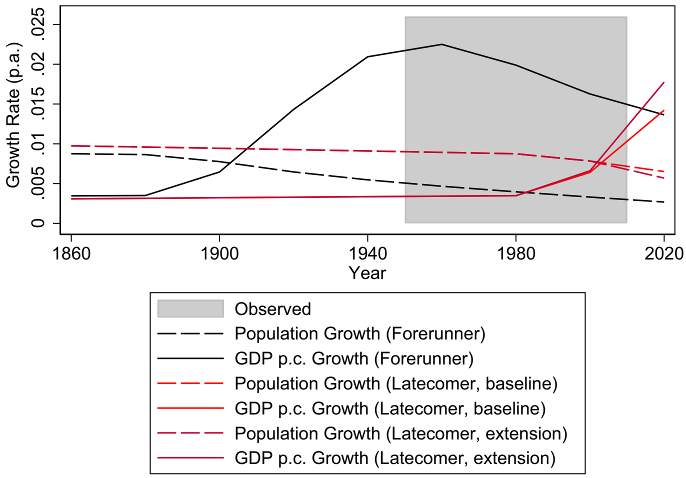


Fig. 10 Simulated development path of two economies

initially increases. Hence, the technological frontier displays a sharp acceleration when the forerunner country undergoes the economic and demographic transition.

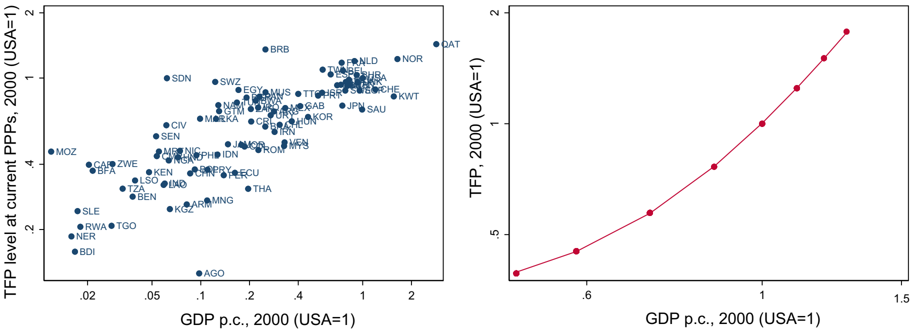
Figure 11 shows the relationship between comparative development differences in incomes and productivity relative to the US. In the simulation, the US is treated as a country with an education transition in 1920. For the cross-section in the year 2000, the figure demonstrates the model's ability to reflect the well-known positive correlation between income and productivity. However, in the model, income and productivity differences are crucially related to delays in the demographic transition.[18]

4.3.2 Cross-country differences in capital intensity and capital returns

Figure 12 depicts the comparative development patterns related to capital intensity per worker. The data in Panel (a) show a strong positive association of capital intensity with economic development. This pattern also emerges in the model simulation, as depicted in Panel (b). It is important to notice in this context that the simulated data reflect the relation along the development trajectory. This provides support for the conjecture that the comparative development patterns are consistent with the non-linear dynamics of a unified growth model.

Finally, Fig. 13 depicts the cross-sectional relationship between income and the marginal product of capital. Panel (a) shows that the marginal product of capital is fairly uncorrelated with output per capita, with rich countries even displaying a moderately higher marginal product on average than the poor countries. Panel (b) documents that the model is also able to generate this fact. Despite the larger capital intensity, the return to capital is non-declining as consequence of the complementarity between physical capital and human capital in the model. This is in line with earlier conjectures that poorer countries exhibit

[18] For the reasons discussed in footnote 15, the effective variation when plotting cross-sectional differences across the 114 simulated model economies corresponds to the differences between eight distinct groups of countries that differ with respect to the onset of their demographic transition.



(a) Data **(b)** Model

Fig. 11 Comparative differences in income and TFP. **a** Data: Penn World Tables (Feenstra et al., 2015, version 9.1). **b** Model: baseline, artificial cross-country panel, evaluated in 2000

lower capital-labor ratios and lower returns to capital as consequence of the low levels of complementary factors such as human capital (see, e.g., Caselli and Feyrer, 2007).

4.3.3 Growth variability and distance to technology frontier

The previous discussion is also relevant for a last stylized fact of long-run growth, according to which the variability of growth in incomes per capita increases with the distance to the world technology frontier (Jones and Romer, 2010). According to this observation, economies that have undergone substantial economic development also exhibit lower variation in growth than countries that are still in the early phases of the development process.

The model also captures the observation that the variation in the growth rate of income per capita is higher the greater the distance to the technology frontier. Intuitively, the fact that growth rates exhibit greater variation among the less developed countries is related to the observation that these countries are either in the early phases of their development process and correspondingly experience low but slowly accelerating growth rates, or they are undergoing the economic and demographic transition with the associated acceleration and overshooting of growth rates above the balanced growth path. The extension of the model to technology diffusion implies an additional momentum that contributes to faster growth as consequence of technology diffusion relative to the frontier economy.

One way to illustrate this is by relating growth rates to relative incomes as shown in Fig. 14. In particular, Panel (a) plots growth rates conditional on relative incomes compared to the frontier for a panel of countries over the period 1950 to 2010 for data from the Penn World Tables. Panel (b) reveals a very similar hump-shaped relation for the corresponding simulated data.[19] The similarities provide support for the conjecture that the frontier country grows faster than pre-transitional countries, whereas countries that are undergoing the transition experience the most rapid growth. These results also complement empirical work that has documented an interaction between education and the distance to the world technology frontier (see, e.g., Madsen, 2014).

[19] For the extension to technology diffusion, the qualitative pattern looks similar, see Fig. A.3 in the "Online Appendix".

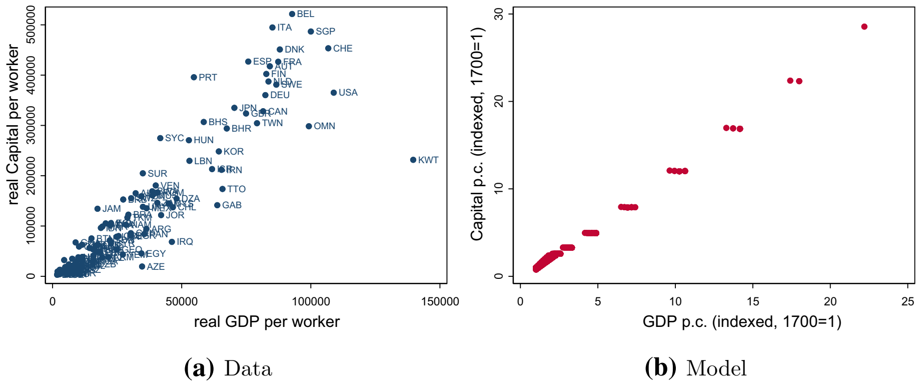


Fig. 12 Comparative development: capital intensity per worker. **a** Data: Penn World Tables (Feenstra et al., 2015, version 9.1). **b** Model: baseline, artificial cross-country panel

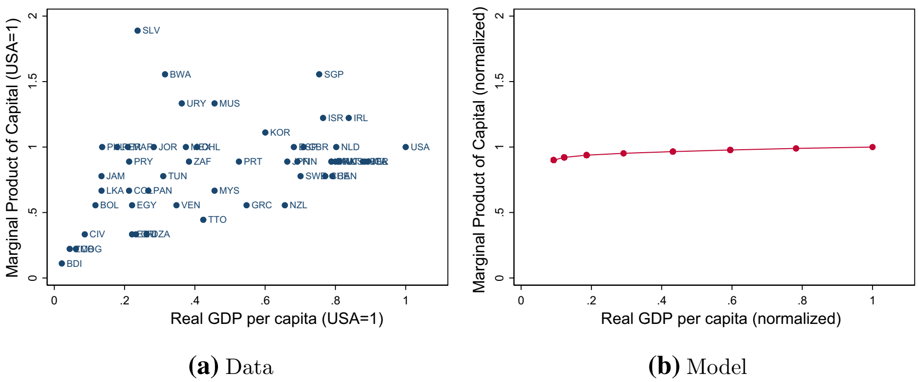
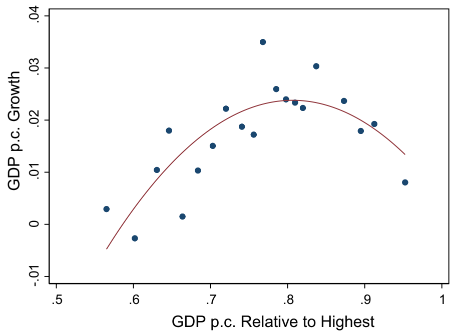


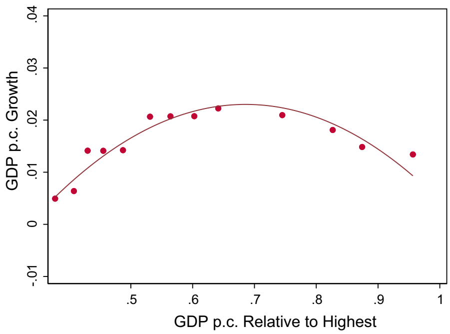
Fig. 13 Comparative differences in the marginal product of capital. **a** Data: Caselli and Feyrer (2007, Table II). Returns after correction for natural capital and price differences (*PMPKL*). **b** Model: baseline, artificial cross-country panel, evaluated in 2000, normalized to forerunner country

5 Implications for cross-country growth empirics

To further explore the comparative development patterns generated by the model and the resulting implications for the interpretation of empirical observations, the following analysis considers its empirical relevance through the lens of reduced-form regressions. In particular, we investigate whether regressions based on synthetic simulated cross-country panel data deliver similar patterns as regressions based on actual data. In this context it should be noted that the simulated data consist of economies that only differ in terms of an initial condition that determines the timing of their economic and demographic transition, but that are otherwise identical.



(a) Data



(b) Model

Fig. 14 Growth relative to frontier: data versus simulation. Note: Binscatter plots, lines correspond to a quadratic fit (both panels). Annualized growth rates over twenty-year periods. **a** Data: Penn World Tables (Feenstra et al., 2015, version 9.1, years 1950, 1970, 1990, 2010.). **b** Model: baseline, artificial cross-country panel, evaluated in 1960, 1980, 2000, 2020

5.1 Data and empirical specification

The empirical analysis is based on cross-country panel data for income, physical capital, and human capital (all in per capita terms) from the Penn World Tables version 9.1 by Feenstra et al. (2015). In these data, physical capital stocks are computed using the perpetual inventory method based on investment data taken from national accounts. Human capital is measured as an index based on average years of schooling from Barro and Lee (2013) and Cohen and Leker (2014). The estimation is conducted for a sample of 114 countries over the period 1950 – 2010, using twenty-year panel data in order to be consistent with the simulation.

We compare the model to the data through the lens of a standard reduced-form empirical growth framework. The reduced-form estimation framework follows the conventional specification used in reduced-form empirical work. In particular, the empirical framework for country i at time t is given by

$$\ln\left(\frac{Y_{i,t}}{N_{i,t}}\right) = a_1 \ln\left(\frac{Y_{i,t-1}}{N_{i,t-1}}\right) + a_2 \ln\left(\frac{K_{i,t}}{L_{i,t-1}}\right) + a_3 h_{i,t} + \mu_i + b_t + u_{i,t}$$

where $\frac{Y_{i,t}}{N_{i,t}}$ corresponds to income per capita, $\frac{K_{i,t}}{L_{i,t-1}}$ is the capital endowment per worker, and $h_{i,t}$ is an index measure of average individual human capital. To account for time-invariant differences across countries that might influence the level of development, such as the initial productivity in the skilled sector, the estimation includes a full set of country fixed effects μ_i. Additionally, to capture time-variant, global dynamics such as the world technology frontier the estimation also includes a full set of common time dummies b_t. The variables a_1, a_2, and a_3 are coefficients to be estimated.

To explore whether the estimation reveals the mechanics behind the development dynamics, in particular the crucial role of the demographic transition for the relevance of human capital, we also estimate an extended specification of the empirical model that accounts for heterogeneity in the coefficients of interest before and after the onset of the demographic transition. In the data, we use the same classification of transition dates by Reher (2004) as in the construction of the synthetic panel discussed above.

5.2 Accounting for cross-country growth empirics

Table 2 presents the estimates. Columns (1) and (2) show the respective estimation results for the empirical data organized as twenty-year panel and for a specification with only physical capital, human capital, a lagged dependent variable and two-way fixed effects, thus exploiting variation within countries. Columns (3) and (4) present the corresponding results for the synthetic panel of simulated countries using the baseline specification of the model without technology diffusion. Columns (5) and (6) show the corresponding results for the extended model with technology diffusion.

Several aspects are noteworthy. First, the data estimates reveal coefficients for physical capital and human capital that roughly correspond to the usual estimates in terms of an elasticity for physical capital and human capital of 0.47 and 0.5 - 0.6, depending on the specification. When allowing for heterogeneous effects of physical capital and human capital before and after the onset of the demographic transition in column (2), the data estimates reveal little evidence for any asymmetry in physical capital. For human capital, on the other hand, the results only document a significant positive effect post-demographic transition.

The corresponding estimates for the simulated data reveal a striking similarity to the estimates obtained with actual data. In particular, regardless of whether one considers the baseline model or the extension, the estimates obtained from the synthetic panel are qualitatively and quantitatively comparable. The effect of physical capital reveals no heterogeneity before and after the onset of the demographic transition, whereas, by construction, human capital only reveals variation that allows for identifying an effect after the transition. The effect of human capital in the most parsimonious specification is slightly smaller in size, but for more extended specifications that account for non-linear dynamics, the overall pattern becomes very comparable. In addition to the similarity in coefficient estimates, the results also reveal a similarity in the relative contributions of the different variables in terms of explanatory power.[20]

Taken together, this evidence suggests that despite the non-linear influence of the demographic transition, the simulated data reveal estimation results that resemble the familiar patterns from analyses in growth empirics. The results are also consistent with a mechanism that works through an acceleration in the accumulation of human capital, which is influenced by the demographic transition as well as by the associated dynamics in the productivity environment, thereby reflecting insights that have been expressed by Hsieh and Klenow (2010). It is worth emphasizing, however, that the striking similarity has been obtained with a synthetic panel of simulated data in which cross-country variation only emerges through the (country-specific and time-invariant) initial condition of skilled productivity, A_0^S. The results show that an empirical specification with two-way fixed effects does not account appropriately for development differences that manifest themselves in delays of non-linear transition dynamics. However, even though the reduced-form empirical estimation model does not explicitly account for the non-linear dynamics featured by the theoretical model, the estimates are very similar. Moreover, the synthetic data have not

[20] See Table A.1 in the Online Appendix for details.

Table 2 Reduced-form estimates reconsidered: data versus simulated data

	Dependent variable: Log [Income p/c]					
	Data 20-year panel		Baseline model		Extension	
	(1)	(2)	(3)	(4)	(5)	(6)
GDP p.c. ($t-1$)	0.090	0.059	−0.059*	0.082***	−0.024	0.14***
	(0.07)	(0.07)	(0.03)	(0.00)	(0.02)	(0.04)
k	0.47***		0.79***		0.72***	
	(0.07)		(0.04)		(0.02)	
h	0.54**		0.22***		0.46***	
	(0.25)		(0.03)		(0.02)	
k (pre-transition)		0.48***		0.67***		0.59***
		(0.07)		(0.00)		(0.04)
k (post-transition)		0.46***		0.60***		0.50***
		(0.07)		(0.00)		(0.05)
h (pre-transition)		0.49		0		0
		(0.73)		(.)		(.)
h (post-transition)		0.60**		0.36***		0.62***
		(0.25)		(0.00)		(0.04)
Year and Country FE	Yes	Yes	Yes	Yes	Yes	Yes
Observations	269	269	342	342	342	342
R^2 (within)	0.78	0.79	1.00	1.00	1.00	1.00

Results of fixed effects regressions. Columns (1) and (2): The dependent variable is (the log of) real GDP at constant 2011 national prices (in mil. 2011 US $) divided by population size (in millions), k is (the log of) capital stock at constant 2011 national prices (in mil. 2011 US $) divided by the number of persons employed (in millions), h is a logged human capital index, normalized to lie between 0 and 1; for details see data description of the PWT 9.1. Sample period: twenty-year panel $\{1950, 1970, ..., 2010\}$. Columns (3)-(6): The dependent variable is (the log of) income per capita, k is (the log of) capital endowment per worker, and h is constructed as an index from the (logged) human capital measure normalized to lie between 0 and 1, replicating the index in the data. Sample period: twenty-year panel $\{1960, 1980, ..., 2020\}$. All specifications include a full set of country fixed effects and common time dummies. Standard errors are clustered at the country level. *, **, *** denote significance at 10%, 5%, and 1% levels

been calibrated in any way to match the estimates. Instead, they were obtained exclusively by matching the transition dates with an otherwise identical parameterization.

In sum, these results can be understood as a proof of concept regarding the suitability of a unified growth model as a data generating process, which, despite the highly non-linear development dynamics, delivers reduced-form estimates that resemble those obtained with empirical data. The evidence is consistent with the conjecture of the demographic transition as a significant turning point in a country's economic development, and with the hypothesis of an acceleration of growth within countries after the onset of the transition. The results demonstrate that the non-linearity of the long-run development trajectory need not necessarily lead to unstable coefficient estimates, which is reassuring for the existing empirical growth literature that is based, at least implicitly, on the notion of a balanced growth path. At the same time, however, the model indicates that comparative development differences can be traced to differences in the timing of the transition, which are themselves the consequence of differences in initial conditions. Moreover, even accounting for technology diffusion across countries does not affect this conclusion. The results therefore illustrate the empirical compatibility of unified growth theories with existing work on growth accounting, while documenting the important role of the timing of the demographic transition for long-run development.

5.3 Implications for technology diffusion

The results of the simulation of the extended model with cross-country technology diffusion also have implications for the interpretation of unified growth frameworks in the context of contemporaneous comparative development patterns. As illustrated in the last section, the incorporation of technology diffusion across countries does not affect the empirical relevance or the qualitative fit of the model in the context of a synthetic panel analysis. In fact, even the quantitative fit of the model is not adversely affected by the incorporation of diffusion in terms of a positive ρ.[21]

The discussion of the central role of the demographic transition for the patterns of comparative development raises additional implications in the context of technology diffusion. In particular, the formulation of the diffusion process suggests that the education expansion might play an important role for the adoption and diffusion of technologies that has not been fully appreciated. Considering the diffusion process as specified in (39) and (40) suggests that the level of education, and hence the timing of the demographic transition, might be an important factor in explaining adoption lags and diverging intensity of technology use.

To illustrate this implication, we replicate the analysis of Comin and Mestieri (2018) and extend their specification to allow for heterogeneity in the timing of the demographic transition to affect delays in technology adoption and the intensity of use. The data contains information about the diffusion of 25 major technologies across countries in a variety of sectors (transportation, communication and IT, industrial, agricultural, and medical sectors). The diffusion of these technologies covers a broad set of countries and the invention dates of these technologies are fairly evenly distributed over the past 200 years, with more than half of the technologies in the data being invented during the 19th century. The analysis focuses on two elements: the lag with which a particular technology is adopted by a country after the technology has first been invented and the intensity with which the technology is used.[22]

The results for technology-country pairs as unit of observation are shown in Table 3. Columns (1) and (4) replicate the main results of Comin and Mestrieri (2018, Table 3) and show that adoption lags decline for later invention dates, while also the intensity of use declined, respectively. In line with the hypothesis, the results in columns (2) and (3) also document that adoption lags are larger for countries that exhibit a later demographic transition.[23] Similarly, the results in columns (5) and (6) show that the intensity of use is systematically lower in countries that exhibit a later onset of the demographic transition. Hence, countries with later transitions exhibit a significantly greater adoption lag and a significantly lower intensity of use, above and beyond the trend relative to the invention dates.

[21] For illustration purposes, we presented a version with no diffusion that corresponds to a simulation of the baseline model ($\rho = 0$) and an extension of the model with diffusion ($\rho = 0.265$), respectively.

[22] Both variables are in logs, see Comin and Mestieri (2018) for details on the construction and coding.

[23] In fact, unreported results reveal that the adoption lags are significantly smaller for countries that had already undergone the demographic transition at the time of invention, while this effect is weaker the later the invention date, as reflected by a positive interaction term.

In other words, the adoption occurs quicker and the intensity of use of adopted technologies is higher in countries that experienced an earlier demographic transition relative to the time of the appearance of new technologies.[24] Moreover, accounting for the demographic transition implies a substantial increase in the explanatory power of the model (in terms of R^2), particularly when considering the intensity of use. In other words, the timing of the demographic transition is an important, yet so far largely neglected, factor in the process of technology diffusion.

Replicating the same estimation on the basis of the synthetic panel data obtained with the model extension to technology diffusion delivers qualitatively identical and quantitatively fairly similar results.[25] This is notable since none of the parameters of the model was calibrated on this dimension of the model.

6 Conclusion

In this article, we developed a comprehensive unified growth framework that is able to replicate quantitatively the stylized facts of long-run growth dynamics and of comparative development. The research contributes to the literature by developing an expanded unified growth model that incorporates endogenous savings and a capital-skill complementarity. This model not only performs well in terms of matching various historical facts of long-run growth dynamics, including ones that have been documented recently, but can also be extended to a multi-country setting by incorporating technology diffusion from the world technology frontier in order to lend insights for comparative development and cross-country growth empirics. The expanded unified growth model accounts for the long-run trajectories of income, physical capital, population, and human capital and their association with the transition from stagnation to growth. The model thereby sheds light on the mechanisms that link the empirical facts of long-run growth and comparative development to each other.[26]

On the quantitative side, the paper documents that a parsimoniously parameterized quantitative version of the model that is calibrated to population and income dynamics in England during the 18th and 19th century is able to account for a large set of disparate empirical regularities. An extension to a multi-country setting in which countries only differ in the timing of the demographic transition is able to account for patterns of comparative development. Analyzing the model performance through the lens of the standard reduced-form empirical growth approach documents that, despite the highly non-linear dynamics and the out-of-sample nature of this exercise, the regression results closely resemble those obtained with standard cross-country panel data. Moreover, model simulations show that the consideration of international technology diffusion does not affect the qualitative results regarding the model dynamics. However, the

[24] Table A.2 in the "Online Appendix" shows that the main result also holds when splitting the sample into Western and Non-Western countries, as in the analysis of Comin and Mestieri (2018).

[25] See Table A.3 in the "Online Appendix".

[26] In this sense, the paper accomplishes a long-standing quest in the growth literature, delivering a comprehensive growth framework that fits the stylized facts and that captures "the endogenous accumulation of, and interaction between, (...) four state variables: ideas, population, and human capital", in addition to physical capital, as requested by Jones and Romer (2010) at the end of their survey of the facts of long-run growth (Jones and Romer, 2010, p. 226).

Table 3 Technology diffusion

Dependent variable	Log [Adoption lag]			Log [Intensity]		
	(1)	(2)	(3)	(4)	(5)	(6)
Invention year-1820	−0.011***	−0.0097***	−0.0096***	−0.0016**	−0.0039***	−0.0038***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Year of demogr. Tr.		0.0074***			−0.015***	
		(0.00)			(0.00)	
Follower country			0.51***			−0.35*
			(0.11)			(0.20)
Trailer country			0.32***			−0.94***
			(0.08)			(0.13)
Latecomer country			0.68***			−1.32***
			(0.08)			(0.19)
Observations	1009	1009	1009	1039	1039	1039
R^2	0.41	0.47	0.48	0.01	0.36	0.38

Columns (1) and (4) replicate the results in Table 3, columns (1) and (4) of Comin and Mestieri (2018). Country labels are as in Reher (2004): Forerunner countries (reference category) experienced the fertility transition before 1935, Followers between 1935 and 1964, Trailers between 1965 and 1979, and Latecomers after 1980. Robust standard errors clustered at the country level are in parentheses. Each observation is re-weighted so that each technology carries equal weight. *, **, *** denote significance at 10%, 5%, and 1% levels

extension has implications for explaining the recently reported regularities regarding the lags and intensity with which new technologies are adopted.

The results of this article provide support for the hypothesis that the unified growth approach is able to rationalize the stylized facts of long-run growth. The findings suggest that the timing of the demographic transition is a critical watershed for long-run growth dynamics that affects patterns contained in cross-country panel data. Consistent with the implications from the unified growth literature, the results also suggest that countries that experience an earlier take-off from stagnation to growth are richer and develop faster. Nevertheless, the momentum of the transition eventually fades out, which provides an explanation for the observed slow-down in growth among advanced economies.

The findings suggest that the usefulness of the insights of unified growth theory for the empirical growth literature have not been adequately appreciated. The intention of this paper was to go beyond the standard methodology of illustrative simulations of the non-linear long-run dynamics of a unified growth model, and use the model instead as a data generating process to study the facts of long-run growth. More work is needed to explore the implications of heterogeneity in the timing of the demographic transition as a determinant of comparative development patterns. In light of the results shown in this paper, this aspect has not received sufficient attention in the existing literature. The results for the extension to technology diffusion indicate the relevance of unified growth mechanisms for understanding the patterns of technology adoption, but more work is needed to assess the quantitative importance of accounting for cross-country spillovers. A more structural approach to other types of spillovers, e.g., related to capital mobility, labor mobility, or trade in intermediate or final goods constitutes an interesting direction for future work.

Appendix

Derivation of generic growth rates

Unskilled sector

Growth in the unskilled sector is given by

$$1 + g_{t+1}^{Y_U} = \frac{Y_{t+1}^U}{Y_t^U}$$

Updating (2) by one period and using (13) gives

$$\begin{aligned} 1 + g_{t+1}^{Y_U} =& \frac{L_{t+1}^{\alpha}\left(A_{t+1}^U X\right)^{1-\alpha}}{L_t^{\alpha}\left(A_t^U X\right)^{1-\alpha}} = \frac{\left(n_t L_t\right)^{\alpha}\left(A_{t+1}^U X\right)^{1-\alpha}}{L_t^{\alpha}\left(A_t^U X\right)^{1-\alpha}} \\ =& n_t^{\alpha}(1 + g_{t+1}^{A_U})^{1-\alpha} \end{aligned}$$

Skilled sector

Growth in the skilled sector is given by

$$1 + g_{t+1}^{Y_S} = \frac{Y_{t+1}^S}{Y_t^S}$$

Updating (3) by one period and using (13), (14), and (15) gives

$$\begin{aligned} 1 + g_{t+1}^{Y_S} =& \frac{H_{t+1}^{\beta}\left(A_{t+1}^S K_{t+1}\right)^{1-\beta}}{H_t^{\beta}\left(A_t^S K_t\right)^{1-\beta}} = \frac{\left(h\left(e_{t+1}\right)L_t\right)^{\beta}\left(A_{t+1}^S s_t L_t\right)^{1-\beta}}{\left(h\left(e_t\right)L_{t-1}\right)^{\beta}\left(A_t^S s_{t-1} L_{t-1}\right)^{1-\beta}} \\ =& \frac{n_{t-1}L_{t-1}}{L_{t-1}}\left(\frac{h\left(e_{t+1}\right)}{h\left(e_t\right)}\right)^{\beta}\left(\frac{A_{t+1}^S}{A_t^S}\right)^{1-\beta}\left(\frac{s_t}{s_{t-1}}\right)^{1-\beta} \\ =& n_{t-1}(1 + g_{t+1}^h)^{\beta}(1 + g_{t+1}^{A_S})^{1-\beta}(1 + g_t^s)^{1-\beta} \end{aligned}$$

Population

Population growth is given by

$$1 + g_{t+1}^N = \frac{N_{t+1}}{N_t}$$

Substituting (16) and using (13) gives

$$1+g^N_{t+1}=\frac{L_t+L_{t+1}}{L_{t-1}+L_t}=\frac{(1+n_t)L_t}{(1+n_{t-1})L_{t-1}}=\frac{(1+n_t)n_{t-1}L_{t-1}}{(1+n_{t-1})L_{t-1}}=\frac{n_{t-1}(1+n_t)}{1+n_{t-1}}$$

Output per capita

Output per capita growth is given by

$$1+g^y_{t+1}=\frac{y_{t+1}}{y_t}\equiv\frac{\frac{Y_{t+1}}{N_{t+1}}}{\frac{Y_t}{N_t}}$$

Re-arranging and using (1) gives

$$1+g^y_{t+1}=\frac{Y_{t+1}}{Y_t}\frac{N_t}{N_{t+1}}=\frac{Y^U_{t+1}+Y^S_{t+1}}{Y_t}\frac{N_t}{N_{t+1}}=\left(\frac{Y^U_{t+1}}{Y^U_t}\frac{Y^U_t}{Y_t}+\frac{Y^S_{t+1}}{Y^S_t}\frac{Y^S_t}{Y_t}\right)\frac{N_t}{N_{t+1}}$$

Using $\frac{Y^U_{t+1}}{Y^U_t}=1+g^{Y_U}_{t+1}$, $\frac{Y^S_{t+1}}{Y^S_t}=1+g^{Y_S}_{t+1}$, $\theta_t\equiv\frac{Y^S_t}{Y_t}$, and $\frac{N_{t+1}}{N_t}=1+g^N_{t+1}$ yields

$$1+g^y_{t+1}=\frac{(1+g^{Y_U}_{t+1})(1-\theta_t)+(1+g^{Y_S}_{t+1})\theta_t}{1+g^N_{t+1}}$$

Growth during the phases of development

Malthusian phase

Notice that since $s_{t+1}>0$ and $n_{t+1}>0$ it must hold generically that

$$\frac{s_{t+1}}{s_t}+\frac{n_{t+1}}{n_t}>\frac{s_{t+1}+n_{t+1}}{s_t+n_t}=\frac{w^L_{t+1}}{w^L_t}$$

where the last equality follows from (7).

To show the claim that savings dynamics depend on the relative sizes of fertility and productivity growth, consider the limit case in which $s_t\to w^L_t$ so that all productivity gains are absorbed by savings and not fertility. Combining (4) and (13), we have

$$\frac{s_{t+1}}{s_t}=\left(\frac{1+g^{A_U}_{t+1}}{n_t}\right)^{1-\alpha}$$

To derive the bound for growth in income per capita, note that combining this condition with $\frac{n_{t+1}}{n_t}=1$ and using (26), (27), as well as (28) delivers the level of fertility consistent with a Malthusian steady state, denoted by $\bar{n}_M$, as

$$\bar{n}_M=\frac{1+g^{A_U}_{t+1}}{(1+g^{A_S}_{t+2})^{\frac{1-\beta}{\beta(1-\alpha)}}}=\frac{\phi_U}{\phi_S^{\frac{1-\beta}{\beta(1-\alpha)}}}$$

In this Malthusian steady state, productivity growth, wage growth, and savings growth exactly balance each other, implying constant fertility.

The per capita growth rate of the unskilled sector is given by

$$\frac{1+g_{t+1}^{Y_U}}{1+g_{t+1}^{N}}=\frac{n_t^{\alpha}(1+g_{t+1}^{A_U})^{1-\alpha}}{1+g_{t+1}^{N}}$$

Inserting (24) yields

$$\frac{1+g_{t+1}^{Y_U}}{1+g_{t+1}^{N}}=n_t^{\alpha}(1+g_{t+1}^{A_U})^{1-\alpha}\frac{1+n_{t-1}}{n_{t-1}(1+n_t)}$$

In the Malthusian steady state $n_{t-1}=n_t=\bar{n}_M=\frac{\phi_U}{\phi_S^{\frac{1-\beta}{\beta(1-\alpha)}}}$ and $1+g_{t+1}^{A_U}=\phi_U$, therefore

$$\frac{1+g_{t+1}^{Y_U}}{1+g_{t+1}^{N}}=\left(\frac{1+g_{t+1}^{A_U}}{\bar{n}_M}\right)^{1-\alpha}=\left(\frac{\phi_S^{\frac{1-\beta}{\beta(1-\alpha)}}}{\phi_U}\phi_U\right)^{1-\alpha}=\phi_S^{\frac{1-\beta}{\beta}}$$

Conversely, the per capita growth rate of the skilled sector is given by

$$\frac{1+g_{t+1}^{Y_S}}{1+g_{t+1}^{N}}=\frac{n_{t-1}(1+g_{t+1}^{h})^{\beta}(1+g_{t+1}^{A_S})^{1-\beta}(1+g_t^{s})^{1-\beta}}{1+g_{t+1}^{N}}$$

Inserting (24) yields

$$\frac{1+g_{t+1}^{Y_S}}{1+g_{t+1}^{N}}=n_{t-1}(1+g_{t+1}^{h})^{\beta}(1+g_{t+1}^{A_S})^{1-\beta}(1+g_t^{s})^{1-\beta}\frac{1+n_{t-1}}{n_{t-1}(1+n_t)}$$

Combining $n_{t-1}=n_t=n_{t+1}=\bar{n}_M$ with $\frac{s_{t+1}}{s_t}=\left(\frac{1+g_{t+1}^{A_U}}{n_t}\right)^{1-\alpha}$ and $1+g_{t+1}^{A_U}=\phi_U$ gives $1+g_t^{s}=\frac{s_t}{s_{t-1}}=\phi_S^{\frac{1-\beta}{\beta}}$ in the Malthusian steady state. Additionally, in the Malthusian steady state $1+g_{t+1}^{h}=1$, $n_{t-1}=n_t=\bar{n}_M$, $1+g_{t+1}^{A_S}=\phi_S$, therefore

$$\frac{1+g_{t+1}^{Y_S}}{1+g_{t+1}^{N}}=\phi_S^{1-\beta}\phi_S^{\frac{(1-\beta)^2}{\beta}}=\phi_S^{\frac{1-\beta}{\beta}}$$

Combining the results gives the growth rate of output per capita in the Malthusian steady state as

$$\begin{aligned}1+g_{t+1}^{y}&=\frac{(1+g_{t+1}^{Y_U})(1-\theta_t)+(1+g_{t+1}^{Y_S})\theta_t}{1+g_{t+1}^{N}}\\&=\frac{1+g_{t+1}^{Y_U}}{1+g_{t+1}^{N}}(1-\theta_t)+\frac{1+g_{t+1}^{Y_S}}{1+g_{t+1}^{N}}\theta_t\\1+g_{t+1}^{y_M}&=\phi_S^{\frac{1-\beta}{\beta}}\end{aligned}$$

Finally, notice that the limit case that $s_t \rightarrow w_t^L$ only applies asymptotically as $A^U \rightarrow \infty$ and $A^S \rightarrow \infty$. This implies that the expressions derived so far constitute an upper bound for savings and a lower bound for fertility. Hence, the expression of growth of income per capita also constitutes an upper bound for growth during the Malthusian phase.

Demographic transition

The per capita growth rate of the unskilled sector is given by

$$\frac{1+g_{t+1}^{Y_U}}{1+g_{t+1}^{N}} = \frac{n_t^{\alpha}(1+g_{t+1}^{A_U})^{1-\alpha}}{1+g_{t+1}^{N}}$$

Inserting (24) and re-arranging yields

$$\frac{1+g_{t+1}^{Y_U}}{1+g_{t+1}^{N}} = \left(\frac{1+g_{t+1}^{A_U}}{n_t}\right)^{1-\alpha} \frac{n_t}{n_{t-1}} \frac{1+n_{t-1}}{(1+n_t)}$$

Note that $\frac{n_t}{n_{t-1}} \frac{1+n_{t-1}}{1+n_t}$ is non-decreasing as long as $n_t \leq n_{t-1}$, i.e., as long as fertility is decreasing or stable over time. If the exit from the Malthusian steady state to the demographic transition occurs in period t the expression becomes

$$\frac{1+g_{t+1}^{Y_U}}{1+g_{t+1}^{N}} = n_t^{\alpha}(1+g_{t+1}^{A_U})^{1-\alpha} \frac{1+\bar{n}_M}{\bar{n}_M(1+n_t)}$$

Inserting $\bar{n}_M = \frac{\phi_U}{\phi_S^{\frac{1-\beta}{\beta(1-\alpha)}}}$, $1+g_{t+1}^{A_U} = \phi_U$, and re-arranging yields

$$\frac{1+g_{t+1}^{Y_U}}{1+g_{t+1}^{N}} = \left(\frac{n_t}{\bar{n}_M}\right)^{\alpha} \frac{1+\bar{n}_M}{1+n_t} \phi_S^{\frac{1-\beta}{\beta}}$$

Note that $\left(\frac{n_t}{\bar{n}_M}\right)^{\alpha} \frac{1+\bar{n}_M}{1+n_t} > 1$ for $n_t < \bar{n}_M$. As a result

$$\frac{1+g_{t+1}^{Y_U}}{1+g_{t+1}^{N}} > \phi_S^{\frac{1-\beta}{\beta}}$$

as long as $n_t \leq n_{t-1}$.

Conversely, the per capita growth rate of the skilled sector is given by

$$\frac{1+g_{t+1}^{Y_S}}{1+g_{t+1}^{N}} = n_{t-1}(1+g_{t+1}^{h})^{\beta}(1+g_{t+1}^{A_S})^{1-\beta}(1+g_t^{s})^{1-\beta}$$

where the growth rate of savings is now given by substituting (19), (4), and using (13)

$$1+g_t^s = \frac{\frac{1-\beta}{\beta}\frac{h(e_{t+1})}{h'(e_{t+1})}w_t^L}{\frac{1-\beta}{\beta}\frac{h(e_t)}{h'(e_t)}w_{t-1}^L} = \frac{h(e_{t+1})}{h(e_t)} \frac{h'(e_t)}{h'(e_{t+1})} \frac{w_t^L}{w_{t-1}^L} = \frac{1+g_{t+1}^{h}}{1+g_{t+1}^{h'}}\left(\frac{1+g_t^{A_U}}{n_{t-1}}\right)^{1-\alpha}$$

Combining the results gives per capita growth in the skilled sector as

$$\frac{1+g_{t+1}^{Y_S}}{1+g_{t+1}^{N}} = \frac{n_{t-1}^{1-(1-\alpha)(1-\beta)}\frac{1+g_{t+1}^{h}}{(1+g_{t+1}^{h'})^{1-\beta}}(1+g_{t+1}^{A_S})^{1-\beta}(1+g_t^{A_U})^{(1-\alpha)(1-\beta)}}{1+g_{t+1}^{N}}$$

Inserting (24) and re-arranging yields

$$\frac{1+g_{t+1}^{Y_S}}{1+g_{t+1}^{N}} = (1+g_{t+1}^{A_S})^{1-\beta}\frac{1+g_{t+1}^{h}}{(1+g_{t+1}^{h'})^{1-\beta}}\left(\frac{1+g_t^{A_U}}{n_{t-1}}\right)^{(1-\alpha)(1-\beta)}\frac{1+n_{t-1}}{1+n_t}$$

Note that $\frac{1+n_{t-1}}{1+n_t}$ is non-decreasing as long as $n_t \leq n_{t-1}$, i.e., as long as fertility is decreasing or stable over time. If the exit from the Malthusian steady state to the demographic transition occurs in period t the expression becomes

$$\frac{1+g_{t+1}^{Y_S}}{1+g_{t+1}^{N}} = (1+g_{t+1}^{A_S})^{1-\beta}\frac{1+g_{t+1}^{h}}{(1+g_{t+1}^{h'})^{1-\beta}}\left(\frac{1+g_t^{A_U}}{\bar{n}_M}\right)^{(1-\alpha)(1-\beta)}\frac{1+\bar{n}_M}{1+n_t}$$

Inserting $\bar{n}_M = \frac{\phi_U}{\phi_S^{\frac{1-\beta}{\beta(1-\alpha)}}}$, $1+g_t^{A_U} = \phi_U$, $1+g_{t+1}^{A_S} = \phi_S$, and re-arranging yields

$$\frac{1+g_{t+1}^{Y_S}}{1+g_{t+1}^{N}} = \frac{1+g_{t+1}^{h}}{(1+g_{t+1}^{h'})^{1-\beta}}\frac{1+\bar{n}_M}{1+n_t}\phi_S^{\frac{1-\beta}{\beta}}$$

After the onset of the demographic transition $g_{t+1}^{h} > 0$. Additionally, $\frac{1+\bar{n}_M}{1+n_t} > 1$ for $n_t < \bar{n}_M$. Lastly, note that $1+g_{t+1}^{h'}$ is non-increasing as e_{t+1} increases for any non-convex human capital production function. As a result

$$\frac{1+g_{t+1}^{Y_S}}{1+g_{t+1}^{N}} > \phi_S^{\frac{1-\beta}{\beta}}$$

as long as $n_t \leq n_{t-1}$ and $e_{t+1} \geq e_t$.

Balanced growth path

Given $e_{t+1} > 0$, re-arranging (21) yields

$$\frac{1-\beta}{\beta}\frac{h\left(e_{t+1}\right)}{h'\left(e_{t+1}\right)} = (1-e_{t+1}) - \frac{\gamma}{w_{t+1}^{H}}\frac{1}{h'\left(e_{t+1}\right)}$$

Inserting the result into (20) gives optimal fertility as

$$n_t = \frac{\gamma}{w_{t+1}^{H}}\frac{w_t^{L}}{h'\left(e_{t+1}\right)}$$

A balanced growth path with stable population is given by $\frac{n_{t+1}}{n_t} = 1$, that is

$$\frac{n_{t+1}}{n_t} = 1 = \frac{\gamma}{w^H_{t+2}} \frac{w^L_{t+1}}{h'\left(e_{t+2}\right)} \frac{w^H_{t+1}}{\gamma} \frac{h'\left(e_{t+1}\right)}{w^L_t} = \frac{w^H_{t+1}}{w^H_{t+2}} \frac{w^L_{t+1}}{w^L_t} \frac{h'\left(e_{t+1}\right)}{h'\left(e_{t+2}\right)}$$

Inserting factor prices and re-arranging yields

$$\begin{aligned}\frac{n_{t+1}}{n_t} = 1 = &\left(\frac{1}{1+g^{A_S}_{t+2}}\right)^{1-\beta} \left(\frac{h'\left(e_{t+2}\right)}{h'\left(e_{t+1}\right)}\right)^{1-\beta} \\ &\left(\frac{n_t}{1+g^{A_U}_{t+1}}\right)^{(1-\alpha)(1-\beta)} \left(\frac{1+g^{A_U}_{t+1}}{n_t}\right)^{(1-\alpha)} \frac{h'\left(e_{t+1}\right)}{h'\left(e_{t+2}\right)} \\ = &\left(\frac{1}{1+g^{A_S}_{t+2}}\right)^{1-\beta} \left(\frac{1+g^{A_U}_{t+1}}{n_t}\right)^{\beta(1-\alpha)} (1+g^{h'}_{t+2})^{\beta}\end{aligned}$$

Inserting $1+g^{A_U}_{t+1} = \phi_U$, $1+g^{A_S}_{t+2} = \phi_S$, evaluating the condition at $\{e_{t+1};e_{t+2}\} \rightarrow e_{max}$, and solving for n_t yields

$$\bar{n}_{BGP} \rightarrow \frac{\phi_U}{\phi_S^{\frac{1-\beta}{\beta(1-\alpha)}}}$$

The per capita growth rate of the unskilled sector is given by

$$\frac{1+g^{Y_U}_{t+1}}{1+g^N_{t+1}} = \left(\frac{1+g^{A_U}_{t+1}}{n_t}\right)^{1-\alpha} \frac{n_t}{n_{t-1}} \frac{1+n_{t-1}}{(1+n_t)}$$

Evaluating the expression at $n_{t-1} = n_t = \bar{n}_{BGP} \rightarrow \frac{\phi_U}{\phi_S^{\frac{1-\beta}{\beta(1-\alpha)}}}$, $1+g^{A_U}_{t+1} = \phi_U$, and simplifying yields

$$\frac{1+g^{Y_U}_{t+1}}{1+g^N_{t+1}} \rightarrow \phi_S^{\frac{1-\beta}{\beta}}$$

Conversely, the per capita growth rate of the skilled sector is given by

$$\frac{1+g^{Y_S}_{t+1}}{1+g^N_{t+1}} = (1+g^{A_S}_{t+1})^{1-\beta} \frac{1+g^h_{t+1}}{(1+g^{h'}_{t+1})^{1-\beta}} \left(\frac{1+g^{A_U}_t}{n_{t-1}}\right)^{(1-\alpha)(1-\beta)} \frac{1+n_{t-1}}{1+n_t}$$

Evaluating the expression at $n_{t-1} = n_t = \bar{n}_{BGP} \rightarrow \frac{\phi_U}{\phi_S^{\frac{1-\beta}{\beta(1-\alpha)}}}$, $1+g^{A_U}_t = \phi_U$, $1+g^{A_S}_{t+1} = \phi_S$, $1+g^h_{t+1} \rightarrow 1, 1+g^{h'}_{t+1} \rightarrow 1$, and simplifying yields

$$\frac{1+g^{Y_S}_{t+1}}{1+g^N_{t+1}} \rightarrow \phi_S^{\frac{1-\beta}{\beta}}$$

Combining the results gives the growth rate of output per capita on the balanced growth path as

$$1 + g^{y}_{t+1} = \frac{(1 + g^{Y_U}_{t+1})(1 - \theta_t) + (1 + g^{Y_S}_{t+1})\theta_t}{1 + g^{N}_{t+1}} = \frac{1 + g^{Y_U}_{t+1}}{1 + g^{N}_{t+1}}(1 - \theta_t) + \frac{1 + g^{Y_S}_{t+1}}{1 + g^{N}_{t+1}}\theta_t$$

$$1 + g^{y_{BGP}}_{t+1} \rightarrow \phi_S^{\frac{1-\beta}{\beta}}$$

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